

# Cannon-Thurston Maps and Bounded Geometry

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## Abstract

This is an expository paper. We prove the Cannon-Thurston property for bounded geometry surface groups with or without punctures. We prove three theorems, due to Cannon-Thurston, Minsky and Bowditch. The proofs are culled out of earlier work of the author.

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## 1 Introduction

Let  $M$  be a closed hyperbolic 3-manifold fibering over the circle with fiber  $F$ . Let  $\tilde{F}$  and  $\tilde{M}$  denote the universal covers of  $F$  and  $M$  respectively. Then  $\tilde{F}$  and  $\tilde{M}$  are quasi-isometric to  $\mathbb{H}^2$  and  $\mathbb{H}^3$  respectively. Now let  $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}_\infty^1$  and  $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}_\infty^2$  denote the standard compactifications. In [7] Cannon and Thurston show that the usual inclusion of  $\tilde{F}$  into  $\tilde{M}$  extends to a continuous map from  $\mathbb{D}^2$  to  $\mathbb{D}^3$ . This was extended to Kleinian surface groups of bounded geometry without parabolics by Minsky [15]. Bowditch [5] [6] proved the Cannon-Thurston property for bounded geometry surface groups with parabolics.

In [16], [17], [18], [20], [19], we have given a different approach to the Cannon-Thurston problem. Though the theorems of Cannon-Thurston, Minsky and Bowditch can be deduced from ours, it might be instructive to write down a complete proof of these results. In some ways, the proof here is easier and more minimalistic. Another reason for writing this paper is that Cannon and Thurston's original result [7] is unpublished. It seems only appropriate that the theorem that motivated all the above results be available.

Much of what follows is true in the setting of hyperbolic metric spaces in the sense of Gromov. We shall often state results in this generality.

## 2 Preliminaries

### 2.1 Hyperbolic Metric Spaces

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [11]. For details, see [8], [10]. Let  $(X, d)$  be a hyperbolic metric space. The **Gromov boundary** of  $X$ , denoted by  $\partial X$ , is the collection of equivalence classes of geodesic rays  $r : [0, \infty) \rightarrow X$  with  $r(0) = x_0$  for some fixed  $x_0 \in X$ , where rays  $r_1$  and  $r_2$  are equivalent if  $\sup\{d(r_1(t), r_2(t))\} < \infty$ . Let  $\widehat{X} = X \cup \partial X$  denote the natural compactification of  $X$  topologized the usual way (cf. [10] pg. 124).

The **Gromov inner product** of elements  $a$  and  $b$  relative to  $c$  is defined by

$$(a, b)_c = 1/2[d(a, c) + d(b, c) - d(a, b)].$$

**Definitions:** A subset  $Z$  of  $X$  is said to be  **$k$ -quasiconvex** if any geodesic joining  $a, b \in Z$  lies in a  $k$ -neighborhood of  $Z$ . A subset  $Z$  is **quasiconvex** if it is  $k$ -quasiconvex for some  $k$ . A map  $f$  from one metric space  $(Y, d_Y)$  into another metric space  $(Z, d_Z)$  is said to be a  **$(K, \epsilon)$ -quasi-isometric embedding** if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \epsilon$$

If  $f$  is a quasi-isometric embedding, and every point of  $Z$  lies at a uniformly bounded distance from some  $f(y)$  then  $f$  is said to be a **quasi-isometry**. A  $(K, \epsilon)$ -quasi-isometric embedding that is a quasi-isometry will be called a  **$(K, \epsilon)$ -quasi-isometry**.

A  **$(K, \epsilon)$ -quasigeodesic** is a  $(K, \epsilon)$ -quasi-isometric embedding of a closed interval in  $\mathbb{R}$ . A  $(K, 0)$ -quasigeodesic will also be called a  **$K$ -quasigeodesic**.

Let  $(X, d_X)$  be a hyperbolic metric space and  $Y$  be a subspace that is hyperbolic with the inherited path metric  $d_Y$ . By adjoining the Gromov boundaries  $\partial X$  and  $\partial Y$  to  $X$  and  $Y$ , one obtains their compactifications  $\widehat{X}$  and  $\widehat{Y}$  respectively.

Let  $i : Y \rightarrow X$  denote inclusion.

**Definition:** Let  $X$  and  $Y$  be hyperbolic metric spaces and  $i : Y \rightarrow X$  be an embedding. A **Cannon-Thurston map**  $\hat{i}$  from  $\widehat{Y}$  to  $\widehat{X}$  is a continuous extension of  $i$ .

The following lemma says that a Cannon-Thurston map exists if for all  $M > 0$  and  $y \in Y$ , there exists  $N > 0$  such that if  $\lambda$  lies outside an  $N$  ball around  $y$  in  $Y$  then any geodesic in  $X$  joining the end-points of  $\lambda$  lies outside the  $M$  ball around  $i(y)$  in  $X$ . For convenience of use later on, we state this somewhat differently. The proof is similar to Lemma 2.1 of [16].

