PROPERTY (T) FOR FIBER PRODUCTS

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ABSTRACT. We study when the fiber product of groups with Property (T) has Property (T).

1. INTRODUCTION

It is well-known and easy to see that the product of two groups $G_1 \times G_2$ has Kazhdan's property (T) [2] if and only if both G_1 and G_2 have Property (T). The aim of this article is to study the following question:

Question 1.1. *Let G_1 and G_2 have Property (T). Let $q_1 : G_1 \to H$ and $q_2 : G_2 \to H$ be homomorphisms. When does $G_1 \times_H G_2$ have Property (T)?*

It turns out that Question 1.1 does not have a uniform answer, and we provide both sufficient conditions and counter-examples. In Proposition 1.6, we show that the existence of a section of $G_1 \to H$ guarantees that $G_1 \times_H G_2$ has Property (T) if both G_1, G_2 do.

We shall pay special attention to the case that $G_1 = G_2$ and $q_1 = q_2$, in which case $G \times_H G$ can be described as a semi-direct product (Lemma 1.3). Section 2 deals exclusively with this case, and deduces equivalent conditions for Property (T) of $G \times_H G$ in terms of

(1) Normalized functions of positive type (Theorem 2.6), and
(2) Functions conditionally of negative type (Theorem 2.11).

Given an abelian locally compact group N and the action of a Property (T) group G on it, we obtain an action of G on the space $\widehat{N}$ of equivalence classes of irreducible unitary representations of N, and hence on the space $\mathcal{M}(\widehat{N})$ of regular probability measures on $\widehat{N}$. Specializing thus to the case of abelian kernels, we deduce a further equivalent condition in terms of the space of G–fixed points $\mathcal{M}(\widehat{N})^G$ (Theorem 2.14).

In Section 3, we provide two different sets of counterexamples:

(1) The case $G \times_H G$, when the kernel of the map $G \to H$ is non-compact and central in G (Proposition 3.1),

(2) A purely group-theoretic counterexample of $G_1 \times_H G_2$ where H is not finitely presented (Theorem 3.3).

In Section 4, we return to the case of $G \times_H G \cong N \rtimes G$. We treat two special cases:

(1) N is nilpotent (Propositions 4.3 and 4.6), where we make essential use of work of Chatterji, Witte-Morris and Shah [4].
(2) Theorem 4.14, where the kernel (of the map $G \to H$) arises as the normal closure of a hyperbolically embedded subgroup in the sense of Dahmani, Guirardel and Osin [8]. The hyperbolically embedded subgroups treated here are free, surface or abelian groups. The proof of Theorem 4.14 uses Chevalley-Weil theory in an essential way.

1.1. Group-theoretic Preliminaries.
We begin with the following well known fact about when an extension of a Property (T) group has Property (T).

Lemma 1.2. [2, Remark 1.7.7] *Given a short exact sequence of groups*

$$1 \to N \to G \to Q \to 1,$$

G has Property (T) if and only if Q has Property (T) and (G, N) has relative Property (T).

Next we observe that the fiber product of two copies of the same map can be written as a semi-direct product. This observation is also made by Bass and Lubotzky in [1, §6].

Lemma 1.3. *Consider two copies of the same surjective homomorphism $f : G \to H$ with kernel N. Then $G \times_H G \cong N \rtimes G$.*

Proof. It is enough to locate a normal subgroup N' of $G \times_H G$ that is isomorphic to N and a subgroup G' which is isomorphic to G, such that $N' \cap G'$ is trivial and $G \times_H G = N'G'$. Take $N' := \{1\} \times N$ and $G' := \{(g_1, g_2) \in G \times_H G : g_1 = g_2\}$. Clearly $N' \cap G' = \{(1,1)\}$. Let $(g_1, g_2) \in G \times_H G$. Then $f(g_1) = f(g_2)$. So there exists $n \in N$ such that $ng_1 = g_2$. Hence $(g_1, g_2) = (1, n)(g_1, g_1)$. Thus $G \times_H G = N'G'$.   □

In the case of fiber product of two copies of the same surjective map $f : G \to H$, with G having Property (T), we thus have the following lemma.

Lemma 1.4. *Let G be a group which has Property (T) and $\phi : G \to \mathrm{Aut}(N)$ be an action of G on N. Then $N \rtimes G$ has Property (T) if and only if $N \rtimes \phi(G)$ has Property (T).*

Proof. Since $N \rtimes \phi(G)$ is a quotient of $N \rtimes G$, therefore $N \rtimes G$ having Property (T) implies the same for $N \rtimes \phi(G)$. On the other hand suppose $N \rtimes \phi(G)$ has Property (T). Consider the short exact sequence

$$1 \to \{1\} \times \ker(\phi) \to N \rtimes G \to N \rtimes \phi(G) \to 1.$$

Since $\{1\} \times \ker(\phi) \subset \{1\} \times G \subset N \rtimes G$, with G having Property (T), therefore $(N \rtimes G, \{1\} \times \ker(\phi))$ has relative Property (T). Thus by Lemma 1.2, $N \rtimes G$ has Property (T).   □

Corollary 1.5. *With notation as in Lemma 1.4, if $\phi(G)$ is finite and N does not have Property (T), then $N \rtimes G$ does not have Property (T).*

Proof. By the above lemma, it is enough to prove that $\phi(G) \rtimes N$ does not have Property (T). If $\phi(G)$ is finite then N is a finite index subgroup of $N \rtimes \phi(G)$. This implies $N \rtimes \phi(G)$ has Property (T) if and only if N has Property (T). Since N does not have Property (T), nor does $\phi(G) \rtimes N$. $\qquad\square$

To conclude this preliminary section, we provide now a simple sufficient condition.

Proposition 1.6. *Let G_1 and G_2 be two groups having Property (T). If either of the surjective homomorphisms $f_1 : G_1 \to H$ and $f_2 : G_2 \to H$ splits, then $G_1 \times_H G_2$ also has Property (T).*

Proof. Let us fix that f_1 splits. The idea is to apply Lemma 1.2 to the short exact sequence
$$1 \to \{1\} \times N_2 \to G_1 \times_H G_2 \to G_1 \to 1.$$
Thus all we need to show is that $(G_1 \times_H G_2, \{1\} \times N_2)$ has relative Property (T). If we have a sequence of subgroups $N < H < G$ with H having Property (T), then (G, N) has Property (T). Now we will use the splitting of f_1 to embed a copy of G_2 in $G_1 \times_H G_2$ containing $\{1\} \times N_2$. This will finish the proof. Let ϕ be the splitting map. Then consider the subgroup $G_2' := \{(\phi(h), g) : h \in H, f_2(g) = h\}$ of $G_1 \times_H G_2$. The restriction of the projection $G_1 \times_H G_2 \to G_2$ to G_2' is an isomorphism and $\{1\} \times N_2 \subset G_2'$. $\qquad\square$

2. NECESSARY AND SUFFICIENT CONDITIONS

In this section, we shall deal with the special case of a fiber product corresponding to two copies of the same surjective homomorphism $q : G \to H$. Then, $G \times_H G \cong N \rtimes G$ by Lemma 1.3, where N is the kernel of q and the action of G on N is via conjugation. For any semidirect product $N \rtimes G$ (not necessarily coming from a fiber product), we assume that, for the purposes of this section, G has Property (T) and want to know when $N \rtimes G$ has Property (T). Property (T) can be expressed in terms of normalized functions of positive type [10, Chapitre 5, Théorème 11] or functions conditionally of negative type [2, Theorem 2.10.4]. In our situation, we show that Property (T) can be expressed in terms of G fixed points in the space of such functions on N. See Theorem 2.6 and Theorem 2.11 below. The idea is that, since G already has Property (T), it should be possible to check whether $N \rtimes G$ has Property (T) purely in terms of unitary representations or affine isometric actions of N and the G-action on N. The relation between unitary representations of a group G and functions of positive type on G is given by the GNS construction for such functions. Functions of positive type which take the value 1 at the identity e are called *normalized* and the set of normalized functions of positive type on G is denoted by $\mathcal{P}_1(G)$.

