CHEEGER INEQUALITIES FOR GRAPH LIMITS

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Abstract. We introduce notions of Cheeger constants for graphons and graphings. We prove Cheeger and Buser inequalities for these. On the way we prove co-area formulae for graphons and graphings.

Contents

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1. Introduction

The Cheeger constant, introduced in Riemannian geometry by Cheeger [Che70] in the early 70's measures the 'most efficient' way to cut a closed Riemannian manifold into two pieces, where efficiency is measured in terms of an isoperimetric constant. Cheeger [Che70] and Buser related this geometric quantity to a spectral quantity–the second eigenvalue of the Laplacian. These are the well-known Cheeger-Buser inequalities in Riemannian geometry (see [Bus10, Section 8.3] for details). A discrete version of the Cheeger constant and the Cheeger-Buser inequalities was then obtained independently by Dodziuk [Dod84] and Alon-Milman [Alo86, AM85] for finite graphs (see [Chu10] for a number of different proofs and [Moh91] for a survey). These ideas and inequalities have also been extended to weighted graphs [FN02] (see also [Chu97, Chapter 2, pg. 24], [Tre11]). In a certain sense, this marked a fertile way of discretizing a notion that arose in the setup of continuous geometry.

More recently, the theory of graph limits, graphons and graphings was developed by Lovasz [Lov12] and others (see especially [BCL$^+$08, BCL$^+$12, BCKL13, Cha17]) giving a method of obtaining measured continua from infinite sequences of finite graphs. From a certain point of view, this gives us a path in the opposite direction: from the discrete to the continuous.

Such continuous limits come in two flavors: dense graphs (graphons) or sparse graphs (graphings). A graphon is relatively easy to describe: it is a bounded (Lebesgue) measurable function $W : I^2 \to I$ that is symmetric: $W(x, y) = W(y, x)$ for all $x, y \in I$. A graphing on the other hand may be thought of as a measure on I^2 that can be locally described as a product of a sub-probability measure on I with the counting measure on a set of uniformly bounded cardinality (see Sections 2.1 and 2.2 for details). Each of the co-ordinate intervals $I \times \{0\}$ and $\{0\} \times I$ may be thought of as the vertex set of the graphon or graphing and is equipped with a Borel measure.

The aim of this paper is to define the notion of a Cheeger constant for graphons and graphings and prove the Cheeger-Buser inequalities for them. For both a graphon W and a graphing G, the Cheeger constants $h(W)$ and $h(G)$ respectively measure (as in Cheeger's original definition) the best way to partition the "vertex set" I into A, A^c such that the isoperimetric constant is minimized. For instance for a graphon W,

$$h(W) = \inf_{A\subseteq I:\ 0<\mu_L(A)<1} \frac{e_W(A, A^c)}{\min\{\mathrm{vol}_W(A), \mathrm{vol}_W(A^c)\}}, \tag{1.1}$$

where $e_W(A, A^c)$ measures the total measure of edges between A, A^c (see Definitions 3.1 and 2.2 below for details). A rather different Cheeger-type inequality for graphings (but not for graphons) involving von Neumann algebras was explored by Elek in [Ele08].

The main theorem of the paper is the following (see Theorems 3.5, 3.7, 4.4 and 4.6):

Theorem 1.1. *Let W be a connected graphon and $\lambda(W)$ denote the least positive eigenvalue of the Laplacian. Then*

$$\frac{h^2(W)}{8} \le \lambda(W) \le 2h(W).$$

Again, let G be a connected graphing and $\lambda(G)$ denote the least positive eigenvalue of the Laplacian. Then

$$\frac{h^2(G)}{8} \leq \lambda(G) \leq 2h(G).$$

Connectedness in the hypothesis of Theorem 1.1 above is a mild technical restriction to ensure that the Cheeger constant is well-defined. The classical Cheeger-Buser inequalities for finite graphs can be obtained (modulo a factor of 4) as an immediate consequence of Theorem 1.1 for graphings using the following canonical graphing that corresponds to a finite graph.

For any finite connected graph F on $\{1, \ldots, n\}$ as the vertex set, we can define a graphing $G = (I, E, \mu)$ as follows: Let $v_i = (2i-1)/2n$ for $1 \leq i \leq n$. Define E as

$$E = \{(v_i, v_j) : \ \{i, j\} \text{ is an edge in } F\}$$

Define μ as $\mu(v_i) = 1/n$ for each i. Thus $\mu(B) = 0$ for all Borel sets which do not contain any of the v_i's. It is easy to check that G is connected, and the Cheeger constant of G and the Cheeger constant for F are equal. The same is true for $\lambda(G)$ and $\lambda(F)$. So we get

$$\frac{h^2(F)}{8} \leq \lambda(F) \leq 2h(F)$$

as a special case of Theorem 1.1 for graphings.

We have stated Theorem 1.1 in the form above to demonstrate the fact that the statements for graphons and graphings are essentially identical. In fact, once the preliminaries about graphons and graphings are dealt with in Section 2, the proof of Theorem 1.1 in the two cases of graphons and graphings follows essentially the same formal route. Thus, though structurally graphons and graphings are quite dissimilar, the proofs of the Cheeger-Buser inequalities have striking parallels. This is quite unlike some of the other spectral theorems exposed in [Lov12] (see in particular the differences in approach in [BCL$^+$08, BCL$^+$12, BCKL13]).

To emphasize this formal similarity of proof-strategy in the two cases, Sections 3 and 4 have been structured in an identical manner. In both cases, we use the formalism and language of differential forms and define the Laplacian $\Delta = d^*d$ on functions after proving that the "exterior derivative" d is continuous. This is adequate for Buser's inequality (Theorems 3.5 and 4.4). The proof of the Cheeger's inequality part of Theorem 1.1 we then furnish (Theorems 3.7 and 4.6) adapts Cheeger's original idea from [Che70]. Thus, we prove **co-area formulae** in the two settings of graphons and graphings (see Theorems 3.6 and 4.5). This might be of independent interest.

Finally, in Section 5 we investigate connectivity. For a finite graph F it is clear that the Cheeger constant of F is positive if and only if F is connected. This is equivalent, via the Cheeger-Buser inequality for finite graphs, to the statement that a graph is connected if and only if the normalized Laplacian has a one dimensional eigenspace corresponding to the zero (lowest) eigenvalue. The analogous statement is not true for either graphons or graphings. We furnish counterexamples in Sections 5.1 and 5.2 respectively. However, for graphons whose degree is bounded away from zero, we prove the following equivalence (see Proposition 5.7):

Proposition 1.2. *Let $\varepsilon > 0$ and W be a graphon such that $d_W(x) \geq \varepsilon$ for all $x \in I$. Then W is connected if and only if $h(W) > 0$.*

We provide two proofs of this theorem, one of which uses the Cheeger-Buser inequality for graphons from Theorem 1.1 and the other a structural lemma about connected graphons proved in [BBCR10].

2. Preliminaries

In this section, we summarize general facts about graphons and graphings that we shall need in the paper. Most of the material is from the book by Lovasz [Lov12], but in the subsection on graphings below we deduce a few elementary consequences as well as a slightly different perspective from that in [Lov12].

2.1. **Preliminaries on graphons.** We summarize the relevant material from [Lov12, Chapter 7]. Let I denote the unit interval $[0,1]$ and μ_L denote the Lebesgue measure on I. A function $W : I^2 \to I$ is said to be a **graphon** if W is measurable and symmetric, that is, $W(x,y) = W(y,x)$ for all $x, y \in I$. Given a graphon W, we define for each $x \in I$ the **degree** of x as

$$d_W(x) = \int_0^1 W(x,y)\ dy \tag{2.1}$$

For two measurable subsets A and B of I, we define

$$e_W(A,B) = \int_{A\times B} W \tag{2.2}$$

Thus, $e_W(A,B)$ is the total weight of edges between A and B. For a measurable subset A of I, the **volume** of W over A is defined as

$$\mathrm{vol}_W(A) = \int_{A\times I} W = e_W(A,I) \tag{2.3}$$

Thus, $\mathrm{vol}_W(A)$ measures the total weight of edges emanating from A.

A graphon is said to be **connected** if for all measurable subsets A of I with $0 < \mu_L(A) < 1$ we have $e_W(A, A^c) \neq 0$. Note that if W is connected then $d_W > 0$ a.e.

2.2. **Preliminaries on graphings.** Let I denote the unit interval $[0,1]$.

Definition 2.1. [Lov12, Chapter 18] *A **bounded degree Borel graph** on I is a pair (I,E), where E is a symmetric measurable subset of I^2 such that there is a positive integer D satisfying*

$$|\{y \in I :\ (x,y) \in E\}| \leq D \tag{2.4}$$

for all $x \in I$.

In other words, the number of neighbors of each point in I is at most D. Given a bounded degree Borel graph (I,E), we have a **degree function** $\deg : I \to \mathbf{R}$ defined as

$$\deg(x) = |\{y \in I :\ (x,y) \in E\}| \tag{2.5}$$

For any measurable subset A of I we define $\deg_A : I \to \mathbf{R}$ as

$$\deg_A(x) = |\{y \in A :\ (x,y) \in E\}| \tag{2.6}$$

It is proved in [Lov12, Lemma 18.4] that the map $\deg_A$ is a measurable function for any measurable set $A \subseteq I$. Note that deg is nothing but $\deg_I$.

Definition 2.2. [Lov12, Chapter 18] *A **graphing** is a triple $G = (I, \mu, E)$ such that (I, E) is a bounded degree Borel graph, and μ is a probability measure on I such that*

$$\int_A \deg_B(x)\ d\mu(x) = \int_B \deg_A(x)\ d\mu(x) \tag{2.7}$$

for all measurable subsets A and B of I.

Given a graphing $G = (I, \mu, E)$, the measure μ allows us to define a measure η on I^2 as follows. For each measurable rectangle $A \times B \subseteq I^2$, we define

$$\eta(A \times B) = \int_A \deg_B(x)\ d\mu(x).$$

Equation 2.7 ensures that $\eta(A \times B) = \eta(B \times A)$. By Caratheodory extension, we get a measure η on the Borel σ-algebra of I^2. As proved in [Lov12, Lemma 18.14], the measure η is concentrated on E.

A fundamental result proved in [Lov12, Theorem 18.21] is that every graphing can be decomposed as a disjoint union of finitely many graphings, each having degree $\deg(x)$ bounded by 1 for all x. More precisely,

Theorem 2.3. *Let $G = (I, \mu, E)$ be a graphing. Then there exist measurable subsets $A_1, \dots, A_k \subseteq I$ and μ-measure preserving involutions $\varphi_i : A_i \to A_i$ such that*

$$E = \bigsqcup_{i=1}^{k} \{(x, \varphi_i(x)) :\ x \in A_i\} \tag{2.8}$$

We can pictorially represent a graphing $G = (I, \mu, E)$ by drawing the edge set E of G in the unit square. Each subset $\{(x, \varphi_i(x)) :\ x \in A_i\}$ can be thought of as a "strand" in I^2. Thus the previous theorem allows us to think of a graphing as a disjoint union of strands in the unit square. When the degree bound of a graphing is 1, we may say that the graphing consists of a *single strand.*

The measure η counts the number of edges in any measurable subset of $S \subseteq I^2$. When $S = A \times B$ is a rectangle, we count the number of strands in S each vertical line cuts, and integrate this count against $d\mu$. This is immediate from the definition of η. This extends to arbitrary S, as the following lemma shows.

Lemma 2.4. *Let S be any measurable subset of I^2. Then*

$$\eta(S) = \int_I \sum_y \chi_{E \cap S}(x, y)\ d\mu(x) \tag{2.9}$$

Proof. First let us see why the integral on the RHS makes sense. Using Theorem 2.3 we know that there exist μ-measure preserving involutions $\varphi_i : A_i \to A_i$, $i = 1, \dots, k$, for some measurable subsets A_i of I, such that

$$E = \bigsqcup_{i=1}^{k} \{(x, \varphi_i(x)) :\ x \in A_i\} \tag{2.10}$$

Hence,

$$\sum_{y\in I}\chi_{E\cap S}(x,y)=\sum_{i=1}^{k}\chi_S(x,\varphi_i(x)) \tag{2.11}$$

Thus the integrand in the RHS of Equation 2.9 is a sum of finitely many non-negative measurable functions $I\to\mathbf{R}$ and therefore the RHS of Equation 2.9 is well-defined.