**Lemma 2.1** *A Cannon-Thurston map from  $\widehat{Y}$  to  $\widehat{X}$  exists if the following condition is satisfied:*

*Given  $y_0 \in Y$ , there exists a non-negative function  $M(N)$ , such that  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  and for all geodesic segments  $\lambda$  lying outside an  $N$ -ball around  $y_0 \in Y$  any geodesic segment in  $\Gamma_G$  joining the end-points of  $i(\lambda)$  lies outside the  $M(N)$ -ball around  $i(y_0) \in X$ .*

*Proof:* Suppose  $i : Y \rightarrow X$  does not extend continuously. Since  $i$  is proper, there exist sequences  $x_m, y_m \in Y$  and  $p \in \partial Y$ , such that  $x_m \rightarrow p$  and  $y_m \rightarrow p$  in  $\widehat{Y}$ , but  $i(x_m) \rightarrow u$  and  $i(y_m) \rightarrow v$  in  $\widehat{X}$ , where  $u, v \in \partial X$  and  $u \neq v$ .

Since  $x_m \rightarrow p$  and  $y_m \rightarrow p$ , any geodesic in  $Y$  joining  $x_m$  and  $y_m$  lies outside an  $N_m$ -ball  $y_0 \in Y$ , where  $N_m \rightarrow \infty$  as  $m \rightarrow \infty$ . Any bi-infinite geodesic in  $X$  joining  $u, v \in \partial X$  has to pass through some  $M$ -ball around  $i(y_0)$  in  $X$  as  $u \neq v$ . There exist constants  $c$  and  $L$  such that for all  $m > L$  any geodesic joining  $i(x_m)$  and  $i(y_m)$  in  $X$  passes through an  $(M + c)$ -neighborhood of  $i(y_0)$ . Since  $(M + c)$  is a constant not depending on the index  $m$  this proves the lemma.  $\square$

The above result can be interpreted as saying that a Cannon-Thurston map exists if the space of geodesic segments in  $Y$  embeds properly in the space of geodesic segments in  $X$ .

We shall be needing the notion of a tree of hyperbolic metric spaces [2].

**Definition:** A tree (T) of hyperbolic metric spaces satisfying the **q(uasi) i(sometrically) embedded condition** is a metric space  $(X, d)$  admitting a map  $P : X \rightarrow T$  onto a simplicial tree  $T$ , such that there exist  $\delta, \epsilon$  and  $K > 0$  satisfying the following:

1. For all vertices  $v \in T$ ,  $X_v = P^{-1}(v) \subset X$  with the induced path metric  $d_v$  is a  $\delta$ -hyperbolic metric space. Further, the inclusions  $i_v : X_v \rightarrow X$  are uniformly proper, i.e. for all  $M > 0$ ,  $v \in T$  and  $x, y \in X_v$ , there exists  $N > 0$  such that  $d(i_v(x), i_v(y)) \leq M$  implies  $d_v(x, y) \leq N$ .

2. Let  $e$  be an edge of  $T$  with initial and final vertices  $v_1$  and  $v_2$  respectively. Let  $X_e$  be the pre-image under  $P$  of the mid-point of  $e$ . Then  $X_e$  with the induced path metric is  $\delta$ -hyperbolic.
3. There exist maps  $f_e : X_e \times [0, 1] \rightarrow X$ , such that  $f_e|_{X_e \times (0,1)}$  is an isometry onto the pre-image of the interior of  $e$  equipped with the path metric.
4.  $f_e|_{X_e \times \{0\}}$  and  $f_e|_{X_e \times \{1\}}$  are  $(K, \epsilon)$ -quasi-isometric embeddings into  $X_{v_1}$  and  $X_{v_2}$  respectively.  $f_e|_{X_e \times \{0\}}$  and  $f_e|_{X_e \times \{1\}}$  will occasionally be referred to as  $f_{v_1}$  and  $f_{v_2}$  respectively.

$d_v$  and  $d_e$  will denote path metrics on  $X_v$  and  $X_e$  respectively.  $i_v, i_e$  will denote inclusion of  $X_v, X_e$  respectively into  $X$ .

A few general lemmas about hyperbolic metric spaces will be useful. We reproduce the proofs here from [17].

### Nearest Point Projections

The following Lemma says nearest point projections in a  $\delta$ -hyperbolic metric space do not increase distances much.