Theorem 2.1. [2, Theorem C.4.10] *There is a one to one correspondence*
$$\begin{Bmatrix} (\pi, V_\pi, v) : (\pi, V_\pi) \text{ is a unitary} \\ \text{cyclic representation of } G \text{ and} \\ v \in V_\pi \text{ a cyclic unit vector} \end{Bmatrix} / \sim \quad \leftrightarrow \quad \mathcal{P}_1(G),$$
where $(\pi, V_\pi, v) \sim (\rho, V_\rho, w)$ if there exists an isometric intertwining operator $T : V_\pi \to V_\rho$ such that $T(v) = w$. The correspondence is given by $(\pi, V_\pi, v) \mapsto (x \mapsto \langle \pi(x)v, v \rangle)$.

The relation between affine isometric actions of a group G and functions conditionally of negative type on G is given by the GNS construction for such functions. In [2] the correspondence is not stated in the form below, but can be easily derived from the relevant results there. We will supply the extra detail. First, in analogy to cyclic representations, we define a *cyclic affine action* to be an affine action $(\alpha, \mathcal{H})$ for which the span of $\{\alpha(x)(0) : x \in G\}$ is dense in $\mathcal{H}$. Also, let functions conditionally of negative type on G which take the value 0 at the identity e, be called *normalized* and let the space of such functions be denoted by $\mathcal{N}_0$.

Theorem 2.2. [2, Proposition 2.10.2] *There is a one to one correspondence*

$$\{(\alpha, \mathcal{H}) : (\alpha, \mathcal{H}) \text{ is a cyclic affine isometric action}\}/ \sim \quad \leftrightarrow \quad \mathcal{N}_0(G),$$

where $(\alpha, \mathcal{H}) \sim (\alpha_1, \mathcal{H}_1)$ *if there exists a G-equivariant orthogonal map $T : \mathcal{H} \to \mathcal{H}_1$. The correspondence is given by* $(\alpha, \mathcal{H}) \mapsto (x \mapsto \|\alpha(x)(0)\|^2)$.

Proof of well-definedness of the correspondence: Let $(\alpha, \mathcal{H})$ and $(\alpha_1, \mathcal{H}_1)$ be two cyclic affine isometric actions of G such that $\|\alpha(x)(0)\|^2 = \|\alpha_1(x)(0)\|^2 =: \psi(x)$, for all $x \in G$. Consider the kernel conditionally of negative type $\Psi : G \times G \to \mathbb{R}$ associated to ψ, that is, $\Psi(x, y) = \psi(y^{-1}x)$. Using the fact that α is a group homomorphism and each element in its image is an affine action, we get $\Psi(x, y) = \|\alpha(y^{-1}x)(0)\|^2 = \|\alpha(y^{-1}x)(0) - \alpha(e)(0)\|^2 = \|\alpha(x)(0) - \alpha(y)(0)\|^2$. Similarly $\Psi(x, y) = \|\alpha_1(x)(0) - \alpha_1(y)(0)\|^2$. We are also given that the spans of $\{\alpha(x)(0) - \alpha(e)(0) : x \in G\}$ and $\{\alpha_1(x)(0) - \alpha_1(e)(0) : x \in G\}$ are dense in $\mathcal{H}$. Now uniqueness of GNS construction for kernels conditionally of negative type [2, Theorem C.2.3] tells us that there exists a unique affine isometry $A : \mathcal{H} \to \mathcal{H}_1$, such that $\alpha_1(x)(0) = A(\alpha(x)(0))$ for all $x \in G$. In particular $A(0) = A(\alpha(e)(0)) = \alpha_1(e)(0) = 0$. An affine map that takes 0 to 0 is an orthogonal operator. Hence A is an orthogonal operator. Also clearly A is G-equivariant on the span of $\{\alpha(x)(0) : x \in G\}$, which is dense in $\mathcal{H}$. Hence by continuity A is G-equivariant on $\mathcal{H}$.   □

Now we state some results that we will need. The first result states that in a group having Property (T), one can choose almost invariant vectors as close to invariant vectors as one wishes.

Proposition 2.3. [10, Chapitre 1, Proposition 16] *Let K be a compact generating set of a group G having Property (T). Given any $0 < \delta \le 2$, there exists $\epsilon > 0$, such that in any unitary representation (π, V_π) of G, if v is a (K, ϵ)-invariant unit vector, then there exists an invariant unit vector $w \in V_\pi$, such that $\|v - w\| < \delta$.*

The next two results are a consequence of [2, Lemma 2.2.7] which says that among all closed balls containing a non-empty bounded subset of a real or complex Hilbert space, there exists a unique one with minimal radius. The center of this unique minimal closed ball is the *center*. The significance of this lemma for group actions via isometry is that if X is a G-invariant bounded subset of a Hilbert space then the center of X is a G-fixed point. When the action is via a unitary representation we have the following consequence.

Lemma 2.4. [10, Chapitre 3, Corollaire 11] *Let (π, V_π) be a unitary representation of a group G. Let $v \in V_\pi$ be such that $\mathrm{Re}\langle \pi(g)v, v \rangle \ge \epsilon$, for some $\epsilon > 0$ and all $g \in G$. Then $\pi(G)$ has a non-zero invariant vector.*

When the action is via affine isometries on a real Hilbert space we have the following consequence.

Lemma 2.5. [2, Proposition 2.2.9] *Let $(\alpha, \mathcal{H})$ be an affine isometric action of a group G. Let b be the corresponding cocycle. Then α has a fixed vector if and only if b is bounded.*

2.1. Normalized functions of positive type.

Theorem 2.6. *Let G be a locally compact group having Property (T), which acts continuously by automorphisms on a σ-compact, locally compact group N. This induces an action of G on the space $\mathcal{P}_1(N)$ of normalized functions of positive type on N. Let $\mathcal{P}_1(N)^G$ be the set of fixed points. Then $N \rtimes G$ has Property (T) if and only if every sequence $\{f_n\} \subset \mathcal{P}_1(N)^G$, that converges uniformly on compact subsets to the constant function 1, converges uniformly on N.*

Remark 2.7. If we put $G = \{1\}$ in the above theorem then we get back the statement of [10, Chapitre 5, Théorème 11].

The proof of [10, Chapitre 5, Théorème 11] can be adapted to our situation using a couple of preliminary Lemmas. The first lemma can be seen as an adaptation of the GNS construction to our case.

Lemma 2.8. *Any $f \in \mathcal{P}_1(N)^G$ is of the form $f(n) = \langle \Pi(n,1)v, v \rangle$, where (Π, V_Π) is a unitary representation of $N \rtimes G$ and $v \in V_\Pi$ is a unit vector which is fixed by $\Pi(\{1\} \times G)$.*

Proof. The G-action on N induces a G-action on both sides of the GNS correspondence for the group N, as given in Theorem 2.1. Denoting the G action on N by ϕ, the induced action on $\mathcal{P}_1(N)$ is given by $(g \cdot f)(x) = f(\phi(g)^{-1}x)$. The action on the equivalence class of cyclic representations is given by $(g \cdot \pi)(n) = \pi(\phi(g^{-1})n)$. Note that the map giving this correspondence is G-equivariant. Thus, if $f \in \mathcal{P}_1(N)^G$ then there exists (π, V_π, v), whose equivalence class is G-invariant, such that $f(n) = \langle \pi(n)v, v \rangle$. The G-invariance of the class of (π, V_π, v) means that for each $g \in G$ there exists a unitary operator $T_g : V_\pi \to V_\pi$ such that $T_g(g \cdot \pi)(n) = \pi(n)T_g$, for all $n \in N$, and $T_g v = v$. Since (π, V_π) is cyclic, therefore T_g is the unique operator satisfying these properties. The uniqueness implies $T_{g_1 g_2} = T_{g_1} T_{g_2}$, for all $g_1, g_2 \in G$. Now define a representation (Π, V_π) of $N \rtimes G$ as $\Pi(n,g) = \pi(n)T_g$. Let us check this is indeed a representation:

$$
\begin{aligned}
\Pi((n_1,g_1)(n_2,g_2)) \quad &= \Pi(n_1\phi(g_1)(n_2), g_1 g_2) \quad &= \pi(n_1\phi(g_1)n_2)T_{g_1 g_2} \\
= \pi(n_1)\pi(\phi(g_1)n_2)T_{g_1}T_{g_2} \quad &= \pi(n_1)(g_1^{-1} \cdot \pi)(n_2)T_{g_1}T_{g_2} \quad &= \pi(n_1)T_{g_1}\pi(n_2)T_{g_2} \\
= \Pi(n_1,g_1)\Pi(n_2,g_2). &&
\end{aligned}
$$