Let $\nu(S)$ be the RHS of Equation 2.4. Let us verify that ν is a measure on the Borel σ-algebra of I^2. So let $S=\bigsqcup_{j=1}^{\infty}S_j$ be a countable disjoint union of measurable sets. Then

$$\begin{aligned}\chi_S(x,\varphi_i(x))&=\sum_{j=1}^{\infty}\chi_{S_j}(x,\varphi_i(x))\\ \Rightarrow\sum_{i=1}^{k}\chi_S(x,\varphi_i(x))&=\sum_{i=1}^{k}\sum_{j=1}^{\infty}\chi_{S_j}(x,\varphi_i(x))=\sum_{j=1}^{\infty}\sum_{i=1}^{k}\chi_{S_j}(x,\varphi_i(x)) \qquad (2.12)\\ \Rightarrow\int_I\sum_{i=1}^{k}\chi_S(x,\varphi_i(x))&=\int_I\sum_{j=1}^{\infty}\sum_{i=1}^{k}\chi_{S_j}(x,\varphi_i(x))\ d\mu(x)\end{aligned}$$

This implies that

$$\begin{aligned}\nu(S)&=\int_I\lim_{n\to\infty}\sum_{j=1}^{n}\left(\sum_{i=1}^{k}\chi_{S_j}(x,\varphi_i(x))\right)d\mu(x)\\ &=\lim_{n\to\infty}\int_I\sum_{j=1}^{n}\left(\sum_{i=1}^{k}\chi_{S_j}(x,\varphi_i(x))\right)d\mu(x) \qquad (2.13)\\ &=\lim_{n\to\infty}\left[\sum_{j=1}^{n}\int_I\left(\sum_{i=1}^{k}\chi_{S_j}(x,\varphi_i(x))\right)d\mu(x)\right]\\ \Rightarrow\nu(S)&=\lim_{n\to\infty}\sum_{j=1}^{n}\nu(S_j)=\sum_{j=1}^{\infty}\nu(S_j)\end{aligned}$$

showing that ν is countably additive and is therefore a measure. Now let $S=A\times B$ be a measurable rectangle. Then

$$\begin{aligned}\nu(S)&=\int_I\sum_y\chi_{E\cap(A\times B)}(x,y)\ d\mu(x)\\ &=\int_A\sum_y\chi_{E\cap(I\times B)}(x,y)\ d\mu(x) \qquad (2.14)\\ &=\int_A\deg_B(x)\ d\mu(x)\\ &=\eta(A\times B)\end{aligned}$$

So ν agrees with η on the measurable rectangles. But since extension of a finitely additive and countably sub-additive measure on the algebra of measurable rectangles to the Borel

σ-algebra is unique (Caratheodory Extension Theorem), we must have that $\nu = \eta$ and we are done. ■

If we have a non-negative map $\psi : I^2 \to \mathbf{R}$, then by definition of integration we have that

$$\int_{I^2} \psi \; d\eta = \lim_{n\to\infty} \int_{I^2} \psi_n \; d\eta \tag{2.15}$$

where (ψ_n) is a sequence of non-negative simple functions such that $\psi_n \uparrow \psi$. This definition gives a theory of integration. We could define another theory of integration by declaring the integral of ψ to be equal to

$$\int_I \sum_{y\in I} \psi(x,y)\chi_E(x,y) \; d\mu(x) \tag{2.16}$$

By Lemma 2.4 these two theories agree on simple functions, and therefore are the same theories of integration. So for any $\psi \in L^1(I^2, \eta)$ we have

$$\int_{I^2} \psi(x,y) \; d\eta(x,y) = \int_I \sum_y \psi(x,y)\chi_E(x,y) \; d\mu(x) \tag{2.17}$$

3. Cheeger Constant for Graphons and the Cheeger-Buser Inequalities

In this section, we define the Cheeger constant for graphons and prove the Cheeger and Buser inequalities for graphons.

3.1. Cheeger Constant for Graphons.

Definition 3.1. *Given a graphon W, we define the **Cheeger constant** of W as*

$$h(W) = \inf_{A\subseteq I:\; 0<\mu_L(A)<1} \frac{e_W(A, A^c)}{\min\{\mathrm{vol}_W(A), \mathrm{vol}_W(A^c)\}} \tag{3.1}$$

It will be convenient to denote the quantity

$$\frac{e_W(A, A^c)}{\min\{\mathrm{vol}_W(A), \mathrm{vol}_W(A^c)\}} \tag{3.2}$$

*as $h_A(W)$. A symmetrized version of the above constant, which we call the **symmetric Cheeger constant** is defined as*

$$g(W) = \inf_{A\subseteq I:\; 0<\mu_L(A)<1} \frac{e_W(A, A^c)}{\mathrm{vol}_W(A)\,\mathrm{vol}_W(A^c)} \tag{3.3}$$

The analogue of $g(W)$ for finite graphs is called the *averaged minimal cut* in [FN02]. Note that the above defined constants exist for connected graphons.

3.2. **Definition of d, d*, and Laplacian.** Let W be a connected graphon. Define

$$E = \{(x, y) \in I^2 : \ y > x\}, \qquad E_W = \{(x, y) \in E : \ W(x, y) > 0\} \tag{3.4}$$

The set E can be thought of as an orientation of all the “edges”. The set E_W disregards the oriented edges which have zero weight. Define a measure ν on I as

$$\nu(A) = \int_A d_W(x) \ dx = \mathrm{vol}_W(A) \tag{3.5}$$

for all measurable subsets A of I. In other words, the Radon-Nikodym derivative of ν with respect to the Lebesgue measure is d_W. Clearly, ν is absolutely continuous with respect to the Lebesgue measure on I. The connectedness of W implies that the Lebesgue measure is also absolutely continuous with respect to ν. Thus we may talk about null sets in I unambiguously. This also says that $L^\infty(I, \nu) = L^\infty(I, \mu_L)$, and thus we write these simply as $L^\infty(I)$.

Similarly, define a measure η on E_W as

$$\eta(S) = \int_S W(x, y) \ dxdy \tag{3.6}$$

for all measurable subsets S of E_W. So the Radon-Nikodym derivative of η with respect to the Lebesgue measure is W. These measures give rise to Hilbert spaces $L^2(I, \nu)$ and $L^2(E_W, \eta)$, the inner products on which will be denoted by $\langle \cdot, \cdot \rangle_v$ and $\langle \cdot, \cdot \rangle_e$ respectively. Explicitly

$$\langle f, g \rangle_v = \int_0^1 f(x)g(x)d_W(x) \ dx \tag{3.7}$$

for all $f, g \in L^2(I, \nu)$, and

$$\langle \varphi, \psi \rangle_e = \int_{E_W} \varphi\psi W = \int_0^1 \int_x^1 \varphi(x, y)\psi(x, y)W(x, y) \ dydx \tag{3.8}$$

for all $\varphi, \psi \in L^2(E_W)$. The standard inner products on $L^2(I, \mu_L)$ and $L^2(I^2, \mu_L \otimes \mu_L)$ will be denoted by $\langle \cdot, \cdot \rangle_{L^2(I)}$ and $\langle \cdot, \cdot \rangle_{L^2(I^2)}$.

Define a map $d : L^2(I, \nu) \to L^2(E_W, \eta)$ as

$$(df)(x, y) = f(y) - f(x) \tag{3.9}$$

for all $f \in L^2(I, \nu)$. The map d can be thought of as a gradient which measures the change in f as we travel from the tail of an edge to the head. We need to check that df actually lands in $L^2(E_W, \eta)$ for any given member of $L^2(I, \nu)$. This and more is proved in the following lemma.

Lemma 3.2. *The map d is continuous.*

Proof. We want to show that d is bounded. Let $f \in L^2(I, \nu)$. Then

$$\begin{aligned}
\|df\|_e^2 &= \int_{E_W} (df)^2 W \\
&= \int_0^1 \int_x^1 (f(y) - f(x))^2 W(x,y)\ dydx \\
&\leq \int_0^1 \int_0^1 (f(y) - f(x))^2 W(x,y)\ dydx \\
&\leq \int_0^1 \int_0^1 f^2(y) W(x,y)\ dydx + \int_0^1 \int_0^1 f^2(x) W(x,y)\ dydx \\
&\qquad + 2\int_0^1 \int_0^1 |f(x)f(y)| W(x,y)\ dydx \\
&= 2\int_0^1 f^2(x) d_W(x)\ dx + 2\int_0^1 \int_0^1 |f(x)f(y)| W(x,y)\ dydx
\end{aligned} \tag{3.10}$$

The first term is the same as $2\|f\|_v^2$. So we need to bound the second term. Let $\alpha, \beta : I^2 \to \mathbf{R}$ be defined as

$$\alpha(x,y) = |f(x)|\sqrt{W(x,y)}, \qquad \beta(x,y) = |f(y)|\sqrt{W(x,y)} \tag{3.11}$$

The fact that $f \in L^2(I, \nu)$ implies that $\alpha, \beta \in L^2(I^2)$. Then we have

$$\int_0^1 \int_0^1 |f(x)f(y)| W(x,y)\ dydx = \langle \alpha, \beta \rangle_{L^2(I^2)} \tag{3.12}$$

But now by Cauchy-Schwarz inequality we have

$$\begin{aligned}
\langle \alpha, \beta \rangle_{L^2(I^2)} &\leq \|\alpha\|_{L^2(I^2)} \|\beta\|_{L^2(I^2)} \\
&= \left(\int_0^1 \int_0^1 f^2(x) W(x,y)\ dydx \right)^{1/2} \left(\int_0^1 \int_0^1 f^2(y) W(x,y)\ dxdy \right)^{1/2} \\
&= \|f\|_v^2
\end{aligned} \tag{3.13}$$

We conclude that $\|df\|_e \leq 2\|f\|_v$. This shows that d is continuous. ■

The above lemma shows that d^*, the adjoint[1] of d, exists. We now calculate it explicitly. Let $f \in L^2(I, \nu)$ and $\varphi \in L^2(E_W, \eta)$ be arbitrary. We have

[1]Here, again, the adjoint is taken with respect to the Hilbert space structure coming from $\langle \cdot, \cdot \rangle_v$ and $\langle \cdot, \cdot \rangle_e$.

(3.14)

$$\begin{aligned}
\langle df, \varphi \rangle_e &= \int_0^1 \int_x^1 df(x,y)\varphi(x,y)W(x,y)\ dydx \\
&= \int_0^1 \int_x^1 (f(y) - f(x))\varphi(x,y)W(x,y)\ dydx \\
&= \int_0^1 \int_x^1 f(y)\varphi(x,y)W(x,y)\ dydx - \int_0^1 \int_x^1 f(x)\varphi(x,y)W(x,y)\ dydx \\
&\overset{\text{Fubini}}{=} \int_0^1 \int_0^y f(y)\varphi(x,y)W(x,y)\ dxdy - \int_0^1 \int_x^1 f(x)\varphi(x,y)W(x,y)\ dydx \\
&= \int_0^1 \int_0^x f(x)\varphi(y,x)W(x,y)\ dydx - \int_0^1 \int_x^1 f(x)\varphi(x,y)W(x,y)\ dydx \\
&= \int_0^1 f(x) \left[\int_0^x \varphi(y,x)W(x,y)\ dy - \int_x^1 \varphi(x,y)W(x,y)\ dy \right] dx
\end{aligned}$$

On the other hand we have

$$\langle f, d^*\varphi \rangle_v = \int_0^1 f(x)d^*\varphi(x)d_W(x)\ dx \tag{3.15}$$

Thus we have

$$(d^*\varphi)(x) = \frac{1}{d_W(x)} \left[\int_0^x \varphi(y,x)W(x,y)\ dy - \int_x^1 \varphi(x,y)W(x,y)\ dy \right] \tag{3.16}$$

wherever $d_W(x) \neq 0$. We set $(d^*\varphi)(x) = 0$ if $d_W(x) = 0$.