**Lemma 2.2** *Let  $(Y, d)$  be a  $\delta$ -hyperbolic metric space and let  $\mu \subset Y$  be a geodesic segment. Let  $\pi : Y \rightarrow \mu$  map  $y \in Y$  to a point on  $\mu$  nearest to  $y$ . Then  $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$  for all  $x, y \in Y$  where  $C_3$  depends only on  $\delta$ .*

*Proof:* Let  $[a, b]$  denote a geodesic edge-path joining vertices  $a, b$ . Recall that the Gromov inner product  $(a, b)_c = 1/2[d(a, c) + d(b, c) - d(a, b)]$ . It suffices by repeated use of the triangle inequality to prove the Lemma when  $d(x, y) = 1$ . Let  $u, v, w$  be points on  $[x, \pi(x)], [\pi(x), \pi(y)]$  and  $[\pi(y), x]$  respectively such that  $d(u, \pi(x)) = d(v, \pi(x)), d(v, \pi(y)) = d(w, \pi(y))$  and  $d(w, x) = d(u, x)$ . Then  $(x, \pi(y))_{\pi(x)} = d(u, \pi(x))$ . Also, since  $Y$  is  $\delta$ -hyperbolic, the diameter of the inscribed triangle with vertices  $u, v, w$  is less than or equal to  $2\delta$  (See [1]).

$$\begin{aligned}
d(u, x) + d(u, v) &\geq d(x, \pi(x)) = d(u, x) + d(u, \pi(x)) \\
&\Rightarrow d(u, \pi(x)) \leq d(u, v) \leq 2\delta \\
&\Rightarrow (x, \pi(y))_{\pi(x)} \leq 2\delta
\end{aligned}$$

Similarly,  $(y, \pi(x))_{\pi(y)} \leq 2\delta$ .

$$\begin{aligned} \text{i.e. } & d(x, \pi(x)) + d(\pi(x), \pi(y)) - d(x, \pi(y)) \leq 4\delta \\ \text{and } & d(y, \pi(y)) + d(\pi(x), \pi(y)) - d(y, \pi(x)) \leq 4\delta \end{aligned}$$

Therefore,

$$\begin{aligned} & 2d(\pi(x), \pi(y)) \\ & \leq 8\delta + d(x, \pi(y)) - d(y, \pi(y)) + d(y, \pi(x)) - d(x, \pi(x)) \\ & \leq 8\delta + d(x, y) + d(x, y) \\ & \leq 8\delta + 2 \end{aligned}$$

Hence  $d(\pi(x), \pi(y)) \leq 4\delta + 1$ . Choosing  $C_3 = 4\delta + 1$ , we are through.  $\square$

**Lemma 2.3** *Let  $(Y, d)$  be a  $\delta$ -hyperbolic metric space. Let  $\mu$  be a geodesic segment in  $Y$  with end-points  $a, b$  and let  $x$  be any vertex in  $Y$ . Let  $y$  be a vertex on  $\mu$  such that  $d(x, y) \leq d(x, z)$  for any  $z \in \mu$ . Then a geodesic path from  $x$  to  $y$  followed by a geodesic path from  $y$  to  $z$  is a  $k$ -quasigeodesic for some  $k$  dependent only on  $\delta$ .*

**Proof:** As in Lemma 2.2, let  $u, v, w$  be points on edges  $[x, y]$ ,  $[y, z]$  and  $[z, x]$  respectively such that  $d(u, y) = d(v, y)$ ,  $d(v, z) = d(w, z)$  and  $d(w, x) = d(u, x)$ . Then  $d(u, y) = d(z, x)_y \leq 2\delta$  and the inscribed triangle with vertices  $u, v, w$  has diameter less than or equal to  $2\delta$  (See [1]).  $[x, y] \cup [y, z]$  is a union of 2 geodesic paths lying in a  $4\delta$  neighborhood of a geodesic  $[x, z]$ . Hence a geodesic path from  $x$  to  $y$  followed by a geodesic path from  $y$  to  $z$  is a  $k$ -quasigeodesic for some  $k$  dependent only on  $\delta$ .  $\square$

**Lemma 2.4** *Suppose  $(Y, d)$  is a  $\delta$ -hyperbolic metric space. If  $\mu$  is a  $(k_0, \epsilon_0)$ -quasigeodesic in  $Y$  and  $p, q, r$  are 3 points in order on  $\mu$  then  $(p, r)_q \leq k_1$  for some  $k_1$  dependent on  $k_0, \epsilon_0$  and  $\delta$  only.*

**Proof:**  $[a, b]$  will denote a geodesic path joining  $a, b$ . Since  $p, q, r$  are 3 points in order on  $\mu$ ,  $[p, q]$  followed by  $[q, r]$  is a  $(k_0, \epsilon_0)$ -quasigeodesic in the  $\delta$ -hyperbolic metric space  $Y$ . Hence there exists a  $k_1$  dependent on  $k_0, \epsilon_0$  and  $\delta$  alone such that  $d(q, [p, r]) \leq k_1$ . Let  $s$  be a point on  $[p, r]$  such that  $d(q, s) = d(q, [p, r]) \leq k_1$ . Then

$$\begin{aligned} (p, r)_q &= 1/2(d(p, q) + d(r, q) - d(p, r)) \\ &= 1/2(d(p, q) + d(r, q) - d(p, s) - d(r, s)) \\ &\leq d(q, s) \leq k_1. \square \end{aligned}$$