Now let us check that Π is continuous. That is, we have to check that for any $w \in V_\pi$, the orbit map $G \to V_\pi, (n,g) \mapsto \Pi(n,g)w$ is continuous. Since Π is unitary it is enough to check the continuity at $(1,1)$. Since $\|\Pi(n,g)w - w\| = \|\Pi(n,1)\Pi(1,g)w - \Pi(1,g)w\| + \|\Pi(1,g)w - w\|$ and $\Pi|_{N \times \{1\}} = \pi$ is already continuous, it suffices to prove that $\Pi|_{\{1\} \times G}$ is continuous. Since the subspace D, which is the linear span of $\{\pi(n)v : n \in N\}$, is dense in V_π, therefore it is enough to show that $g \mapsto \Pi(1,g)w$ is continuous for all $w \in D$. Any $w \in D$ is of the form $\sum_{i=1}^k \lambda_i \pi(n_i)v$, where $\lambda_i \in \mathbb{C}$ and $n_i \in N$. Now the continuity of $\Pi|_{\{1\} \times G}$ follows from the continuity

of ϕ, π and the following inequality:

$$\|\Pi(1,g)(\sum_{i=1}^{k}\lambda_i\pi(n_i)v) - \sum_{i=1}^{k}\lambda_i\pi(n_i)v\| = \|\sum_{i=1}^{k}\lambda_i T_g\pi(n_i)v - \sum_{i=1}^{k}\lambda_i\pi(n_i)v\|$$

$$= \|\sum_{i=1}^{k}\lambda_i\pi(\phi(g)n_i)T_g v - \sum_{i=1}^{k}\lambda_i\pi(n_i)v\|$$

$$\leq \sum_{i=1}^{k}|\lambda_i|\|\pi(\phi(g)n_i)v - \pi(n_i)v\| \qquad \square$$

The next lemma tells us that if G has Property (T) then in a representation of $N \rtimes G$ which has almost invariant vectors, the almost invariant vectors can be chosen to be G-invariant.

Lemma 2.9. *Let (π, V_π) be a unitary representation of $N \rtimes G$ which has almost invariant vectors. Let V_π^G denote the subspace of $\pi(\{1\} \times G)$-invariant vectors. If G has Property (T) then for any compact $K \subset N \rtimes G$ and $\delta > 0$, there exists a unit vector $w \in V_\pi^G$ which is (K, δ)-invariant.*

Proof. Without loss of generality we may assume that K contains a compact generating set of $\{1\} \times G$ and $\delta \leq 2$. By Proposition 2.3, there exists $\epsilon > 0$ such that given a (K, ϵ)-invariant unit vector v there exists a unit vector $w \in V_\pi^G$ such that $\|v - w\| < \delta/3$. We may assume that v is a $(K, \min(\epsilon, \delta/3))$-invariant unit vector. Then for all $k \in K$, we have

$$\|\pi(k)w - w\| \leq \|\pi(k)w - \pi(k)v\| + \|\pi(k)v - v\| + \|v - w\|$$

$$\leq \|\pi(k)\|\|v - w\| + \|\pi(k)v - v\| + \|v - w\|$$

$$< \delta/3 + \delta/3 + \delta/3 = \delta. \qquad \square$$

Proof of Theorem 2.6. ($\Rightarrow$) Let $\{f_n\}$ be a sequence in $\mathcal{P}_1(N)^G$ which converges uniformly on compact subsets to 1. Given $\delta > 0$, we wish to show that for all large n, $|f_n(x) - 1| < \delta$ for all $x \in N$. By Lemma 2.8, for each $n \in \mathbb{N}$, there exists a representation (π_n, V_n) of $N \rtimes G$ and $v_n \in V_n$ such that $f_n(x) = \langle \pi_n(x, 1)v_n, v_n \rangle$ and $\pi_n(\{1\} \times G)v_n = v_n$. By hypothesis, $N \rtimes G$ has Property (T), hence it has a compact generating set K. By Proposition 2.3, there exists $\epsilon > 0$ such that if any representation (π, V_π) of $N \rtimes G$ has a (K, ϵ)-invariant unit vector v, then there exists a unit invariant vector w such that $\|v - w\| < \delta/2$. We will apply this to the representations π_n. Let p be the projection onto the first coordinate of $N \rtimes G$. By definition of convergence on compact subsets, for large n, $|1 - f_n(x)| < \epsilon^2/2$ for all $x \in p(K)$. Then $\|\pi_n(x, 1)v_n - v_n\|^2 \leq 2|1 - \langle \pi_n(x, 1)v_n, v_n \rangle| < \epsilon^2$, for large n and $x \in p(K)$. So for any $(x, g) \in K$ and large enough n, we have $\|\pi_n(x, g)v_n - v_n\| = \|\pi_n(x, 1)\pi_n(1, g)v_n - v_n\| = \|\pi_n(x, 1)v_n - v_n\| < \epsilon$. Thus there exist unit invariant vectors $w_n \in V_n$ such that $\|v_n - w_n\| < \delta/2$. Now for large n and all $x \in N$, we have

$$|f_n(x) - 1| = |\langle \pi_n(x, 1)v_n, v_n \rangle - \langle \pi_n(x, 1)w_n, w_n \rangle|$$

$$= |\langle \pi_n(x, 1)v_n, v_n \rangle - \langle \pi_n(x, 1)v_n, w_n \rangle + \langle \pi_n(x, 1)v_n, w_n \rangle$$

$$- \langle \pi_n(x, 1)w_n, w_n \rangle|$$

$$\leq |\langle \pi_n(x, 1)v_n, v_n - w_n \rangle| + |\langle \pi_n(x, 1)(v_n - w_n), w_n \rangle|$$

$$\leq \|\pi_n(x, 1)\|\|v_n\|\|v_n - w_n\| + \|\pi_n(x, 1)\|\|v_n - w_n\|\|w_n\|$$

$$\leq 2\|v_n - w_n\| < \delta.$$

($\Leftarrow$) Given a representation π of $N \rtimes G$ which has almost invariant vectors, we will show that it must have an invariant vector. Since N is σ-compact, there exists a sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets of N, such that $\cup_i K_i = N$. By Lemma 2.9, for each n, there exists $v_n \in V_\pi^G$ which is $(K_n, 1/n)$-invariant. Define functions f_n on N as $f_n(x) := \langle \pi(x, 1)v_n, v_n \rangle$. Note that the sequence $\{f_n\} \subset \mathcal{P}_1(N)^G$ and converges uniformly to 1 on compact subsets of N. To see this, take any compact $K \subset N$, then $K \subset K_n$ for all large n and hence for all $k \in K$, we have $|f_n(k) - 1| = |\langle \pi(k, 1)v_n, v_n \rangle - 1| \leq \|\pi(k, 1)v_n - v_n\| < 1/n$. By hypothesis, $f_n \to 1$ uniformly on N. Then for large n, we have $\sup_{x \in N} |f_n(x) - 1| \leq 1/2$. This implies $\inf_{x \in N} \operatorname{Re}\langle \pi(x, 1)v_n, v_n \rangle \geq 1/2$. Then for any $(x, g) \in N \rtimes G$, we have $\operatorname{Re}\langle \pi(x, g)v_n, v_n \rangle = \operatorname{Re}\langle \pi(x, 1)\pi(1, g)v_n, v_n \rangle = \operatorname{Re}\langle \pi(x, 1)v_n, v_n \rangle \geq 1/2$. Now Lemma 2.4 implies that π has an invariant vector. $\square$

For applications to finitely generated groups we record the discrete version of Theorem 2.6.

Corollary 2.10. *Let G be a group having Property (T), which acts by automorphisms on a countable discrete group N. This induces an action of G on the space $\mathcal{P}_1(N)$ of normalized functions of positive type on N. Let $\mathcal{P}_1(N)^G$ be the set of fixed points. Then $N \rtimes G$ has Property (T) if and only if every sequence $\{f_n\} \subset \mathcal{P}_1(N)^G$, that converges pointwise to the constant function 1, converges uniformly on N.*

2.2. **Functions conditionally of negative type.** In this subsection, we shall continue to deal with the special case of a fiber product corresponding to two copies of the same surjective homomorphism $f : G \to H$. By Lemma 1.3, it suffices to deal with semi-direct products. The Delorme-Guichardet Theorem [2, Theorem 2.12.4] says that, for σ-compact locally compact groups, Property (T) is equivalent to Property (FH). Recall that a group G is said to have Property (FH) if any affine isometric action of G on a real Hilbert space has a fixed point. Property (FH) can be expressed in terms of functions conditionally of negative type [2, Theorem 2.10.4]. We now derive an analogous statement in the case of a semidirect product $N \rtimes G$, where G already has Property (FH).