Remark 3.3. *We have adapted the language of differential forms above so that we think of the map*

$$d : C^0(W) \to C^1(W)$$

as an exterior derivative from $0-$*forms (i.e. functions on the vertex set) to* $1-$*forms (i.e. functions on the set of directed edges). Then* d^* *is the adjoint map using the Hodge* $*$*:*

$$d^* : C^1(W) \to C^0(W).$$

Alternately, in the presence of inner products on both the vertex and edge-spaces (the situation here) we may think of d *as an analog of the gradient operator (*grad *or* ∇*) in classical vector calculus and* d^* *as an analog of the divergence operator* div.

Define the **Laplacian** of W as $\Delta_W = d^*d$. We may drop the subscript when there is no confusion. For $f \in L^2(I,\nu)$, we calculate $(\Delta_W f)(x)$.

$$\begin{aligned}(\Delta_W f)(x) &= (d^*df)(x)\\ &= \frac{1}{d_W(x)}\left[\int_0^x df(y,x)W(x,y)\ dy - \int_x^1 df(x,y)W(x,y)\ dy\right]\\ &= \frac{1}{d_W(x)}\left[\int_0^1 (f(x)-f(y))W(x,y)\ dy\right]\\ &= f(x) - \frac{1}{d_W(x)}T_W f(x)\end{aligned} \tag{3.17}$$

where $T_W : L^2(I,\nu) \to L^2(I,\nu)$ is a linear map defined as

$$(T_W f)(x) = \int_0^1 W(x,y)f(y)\ dy \tag{3.18}$$

The map T_W is well-defined. Indeed, the integral on the RHS of Equation 3.18 exists. To see this, let $\mathbb{1}$ denote the constant map $I \to \mathbf{R}$ which takes all points to 1. Then $\mathbb{1} \in L^2(I,\nu)$, and thus

$$\begin{aligned}\langle |f|, \mathbb{1}\rangle_v &= \int_0^1 |f(y)|d_W(y)\ dy < \infty\\ \Rightarrow \int_0^1 |f(y)|&\left[\int_0^1 W(x,y)\ dx\right] dy < \infty\\ \Rightarrow \int_0^1 &\left[\int_0^1 |f(y)|W(x,y)\ dy\right] dx < \infty\end{aligned} \tag{3.19}$$

Therefore $\int_0^1 |f(y)|W(x,y)\ dy$ is almost everywhere finite and consequently $(T_W f)(x)$ exists. It is also easy to check (using Cauchy-Schwarz) that $T_W f$ lies in $L^2(I,\nu)$. Therefore we have $\Delta_W = I - \frac{1}{d_W}T_W$.

3.3. **Buser Inequality for Graphons.** Let us see what is the multiplicity of the eigenvalue 0 of the Laplacian of a connected graphon W. For $f \in L^2(I,\nu)$, we have $\Delta f = 0$ if and only if $df = 0$. We claim that $df = 0$ if and only if f is constant (up to a set of measure zero). Clearly, if f is constant, then $df = 0$. Conversely, assume that $df = 0$. Thus

$$\int_{E_W} (df)^2 W = \int_E (df)^2 W = \int_0^1 \int_x^1 (f(y)-f(x))^2 W(x,y)\ dydx = 0 \tag{3.20}$$

which implies that

$$\int_0^1 \int_0^1 (f(y)-f(x))^2 W(x,y)\ dydx = 0 \tag{3.21}$$

For each $t \in \mathbf{R}$, let $S_t = f^{-1}(t,\infty)$. From the last equation we have

$$\int_{S_t^c \times S_t} (f(y)-f(x))^2 W(x,y)\ dydx = 0 \tag{3.22}$$

which implies that $(f(y)-f(x))^2W(x,y)$ is a.e. 0 on $S_t^c \times S_t$. But $f(y)-f(x) \neq 0$ for all $(x,y) \in S_t^c \times S_t$, which means that $W=0$ a.e. on $S_t^c \times S_t$. The connectedness of W then implies that either S_t or S_t^c has measure 0. So our claim follows from the following lemma.

Lemma 3.4. *Let $f : I \to \mathbf{R}$ be a measurable function such that for all $t \in \mathbf{R}$ we have either $f^{-1}(-\infty, t]$ or $f^{-1}(t,\infty)$ has measure 0. Then f is essentially constant.*

Proof. Let

$$t_0 = \inf\{t \in \mathbf{R} : \ f^{-1}(-\infty, t] \text{ is full measure}\} \tag{3.23}$$

Then $t_0 \neq -\infty$. This is because $I = \bigsqcup_{n \in \mathbf{Z}} f^{-1}(n, n+1]$. Thus $f^{-1}(n,n+1]$ has positive measure for some integer n, and this n cannot exceed t_0. Also, by definition of t_0, we have that $f^{-1}(-\infty, t_0 - 1/n]$ has measure 0 for each n. Thus $f^{-1}(-\infty, t_0)$ also has measure 0. Again, by definition of t_0 we have that $f^{-1}(t_0 + 1/n, \infty)$ has measure zero for all n, and thus $f^{-1}(t_0, \infty)$ has measure zero. So we conclude that $f^{-1}(t_0)$ has full measure. ■

So we have shown that the eigenfunctions corresponding to 0 are precisely the essentially constant functions. In other words, the eigenspace of Δ corresponding to 0 is generated by $\mathbb{1}$, the constant function taking value 1 everywhere. The **second smallest eigenvalue** denoted $\lambda(W)$ is therefore given by the following Rayleigh quotient:

$$\lambda(W) = \inf_{f \in \mathbb{1}_v^\perp, f \neq 0} \frac{\langle f, \Delta f \rangle_v}{\langle f, f \rangle_v} = \inf_{f \in \mathbb{1}_v^\perp, f \neq 0} \frac{\|df\|_e^2}{\|f\|_v^2} \tag{3.24}$$

(Here, $\mathbb{1}_v^\perp$ denotes the orthogonal complement of $\mathbb{1}$ with respect to the inner product $\langle \cdot, \cdot \rangle_v$).

Theorem 3.5 (Buser Inequality)**.** *Let W be a connected graphon. Then*

$$\lambda(W) \leq 2h(W) \text{ and } \lambda(W) \leq g(W) \tag{3.25}$$

Proof. We are now in a position to adapt the proof of Lemma 2.1 in [Chu97]. Let $A \sqcup B$ form a measurable partition of I with $0 < \mu_L(A) < 1$. Define $f : I \to \mathbf{R}$ as

$$f(x) = \begin{cases} \dfrac{1}{\mathrm{vol}(A)} & \text{if } x \in A \\ -\dfrac{1}{\mathrm{vol}(B)} & \text{if } x \in B \end{cases} \tag{3.26}$$

Then $f \in \mathbb{1}_v^\perp$. Now

$$
\begin{aligned}
\lambda(W) &\leq \frac{\|df\|_e^2}{\|f\|_v^2} \\
&= \frac{\int_E (f(x)-f(y))^2 W(x,y)\ dydx}{\int_0^1 f(x)^2 d_W(x)\ dx} \\
&= \frac{\int_{I\times I} (f(x)-f(y))^2 W(x,y)\ dydx}{2\int_0^1 f(x)^2 d_W(x)\ dx} \\
&= \frac{\int_{A\times B} (f(x)-f(y))^2 W(x,y)\ dydx + \int_{B\times A} (f(x)-f(y))^2 W(x,y)\ dydx}{2\left[\int_A f(x)^2 d_W(x)\ dx + \int_B f(x)^2 d_W(x)\ dx\right]} \qquad (3.27) \\
&= \frac{\left(\frac{1}{\mathrm{vol}(A)} + \frac{1}{\mathrm{vol}(B)}\right)^2 \left(\int_{A\times B} W(x,y)\ dxdy + \int_{B\times A} W(x,y)\ dxdy\right)}{2\left[\frac{1}{\mathrm{vol}(A)^2}\mathrm{vol}(A) + \frac{1}{\mathrm{vol}(B)^2}\mathrm{vol}(B)\right]} \\
&= \left(\frac{1}{\mathrm{vol}(A)} + \frac{1}{\mathrm{vol}(B)}\right)\int_{A\times B} W(x,y)\ dxdy \\
&\leq 2\frac{\int_{A\times B} W}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}
\end{aligned}
$$

Since $B = A^c$, and since the above holds for all choices of A with $0 < \mu_L(A) < 1$, we have $\lambda(W) \leq 2h(W)$. From the penultimate inequality above we also get

$$\lambda(W) \leq \frac{\int_{A\times B} W(x,y)\ dxdy}{\mathrm{vol}(A)\,\mathrm{vol}(B)} \qquad (3.28)$$

since $\mathrm{vol}(A) + \mathrm{vol}(B) \leq 1$. This leads to $\lambda(W) \leq g(W)$. ■

3.4. **The Co-area Formula for Graphons.** Consider a finite graph $G = (V, E)$ and let $f : V \to \mathbf{R}$ be any map. Orient the edges of G in such a way that for each oriented edge e we have $f(e^+) \geq f(e^-)$. Let $\gamma_0 < \gamma_1 < \cdots < \gamma_k$ be all the reals in the image of f. Define $S_i = \{v \in V : \ f(v) \geq \gamma_i\}$. Then we have

$$\sum_{e\in E} df(e) = \sum_{i=1}^{m} (\gamma_i - \gamma_{i-1})|E(S_i^c, S_i)| \qquad (3.29)$$

where $E(S_i^c, S_i)$ denotes the set of all the edges in G which have their tails in S_i^c and heads in S_i. To see why Equation 3.29 is true, we fix an edge e and see how much it contributes to the sum on the RHS. We add $\gamma_i - \gamma_{i-1}$ for each i such that $e^- \in S_i^c$ and $e^+ \in S_i$. This adds up to a total of $df(e)$, which is the same as the contribution of e to the LHS.

If G were a weighted graph with weight function $w : E \to \mathbf{R}^+$, Equation 3.29 takes the form

$$\sum_{e\in E} df(e)w(e) = \sum_{i=1}^{m} (\gamma_i - \gamma_{i-1})e_w(S_i^c, S_i) \qquad (3.30)$$

where $e_w(S_i^c, S_i)$ denotes the sum of weights of all the edges which have their tails in S_i^c and heads in S_i.

Let us see how Equation 3.30 generalizes for graphons. Let W be a graphon and $f : I \to \mathbf{R}$ be in $L^2(I, \nu)$. Define E_f to be the set $\{(x, y) \in I^2 : \ f(y) > f(x)\}$. Let S_t denote the set $f^{-1}(t, \infty)$. Then

$$\int_{E_f} df(x,y)W(x,y)\ dxdy = \int_{-\infty}^{\infty} e_W(S_t^c, S_t)\ dt \tag{3.31}$$

This can be easily proved using Fubini's theorem. We shall however need a slight variant of this formula in order to establish Cheeger's inequality.