## 2.2 Stability of Tripods, or NPP's and QI's almost commute

A crucial property of hyperbolic metric spaces is *stability of quasigeodesics*, i.e. any quasigeodesic (in particular an image of a geodesic under a quasi-isometry) lies in a bounded neighborhood of a geodesic. This property can be easily extended to quasiconvex sets [10] [8]. Here we are interested in a particular kind of quasiconvex set, the **tripod**. In general, a tripod is a union of three geodesic segments, all of which share a common end-point. It is easy to see (for instance by thinness of triangles) that a tripod is quasiconvex. We shall be interested in a special kind of a tripod. Let  $[a, b]$  be a geodesic in a hyperbolic metric space  $X$ . Let  $x \in X$  be some point. Let  $p$  be a nearest point projection of  $x$  onto  $[a, b]$ . We shall look at tripods of the form  $[a, b] \cup [x, p]$ . We shall show that such tripods are stable under quasi-isometries.

However, we shall interpret this differently to say that nearest point projections and quasi-isometries in hyperbolic metric spaces ‘almost commute’. The following Lemma says precisely this: nearest point projections and quasi-isometries in hyperbolic metric spaces ‘almost commute’.

**Lemma 2.5** *Suppose  $(Y, d)$  is  $\delta$ -hyperbolic. Let  $\mu_1$  be some geodesic segment in  $Y$  joining  $a, b$  and let  $p$  be any vertex of  $Y$ . Also let  $q$  be a vertex on  $\mu_1$  such that  $d(p, q) \leq d(p, x)$  for  $x \in \mu_1$ . Let  $\phi$  be a  $(K, \epsilon)$ -quasiisometry from  $Y$  to itself. Let  $\mu_2$  be a geodesic segment in  $Y$  joining  $\phi(a)$  to  $\phi(b)$  for some  $g \in S$ . Let  $r$  be a point on  $\mu_2$  such that  $d(\phi(p), r) \leq d(\phi(p), x)$  for  $x \in \mu_2$ . Then  $d(r, \phi(q)) \leq C_4$  for some constant  $C_4$  depending only on  $K, \epsilon$  and  $\delta$ .*

**Proof:** Since  $\phi(\mu_1)$  is a  $(K, \epsilon)$ -quasigeodesic joining  $\phi(a)$  to  $\phi(b)$ , it lies in a  $K'$ -neighborhood of  $\mu_2$  where  $K'$  depends only on  $K, \epsilon, \delta$ . Let  $u$  be a point in  $\phi(\mu_1)$  lying at a distance at most  $K'$  from  $r$ . Without loss of generality suppose that  $u$  lies on  $\phi([q, b])$ , where  $[q, b]$  denotes the geodesic subsegment of  $\mu_1$  joining  $q, b$ . [See Figure 1 below.]





$$\begin{aligned}
d(r, s) &= (\phi(p), s)_r + (\phi(p), r)_s \\
&\leq K_1 + 2K' + 2\delta \\
d(r, \phi(q)) &\leq K_1 + 2K' + 2\delta + d(s, \phi(q)) \\
&\leq K_1 + 2K' + 2\delta + K'
\end{aligned}$$

Let  $C_4 = K_1 + 3K' + 2\delta$ . Then  $d(r, \phi(q)) \leq C_4$  and  $C_4$  is independent of  $a, b, p$ .  $\square$

### 3 Rays of Spaces and Hyperbolic Ladders

#### 3.1 Trees of Hyperbolic Metric spaces

In this section, we shall consider trees  $T$  of hyperbolic metric spaces satisfying the qi-embedded condition.  $(X, d_X)$  will denote the space that is a tree of hyperbolic metric spaces. Our trees will be rather special.

•  $T$  will either be a ray  $[0, \infty)$  or a bi-infinite geodesic  $(-\infty, \infty)$ .

The vertices of  $T$  will be the integer points  $j \in \{0\} \cup \mathbb{N}$  or  $\mathbb{Z}$  according as  $T$  is  $[0, \infty)$  or  $(-\infty, \infty)$ . The edges will be of the form  $[j, j + 1]$ . All the vertex and edge spaces will be abstractly isometric and identified with a fixed hyperbolic metric space  $Y$ . The vertex space over  $j$  will be denoted as  $Y_j$ , and the edge space over  $[j, j + 1]$  will be denoted as  $Y_{ej}$ . There are two qi-embeddings of  $Y_{ej}$ , one into  $Y_j$  (denoted  $i_{j0}$ ), and the other into  $Y_{j+1}$  (denoted  $i_{j1}$ ). We shall demand:

1.  $i_{j0}$  is the **identity** map for all  $j$ , i.e the identification of  $Y_j$  and  $Y_{ej}$  with  $Y$  is the same
2. There exist  $K, \epsilon$  such that for all  $j$ ,  $i_{j1}$  is a  $(K, \epsilon)$  quasi-isometry between  $Y_{ej}$  and  $Y_{j+1}$ .

Thus, the tree  $T$  is assumed rooted at 0, and maps are described on this basis.