Theorem 2.11. *Let G be a topological group having Property (FH), which acts continuously by automorphisms on a topological group N. This induces an action of G on the space of functions conditionally of negative type on N. Then $N \rtimes G$ has Property (FH) if and only if every G-invariant function conditionally of negative type on N is bounded.*

Remark 2.12. Putting $G = \{e\}$ in the above theorem give us back the statement of [2, Theorem 2.10.4, (i)$\Leftrightarrow$(iii)].

As before, we prove a GNS construction lemma that will allow us to adapt the proof of [2, Theorem 2.10.4] to a proof of Theorem 2.11.

Lemma 2.13. *Let ψ be a G-invariant function conditionally of negative type on N. Then there exists an affine isometric action $(A, \mathcal{H})$ of $N \rtimes G$ such that $\psi(x) = \|A(x, 1)(0)\|^2$.*

The proof is very similar to that of Lemma 2.8, so we omit it.

Proof of Theorem 2.11. ($\Rightarrow$) Let ψ be a G-invariant function conditionally of negative type on N. By Lemma 2.13, there exists an affine isometric action $(\alpha, \mathcal{H})$ of $N \rtimes G$ such that $\psi(x) = \|\alpha(x, 1)(0)\|^2$. Since $N \rtimes G$ has Property (FH) therefore α has a fixed point. By Lemma 2.5, the corresponding 1-cocycle $(x, g) \mapsto \alpha(x, g)(0)$

is bounded. Since ψ is the restriction of the norm square of this map to $N \times \{1\}$, therefore ψ is also bounded.

($\Leftarrow$) Let $(\alpha, \mathcal{H})$ be an affine isometric action of $N \rtimes G$. We wish to show that α has a fixed point. By Lemma 2.5, it is enough to show that the cocycle corresponding to α is bounded. That is, the map $N \rtimes G \to \mathcal{H}, (x, g) \mapsto \alpha(x, g)(0)$ is bounded. The restriction of α to $\{1\} \times G$ is a affine isometric action of G. Since G has Property (FH), there exists a $v \in \mathcal{H}$ such $\alpha(1, g)v = v$, for all $g \in G$. The function $\psi : N \to \mathbb{R}$ given by $\psi(x) = \|\alpha(x, 1)v - v\|^2$ is a function conditionally of negative type on N. The following calculation tells us that ψ is G-invariant.

$$\begin{aligned}
\psi(\phi(g)x) &= \|\alpha(\phi(g)x, 1)v - v\|^2 \\
&= \|\alpha((1, g)(x, 1)(1, g^{-1}))v - v\|^2 \\
&= \|\alpha(1, g)\alpha(x, 1)v - v\|^2 \\
&= \|\alpha(1, g)\alpha(x, 1)v - \alpha(1, g)v\|^2 \\
&= \|\alpha(x, 1)v - v\|^2
\end{aligned}$$

For any $(x, g) \in N \rtimes G$, $\|\alpha(x, g)v - v\|^2 = \|\alpha(x, 1)\alpha(1, g)v - v\|^2 = \|\alpha(x, 1)v - v\|^2 = \psi(x)$. Thus $\|\alpha(x, g)(0)\| \le \|\alpha(x, g)(0) - \alpha(x, g)(v)\| + \|\alpha(x, g)v - v\| + \|v\| \le \sqrt{\psi(x)} + 2\|v\|$ and the last quantity is bounded by assumption. $\square$

2.3. **Abelian kernel: Invariant Probability Measures.** Theorem 2.6 gives an equivalent condition for $N \rtimes G$ to have Property (T), when G has Property (T), in terms of G-invariant functions of positive type on N. Due to availability of spectral theory, in the case N is abelian, it is possible to give an equivalent condition in terms of G-invariant probability measures on $\widehat{N}$.

Theorem 2.14. *Let G be a locally compact group having Property (T), which acts continuously by automorphisms on a second countable locally compact abelian group N. This induces an action of G on the space $\widehat{N}$ of equivalence classes of irreducible unitary representations of N, and hence on the space $\mathcal{M}(\widehat{N})$ of regular probability measures on $\widehat{N}$. We put the subspace topology on $\mathcal{M}(\widehat{N})$ induced by the inclusion $\mathcal{M}(\widehat{N}) \subset C_c(\widehat{N})^*$, where $C_c(\widehat{N})^*$ has the weak* topology. Let $\delta_1 \in \mathcal{M}(\widehat{N})$ be the Dirac mass at 1. Let $\mathcal{M}(\widehat{N})^G$ be the set of G fixed points. Then $N \rtimes G$ has Property (T) if and only if there is no sequence $\{\mu_n\} \subset \mathcal{M}(\widehat{N})^G$, such that $\mu_n \to \delta_1$ and $\mu_n(1) = 0$, for all $n \in \mathbb{N}$.*

This result is very similar to [13, Theorem 5.1] (for discrete groups) and [7, Theorem 1, ($\neg$T)$\Leftrightarrow$(P)]. Their result gives an equivalent condition for $(N \rtimes G, N)$ to have relative Property (T) in terms of sequences in $\mathcal{M}(\widehat{N})$. Since they do not assume that G has Property (T), they put an appropriate condition on the sequences that reflects the existence of almost invariant vectors of G. Since we assume that G has Property (T), considering G-invariant measures is enough for us. The remaining two conditions are the same. The proof of [7, Theorem 1] can be adapted to our case with the necessary modifications provided by Lemma 2.9 and Lemma 2.16. For the convenience of reader we give a complete proof below.

We first prove the analogues of Theorem 2.1 and Lemma 2.8.

Lemma 2.15. *Let G be a locally compact abelian group. Let $\widehat{G}$ be its unitary dual and let $\mathcal{M}(\widehat{G})$ be the space of regular probability measures on $\widehat{G}$. There is a one to one correspondence*

$$\{(\pi, V_\pi, v) : (\pi, V_\pi) \text{ is a unitary}$$
$$\text{cyclic representation of } G \text{ and} \quad \leftrightarrow \quad \mathcal{M}(\widehat{G}),$$
$$v \in V_\pi \text{ a cyclic unit vector}\} / \sim$$

where $(\pi, V_\pi, v) \sim (\rho, V_\rho, w)$ if there exists an isometric intertwining operator $T : V_\pi \to V_\rho$ such that $T(v) = w$.

Proof. Let us describe the map which gives the correspondence. First note that any regular measure μ on $\widehat{N}$ gives us a representation $(\rho, L^2_\mu(\widehat{N}))$ of N, where $\rho(x)(f)(\chi) := \chi(x)f(\chi)$. We claim that the constant function 1 is a cyclic vector. It is enough to check that $\rho(L^1(N))1$ is dense in $L^2_\mu(\widehat{N})$. But $\rho(L^1(N))1$ is the image of $L^1(N)$ under the Fourier transform and hence is dense in $C_0(\widehat{N})$, which is itself dense in $L^2_\mu(\widehat{N})$, since μ is finite regular. The direct integral decomposition of unitary representations of locally compact abelian groups [11, Theorem 7.28] imply that any equivalence class in the left hand side of the above correspondence contains a pair $(\rho, L^2_\mu(\widehat{N}), 1)$, for some regular probability measure μ. This measure must be unique, since if there is a intertwining map $L^2_{\mu_1}(\widehat{N}) \to L^2_{\mu_2}(\widehat{N})$ sending 1 to 1, then it must be identity on $C_0(\widehat{N})$ which implies $\mu_1 = \mu_2$ by Riesz Representation Theorem. The correspondence is given by the map which sends the equivalence class of (π, V_π, v) to μ, where $(\rho, L^2_\mu(\widehat{N}), 1)$ is the unique such representation in that class. This map is a bijection by the above discussion. $\square$

Corollary 2.16. *Given $\mu \in \mathcal{M}(\widehat{N})^G$, there exists a representation $(\Pi, L^2_\mu(\widehat{N}))$ of $N \rtimes G$ such that $\Pi(\{1\} \times G)$ fixes the constant function 1 in $L^2_\mu(\widehat{N})$.*

Proof. Consider the set of equivalence classes on the left hand side of the correspondence in Lemma 2.15. In the proof of Lemma 2.8 we showed that if the class of (π, V_π, v) is fixed by G, then π can be extended to a representation Π of $N \rtimes G$ in such a way that v is fixed by $\Pi(\{1\} \times G)$. Thus it is enough to show that the bijective map between the two sets in Lemma 2.15 is G-equivariant. We need to check that if μ is the measure corresponding to (π, V_π, v) then $g \cdot \mu$, the pushforward of μ by g, is the measure corresponding to $(g \cdot \pi, V_\pi, v)$. For this we note that the measure μ is in fact $P_{v,v}$, where P is the V_π-projection valued measure associated to the representation (π, V_π) [11, Theorem 4.45]. The projection valued measure associated to $(g \cdot \pi, V_\pi)$ is $g \cdot P$. Hence $(g \cdot P)_{v,v} = g \cdot P_{v,v} = g \cdot \mu$. $\square$

Next we characterize the existence of a non-zero invariant vector in $L^2_\mu(\widehat{N})$ in terms of the measure μ. This is the reason for putting the condition $\mu_n(\{1\}) = 0$ in Theorem 2.14. The argument given here is a part of the proof of [7, Theorem 7].