Theorem 3.6 (Co-area formula for graphons)**.** *Let W be a graphon and $f : I \to \mathbf{R}$ be an arbitrary map in $L^2(I, \nu)$. Define $f_+ : I \to \mathbf{R}$ and $f_- : I \to \mathbf{R}$ as $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$. Let $S_t = f^{-1}(t, \infty)$. Then*

$$\begin{aligned} \int_{E_f} |df_+^2|W &= \int_0^{\infty} e_W(S_{\sqrt{t}}^c, S_{\sqrt{t}})\ dt = \int_0^{\infty} 2te_W(S_t^c, S_t)\ dt, \quad \text{and} \\ \int_{E_f} |df_-^2|W &= \int_0^{\infty} e_W(S_{-\sqrt{t}}^c, S_{-\sqrt{t}}) = \int_0^{\infty} 2te_W(S_{-t}^c, S_{-t})\ dt \end{aligned} \tag{3.32}$$

Proof. We prove the first one. The second one is similar. We have by change of variables that

$$\int_0^{\infty} e_W(S_{\sqrt{t}}^c, S_{\sqrt{t}})\ dt = \int_0^{\infty} 2te_W(S_t^c, S_t)\ dt \tag{3.33}$$

Now

$$\begin{aligned} \int_0^{\infty} 2te_W(S_t^c, S_t)\ dt &= \int_0^{\infty} 2t \left[\int_{S_t^c \times S_t} W(x,y)\ dxdy\right] dt \\ &= \int_0^{\infty} \left[\int_{I^2} 2t\chi_{S_t^c \times S_t}(x,y)W(x,y)\ dxdy\right] dt \\ &= \int_{I^2} \left[\int_0^{\infty} 2t\chi_{S_t^c \times S_t}(x,y)W(x,y)\ dt\right] dxdy \\ &= \int_{I^2} \left[\int_0^{\infty} 2t\chi_{S_t^c \times S_t}(x,y)\ dt\right] W(x,y)dxdy \\ &= \int_{E_f} \left[\int_0^{\infty} 2t\chi_{S_t^c \times S_t}(x,y)\ dt\right] W(x,y)dxdy \\ &= \int_{E_f} \left[\int_{f_+(x)}^{f_+(y)} 2t\ dt\right] W(x,y)dxdy \\ &= \int_{E_f} (f_+^2(y) - f_+^2(x))W(x,y)\ dxdy \\ &= \int_{E_f} |df_+^2|W \end{aligned} \tag{3.34}$$

as desired. ∎

3.5. **Cheeger's Inequality for Graphons.** In this subsection we will prove the following.

Theorem 3.7. *Let W be a connected graphon. Then*

$$\lambda(W) \geq \frac{h^2(W)}{8} \tag{3.35}$$

Before we prove Cheeger's inequality above, we first obtain a more convenient formula (Lemma 3.8 below) for $\lambda(W)$. Consider the map $\mathcal{I} : L^2(I,\nu) \to \mathbf{R}$ defined as

$$\mathcal{I}(f) = \int_0^1 f(x)d_W(x)\ dx = \langle f, \mathbb{1}\rangle_v \tag{3.36}$$

We show that $L^\infty(I) \cap \mathbb{1}_v^\perp$ is dense in $\mathbb{1}_v^\perp$. Let $P : L^2(I,\nu) \to L^2(I,\nu)$ be the map defined as $P(f) = f - \mathcal{I}(f)$. Then P is a bounded linear operator. Also, we have

$$P^2(f) = P(f - \mathcal{I}(f)) = f - \mathcal{I}(f) = P(f) \tag{3.37}$$

So $P^2 = P$. Further,

$$\begin{aligned} \langle Pf, g\rangle_v - \langle f, Pg\rangle_v &= \langle f - \mathcal{I}(f), g\rangle_v - \langle f, g - \mathcal{I}(g)\rangle_v \\ &= -\langle \mathcal{I}(f), g\rangle_v + \langle f, \mathcal{I}(g)\rangle_v \\ &= -\mathcal{I}(f)\mathcal{I}(g) + \mathcal{I}(f)\mathcal{I}(g) \\ &= 0 \end{aligned} \tag{3.38}$$

Therefore P is self-adjoint. This means that P is the orthogonal projection onto its image. It is clear that $\mathrm{Im}(P) \subseteq \mathbb{1}_v^\perp$, and also that P behaves as the identity when restricted to $\mathbb{1}_v^\perp$. Therefore P is the orthogonal projection onto $\mathbb{1}_v^\perp$. It is also clear that $P(L^\infty(I)) \subseteq L^\infty(I)$. We conclude that $L^\infty(I) \cap \mathbb{1}_v^\perp$ is dense in $\mathbb{1}_v^\perp$.

Now let $g \in \mathbb{1}_v^\perp$ be an arbitrary nonzero vector. Then both $\|dg\|_e$ and $\|g\|_v$ are nonzero.[2] Let $M > 0$ be such that $\|dg\|_e, \|g\|_v \geq M$. Let $\varepsilon > 0$ be arbitrary and choose $g' \in L^\infty(I) \cap \mathbb{1}_v^\perp$ such that $\|g - g'\|_v < \varepsilon M$. We had shown in the proof of Lemma 3.2, $\|df\|_e \leq 2\|f\|_v$ for all $f \in L^2(I,\nu)$. So

$$\|d(g - g')\|_e = \|dg - dg'\|_e < 2\varepsilon M.$$

Thus we have

$$|\|g\|_v - \|g'\|_v| < \varepsilon M \text{ and } |\|dg\|_e - \|dg'\|_e| < 2\varepsilon M$$

Hence we can approximate $\|dg\|_e/\|g\|_v$ arbitrarily well by the expressions of the form $\|dg'\|_e/\|g'\|_v$ by choosing a suitable $g' \in L^\infty(I) \cap \mathbb{1}_v^\perp$. We have proved

Lemma 3.8. *Let W be a connected graphon. Then*

$$\lambda(W) = \inf_{g \in \mathbb{1}_v^\perp :\ g \neq 0} \frac{\|dg\|_e^2}{\|g\|_v^2} = \inf_{\substack{g \in \mathbb{1}_v^\perp :\ g \neq 0, \\ g \in L^\infty(I)}} \frac{\|dg\|_e^2}{\|g\|_v^2} \tag{3.39}$$

---

[2]If $\|dg\|_e$ were equal to 0 then g would be an essentially constant function, which would force $g = 0$ since $g \in \mathbb{1}_v^\perp$.

Now we are ready to prove Theorem 3.7. Let $g : I \to \mathbf{R}$ be an arbitrary map in $L^\infty(I)$ with $\|g\|_v = 1$ and $\langle g, \mathbb{1}\rangle_v = 0$. To prove Theorem 3.7 it suffices to show that $\|dg\|_e^2 \geq \frac{1}{8}h^2(W)$. Let

$$t_0 = \sup\{t \in \mathbf{R} : \ \mathrm{vol}_W(g^{-1}(-\infty, t)) \leq \frac{1}{2}\mathrm{vol}_W(I)\} \tag{3.40}$$

The number t_0 exists since g is L^∞. Define $f = g - t_0$. Then both the sets $\{f < 0\}$ and $\{f > 0\}$ have volumes at most half of $\mathrm{vol}_W(I)$. Also

$$\|f\|_v^2 = \|g - t_0\|_v^2 = \|g\|_v^2 + \|t_0\|_v^2 - 2t_0\langle g, \mathbb{1}\rangle_v = 1 + \|t_0\|_v^2 \geq 1 \tag{3.41}$$

Clearly, $df = dg$. Therefore

$$\|dg\|_e^2 \geq \frac{\|df\|_e^2}{\|f\|_v^2} \tag{3.42}$$

Lemma 3.9.

$$\|df\|_e^2 \geq \frac{1}{8\|f\|_v^2}\left[\int_{E_f} |df_+^2|W + \int_{E_f} |df_-^2|W\right]^2 \tag{3.43}$$

Proof. Note that

$$\|df\|_e^2 = \int_E |df|^2 W = \int_{E_f} |df|^2 W \tag{3.44}$$

where $E_f = \{(x, y) \in I^2 : \ f(y) > f(x)\}$. Also

$$\int_{E_f} |df|^2 W \geq \int_{E_f} |df_+|^2 W + \int_{E_f} |df_-|^2 W \tag{3.45}$$

This is because $|df|^2 \geq |df_+|^2 + |df_-|^2$ is true pointwise in E_f. By Cauchy-Schwarz we have

$$\left[\int_{E_f} |df_+|^2 W\right]^{1/2} \left[\int_{I^2} (|f(x)| + |f(y)|)^2 W(x, y)\ dxdy\right]^{1/2} \geq \int_{E_f} |df_+^2|W \tag{3.46}$$

Using Cauchy-Schwarz again, we can show that

$$4\|f\|_v^2 \geq \int_{I^2} (|f(x)| + |f(y)|)^2 W(x, y)\ dxdy \tag{3.47}$$

which gives

$$\begin{aligned} 2\|f\|_v \left[\int_{E_f} |df_+|^2 W\right]^{1/2} &\geq \int_{E_f} |df_+^2|W \\ \Rightarrow \int_{E_f} |df_+|^2 W &\geq \frac{1}{4\|f\|_v^2}\left(\int_{E_f} |df_+^2|W\right)^2 \end{aligned} \tag{3.48}$$

Similarly,

$$\int_{E_f} |df_-|^2 W \geq \frac{1}{4\|f\|_v^2}\left(\int_{E_f} |df_-^2|W\right)^2 \tag{3.49}$$

Using these in Equation 3.45 gives

$$\begin{aligned}\int_{E_f} |df|^2 W &\geq \frac{1}{4\|f\|_v^2}\left[\left(\int_{E_f} |df_+^2|W\right)^2 + \left(\int_{E_f} |df_-^2|W\right)^2\right] \\ &\geq \frac{1}{8\|f\|_v^2}\left[\int_{E_f} |df_+^2|W + \int_{E_f} |df_-^2|W\right]^2\end{aligned} \tag{3.50}$$

and we have proved the lemma. ■

We now proceed to complete the proof of Theorem 3.7.

Proof of Theorem 3.7:
The Co-area Formula Theorem 3.6 gives

$$\int_{E_f} |df_+^2|W = \int_0^\infty 2te_W(S_t^c, S_t)\ dt \quad \text{and} \quad \int_{E_f} |df_-^2|W = \int_0^\infty 2te_W(S_{-t}^c, S_{-t})\ dt \tag{3.51}$$

But

$$\begin{aligned}\int_0^\infty 2te_W(S_t^c, S_t)\ dt &\geq h(W)\int_0^\infty 2t\,\mathrm{vol}(S_t)\ dt \\ &= h(W)\int_0^\infty 2t\left[\int_{I^2} \chi_{I\times S_t}(x,y)W(x,y)\ dxdy\right] dt \\ &= h(W)\int_{I^2}\left[\int_0^\infty 2t\chi_{I\times S_t}(x,y)\ dt\right] W(x,y)dxdy \\ &= h(W)\int_{I^2}\left[\int_0^\infty 2t\chi_{I\times S_t}(x,y)\ dt\right] W(x,y)dxdy \\ &= h(W)\int_{I^2}\left[\int_0^{f_+(y)} 2t\ dt\right] W(x,y)\ dxdy \\ &= h(W)\int_{I^2} f_+^2(y)W(x,y)\ dxdy\end{aligned} \tag{3.52}$$

Similarly

$$\begin{aligned}\int_0^\infty 2te_W(S^c_{-t}, S_{-t})\ dt &\geq h(W)\int_0^\infty 2t\,\mathrm{vol}(S^c_{-t})\ dt \\ &= h(W)\int_0^\infty 2t\left[\int_{I^2}\chi_{I\times S^c_{-t}}(x,y)W(x,y)\ dxdy\right]dt \\ &= h(W)\int_{I^2}\left[\int_0^\infty 2t\chi_{I\times S^c_{-t}}(x,y)\ dt\right]W(x,y)dxdy \\ &= h(W)\int_{I^2}\left[\int_0^\infty 2t\chi_{I\times S^c_{-t}}(x,y)\ dt\right]W(x,y)dxdy \\ &= h(W)\int_{I^2}\left[\int_0^{f_-(y)} 2t\ dt\right]W(x,y)dxdy \\ &= h(W)\int_{I^2} f_-^2(y)W(x,y)\ dxdy\end{aligned} \tag{3.53}$$

Therefore

$$\begin{aligned}\int_{E_f}|df_+^2|W + \int_{E_f}|df_-^2|W &\geq h(W)\left[\int_{I^2} f_+^2(y)W(x,y)\ dxdy + \int_{I^2} f_-^2(y)W(x,y)\ dxdy\right] \\ &= h(W)\left[\int_{I^2}(f_+^2(y)+f_-^2(y))W(x,y)\ dxdy\right] \\ &= h(W)\left[\int_{I^2} f^2(y)W(x,y)\ dxdy\right] \\ &= h(W)\left[\int_I f^2(y)d_W(y)\ dy\right] = h(W)\|f\|_v^2\end{aligned} \tag{3.54}$$

Combining this with Lemma 3.9, we have

$$\|df\|_e^2 \geq \frac{1}{8\|f\|_v^2}h^2(W)\|f\|_v^4 \tag{3.55}$$

and thus

$$\frac{\|df\|_e^2}{\|f\|_v^2} \geq \frac{1}{8}h^2(W) \tag{3.56}$$

Lastly, using Equation 3.42 we have

$$\|dg\|_e^2 \geq \frac{1}{8}h^2(W) \tag{3.57}$$

and we are done. ■

4. Cheeger Constant for Graphings and the Cheeger-Buser Inequalities

We now turn to graphings. For the purposes of this section $G = (I, \mu, E)$ will denote a graphing. As discussed in Section 2.2, graphings are substantially different from graphons in terms of their structure. In spite of this difference, Lemma 2.4 will allow us to furnish proofs that are, at least at a formal level, extremely similar to the proofs in Section 3 above. However, the actual intuition and idea behind the proofs will really go back to Theorem 2.3. In this section, we shall therefore try to convey to the reader both the formal similarity with the proofs in Section 3 as well as the actual structural idea going back to Theorem 2.3.