This induces a map  $\phi_j$  from  $Y_j$  to  $Y_{j+1}$  which is a uniform  $(K, \epsilon)$  quasi-isometry for all  $j$ . Let  $\Phi_j$  denote the induced map on geodesics. So  $\Phi_j([a, b])$  is a geodesic in  $Y_{j+1}$  joining  $\phi_j(a), \phi_{j+1}(b)$ .  $\phi_j^{-1}$  and  $\Phi_j^{-1}$  will denote the quasi-isometric inverse of  $\phi_j$  and the induced map on geodesics respectively. We shall assume that the quasi-isometric inverse  $\phi_j^{-1}$  is also a  $(K, \epsilon)$  quasi-isometry.

- The space  $Y$  in question will also be quite special.  **$Y$  will either be the universal cover of a closed hyperbolic surface (hence the hyperbolic plane), or the universal cover of a finite volume hyperbolic surface minus cusps.**

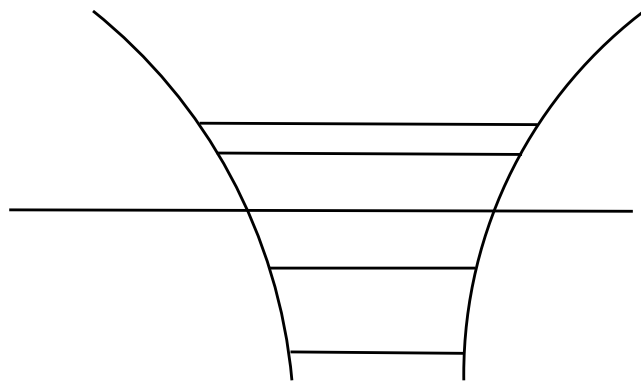
The first case, where  $Y$  is  $\mathbb{H}^2$  will be necessary when we prove the results of Cannon-Thurston[7] and Minsky [15]. The second case, where  $Y$  is  $\mathbb{H}^2$  minus an equivariant collection of horodisks, will be useful while proving a result of Bowditch [5].

### 3.2 Hyperbolic Ladders

Given a geodesic  $\lambda = \lambda_0 \subset Y_0$ , we shall now construct a hyperbolic ladder  $B_\lambda \subset X$  containing  $\lambda_0$ . We shall then prove that  $B_\lambda$  is uniformly quasiconvex (independent of  $\lambda$ ). Inductively define:

$$\begin{aligned} \lambda_{j+1} &= \Phi_j(\lambda_j) \text{ for } j \geq 0 \\ \lambda_{j-1} &= \Phi_j^{-1}(\lambda_j) \text{ for } j \leq 0 \\ B_\lambda &= \bigcup_j \lambda_j \end{aligned}$$

[See Figure 2 below.]



*Figure 2: The Hyperbolic Ladder*

To prove quasiconvexity, we construct a retraction  $\Pi_\lambda$  from  $\bigcup_j Y_j$  onto  $B_\lambda$  that fixes  $B_\lambda$  and does not stretch distances much. So  $\Pi_\lambda$  is a ‘quasi-Lipschitz’ map.

On  $Y_j$  define  $\pi_j(y)$  to be a nearest-point projection of  $y$  onto  $\lambda_j$ . Thus,  $d_Y(y, \pi_j(y)) \leq d_Y(y, z)$  for all  $z \in \lambda_j$ .

Next, define

$$Pi_\lambda(y) = \pi_j(y) \text{ for } y \in Y_j.$$

We shall choose a collection of **admissible paths** in  $X$ , such that any path in  $X$  can be uniformly approximated by these. An admissible path in  $X$  is a union of paths of the following forms:

1. Paths lying in  $Y_j$  for some  $j$ .
2. Paths of length one joining  $y \in Y_j$  to  $\phi_j(y) \in Y_{j+1}$  for  $j \geq 0$
3. Paths of length one joining  $y \in Y_j$  to  $\phi_j^{-1}(y) \in Y_{j-1}$  for  $j \leq 0$

**Note:** Any path in  $X$  can be uniformly approximated by an admissible path.

**Definition:** An admissible geodesic is an admissible path that minimises distances amongst admissible path.

We are now in a position to prove the main technical theorem of this paper.

**Theorem 3.1** *There exists  $C_0 \geq 0$  such that  $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d_X(x, y)$  for  $x, y \in \bigcup_i Y_i$ . Further, given  $\delta \geq 0$ , there exists  $C \geq 0$  such that if  $X$  is  $\delta$ -hyperbolic, then  $B_\lambda$  is  $C$ -quasiconvex.*

**Proof:** It suffices to prove the the theorem when  $d_X(x, y) = 1$ .

*Case (a):*  $x, y \in Y_j$  for some  $j$ :

From Lemma 2.2, there exists  $C_3$  such that  $d_Y(x, y) \leq C_3$ . Since embeddings of  $Y_j$  in  $X$  are uniform,  $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_3$ .