Lemma 2.17. *Let N be a second countable locally compact group and let μ be a regular probability measure on $\widehat{N}$. Let $(\rho, L^2_\mu(\widehat{N}))$ be the representation of N given by $\rho(x)(f)(\chi) := \chi(x)f(\chi)$. Then ρ has a non-zero invariant vector if and only if $\mu(\{1\}) \neq 0$.*

Proof. ($\Leftarrow$) The characteristic function $\mathbb{I}_{\{1\}}$ is a non-zero invariant vector.
($\Rightarrow$) Let f be an invariant vector. We wish to show that $f = 0$ μ-almost everywhere. That is, if $Z := \{\chi \in \widehat{G} : f(\chi) \neq 0\}$, then we wish to show that $\mu(Z) = 0$. Invariance of f means, for any $x \in N$, the function $\chi \mapsto \chi(x)f(\chi) - f(\chi)$ vanishes μ-almost everywhere. Hence there exists measurable $E_x \subset \widehat{G}$, with $\mu(E_x) = 1$, such that $(\chi(x) - 1)f(\chi) = 0$ for all $\chi \in E_x$. Let K_x be the closed set $\{\chi \in \widehat{G} : \chi(x) = 1\}$.

Then $\chi \in E_x \cap K_x^c$ implies $f(\chi) \neq 0$, that is, $\chi \in Z^c$. Hence $Z \subset E_x^c \cup K_x$ for all $x \in N$. Since $\chi \neq 1$ means there exists $x \in N$ such that $\chi(x) \neq 1$, we have $\{1\}^c = \cup_{1 \neq x \in N} K_x^c$. Since N is second countable, there exists a countable subset $\{x_i\}_i \subset N$ such that $\{1\}^c = \cup_i K_{x_i}^c$. Therefore by hypothesis, $\mu(\cap_i K_{x_i}) = \mu(\{1\}) = 0$. Since $Z \subset \cap_i(E_{x_i}^c \cup K_{x_i}) \subset \cap_i((\cup_i E_{x_i}) \cup K_{x_i}) = (\cup_i E_{x_i}) \cup (\cap_i K_{x_i})$, we have $\mu(Z) \leq \sum_i \mu(E_{x_i}) + \mu(\cap_i K_{x_i}) = 0$, that is, $\mu(Z) = 0$. $\qquad\square$

The next result reformulates the convergence $\mu_n \to \delta_1$ in weak* topology on $\mathcal{M}(\widehat{N}) \subset C_c(\widehat{N})^*$ in terms of integration of the evaluation maps at each $x \in N$.

Lemma 2.18. [7, Lemma 11] *If N is σ-compact, then $\mu_n \to \delta_1$ in weak* topology if and only if for all $x \in N$, $\int_{\widehat{N}} \chi(x) d\mu(\chi) \to 1$, uniformly on compact subsets of N.*

Now we prove the main result of this subsection.

Proof of Theorem 2.14. ($\Rightarrow$) Let $\{\mu_n\}$ be a sequence in $\mathcal{M}(\widehat{N})^G$ such that $\mu_n \to \delta_1$ and $\mu_n(\{1\}) = 0$, for all $n \in \mathbb{N}$. We will show that $N \rtimes G$ does not have Property (T). For that we will produce a unitary representation Π of $N \rtimes G$, which has almost invariant vectors but no invariant vector. By Corollary 2.16, for each $n \in \mathbb{N}$, there exists a representation $(\Pi_n, L^2_{\mu_n}(\widehat{N}))$ of $N \rtimes G$ such that $\Pi_n(\{1\} \times G)1 = 1$. Let $\Pi := \oplus_n \Pi_n$. By Lemma 2.17, Π_n does not have a non-zero N-invariant vector. Hence Π does not have a non-zero N-invariant vector. In particular, Π does not have a non-zero $N \rtimes G$ invariant vector. On the other hand we have

$$\|\Pi_n(x,1)(1) - 1\|^2 = \int_{\widehat{N}}(\chi(x) - 1)(\overline{\chi(x)} - 1)d\mu_n(\chi) = 2\mathrm{Re}\big(1 - \int_{\widehat{N}}\chi(x)d\mu_n(\chi)\big).$$

Since $\mu_n \to \delta_1$, Lemma 2.18 implies that the right hand side of the above equation converges to 0 uniformly on compact subsets of N as $n \to \infty$. Together with the fact that $1 \in L^2_{\mu_n}(\widehat{G})$ is $\{1\} \times G$ invariant, this implies $\Pi = \oplus_n \Pi_n$ has almost invariant vectors.

($\Leftarrow$) Given a representation π of $N \rtimes G$ which has almost invariant vectors, but no invariant vector, we will produce a sequence $\{\mu_n\}$ of regular G-invariant probability measures on $\widehat{N}$ such that $\mu_n \to \delta_1$ and $\mu_n(\{1\}) = 0$, for all $n \in \mathbb{N}$. Let $K_1 \subset K_2 \subset \cdots$ be a sequence of compact sets such that $\cup_i K_i = N$. By Lemma 2.9, for each n, there exists a unit vector $v_n \in V_\pi^G$ which is $(K_n, 1/n)$-invariant. Consider the cyclic subrepresentation π_n of $\pi|_N$ generated by v_n. Via the correspondence defined in Lemma 2.15, there exists a regular probability measure μ_n such that the pair (π_n, v_n) is equivalent to the pair $(\rho_n, 1)$, where $(\rho_n, L^2_{\mu_n}(\widehat{N}))$ is the representation given by $(\rho_n(x)f)(\chi) := \chi(x)f(\chi)$. Moreover $\pi(g)$ intertwines $g \cdot \pi_n$ and π_n, while sending v_n to v_n. Thus the class of (π_n, v_n) is a fixed point under G-action, hence $\mu_n \in \mathcal{M}(\widehat{N})^G$. Since π has no invariant vector, π_n has no invariant vector and hence ρ_n has no invariant vector. Therefore, by Lemma 2.17, $\mu_n(\{1\}) = 0$. On the other hand $1/n^2 > \|\pi(x)v_n - v_n\|^2 = \|\rho_n(x)1 - 1\|^2 = \int_{\widehat{N}}|\chi(x) - 1|^2 d\mu_n(\chi)$ for all $x \in K_n$. This implies that $\int_{\widehat{N}}\chi(x)d\mu_n(\chi) \to 1$ uniformly on compact subsets of N. Hence, by Lemma 2.18, $\mu_n \to \delta_1$ in the weak* topology. $\qquad\square$

3. Counterexamples

Now we give examples of fiber products where both groups have Property (T) but the fiber product does not.

3.1. **Counterexample: non-compact center.** Let H be any simple Lie group with trivial center associated to one of the higher rank Hermitian symmetric spaces of non-compact type. Let G be the universal cover of H. Then $G \to H$ is an infinite cover and G has Property (T). See [2, Remark 3.5.5 (iii)]. Now it follows from the following proposition that $G \times_H G$ does not have Property (T). For an example with G discrete, replace H with a lattice in it and G with the pre-image of the lattice in G.

Proposition 3.1. *Let $G \to H$ be a surjective group homomorphism whose kernel N is contained in the center Also assume that N is not compact. Then $G \times_H G$ does not have Property (T).*

Proof. By Lemma 1.3, $G \times_H G \cong N \rtimes G$. Since N is contained in the center of G, therefore the G-action on N is trivial. Thus $G \times_H G \cong N \times G$, which cannot have Property (T) since its quotient N being non-compact abelian, does not have Property (T). $\square$

3.2. **Counterexample: Infinite presentation.** The basic idea for the counterexamples of this section is to show the existence of fiber products of discrete groups with Property (T), which are not finitely generated and hence do not have Property (T). Our main result is the following.