4.1. **Cheeger Constant for Graphings.** For two measurable subsets A and B of I, we define

$$e_G(A,B) = \eta(A,B) = \int_A \deg_B(x)\ d\mu(x) \tag{4.1}$$

For a measurable subset A of I, the **volume** of G over A is defined as

$$\mathrm{vol}(A) = \int_A \deg(x)\ d\mu(x) = e_G(A,I) \tag{4.2}$$

A graphing is said to be **connected** if for all measurable subsets A of I with $0 < \mu(A) < 1$ we have $e_G(A,A^c) \neq 0$. Note that if G is connected then $\deg > 0$ a.e.

Given a graphing G, we define the **Cheeger constant** of G as

$$h(G) = \inf_{A\subseteq I:\ 0<\mu(A)<1} \frac{e_G(A,A^c)}{\min\{\mathrm{vol}(A),\mathrm{vol}(A^c)\}} \tag{4.3}$$

A symmetrized version of the above constant which we will be referred to as the **symmetric Cheeger constant** is defined as

$$g(G) = \inf_{A\subseteq I:\ 0<\mu(A)<1} \frac{e_G(A,B)}{\mathrm{vol}(A)\,\mathrm{vol}(A^c)} \tag{4.4}$$

Note that the above defined constants exist for connected graphings.

4.2. **Definition of d, d*, and the Laplacian.** While the definitions in this subsection are (intentionally) formally similar to the corresponding ones for graphons, the details are somewhat different. Let G be a connected graphing. Define

$$E^+ = \{(x,y)\in E:\ y > x\}, \qquad E^- = \{(x,y)\in E:\ x>y\} \tag{4.5}$$

Remark 4.1. *A word about orientation of edges is in order: the definition of E^+ implicitly assigns an orientation to edges (x,y) where edges in E^+ (resp. E^-) are positively (resp. negatively) oriented. This is just a convention. A measurable automorphism of I could be used to change orientations on edges for instance.*

Define bilinear maps $\langle\cdot,\cdot\rangle_v$ on $L^2(I,\mu)$ and $\langle\cdot,\cdot\rangle_e$ on $L^2(E^+,\eta)$ as follows:

$$\langle f,g\rangle_v = \int_0^1 f(x)g(x)\deg(x)\ d\mu(x) \tag{4.6}$$

for all $f,g\in L^2(I,\mu)$, and

$$\langle \theta,\psi\rangle_e = \int_{E^+} \theta(x,y)\psi(x,y)\ d\eta(x,y) \tag{4.7}$$

for all $\theta,\psi\in L^2(E^+,\eta)$. When the graphing has a single strand given by a measure preserving involution $\varphi: A\to A$, then Equation 4.7 becomes

$$\langle \theta,\psi\rangle_e = \int_R \theta(x,\varphi(x))\psi(x,\varphi(x))\ d\mu(x) \tag{4.8}$$

where $R = \{x \in I : \varphi(x) > x\}$.

The bounded degree condition ensures that $\langle f, g\rangle_v$ exists for all $f, g \in L^2(I,\mu)$. Further, the connectedness of G implies that $\langle \cdot,\cdot\rangle_v$ is in fact an inner product. The subscript 'v' is put to distinguish this inner product from the standard inner product on $L^2(I,\mu)$. The standard inner product will be denoted by $\langle \cdot,\cdot\rangle_{L^2(I,\mu)}$. The bilinear map $\langle \cdot,\cdot\rangle_e$ on $L^2(E^+,\eta)$ is the same as the standard inner product. Even so, we will continue to put the subscript 'e'.

Since G is a connected graphing, we have $\deg(x) \geq 1$ a.e., and thus $\|f\|_v \geq \|f\|_{L^2(I,\mu)}$. The bounded degree condition yields that $D\|f\|_{L^2(I,\mu)} \geq \|f\|_v$. Thus the standard L^2-norm on $L^2(I,\mu)$ and the norm coming from $\langle \cdot,\cdot\rangle_v$ are comparable, and therefore the topology on $L^2(I,\mu)$ induced by $\|\cdot\|_v$ is same as the standard topology.

Define a map $d : L^2(I,\mu) \to L^2(E^+,\eta)$ as

$$(df)(x,y) = f(y) - f(x) \tag{4.9}$$

for all $f \in L^2(I,\mu)$. We check that $df \in L^2(E^+,\eta)$ and that d is a bounded operator.

Lemma 4.2. *The map d is continuous.*

Proof. We want to show that d is a bounded operator. Let $f \in L^2(I,\mu)$. Once we observe that

$$\begin{aligned}\|df\|_e^2 &= \int_{E^+} (df)^2 \, d\eta \\ &\leq \int_{I^2} (df)^2 \, d\eta,\end{aligned} \tag{4.10}$$

the proof of Lemma 3.2, followed line-by-line, gives:

$$\|df\|_e^2 \leq 4 \int_{I^2} f(x)^2 \, d\eta(x,y). \tag{4.11}$$

By Equation 2.17 the last term is same as $4\|f\|_v^2$. ■

We compute the adjoint of d. Let $f \in L^2(I,\mu)$ and $\psi \in L^2(E^+,\eta)$ be arbitrary. We have

(4.12)

$$\begin{aligned}
\langle df, \psi\rangle_e &= \int_{E^+} df(x,y)\psi(x,y)\ d\eta(x,y) = \int_{I^2} df(x,y)\psi(x,y)\chi_{E^+}(x,y)\ d\eta(x,y) \\
&= \int_{I^2} (f(y)-f(x))\psi(x,y)\chi_{E^+}(x,y)\ d\eta(x,y) \\
&= \int_{I^2} f(y)\psi(x,y)\chi_{E^+}(x,y)\ d\eta(x,y) - \int_{I^2} f(x)\psi(x,y)\chi_{E^+}(x,y)\ d\eta(x,y) \\
&= \int_0^1 \sum_x f(y)\psi(x,y)\chi_{E^+}(x,y)\ d\mu(y) - \int_0^1 \sum_y f(x)\psi(x,y)\chi_{E^+}(x,y)\ d\mu(x) \\
&= \int_0^1 \sum_y f(x)\psi(y,x)\chi_{E^+}(y,x)\ d\mu(x) - \int_0^1 \sum_y f(x)\psi(x,y)\chi_{E^+}(x,y)\ d\mu(x) \\
&= \int_0^1 f(x)\left[\sum_y \psi(y,x)\chi_{E^+}(y,x) - \sum_y \psi(x,y)\chi_{E^+}(x,y)\right] d\mu(x)
\end{aligned}$$

On the other hand we have

$$\langle f, d^*\psi\rangle_v = \int_0^1 f(x) d^*\psi(x)\deg(x)\ d\mu(x) \tag{4.13}$$

Thus we have

$$(d^*\psi)(x) = \frac{1}{\deg(x)}\left[\sum_y \psi(y,x)\chi_{E^+}(y,x) - \sum_y \psi(x,y)\chi_{E^+}(x,y)\right] \tag{4.14}$$

whenever $\deg(x) \neq 0$. We set $(d^*\psi)(x) = 0$ if $\deg(x) = 0$. Define the **Laplacian** of G as $\Delta_G = d^*d$.

4.3. **Buser Inequality for Graphings.** We first observe that the multiplicity of the eigenvalue 0 of the Laplacian of a connected graphing (I,μ,E) is 1. For $f \in L^2(I,\mu)$, we have $\Delta f = 0$ if and only if $df = 0$. As in the case of graphons it now suffices to show the following: $df = 0$ if and only if f is constant (up to a set of measure zero). Of course, $df = 0$ for constant f. Conversely, assume that $df = 0$. Then

$$\int_{E^+} (df)^2\ d\eta = 0 \tag{4.15}$$

which implies that

$$\int_{E^+} (f(y)-f(x))^2\ d\eta(x,y) = 0. \tag{4.16}$$

Since

$$\int_{E^+} (f(y)-f(x))^2\ d\eta(x,y) = \int_{E^-} (f(y)-f(x))^2\ d\eta(x,y). \tag{4.17}$$

it follows that

$$\int_{I^2} (f(y) - f(x))^2 \, d\eta(x, y) = 0 \tag{4.18}$$

For each $t \in \mathbf{R}$, let $S_t = f^{-1}(t, \infty)$. Therefore,

$$\int_{S_t^c \times S_t} (f(y) - f(x))^2 \, d\eta(x, y) = 0. \tag{4.19}$$

It follows that $(f(y) - f(x))^2$ is η-a.e. 0 on $S_t^c \times S_t$. But $f(y) - f(x) \neq 0$ for all $(x, y) \in S_t \times S_t^c$. So $\eta(S_t^c, S_t) = 0$ for all t. The connectedness of G then implies that either S_t or S_t^c has μ-measure 0. So our claim follows from the following lemma whose proof is an exact replica of Lemma 3.4 and we omit it.

Lemma 4.3. *Let $f : I \to \mathbf{R}$ be a measurable function such that for all $t \in \mathbf{R}$ we have either $f^{-1}(-\infty, t]$ or $f^{-1}(t, \infty)$ has μ-measure 0. Then f is constant μ-a.e.*

The eigenfunctions corresponding to 0 are thus the essentially constant functions: the $0-$eigenspace of Δ is generated by $\mathbb{1}$. Define

$$\lambda(G) = \inf_{g \in \mathbb{1}_v^{\perp}:\ g \neq 0} \frac{\langle g, \Delta g \rangle_v}{\langle g, g \rangle_v} = \inf_{g \in \mathbb{1}_v^{\perp}:\ g \neq 0} \frac{\|dg\|_e^2}{\|g\|_v^2} \tag{4.20}$$

(Here, $\mathbb{1}_v^{\perp}$ denotes the orthogonal complement of $\mathbb{1}$ with respect to the inner product $\langle \cdot, \cdot \rangle_v$).

Theorem 4.4 (Buser Inequality)**.** *Let $G = (I, \mu, E)$ be a connected graphing. Then*

$$\lambda(G) \leq 2h(G) \text{ and } \lambda(G) \leq g(G) \tag{4.21}$$

The proof below exploits the fact that one can decompose a graphing into finitely many matchings (Theorem 2.3). An alternate proof can also be given following that of Theorem 3.5 replacing $W(x, y)dxdy$ formally with $d\eta(x, y)$.