*Case (b):*  $x \in Y_j$  and  $y = \phi_j(x) \in Y_{j+1}$  for some  $j \geq 0$ :

By Lemma 2.5, there exists  $C_4 \geq 0$  such that

$$d_Y(\phi_j(\pi_j(x)), \pi_{j+1}(\phi_j(x))) \leq C_4$$

Unravelling definitions, and using the uniformity of the embedding of  $Y_j$  in  $X$ ,  
 $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_4 + 1$ .

*Case (c):*  $x \in Y_j$  and  $y = \phi_j^{-1}(x) \in Y_{j-1}$  for some  $j \leq 0$ :

The argument is just as in Case(b) above.

Finally, to prove quasiconvexity of  $B_\lambda$ , we consider any two points  $a, b \in B_\lambda$ . Let  $\mu$  be a geodesic in  $X$  joining  $a, b$ . Then by the above argument,  $\Pi_\lambda(\mu \cap \bigcup_j Y_j)$  is a ‘dotted  $(K, \epsilon)$ -quasigeodesic’ for some  $K, \epsilon$ , i.e. a quasigeodesic which is not necessarily connected. Note that by our definition of a quasigeodesic (viz. a quasi-isometric image of an interval in  $\mathbb{R}$ ) a dotted quasigeodesic is a quasigeodesic. Since  $X$  is  $\delta$ -hyperbolic, there exists  $C \geq 0$  such that the geodesic  $\mu$  lies in a  $C$  neighborhood of  $\Pi_\lambda(\mu \cap \bigcup_j Y_j)$ . Further, since the latter lies on  $B_\lambda$ , we conclude that  $B_\lambda$  is  $C$ -quasiconvex.  $\square$

**Note:** Suppose instead of being geodesics  $\lambda_i$  are all  $(K_Y, \epsilon_Y)$ -quasigeodesics in each  $Y_i$ . Then observe that the Theorem 3.1 still goes through, and the union  $B_\lambda$  (in this case of **quasigeodesics**) of  $\lambda_i$  is still a retract of  $\bigcup_i Y_i$ .

The following lemma gives us a method of finding an admissible path from  $\lambda_j$  to  $\lambda_0$ , whose length is of the order of  $j$ .

**Lemma 3.2** *There exists  $A > 0$ , such that if  $a \in \lambda_j$  for some  $j$  then there exists  $b \in (\lambda) = \lambda_0$  such that  $d_X(a, b) \leq Aj$ .*

**Proof:** It suffices to prove that for  $j \geq 1$ , there exists  $A > 0$  independent of  $j$  such that if  $p \in (\lambda_j)$ , there exists  $q \in (\lambda_{j-1})$  with  $d(p, q) \leq A$ . A symmetric argument works for  $j \leq -1$ .

By construction,  $\lambda_j = \Phi_j(\lambda_{j-1})$ . Since  $\phi_j$  is a  $(K, \epsilon)$  quasi-isometry for all  $j$ , there exists  $C_1$  such that  $\phi_j^{-1}(p)$  lies in a  $C_1$  neighborhood of  $\lambda_{j-1}$ . Hence, there exists  $q \in \lambda_{j-1}$  such that  $d(q, p) \leq 1 + C_1$ . Choosing  $A = C_1 + 1$  we are through.  $\square$

**Remark:** Note that Lemma 3.2 also goes through when each  $\lambda_i$  is a  $(K_Y, \epsilon_Y)$ -quasigeodesic in  $Y_i$  for all  $i$ .

The main theorem of this section follows:

**Theorem 3.3** *Let  $(X, d)$  be a tree  $(T)$  of hyperbolic metric spaces satisfying the  $qi$ -embedded condition. Suppose in addition that  $T$  is either  $\mathbb{R}$  or  $[0, \infty)$  with the usual vertex and edge sets  $Y_j$ , for  $j$  integers. If  $X$  is hyperbolic, then  $i : Y_0 \rightarrow X$  extends continuously to  $\hat{i} : \widehat{Y}_0 \rightarrow \widehat{X}$ .*

**Proof:** To prove the existence of a Cannon-Thurston map, it suffices to show (from Lemma 2.1) that for all  $M \geq 0$  and  $y_0 \in Y_0$  there exists  $N \geq 0$  such that if a geodesic segment  $\lambda$  lies outside the  $N$ -ball around  $y_0 \in Y_0$ ,  $B_\lambda$  lies outside the  $M$ -ball around  $y_0 \in X$ .

To prove this, we show that if  $\lambda$  lies outside the  $N$ -ball around  $y_0 \in Y_0$ ,  $B_\lambda$  lies outside a certain  $M(N)$ -ball around  $y_0 \in Y_0 \subset X$ , where  $M(N)$  is a proper function from  $\mathbb{N}$  into itself.

Since  $Y_0$  is properly embedded in  $X$  there exists  $f(N)$  such that  $\lambda$  lies outside the  $f(N)$ -ball around  $y_0$  in  $X$  and  $f(N) \rightarrow \infty$  as  $N \rightarrow \infty$ .