Theorem 3.2. *For $i = 1, 2$, let $f_i : G_i \to H$, be surjective homomorphisms from finitely presented groups G_i, having Property (T), to a group H which is not finitely presented. Then $G_1 \times_H G_2$ does not have Property (T).*

This will follow immediately from:

Theorem 3.3. *For $i = 1, 2$, let $f_i : G_i \to H$, be surjective homomorphisms from finitely presented groups G_i to a group H which is not finitely presented. Then $G_1 \times_H G_2$ is not finitely generated.*

To produce examples of surjective homomorphisms from finitely presented groups with Property (T) to groups that are not finitely presented, we use a result of Shalom [2, Theorem 3.4.5] which asserts that any discrete group with Property (T) is the quotient of a finitely presented group with Property (T). In particular, if we start with a group H which has Property (T) but is not finitely presented, this result guarantees the existence of a finitely presented group G with Property (T) and a surjective homomorphism $f : G \to H$. Examples of groups having Property (T) but which are not finitely presented are due to Margulis, $H = \mathrm{SL}_3(\mathbb{F}_p[X])$, Cornulier, $H = \mathrm{Sp}_4(\mathbb{Z}[1/p]) \ltimes \mathbb{Z}[1/p]^4$, p prime, and Gromov, H is infinite torsion quotient of uniform lattices in $\mathrm{Sp}(n, 1), n \geq 2$. For more details see [2, §3.4].

To prepare for the proof of Theorem 3.3 we need some lemmas.

Lemma 3.4. *Let $f_i : G_i \to H$ be a surjective homomorphism from a finitely generated group G_i, $i = 1, 2$. Then there exists a finitely generated free group F and surjective homomorphisms $p_i : F \to G_i$, $i = 1, 2$, such that $f_1 \circ p_1 = f_2 \circ p_2$.*

Proof. Let $\{a_1, \cdots, a_m\}$ be a generating set of G_1. Then $\{f_1(a_1), \cdots, f_1(a_m)\}$ is a generating set of H. For each $1 \leq i \leq m$, choose $c_i \in G_2$ such that $f_1(a_i) = f_2(c_i)$. Let $\{b_1, \cdots, b_n\}$ be a generating set of G_2. For each $1 \leq j \leq n$, there exists a word w_j such that $f_2(b_j) = w_j(f_1(a_1), \cdots, f_1(a_m))$. Since $f_2(w_j(c_1, \cdots, c_m)) = f_2(b_j)$, there exists $d_j \in \ker f_2$, such that $b_j = w_j(c_1, \cdots, c_m)d_j$. Thus $\{c_1, \cdots, c_m, d_1, \cdots, d_n\}$ is

another generating set of G_2. Let F be a free group with $m + n$ generators. Define $p_1 : F \to G_1$ by sending the first m generators to $a_1, \cdots, a_m$, respectively, and the remaining to identity. Define $p_2 : F \to G_2$ by sending the first m generators to $c_1, \cdots, c_m$, respectively, and the last n to $d_1, \cdots, d_n$, respectively. Clearly $f_1 \circ p_1 = f_2 \circ p_2$. $\square$

Remark 3.5. The choice of generators in the above proof is same as that in [9, Lemma 9.5].

Lemma 3.6. *Let G be a finitely generated group with subgroups N, H, such that H a subset of the normalizer of N and $G = NH$. Suppose $N \cap H$ is a normally finitely generated subgroup of H. Let $\phi : H \to \mathrm{Aut}(N)$ be the conjugation action of H on N. Then there exists a finite subset $S \subset N$ such that $\phi(H)(S)$ generates N.*

Proof. Generating sets will always be assumed to be symmetric. If A is a symmetric subset of a group then the subgroup generated by A will be denoted by $\langle A \rangle$. We have the following short exact sequence of groups:

$$1 \to N \cap H \to N \rtimes H \to G \to 1.$$

The first homomorphism is given by $x \mapsto (x, x^{-1})$ and the second is given by $(n, h) \mapsto nh$. Let T be a finite generating set of G. Let $\widetilde{T}$ be a lift of T in $N \rtimes H$ (again symmetric). Let $S_1 := \{n \in N : (n, h) \in \widetilde{T}\}$ and let S_2 be a symmetric finite set in $N \cap H$ whose normal closure in H is $N \cap H$. We will show that $S := S_1 \cup S_2$ satisfies $\langle \phi(H)(S) \rangle = N$. Let $n \in N$. Then $(n, 1)$ is of the form $(n', h')(x, x^{-1})$, where $(n', h') \in \langle \widetilde{T} \rangle$ and $x \in N \cap H$. We claim that the first coordinate of any element of $\langle \widetilde{T} \rangle$ belongs to $\langle \phi(H)(S_1) \rangle$. By induction on word length in $\widetilde{T}$, let us assume that $(n_1, h_1) \in \langle \widetilde{T} \rangle$ such that $n_1 \in \langle \phi(H)(S_1) \rangle$. For $(n_2, h_2) \in \widetilde{T}$, the first coordinate of $(n_1, h_1)(n_2, h_2)$ is $n_1 \phi(h_1)(n_2)$ which is indeed in $\langle \phi(H)(S_1) \rangle$, finishing the induction argument. Thus $n' \in \langle \phi(H)(S_1) \rangle$. On the other hand $x \in \langle \phi(H)(S_2) \rangle$. Hence $n = n'\phi(h')(x) \in \langle \phi(H)(S_1 \cup S_2) \rangle$. $\square$

The following lemma is standard. We include a proof for completeness.

Lemma 3.7. *Let N be a normally finitely generated subgroup of a finitely presented group G. Then G/N is finitely presented.*

Proof. First note that the statement is true by definition if G is a finitely generated free group. For a general finitely presented group G there exists a finitely generated free group F and a surjective homomorphism $p : F \to G$. Consider the composition, $F \to G \to G/N$, with the quotient homomorphism $G \to G/N$. Then it is enough to show that $p^{-1}(N)$ is a normally finitely generated subgroup of F. Let $S \subset N$ be a finite set whose normal closure in G is N. Let $\widetilde{S}$ be a lift of S in F. Since G is finitely presented, there exists a finite subset T of $R := \ker(p)$ such that the normal closure of T in F is R. We will show that $p^{-1}(N)$ is the normal closure of $\widetilde{S} \cup T$. Let $w \in p^{-1}(N)$. Then w is of the form $w'r$, for some w' in the normal closure of $\widetilde{S}$ and $r \in R$. Since r is in the normal closure of T, w is in that of $\widetilde{S} \cup T$. $\square$

Proof of Theorem 3.3. By Lemma 3.4 there exists a finitely generated free group F and surjective homomorphisms $p_i : F \to G_i$, $i = 1, 2$, such that $f_1 \circ p_1 = f_2 \circ p_2$. Let $R_i = \ker(p_i), i = 1, 2$ and $R = \ker(f_1 \circ p_1) = \ker(f_2 \circ p_2)$. For $i = 1, 2$, since G_i is finitely presented, R_i is normally finitely generated in F. To use Lemma 3.6 we will define two subgroups of $G_1 \times_H G_2$, one of which is normal. The normal

subgroup is $R/R_1 \times \{1\}$. The other subgroup is the image of the homomorphism $\Delta : F \to G_1 \times_H G_2$ sending w to $(p_1(w), p_2(w))$. Since $R/R_1 = \ker(f_1)$, we have $(R/R_1 \times \{1\}) \Delta F = G_1 \times_H G_2$. The intersection of the two subgroups is ΔR_2. Note that ΔR_2 is normally finitely generated in ΔF since R_2 is so in F. Thus the short exact sequence of groups

$$1 \to \Delta R_2 \to (R/R_1 \times \{1\}) \rtimes \Delta F \to G_1 \times_H G_2 \to 1,$$

satisfy the hypothesis of Lemma 3.6, if we assume that $G_1 \times_H G_2$ is finitely generated. Assuming this, there exists a finite $S \subset R/R_1 \times \{1\}$ such that $\phi(\Delta F)(S)$ generates $R/R_1 \times \{1\}$, where ϕ is the conjugating action of ΔF on $R/R_1 \times \{1\}$ in $G_1 \times_H G_2$. We observe that the action ϕ factors through the conjugating action of $G_1 \cong F/R_1$ on its normal subgroup R/R_1. That is, if $\pi_1 : \Delta F \to G_1$ is the projection onto the first coordinate (which is surjective) and ψ is the conjugating action of G_1 on R/R_1 then $\phi = \psi \circ \pi_1$, when we identify R/R_1 with $R/R_1 \times \{1\}$. With this identification, R/R_1 is generated by $\psi(G_1)(S)$. Now we apply Lemma 3.7 to conclude that $H \cong G_1/(R/R_1)$ is finitely presented. This is a contradiction. $\qquad\square$

Remark 3.8. In [1, Lemma 6.2] Bass and Lubotzky show that in the special case of fiber product of two copies of the same map $G \to H$, if G is finitely generated and H is finitely presented then $G \times_H G \cong N \rtimes G$ is finitely generated. In this special case, Lemmas 3.6 and 3.7 say that the converse is true if G is additionally assumed to be finitely presented.