Proof. Let $\varphi_i : C_i \to C_i$, $1 \leq i \leq k$, be μ-measure preserving involutions such that

$$E = \bigsqcup_{i=1}^{k} \{(x, \varphi_i(x)) : \ x \in C_i\} \tag{4.22}$$

Extend each φ_i to a map $\varphi_i : I \to I$ by declaring $\varphi_i(x) = x$ for all $x \notin C_i$. Let $A \sqcup B$ form a measurable partition of I with $0 < \mu(A) < 1$. Define $f : I \to \mathbf{R}$ as

$$f(x) = \begin{cases} \dfrac{1}{\mathrm{vol}(A)} & \text{if } x \in A \\ -\dfrac{1}{\mathrm{vol}(B)} & \text{if } x \in B \end{cases} \tag{4.23}$$

Then $f \in \mathbb{1}_v^{\perp}$. Now

$$\begin{aligned}
\lambda(G) &\leq \frac{\|df\|_e^2}{\|f\|_v^2} = \frac{\int_{E^+}(f(x)-f(y))^2\ d\eta(x,y)}{\int_0^1 f(x)^2 \deg(x)\ d\mu(x)} = \frac{\int_{I^2}(f(x)-f(y))^2\ d\eta(x,y)}{2\int_0^1 f(x)^2 \deg(x)\ d\mu(x)} \\
&= \frac{\sum_{i=1}^k \int_I (f(x)-f(\varphi_i(x)))^2\ d\mu(x)}{2\int_0^1 f(x)^2 \deg(x)\ d\mu(x)} \\
&= \frac{\sum_{i=1}^k \left[\int_A (f(x)-f(\varphi_i(x)))^2\ d\mu(x) + \int_B (f(x)-f(\varphi_i(x)))^2\ d\mu(x)\right]}{2\int_0^1 f(x)^2 \deg(x)\ d\mu(x)} \\
&= \frac{\sum_{i=1}^k \left[\int_A (f(x)-f(\varphi_i(x)))^2 \chi_B(\varphi_i(x))\ d\mu(x) - \int_B (f(x)-f(\varphi_i(x)))^2 \chi_A(\varphi_i(x))\ d\mu(x)\right]}{2\int_0^1 f(x)^2 \deg(x)\ d\mu(x)} \\
&= \frac{\sum_{i=1}^k \left[\int_A \left(\frac{1}{\mathrm{vol}(A)} + \frac{1}{\mathrm{vol}(B)}\right)^2 \chi_B(\varphi_i(x))\ d\mu(x) + \int_B \left(\frac{1}{\mathrm{vol}(B)} + \frac{1}{\mathrm{vol}(A)}\right)^2 \chi_A(\varphi_i(B))\ d\mu(x)\right]}{2\int_0^1 f(x)^2 \deg(x)\ d\mu(x)} \\
&= \frac{\left(\frac{1}{\mathrm{vol}(A)} + \frac{1}{\mathrm{vol}(B)}\right)^2 \sum_{i=1}^k \left[\int_A \chi_B(\varphi_i(x))\ d\mu(x) + \int_B \chi_A(\varphi_i(x))\ d\mu(x)\right]}{2\left[\int_A f(x)^2 \deg(x)\ d\mu(x) + \int_B f(x)^2 \deg(x)\ d\mu(x)\right]} \\
&= \frac{\left(\frac{1}{\mathrm{vol}(A)} + \frac{1}{\mathrm{vol}(B)}\right)^2 \left[\int_A \deg_B(x)\ d\mu(x) + \int_B \deg_A(x)\ d\mu(x)\right]}{2\left[\frac{1}{\mathrm{vol}(A)^2}\mathrm{vol}(A) + \frac{1}{\mathrm{vol}(B)^2}\mathrm{vol}(B)\right]} \\
&= \left(\frac{1}{\mathrm{vol}(A)} + \frac{1}{\mathrm{vol}(B)}\right) e_G(A,B) \\
&\leq 2\frac{e_G(A,B)}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}, \frac{e_G(A,B)}{\mathrm{vol}(A)\,\mathrm{vol}(B)}
\end{aligned}$$

Since $B = A^c$, and since the above holds for all choices of A with $0 < \mu(A) < 1$, we have $\lambda(G) \leq 2h(G)$ and $\lambda(G) \leq g(G)$. ■

Alternate Proof. In the proof of Theorem 3.5 replace $W(x,y)dxdy$ formally with $d\eta(x,y)$. ■

4.4. **Co-area Formula for Graphings.** Let $G = (I, E, \mu)$ be a graphing and $f : I \to \mathbf{R}$ be any L^∞-map. Let E_f be defined as

$$E_f = \{(x,y) \in E :\ f(y) > f(x)\} \tag{4.24}$$

The set E_f will be referred to as the ***f*-oriented edges** of G. Let S_t denote the set $f^{-1}(t,\infty)$. Then

$$\int_{E_f} df\, d\eta = \int_{-\infty}^{\infty} e_G(S_t^c, S_t)\ dt \tag{4.25}$$

Let us see the proof in the special case when E is given by a single measure preserving involution $\varphi : A \to A$ where A is a measurable subset of I. Define $R = \{x \in A :$

$(x, \varphi(x)) \in E_f\}$. Then the RHS of the above equation is

$$\begin{aligned}
\int_{-\infty}^{\infty} e_G(S_t^c, S_t)\ dt &= \int_{-\infty}^{\infty} \int_{S_t^c} \deg_{S_t}(x)\ d\mu(x)\ dt = \int_{-\infty}^{\infty} \int_0^1 \chi_{S_t^c}(x) \deg_{S_t}(x)\ d\mu(x)\ dt \\
&= \int_0^1 \int_{-\infty}^{\infty} \chi_{S_t^c}(x) \deg_{S_t}(x)\ dt\ d\mu(x) = \int_0^1 \int_{f(x)}^{\infty} \deg_{S_t}(x)\ dt\ d\mu(x) \\
&= \int_R \int_{f(x)}^{\infty} \deg_{S_t}(x)\ dt\ d\mu(x) = \int_R (f \circ \varphi(x) - f(x))\ d\mu(x)
\end{aligned} \tag{4.26}$$

On the other hand the LHS of Equation 4.25 is

$$\begin{aligned}
\int_{E_f} df\ d\eta = \int_{I^2} df \chi_{E_f}\ d\eta &= \int_I \sum_y df(x,y) \chi_{E_f}(x,y)\ d\mu(x) \\
&= \int_R df(x, \varphi(x))\ d\mu(x) \\
&= \int_R (f \circ \varphi(x) - f(x))\ d\mu(x)
\end{aligned} \tag{4.27}$$

and therefore Equation 4.25 holds. Just as in the case of graphons, we need a slightly different lemma

Theorem 4.5. *Let $G = (I, E, \mu)$ be a graphing and $f : I \to \mathbf{R}$ be an arbitrary L^2-map. Define $f_+ : I \to \mathbf{R}$ and $f_- : I \to \mathbf{R}$ as the map $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$. Let $S_t = f^{-1}(t, \infty)$. Then*

$$\begin{aligned}
\int_{E_f} |df_+^2|\ d\eta &= \int_0^{\infty} e_G(S_{\sqrt{t}}^c, S_{\sqrt{t}})\ dt = \int_0^{\infty} 2t e_G(S_t^c, S_t)\ dt, \quad \text{and} \\
\int_{E_f} |df_-^2|\ d\eta &= \int_0^{\infty} e_G(S_{-\sqrt{t}}^c, S_{-\sqrt{t}}) = \int_0^{\infty} 2t e_G(S_{-t}^c, S_{-t})\ dt
\end{aligned} \tag{4.28}$$

Proof. We prove only the first one. Further, as in the proof of Lemma 2.4 (see also Equation 2.17) we first assume that the edge set E is determined by a single μ-measure preserving involutions $\varphi : A \to A$ defined on a measurable subset A of I. Define $E_f = \{(x, y) \in E : f(y) > f(x)\}$. Let $R = \{x \in I : (x, \varphi(x)) \in E_f\}$. We have by change of variables that

$$\int_0^{\infty} e_G(S_{\sqrt{t}}^c, S_{\sqrt{t}})\ dt = \int_0^{\infty} 2t e_G(S_t^c, S_t)\ dt \tag{4.29}$$

Now

(4.30)

$$\begin{aligned}\int_0^\infty 2te_G(S_t^c, S_t)\ dt &= \int_0^\infty 2t\eta(S_t^c \times S_t)\ dt = \int_0^\infty 2t\left[\int_0^1 \chi_{S_t^c}(x)\deg_{S_t}(x)\ d\mu(x)\right] dt \\ &= \int_0^\infty \left[\int_0^1 2t\chi_{S_t^c}(x)\deg_{S_t}(x)\ d\mu(x)\right] dt \\ &= \int_0^1 \left[\int_0^\infty 2t\chi_{S_t^c}(x)\deg_{S_t}(x)\ dt\right] d\mu(x) \\ &= \int_0^1 \left[\int_{f_+(x)}^\infty 2t\deg_{S_t}(x)\ dt\right] d\mu(x) \\ &= \int_R \left[\int_{f_+(x)}^\infty 2t\deg_{S_t}(x)\ dt\right] d\mu(x) \\ &= \int_R \left[\int_{f_+(x)}^{f_+(\varphi(x))} 2t\ dt\right] d\mu(x) \\ &= \int_R (f_+^2(\varphi(x)) - f_+^2(x)) d\mu(x)\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{E_f} |df_+^2|\ d\eta &= \int_{I^2} |df_+^2(x,y)|\chi_{E_f}(x,y)\ d\eta(x,y) \\ &= \int_I \sum_y |df_+^2(x,y)|\chi_{E_f}(x,y)\ d\mu(x) \\ &= \int_R |df_+^2(x,\varphi(x))|\ d\mu(x) \\ &= \int_R (f_+^2(\varphi(x)) - f_+^2(x))\ d\mu(x),\end{aligned} \tag{4.31}$$

completing the proof for the special case of a single strand.

We now deal with the general case where there may be multiple strands. Let $\varphi_i : A_i \to A_i$, $1 \le i \le k$, be μ-measure preserving involutions such that $E = \bigsqcup_{i=1}^k \{(x, \varphi_i(x)) :\ x \in A_i\}$. Let G_i be the graphing corresponding to φ_i. So $G = \bigsqcup_{i=1}^k G_i$. Then

$$\int_0^\infty 2te_G(S_t^c, S_t)\ dt = \sum_{i=1}^k \int_0^\infty 2te_{G_i}(S_t^c, S_t)\ dt \tag{4.32}$$

and

$$\int_{E_f} |df_+^2|\ d\eta = \sum_{i=1}^k \int_{E_f^i} |df_+^2|\ d\eta_i \tag{4.33}$$

where E_f^i are the f-oriented edges of G_i and η_i is the edge measure of G_i. Thus the general case follows from the special case. ∎

4.5. **Cheeger Inequality for Graphings.** In this subsection we will prove the following Cheeger inequality for graphings.