Let  $p$  be any point on  $B_\lambda$ . Then  $p \in Y_j$  for some  $j$ . There exists  $y \in \lambda = \lambda_0$  such that  $d_X(y, p) \leq Aj$  by Lemma 3.2. Therefore,

$$\begin{aligned} d_X(y_0, p) &\geq d_X(y_0, y) - Aj \\ &\geq f(N) - Aj \end{aligned}$$

By our choice of metric on  $X$ ,

$$d_X(y_0, p) \geq j$$

Hence

$$\begin{aligned} d(x_0, p) &\geq \max(f(N) - Aj, j) \\ &\geq \frac{f(N)}{A+1} \end{aligned}$$

From Theorem 3.1 there exists  $C$  independent of  $\lambda$  such that  $B_\lambda$  is a  $C$ -quasiconvex set containing  $\lambda$ . Therefore any geodesic joining the end-points of  $\lambda$  lies in a  $C$ -neighborhood of  $B_\lambda$ .

Hence any geodesic joining end-points of  $\lambda$  lies outside a ball of radius  $M(N)$  where

$$M(N) = \frac{f(N)}{A+1} - C$$

Since  $f(N) \rightarrow \infty$  as  $N \rightarrow \infty$  so does  $M(N)$ .  $\square$

## 4 Closed Surface Groups of Bounded Geometry:

### Theorems of Cannon-Thurston and Minsky

#### 4.1 Three Manifolds Fibered Over the Circle

Now, let  $M$  be a closed hyperbolic 3-manifold, fibering over the circle with fiber  $F$ . Then the universal cover  $\tilde{M}$  may be regarded as a tree  $T$  of spaces, where  $T = \mathbb{R}$ , and where each vertex and edge set is  $\tilde{F}$ . Thus  $X$  is  $\tilde{M}$ , which is quasi-isometric to  $\mathbb{H}^3$  and  $Y$  is  $\tilde{F}$  which is quasi-isometric to  $\mathbb{H}^2$ . Further,  $\phi_i$  can all be identified with  $\tilde{\phi}$ , where  $\phi$  is the pseudo-anosov monodromy of the fibration. Then as a direct consequence of Theorem 3.3, we obtain the following theorem of Cannon and Thurston, which motivated much of the present work.

**Theorem 4.1 (Cannon-Thurston)** [7]  *$M$  be a closed hyperbolic 3-manifold, fibering over the circle with fiber  $F$ . Let  $i : \mathbb{H}^2 \rightarrow \mathbb{H}^3$  denote the inclusion of  $\tilde{F}$  into  $\tilde{M}$  (where  $\tilde{F}$  and  $\tilde{M}$  are identified with  $\mathbb{H}^2$  and  $\mathbb{H}^3$  respectively). Then  $i$  extends continuously to a map  $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$ , where  $\mathbb{D}^2$  and  $\mathbb{D}^3$  denote the compactifications of  $\mathbb{H}^2$  and  $\mathbb{H}^3$  respectively.*

#### 4.2 Closed Surface Kleinian Groups

In this section we apply Theorem 3.3 to geometrically tame closed groups of bounded geometry.

The *convex core* of a hyperbolic 3-manifold  $N$  (without cusps) is the smallest convex submanifold  $C(N) \subset N$  for which inclusion is a homotopy equivalence. If an  $\epsilon$  neighborhood of  $C(N)$  has finite volume,  $N$  is said to be *geometrically finite*. Suppose  $N = \mathbb{H}^3 / \rho(\pi_1(S))$  for a closed surface  $S$ . We say that an end of  $N$  is *geometrically finite* if it has a neighborhood missing  $C(N)$ .  $N$  is *simply degenerate* if it has only one end  $E$ ; if a neighborhood of  $E$  is homeomorphic to  $S \times \mathbb{R}$ ; and if, in addition, there is a sequence of pleated surfaces homotopic in this neighborhood to the inclusion of  $S$ , and exiting every compact set (This last condition is automatic by Bonahon [3]).  $N$  is called *doubly degenerate* if it has two ends, both of which are simply degenerate. For a more detailed discussion of pleated surfaces and geometrically tame ends, see [21] or [14].

Let  $\text{inj}_N(x)$  denote the injectivity radius at  $x \in N$ .  $N$  is said to have **bounded geometry** if there exists  $\epsilon_0 > 0$  such that  $\text{inj}_N(x) > \epsilon_0$  for all  $x \in N$ . In order to apply Theorem 3.3 we need some preliminary Lemmas.

Let  $E$  be a simply degenerate end of  $N$ . Then  $E$  is homeomorphic to  $S \times [0, \infty)$  for some closed surface  $S$  of genus greater than one.

**Lemma 4.2** [21] *There exists  $D_1 > 0$  such that for all  $x \in N$ , there exists a pleated surface  $g : (S, \sigma) \rightarrow N$  with  $g(S) \cap B_{D_1}(x) \neq \emptyset$ .*

The following Lemma follows easily from the fact that  $\text{inj}_N(x) > \epsilon_0$ :

**Lemma 4.3** [3],[21] *There exists  $D_2 > 0$  such that if  $g : (S, \sigma) \rightarrow N$  is a pleated surface, then  $\text{dia}(g(S)) < D_2$ .*

The following Lemma due to Minsky [14] follows from compactness of pleated surfaces.