4. Special kernels

4.1. **Finitely generated abelian and nilpotent kernels.** We saw in the previous section that if N is normally infinitely generated in G then $G \times_H G$ does not have Property (T). Let us assume now that N is a finitely generated abelian group. Let N_f and N_t denote the free and torsion parts of N respectively. Since G acts on N_f and N_t separately, we have $G \times_H G \cong N \rtimes G \cong (N_f \times N_t) \rtimes G \cong (N_f \rtimes G) \times (N_f \rtimes G)$. Since N_t is finite, $N_t \rtimes G$ has Property (T). Thus $N \rtimes G$ has Property (T) if and only if $N_f \rtimes G$ has Property (T). Thus, when N is finitely generated abelian, the question of $N \rtimes G$ having Property (T) depends only on the rank of N. If rank of N is n, then $N_f \cong \mathbb{Z}^n$ and $\mathrm{Aut}(N_f) \cong \mathrm{GL}(n, \mathbb{Z})$. With this identification, if ϕ denotes the action of G on N_f, then $\phi(G) \subset \mathrm{GL}(n, \mathbb{Z})$. We know that $\mathrm{GL}(n, \mathbb{Z})$ has Property (T) if and only if $n > 2$. This fact is reflected in Proposition 4.3 below. First we need two facts that we now quote. They are special cases of the cited results.

Theorem 4.1. [6, Theorem 2] *Any subgroup of* $\mathrm{SL}(2, \mathbb{Z})$ *has Haagerup property.*

Let H, N be subgroups of a group G, then the triple (G, H, N) is said to have relative Property (T) if any unitary representation of G, whose restriction to H has almost invariant vectors, has an N invariant vector. Note that if (G, H, N) has relative Property (T), then so does the pair (G, N).

Proposition 4.2. [15, Corollary 3.1] *Let G be a subgroup of* $\mathrm{GL}(n, \mathbb{R})$. *If $\mathbb{R}^n$ is G-irreducible then either* $(\mathbb{R}^n \rtimes G, G, \mathbb{R}^n)$ *has relative Property (T) or G is contained in a compact extension of a diagonalizable group.*

Now we state our main result in the case where N is finitely generated abelian.

Proposition 4.3. *Let G have Property (T). Let $f : G \to H$ be a surjective group homomorphism whose kernel N is a finitely generated abelian subgroup of rank n. Let $\phi : G \to \mathrm{GL}(n, \mathbb{Z})$ be the conjugation action of G on the free part of N.*

    (1) *If $n = 1$ or 2, then $G \times_H G$ does not have Property (T).*
    (2) *If $n > 2$ and $\phi(G)$ is Zariski dense, then $G \times_H G$ has Property (T).*

Proof. (1) By the discussion preceding Theorem 4.1, we may assume $N = \mathbb{Z}$ or $\mathbb{Z}^2$. If $N = \mathbb{Z}$, then $\mathrm{Aut}(N) = \mathrm{GL}(1, \mathbb{Z})$ is finite and hence so is $\phi(G)$. Thus Corollary 1.5 applies. If $N = \mathbb{Z}^2$, then $\mathrm{Aut}(N) = \mathrm{GL}(2, \mathbb{Z})$. By Theorem 4.1, $\phi(G)$ has the Haagerup property. On the other hand $\phi(G)$, being a quotient of a Property (T) group, has Property (T). Therefore $\phi(G)$ is finite. Again Corollary 1.5 applies.

(2) By Lemma 1.4, it is enough to prove that $\mathbb{Z}^n \rtimes \phi(G)$ has Property (T). Further, since $(\mathbb{R}^n \rtimes \phi(G))/(\mathbb{Z}^n \rtimes \phi(G)) \cong \mathbb{R}^n/\mathbb{Z}^n$ is compact, it is enough to prove that $\mathbb{R}^n \rtimes \phi(G)$ has Property (T). Since $\phi(G)$ is Zariski dense in $\mathrm{GL}(n, \mathbb{R})$ and $\mathbb{R}^n$ is $\mathrm{GL}(n, \mathbb{R})$-irreducible, therefore $\mathbb{R}^n$ is also $\phi(G)$-irreducible. Hence Proposition 4.2 applies. If $\phi(G)$ is contained in a compact extension of a diagonalizable group, then it is amenable, since closed subgroups of amenable groups are amenable. On the other hand $\phi(G)$ has Property (T), hence $\phi(G)$ is finite. But finite subgroups are not Zariski dense. Hence $(\mathbb{R}^n \rtimes \phi(G), \mathbb{R}^n)$ has relative Property (T). Since $\phi(G)$ has Property (T), therefore by Lemma 1.2, $\mathbb{R}^n \rtimes \phi(G)$ has Property (T). $\qquad\square$

Remark 4.4. In the proof of (2), we only used $n \geq 2$. But when $n = 2$, Theorem 4.1 excludes the possibility of a homomorphism ϕ from a Property (T) group to $\mathrm{GL}(2, \mathbb{Z})$ having Zariski dense image.

Any result on relative Property (T) with abelian normal subgroup can be upgraded to one involving nilpotent normal subgroups via the following result. We denote the abelianization of N by N_{ab}.

Theorem 4.5. [4, Theorem 1.2] *Let N be a closed nilpotent normal subgroup of a locally compact group G. Then (G, N) has relative Property (T) if and only if $(G/[N, N], N_{ab})$ has relative Property (T).*

Thus we have the following result.

Proposition 4.6. *Let G have Property (T). Let $f : G \to H$ be a surjective group homomorphism whose kernel N is nilpotent. Suppose N_{ab} is finitely generated with rank n. Let $\phi : G \to \mathrm{GL}(n, \mathbb{Z})$ be the induced action of G on the free part of N_{ab}.*

    (1) *If $n = 1$ or 2, then $G \times_H G$ does not have Property (T).*
    (2) *If $n > 2$ and $\phi(G)$ is Zariski dense, then $G \times_H G$ has Property (T).*

Proof. Since G has Property (T), by Lemma 1.2, $(N \rtimes G, N)$ has relative Property (T) if and only if $N \rtimes G$ has Property (T). Similarly, $(N_{ab} \rtimes G, N_{ab})$ has relative Property (T) if and only if $N_{ab} \rtimes G$ has Property (T). Thus Theorem 4.5 tells us that $N \rtimes G$ has Property (T) if and only if $N_{ab} \rtimes G$ has Property (T). Now the statement follows directly from Proposition 4.3. $\qquad\square$

4.2. **Non-abelian free kernels.** We give a simple sufficient condition for $N \rtimes G$ to not have Property (T).

Lemma 4.7. *Let G be a group acting by automorphisms on another group N. Let ϕ be the action. Denote by $[G, N]$ the subgroup of N generated by elements of the*

form $\phi(g)(n)n^{-1}$, where $g \in G$ and $n \in N$. If $N/[G,N]$ is non-compact then $N \rtimes G$ does not have Property (T).

Proof. We will show that if $N \rtimes G$ has Property (T) then $N/[G,N]$ must be compact. This follows from the fact that the abelianization of a group with Property (T) must be be compact and the observation that the abelianization of $N \rtimes G$ is $(N/[G,N]) \times G_{ab}$. □

We have the following application of this observation.