Theorem 4.6. *Let G be a connected graphing. Then*

$$\lambda(G) \geq \frac{h^2(G)}{8} \tag{4.34}$$

The proof of Lemma 3.8 goes through mutatis mutandis to give:

Lemma 4.7. *Let G be a connected graphing. Then*

$$\lambda(G) = \inf_{g \in \mathbb{1}_v^{\perp}:\ g \neq 0} \frac{\|dg\|_e^2}{\|g\|_v^2} = \inf_{\substack{g \in \mathbb{1}_v^{\perp}:\ g \neq 0, \\ g \in L^\infty(I,\mu)}} \frac{\|dg\|_e^2}{\|g\|_v^2} \tag{4.35}$$

We proceed with the proof of Theorem 4.6 for graphings. Let $g : I \to \mathbf{R}$ be an arbitrary L^∞-map with $\|g\|_v = 1$ and $\langle g, \mathbb{1}\rangle_v = 0$. To prove Cheeger's inequality it is enough to show that $\|dg\|_e^2 \geq \frac{1}{8}h^2(G)$. Let

$$t_0 = \sup\{t \in \mathbf{R} :\ \mathrm{vol}_G(g^{-1}(-\infty, t)) \leq \frac{1}{2}\mathrm{vol}_G(I)\} \tag{4.36}$$

and define $f = g - t_0$. Then both the sets $\{f < 0\}$ and $\{f > 0\}$ have volumes at most half of $\mathrm{vol}_G(I)$. Also

$$\|f\|_v^2 = \|g - t_0\|_v^2 = \|g\|_v^2 + \|t_0\|_v^2 - 2t_0\langle g, \mathbb{1}\rangle_v = 1 + \|t_0\|_v^2 \geq 1 \tag{4.37}$$

Clearly, $df = dg$. Therefore

$$\|dg\|_e^2 \geq \frac{\|df\|_e^2}{\|f\|_v^2} \tag{4.38}$$

Lemma 4.8.

$$\int_{E_f} |df|^2\ d\eta \geq \frac{1}{8\|f\|_v^2}\left[\int_{E_f} |df_+^2|\ d\eta + \int_{E_f} |df_-^2|\ d\eta\right]^2 \tag{4.39}$$

Proof. As in Lemma 3.9, we start by observing that

$$\int_{E^+} |df|^2\ d\eta = \int_{E_f} |df|^2\ d\eta \tag{4.40}$$

The proof of Lemma 4.8 is now an exact replica of that of Lemma 3.9: the only extra point to note being that

$$\int_{I^2} f(x)^2\ d\eta(x,y) = \int_I \sum_y f(x)^2 \chi_E(x,y)\ d\mu(x) = \int_I f(x)^2 \deg(x)\ d\mu(x) = \|f\|_v^2 \tag{4.41}$$

We omit the details. ∎

The rest of the proof of Theorem 4.6 is quite similar to that of Theorem 3.7. However, since this is one of the main theorems of this paper, we include the details for completeness. The Co-area Formula Theorem 4.5 gives:

(4.42)
$$\int_{E_f} |df_+^2|\ d\eta = \int_0^\infty 2te_G(S_t^c, S_t)\ dt, \quad \text{and} \quad \int_{E_f} |df_-^2|\ d\eta = \int_0^\infty 2te_G(S_{-t}^c, S_{-t})\ dt$$

But

(4.43)
$$\begin{aligned}\int_0^\infty 2te_G(S_t, S_t)\ dt \geq h(G)\int_0^\infty 2t\,\mathrm{vol}(S_t)\ dt &= h(G)\int_0^\infty 2t\left[\int_{S_t}\deg(x)\ d\mu(x)\right]\ dt\\ &= h(G)\int_0^\infty 2t\left[\int_I \chi_{S_t}(x)\deg(x)\ d\mu(x)\right]\ dt = h(G)\int_0^\infty\int_I 2t\chi_{S_t}(x)\deg(x)\ d\mu(x)dt\\ &= h(G)\int_I\int_0^\infty 2t\chi_{S_t}(x)\deg(x)\ dtd\mu(x) = h(G)\int_I\left[\int_0^\infty 2t\chi_{S_t}(x)\ dt\right]\deg(x)\ d\mu(x)\\ &= h(G)\int_I\left[\int_0^{f_+(x)} 2t\ dt\right]\deg(x)d\mu(x)\\ &= h(G)\int_I f_+^2(y)\deg(x)\ d\mu(x)\end{aligned}$$

Similarly

(4.44)
$$\begin{aligned}\int_0^\infty 2te_G(S_{-t}^c, S_{-t})\ dt \geq h(G)\int_0^\infty 2t\,\mathrm{vol}(S_{-t}^c)\ dt &= h(G)\int_0^\infty 2t\left[\int_{S_{-t}^c}\deg(x)\ d\mu(x)\right]dt\\ &= h(G)\int_0^\infty 2t\left[\int_I \chi_{S_{-t}^c}(x)\deg(x)\ d\mu(x)\right]dt = h(G)\int_0^\infty\int_I 2t\chi_{S_{-t}^c}(x)\deg(x)\ d\mu(x)dt\\ &= h(G)\int_I\int_0^\infty 2t\chi_{S_{-t}^c}(x)\deg(x)\ dtd\mu(x) = h(G)\int_I\left[\int_0^\infty 2t\chi_{S_{-t}^c}(x)\ dt\right]\deg(x)d\mu(x)\\ &= h(G)\int_I\left[\int_0^{f_-(x)} 2t\ dt\right]\deg(x)d\mu(x)\\ &= h(G)\int_I f_-^2(x)\deg(x)\ d\mu(x,y)\end{aligned}$$

Therefore

(4.45)
$$\begin{aligned}\int_{E_f}|df_+^2|\ d\eta + \int_{E_f}|df_-^2|\ d\eta &\geq h(G)\left[\int_I f_+^2(x)\deg(x)\ d\mu(x) + \int_I f_-^2(x)\deg(x)\ d\mu(x)\right]\\ &= h(G)\int_I (f_+^2(x) + f_-^2(x))\deg(x)\ d\mu(x)\\ &= h(G)\int_I f(x)^2\deg(x)\ d\mu(x)\\ &= h(G)\|f\|_v^2\end{aligned}$$

Combining this with Equation 4.39, we have

$$\int_{E_f} |df|^2 \, d\eta \geq \frac{1}{8\|f\|_v^2} h^2(G)\|f\|_v^4 \tag{4.46}$$

This along with Equation 4.40 gives

$$\frac{\|df\|_e^2}{\|f\|_v^2} = \frac{\int_E |df|^2 \, d\eta}{\|f\|_v^2} \geq \frac{1}{8} h^2(G) \tag{4.47}$$

Lastly, using Equation 4.38 we have

$$\|dg\|_e^2 \geq \frac{1}{8} h^2(G) \tag{4.48}$$

and Theorem 4.6 follows. □

5. Cheeger constant and connectedness

5.1. **A connected graphon with zero Cheeger constant.** Recall that a graphon is connected if for all measurable subsets A of I with $0 < \mu_L(A) < 1$ we have $e_W(A, A^c) \neq 0$. More generally, given a graphon W and a measurable subset S of I, we say that the restriction $W|_S$ is **connected** if for all measurable subsets A of S with $0 < \mu_L(A) < \mu_L(S)$, we have

$$\int_{A \times (S \setminus A)} W > 0 \tag{5.1}$$

Lemma 5.1. *Let W be a graphon and S and T be measurable subsets of I such that $S \cup T = I$ and $S \cap T$ has positive measure. Further assume that $W|_S$ and $W|_T$ are connected. Then W is connected.*

Proof. Assume that W is disconnected and let A be a measurable subset of I such that $0 < \mu_L(A) < 1$ and $e_W(A, A^c) = 0$. Then in particular we have

$$\int_{(A \cap S) \times (A^c \cap S)} W = 0 \text{ and } \int_{(A \cap T) \times (A^c \cap T)} W = 0 \tag{5.2}$$

The connectedness of $W|_S$ implies that either $A \cap S$ or $A^c \cap S$ has full measure in S. Without loss of generality assume that $A \cap S$ has full measure in S. Again, by the connectedness of $W|_T$ we have that $A \cap T$ or $A^c \cap T$ has full measure in T. In the former case we would have that A has full measure in I since $I = S \cup T$. In the latter case the measure of $S \cap T$ would be 0. In any case we get a contradiction. ■

Example 5.2. Consider the graphon W given by the following figure.

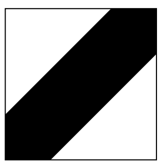

Figure 1. Example of a connected graphon.

The graphon takes the value 1 at all the shaded points and 0 at all other points. It follows by repeated use of Lemma 5.1 that W is connected. ♢

Let W be a graphon taking values in $\{0,1\}$. We call such a graphon a **neighborhood graphon** if $W^{-1}(1)$ contains an open neighborhood of the diagonal of the *open* square $(0,1) \times (0,1)$. Note that the graphon in Example 5.2 contains instead an open neighborhood of the diagonal of the *closed* square I^2.

Lemma 5.3. *Every neighborhood graphon is connected.*

Proof. Let W be a neighborhood graphon. For each $n > 2$ let S_n denote the interval $(1/n, 1-1/n)$. Each $W|_{S_n}$ is connected. This is because a copy of the graphon shown in Figure 1 is embedded in the restriction of W over $S_n \times S_n$.

We argue by contradiction. Suppose that W is disconnected. The there is $A \subseteq I$ with $0 < \mu_L(A) < 1$ such that $e_W(A, A^c) = 0$. Therefore for each n we have $\int_{(A\cap S_n)\times(S_n\setminus A)} W = 0$. By connectedness of $W|_{S_n}$, we must have that for any $n > 2$, either $\mu_L(A \subseteq S_n) = \mu_L(S_n)$ or $\mu_L(A \cap S_n) = 0$. If the former happens for some n, then it must happen for all n, and consequently A is of full measure in I. The other possibility is that $\mu_L(A \cap S_n) = 0$ for all n, but then A has measure 0. So in any case, we have a contradiction. ■

Example 5.4. A particular way of constructing a neighborhood graphon is the following. Let $f : I \to I$ be a continuous map such that $f(x) > x$ for all $0 < x < 1$. Define a graphon W_f as

$$W_f(x,y) = \begin{cases} 1 & \text{if } x \leq y \leq f(x) \\ 0 & \text{if } f(x) < y \\ W(y,x) & \text{if } x > y \end{cases}$$

In other words, W_f takes the value 1 in the region trapped between the graph of f and the reflection of the graph of f about the $y = x$ line, and is 0 everywhere else. For example, let $f(x) = \sqrt{x}$. Then the following diagram illustrates what W_f looks like.

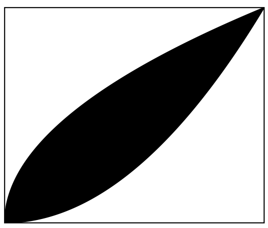

FIGURE 2. An example of a neighborhood graphon: W_f corresponding to $f(x) = \sqrt{x}$.

The graphon shown in Figure 1 is also an example of a neighborhood graphon arising as W_f for a suitably chosen continuous map $f : I \to I$. ♢

Example 5.5. Unlike in the case of finite graphs, the Cheeger constant of a connected graphon may be zero, as illustrated by Figure 3. The graphon takes the value 1 at all the shaded points, either gray or black, and the value 0 at all the unshaded points. Call this graphon W.

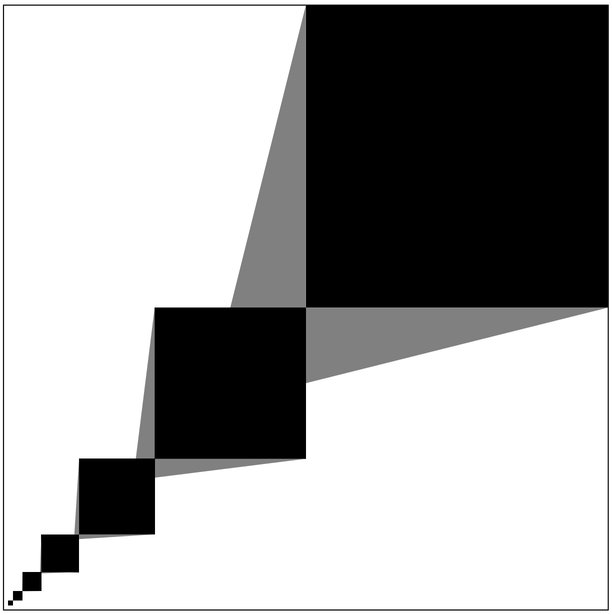

FIGURE 3. Example of a connected graphon whose Cheeger constant vanishes.

The bottom left endpoints of the black squares in the above figure have coordinates $(1/2^n, 1/2^n)$, $n = 1, 2, 3, \ldots$. The lengths of horizontal edges of the gray triangles above the $y = x$ line are $1/2^{2n+1}$, $n = 1, 2, 3, \ldots$. Let A_n be the interval $[0, 1/2^n]$. Then $e(A_n, A_n^c)$ equals the measure of the gray region in $A^n \times A_n^c$. But there is only one gray triangle in this region. The sides of this right triangle (other than the hypotenuse) have lengths $1/2^{2n+1}$ and $1/2^n$. Thus $e(A_n, A_n^c) = 1/2^{3n+2}$. Let V denote the total measure of the points shaded black. Then $\text{vol}(A_n) \geq V/4^n + e(A_n, A_n^c)$ because the measure of the black region inside $A_n \times I$ is exactly $V/4^n$. For large n we have $\text{vol}(A_n)$ is at most half the total volume. Thus for large n we have

$$h_{A_n}(W) = \frac{e(A_n, A_n^c)}{\text{vol}(A_n)} \leq \frac{1/2^{3n+2}}{V/4^n + 1/2^{3n+2}} = \frac{1/2^{n+2}}{V + 1/2^{n+2}} \tag{5.3}$$

This is zero in the limit and thus the Cheeger constant of this graphon is zero. This graphon is connected by Lemma 5.3 because it is a neighborhood graphon. ♢

5.2. **A connected graphing with zero Cheeger constant.** We prove in this section that the *irrational cyclic graphing* [Lov12, Example 18.17] has zero Cheeger constant.

Let a_0 be an irrational number. We get a bounded degree Borel graph (I, E) on I by joining two points x and y if $|x - y| = a_0$. The triple (I, E, μ_L) then becomes a graphing (recall that μ_L denotes the Lebesgue measure).

An equivalent way of thinking of this graphing is as follows: Let $T : S^1 \to S^1$ be the rotation of the unit circle by an angle which is an irrational multiple of 2π. We get a Borel graph on S^1 by declaring $(x, y) \in S^1 \times S^1$ to be an edge if and only if $T(x) = y$ or $T^{-1}(x) = y$. Equipping the circle with the Haar measure μ_H, this Borel graph is in fact a graphing [Lov12, Example 18.17]. We will denote this graphing by G. This is a connected graphing because if A is a measurable subset of S^1 such that $e_G(A, A^c) = 0$, then we would have that $T^{-1}(A) \cap A^c$ has measure 0 and hence A is $T-$invariant. By ergodicity of the action of T on (S^1, μ_H), we infer that A is either of zero or full measure.

We show that the Cheeger constant of this graphing is zero. Let X be a small arc of the circle with one end-point at $(1, 0)$.

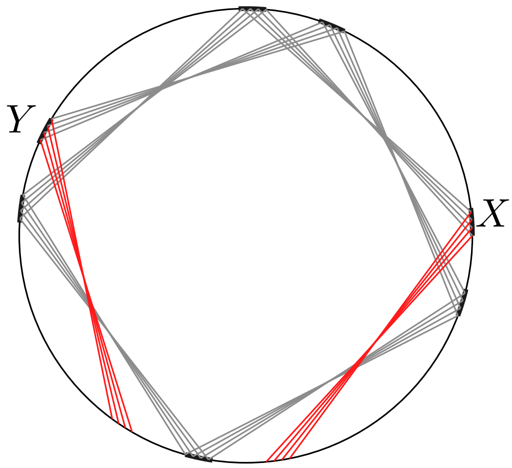

FIGURE 4. Graphing corresponding to an irrational rotation of the circle.

Given $N > 0$, we can choose X small enough so that

(1) $T^i(X) \cap T^j(X) = \emptyset$, for $0 \leq i < j \leq N$,

(2) For $A := X \sqcup T(X) \sqcup \cdots \sqcup T^N(X)$, $\mu_H(A) \leq \frac{1}{2}$.

Write $Y = T^N(X)$. The only edges that contribute to $e_G(A, A^c)$ are the ones going from Y to $T(Y)$ and the ones going from $T^{-1}(X)$ to X. These are shown in red in the above figure. Thus $e_G(A, A^c) \leq 2\mu_H(X)$. Therefore we have

$$h(G) \leq h_A(G) \leq \frac{e_G(A, A^c)}{\mathrm{vol}_G(A)} \leq 2\frac{\mu_H(A)}{(N+1)\mu_H(A)} = 2/(N+1) \tag{5.4}$$

Since N is arbitrary, we conclude that $h(G) = 0$.

Remark 5.6. *An example of a connected graphing G with positive $\lambda(G)$ and hence (by Theorem 4.6) positive Cheeger constant $h(G)$ has been described by Lovasz in* [Lov12, Example 21.5] *under the rubric of 'expander graphings'.*

5.3. **A necessary and sufficient condition for connectedness of a graphon.** In the special case that a graphon W has degree of every vertex uniformly bounded below, we shall now proceed to give a necessary and sufficient condition in terms of the Cheeger constant for W to be connected. This is analogous to the statement that a finite graph is connected if and only if its Cheeger constant is positive.

Proposition 5.7. *Let $\varepsilon > 0$ and W be a graphon such that the degree $d_W(x) \geq \varepsilon$ for all $x \in I$. Then W is connected if and only if $h(W) > 0$.*

We provide two proofs of the above result. The first is an application of Theorem 3.5 the Buser inequality for graphons and is essentially self-contained using some basic facts about compact operators. The second proof uses a structural lemma about connected graphons proved in [BBCR10].

Definition. [Lim96, pg. 196] We say that $\lambda \in \mathbf{R}$ is an **approximate eigenvalue** of a bounded linear operator $T : H \to H$ of a Hilbert space H if the image of $T - \lambda I$ is not bounded below.

Lemma 5.8. [Lim96, Lemma 27.5(a)] *For any bounded linear self-adjoint operator $T : H \to H$ on a Hilbert space H, we have that*

$$\inf_{x \in H:\ \|x\|=1} \langle Tx, x\rangle \tag{5.5}$$

is an approximate eigenvalue of T.

Lemma 5.9. [Lim96, Lemma 28.4] *If $T : H \to H$ is a compact operator then every approximate eigenvalue of T is an actual eigenvalue of T.*

Lemma 5.10. *Let W be a connected graphon with d_W bounded below by a positive real. Then the map $(1/d_W)T_W : L^2(I,\nu) \to L^2(I,\nu)$ is a compact operator.*

Proof. Since d_W is bounded below, it follows that $L^2(I,\mu_L)$ and $L^2(I,\nu)$ have comparable norms. Therefore $I : L^2(I,\mu_L) \to L^2(I,\nu)$ is a bounded linear isomorphism. The operator $T_W : L^2(I,\mu_L) \to L^2(I,\mu_L)$ is compact [Lov12, Section 7.5]. Since d_W is bounded below, i.e. $1/d_W$ is L^∞, it follows that the operator $(1/d_W)T_W : L^2(I,\mu_L) \to L^2(I,\mu_L)$ is also compact.

Take any bounded sequence (f_n) in $L^2(I,\nu)$. Then (f_n) is bounded in $L^2(I,\mu_L)$ too because of comparability of norms. Since $(1/d_W)T_W : L^2(I,\mu_L) \to L^2(I,\mu_L)$ is compact, there exists a subsequence (f_{n_k}) such that $(1/d_W)(T_W f_{n_k})$ converges in $L^2(I,\mu_L)$. Again, the comparability of norms give that $(1/d_W)(T_W f_{n_k})$ converges in $L^2(I,\nu)$ as required. ∎

Note that for any graphon W, the Laplacian $\Delta_W : L^2(I,\nu) \to L^2(I,\nu)$ restricts to a linear operator $\Delta_W : \mathbb{1}_v^\perp \to \mathbb{1}_v^\perp$.

Proof (of Proposition 5.7)**.** If $h(W) > 0$ then clearly W is connected. So we need to prove the other direction. Let W be a connected graphon with $d_W(x) \geq \varepsilon$ for all

$x \in I$. Lemma 5.10 ensures that $(1/d_W)T_W$ is a compact operator on $L^2(I, \nu)$ and it is easy to check that it restricts to a linear operator from $\mathbb{1}_v^\perp$ to itself. Throughout we will think of Δ_W and $(1/d_W)T_W$ as linear operators in $\mathbb{1}_v^\perp$. Now by Lemma 3.39 $\lambda(W)$ is an approximate eigenvalue of Δ_W. Thus the image of $\Delta_W - \lambda(W)I = (1-\lambda(W))I - (1/d_W)T_W$ is not bounded below in $\mathbb{1}_v^\perp$. Hence $1-\lambda(W)$ is an approximate eigenvalue of $(1/d_W)T_W$. But $(1/d_W)T_W : \mathbb{1}_v^\perp \to \mathbb{1}_v^\perp$ is a compact operator, and thus by Lemma 5.9 we have that $1-\lambda(W)$ is in fact an eigenvalue of $(1/d_W)T_W$. Therefore $\lambda(W)$ is an eigenvalue of Δ_W. Let $f \in \mathbb{1}_v^\perp$ be nonzero such that $\Delta_W f = \lambda(W) f$. If $h(W)$ were equal to 0, then by the Buser inequality (Theorem 3.5) we have that $\lambda(W) = 0$. Thus $\Delta_W f = 0$, which is equivalent to saying that $df = 0$. But as observed in the first paragraph of Section 3.3 we then have that f is a constant function and hence the only way it can belong to $\mathbb{1}_v^\perp$ is that $f = 0$–a contradiction. ■

Now we give the second proof of Proposition 5.7.

Definition 5.11. [BBCR10] *Let W be a graphon and $0 < a, b < 1$ be real numbers. An (a,b)-**cut** in W is a partition $\{A, A^c\}$ of I with $a < \mu_L(A) < 1-a$ such that $e_W(A, A^c) \leq b$.*

Note that a graphon is connected if and only if it admits no $(a,0)$-cut for any $0 < a \leq 1/2$.

Lemma 5.12. [BBCR10, Lemma 7] *Let W be a connected graphon and $0 < a < 1/2$. Then there is some $b > 0$ such that W admits no (a,b)-cut.*

Alternate Proof (of Proposition 5.7 using Lemma 5.12)**.** Let W be a graphon with $d_W(x) \geq \varepsilon$ for all x. We prove the non-trivial direction. Assume W is connected. We will show that $h(W) > 0$. Assume on the contrary that $h(W) = 0$. Then for each $n \geq 1$ there is a measurable subset A_n of I with $0 < \mu_L(A_n) < 1$ such that $h_{A_n}(W) < 1/n$. After passing to a subsequence, there are two cases to consider.

Case 1: $\mu_L(A_n) \to 0$ as $n \to \infty$.

In this case we have for large n that

$$\begin{aligned} h_{A_n}(W) &= \frac{e(A_n, A_n^c)}{\mathrm{vol}(A_n)} = \frac{\mathrm{vol}(A_n) - \int_{A_n \times A_n} W}{\mathrm{vol}(A_n)} \\ &= 1 - \frac{\int_{A_n \times A_n} W}{\mathrm{vol}(A_n)} \geq 1 - \frac{\mu_L(A_n)^2}{\varepsilon \mu_L(A_n)} \\ &\geq 1 - \mu_L(A_n)/\varepsilon \end{aligned} \tag{5.6}$$

But this contradicts the assumption that $h_{A_n}(W) < 1/n$ for all n.

Case 2: $\mu_L(A_n) \to t$ for some $t > 0$.

Let $0 < a < 1/2$ be such that $a < t < 1-a$. Now by Lemma 5.12, there is a $b > 0$ such that W admits no (a,b)-cut. Therefore for all n large enough we have

$$\begin{aligned} h_{A_n}(W) &= \frac{e(A_n, A_n^c)}{\min\{\mathrm{vol}(A_n), \mathrm{vol}(A_n^c)\}} \\ &\geq \frac{b}{\varepsilon \min\{\mu_L(A_n), \mu_L(A_n^c)\}} \geq \frac{b}{\varepsilon(1-a)} \end{aligned} \tag{5.7}$$

which again contradicts the assumption that $h_{A_n}(W) < 1/n$ for all n. ■

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