Definition 4.8. [16, Definition 2.2] Let G be a group, H be a subgroup and N be a normal subgroup of H. We denote the normal closure of N in G by $\langle\!\langle N \rangle\!\rangle$. The triple (G, H, N) is said to have the *Cohen-Lyndon property* if there exists a set T of left coset representatives of $H\langle\!\langle N \rangle\!\rangle$ in G such that $\langle\!\langle N \rangle\!\rangle$ is the free product of its subgroups $tNt^{-1}, t \in T$.

Proposition 4.9. *Let (G, H, N) have the Cohen-Lyndon property. Let $N/[H,N]$ be non-compact. Then the fiber product of two copies of the quotient map $G \to G/\langle\!\langle N \rangle\!\rangle$ does not have Property (T).*

Proof. By Lemma 4.7, it is enough to show that there exists a surjective homomorphism $\langle\!\langle N \rangle\!\rangle/[G,\langle\!\langle N \rangle\!\rangle] \to N/[H,N]$. To construct this homomorphism we will first define a homomorphism from each free factor of $\langle\!\langle N \rangle\!\rangle$ to $N/[H,N]$. By the universal property of free products, this will define a unique homomorphism $\langle\!\langle N \rangle\!\rangle \to N/[H,N]$. It will be surjective by construction. Finally we will show that $[G,\langle\!\langle N \rangle\!\rangle]$ belongs to the kernel of this homomorphism. For each $t \in T$, define the homomorphism $tNt^{-1} \to N/[H,N]$ to be composition of the natural isomorphism, $tNt^{-1} \to N, m \mapsto t^{-1}mt$, and the quotient map $N \to N/[H,N]$. This defines a surjective homomorphism $\phi : \langle\!\langle N \rangle\!\rangle \to N/[H,N]$. Note that, for any $n \in N$ and $t \in T$,

$$\phi(tnt^{-1}) = \phi(n).$$

This equation will be used repeatedly in calculations without comment. To show that $[G,\langle\!\langle N \rangle\!\rangle] \subset \ker \phi$, it is enough to show that each of the generators of $[G,\langle\!\langle N \rangle\!\rangle]$ goes to 1 under ϕ. Any generator of $[G,\langle\!\langle N \rangle\!\rangle]$ is of the form $gwg^{-1}w^{-1}$, where $g \in G$ and $w \in \langle\!\langle N \rangle\!\rangle$. Thus we have to show that for each $g \in G$ and $w \in \langle\!\langle N \rangle\!\rangle$,

(1) $$\phi(g)\phi(w)\phi(g)^{-1} = \phi(w).$$

Let us first reduce (1) to the case where $w \in N$ and $g \in \langle\!\langle N \rangle\!\rangle, H$ or T. Any $w \in \langle\!\langle N \rangle\!\rangle$ is of the form $\prod_{i=1}^{k} t_i n_i t_i^{-1}$, where $n_i \in N$ and $t_i \in T$. Then $\phi(g)\phi(w)\phi(g)^{-1} = \phi(g)(\prod_{i=1}^{k} \phi(t_i n_i t_i^{-1}))\phi(g)^{-1} = \phi(g)(\prod_{i=1}^{k} \phi(n_i))\phi(g)^{-1} = \prod_{i=1}^{k} \phi(g)\phi(n_i)\phi(g)^{-1}$. Thus it is enough to show that (1) is true for $w \in N$. Since T is a set of left coset representatives of $H\langle\!\langle N \rangle\!\rangle$, therefore any $g \in G$ is of the form $g = thu$, where $t \in T, h \in H$ and $u \in \langle\!\langle N \rangle\!\rangle$. Then $\phi(g)\phi(w)\phi(g)^{-1} = \phi(t)[\phi(h)[\phi(u)\phi(w)\phi(u)^{-1}]\phi(h)^{-1}]\phi(t)^{-1}$. Hence it is enough to prove (1) for g belonging to $\langle\!\langle N \rangle\!\rangle, H$ or T, separately. When $w \in N$ and $g \in T$ or H, (1) is immediate. So let us assume that $w \in N$ and $g \in \langle\!\langle N \rangle\!\rangle$. The element g is of the form $\prod_{i=1}^{k} t_i n_i t_i^{-1}$, where $n_i \in N$ and $t_i \in T$. By induction on the word length k of g, we are reduced to the case $g = tmt^{-1}$, where $t \in T$ and $m \in N$. Then $\phi(g)\phi(w)\phi(g^{-1}) = \phi(tmt^{-1})\phi(w)\phi(tmt^{-1})^{-1} = \phi(m)\phi(w)\phi(m)^{-1}$. But $mwm^{-1}w^{-1} \in [N,N] \subset [H,N]$. Hence $\phi(m)\phi(w)\phi(m)^{-1} = \phi(w)$. □

Remark 4.10. The homomorphism $\phi : \langle\!\langle N \rangle\!\rangle /[G, \langle\!\langle N \rangle\!\rangle] \to N/[H, N]$, in the above proof, is in fact an isomorphism. As a candidate for the inverse homomorphism, consider the map $\psi : N/[H, N] \to \langle\!\langle N \rangle\!\rangle /[G, \langle\!\langle N \rangle\!\rangle]$ induced by the natural inclusion $N \to \langle\!\langle N \rangle\!\rangle$. Since $H \subset G$ and $N \subset \langle\!\langle N \rangle\!\rangle$, therefore $[H, N] \subset [G, \langle\!\langle N \rangle\!\rangle]$, and hence ψ is well defined. That $\phi\psi = \mathrm{id}$ is immediate. To see the other direction, we claim that the pre-composition of $\psi\phi$ by the quotient $q : \langle\!\langle N \rangle\!\rangle \to \langle\!\langle N \rangle\!\rangle /[G, \langle\!\langle N \rangle\!\rangle]$ is equal to q. By the universal property of quotient maps this will imply that $\psi\phi = \mathrm{id}$. The equality follows from comparing the two maps on each free factor.

To apply Proposition 4.9, we shall require the following consequence of Chevalley-Weil theory (in the case of free groups, the theorem is due to Gaschütz). See [5, 12, 14]. In the form we reproduce this below, it is an immediate consequence of Theorems 3.3 and 3.4 of [3]. Let Q denote the finite group H/N. The group $N/[N, N]$ can then be regarded as a Q-module. Then $N/[H, N]$ is the space of co-invariants of the Q action on $N/[N, N]$. Chevalley-Weil theory gives a non-trivial infinite summand of $N/[N, N]$ corresponding to the trivial representation. It follows that $N/[H, N]$ is infinite. We summarize this below.

Theorem 4.11. [5, 12, 14],[3, Theorems 3.3 and 3.4] *Let H be either the free group F_n $(n > 1)$ or the fundamental group of a closed surface of genus greater than one. Let $N < H$ be a proper finite index normal subgroup. Then $N/[H, N]$ is infinite.*

Remark 4.12. If H is infinite abelian and $N < H$ is a proper finite index (necessarily normal) subgroup, then $N/[H, N]$ is equal to N and is therefore infinite.

We refer the reader to [8] for the notion of **hyperbolically embedded** subgroups. Examples include quasiconvex malnormal subgroups of hyperbolic groups and peripheral subgroups of (strongly) relatively hyperbolic groups. One says [8, 16] that a property $\mathcal{P}$ holds for **all sufficiently deep** normal subgroups $N < H$ if there exists a finite set $\mathcal{F} \subset H \smallsetminus \{1\}$ such that $\mathcal{P}$ holds for all normal subgroups $N < H$ with $N \cap \mathcal{F} = \varnothing$. We shall need the following (see also [8, Theorem 2.27]):

Theorem 4.13. [16, Theorem 2.5] *Suppose that G is a group with a hyperbolically embedded subgroup H. Then (G, H, N) has the Cohen-Lyndon property for all sufficiently deep $N < H$.*

Combining Proposition 4.9, Theorems 4.11 and 4.13, and Remark 4.12 we immediately have the following:

Theorem 4.14. *Let G have Property (T) and $H < G$ be hyperbolically embedded such that H is one of the following:*

(1) *the free group F_n $(n > 1)$,*
(2) *the fundamental group of a closed surface of genus greater than one,*
(3) *infinite abelian.*

Then, for all sufficiently deep finite index normal subgroups $N < H$, the fiber product of two copies of the quotient map $G \to G/\langle\!\langle N \rangle\!\rangle$ does not have Property (T).

We conclude with the following question:

Question 4.15. *In Theorem 4.14, if H does not have Property (T), does the conclusion continue to hold? Shalom's generalisation of the Delorme-Guichardet theorem in terms of reduced first cohomology [2, §3.2] might be relevant here.*

Acknowledgments

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