**Lemma 4.4** [14] *Fix  $S$  and  $\epsilon > 0$ . Given  $a > 0$  there exists  $b > 0$  such that if  $g : (S, \sigma) \rightarrow N$  and  $h : (S, \rho) \rightarrow N$  are homotopic pleated surfaces which are isomorphisms on  $\pi_1$  and  $\text{inj}_N(x) > \epsilon$  for all  $x \in N$ , then*

$$d_N(g(S), h(S)) \leq a \Rightarrow d_{\text{Teich}}(\sigma, \rho) \leq b,$$

where  $d_{\text{Teich}}$  denotes Teichmuller distance.

**Definition:** *The **universal curve** over  $X \subset \text{Teich}(S)$  is a bundle whose fiber over  $x \in X$  is  $x$  itself. [13]*

**Lemma 4.5** *There exist  $K, \epsilon$  and a homeomorphism  $h$  from  $E$  to the universal curve  $S_\gamma$  over a Lipschitz path  $\gamma$  in Teichmuller space, such that  $\tilde{h}$  from  $\tilde{E}$  to the universal cover of  $S_\gamma$  is a  $(K, \epsilon)$ -quasi-isometry.*

**Proof:** We can assume that  $S \times \{0\}$  is mapped to a pleated surface  $S_0 \subset N$  under the homeomorphism from  $S \times [0, \infty)$  to  $E$ . We shall construct inductively a sequence of ‘equispaced’ pleated surfaces  $S_i \subset E$  exiting the end. Assume that  $S_0, \dots, S_n$  have been constructed such that:

1. If  $E_i$  be the non-compact component of  $E \setminus S_i$ , then  $S_{i+1} \subset E_i$ .

2. Hausdorff distance between  $S_i$  and  $S_{i+1}$  is bounded above by  $3(D_1 + D_2)$ .
3.  $d_N(S_i, S_{i+1}) \geq D_1 + D_2$ .
4. From Lemma 4.4 and condition (2) above there exists  $D_3$  depending on  $D_1, D_2$  and  $S$  such that  $d_{Teich}(S_i, S_{i+1}) \leq D_3$

Next choose  $x \in E_n$ , such that  $d_N(x, S_n) = 2(D_1 + D_2)$ . Then by Lemma 4.2, there exists a pleated surface  $g : (S, \tau) \rightarrow N$  such that  $d_N(x, g(S)) \leq D_1$ . Let  $S_{n+1} = g(S)$ . Then by the triangle inequality and Lemma 4.3, if  $p \in S_n$  and  $q \in S_{n+1}$ ,

$$D_1 + D_2 \leq d_N(p, q) \leq 3(D_1 + D_2).$$

This allows us to continue inductively. The Lemma follows.  $\square$

Note that in the above Lemma, pleated surfaces are not assumed to be embedded. This is because immersed pleated surfaces with a uniform lower bound on injectivity radius are uniformly quasi-isometric to the corresponding Riemann surfaces.

Since there are exactly one or two ends  $E_i$ , we have thus shown:

**Lemma 4.6** *The hyperbolic metric space  $\widetilde{C(N)}$  is quasi-isometric to a tree  $(T)$  of hyperbolic metric spaces satisfying the qi-embedded condition, where  $T$  is either  $[0, \infty)$  (simply degenerate) or  $\mathbb{R}$  (doubly degenerate).*

Applying Theorem 3.3, we obtain the following theorem of Minsky:

**Theorem 4.7 Minsky [15]** *Let  $\Gamma = \rho(\pi_1(S))$  be a closed surface Kleinian group, such that  $\mathbb{H}^3/\Gamma = M$  has injectivity radius uniformly bounded below by some  $\epsilon > 0$ . Then there exists a continuous map from the Gromov boundary of  $\Gamma$  (regarded as an abstract group) to the limit set of  $\Gamma$  in  $\mathbb{S}_\infty^2$ .*

Since a continuous image of a compact locally connected set is locally connected [12], and since the limit set of  $\Gamma$  is a continuous image of the circle, by Theorem 3.3, we have:

**Corollary 4.8** *Suppose  $\Gamma$  is a closed surface Kleinian group, such that  $N = \mathbb{H}^3/\Gamma$  has bounded geometry, i.e.  $inj_N(x) > \epsilon_0$  for all  $x \in N$ . Then the limit set of  $\Gamma$  is locally connected.*



# 5 Punctured Surface Groups of Bounded Geometry:

## A Theorem of Bowditch

### 5.1 Outline of Proof

In [5] [6], Bowditch proved the existence of Cannon-Thurston maps for punctured surface groups of bounded geometry using some of the ideas from [17]. We give below a different proof of the result, which is in some ways simpler. Furst, a sketch.

Let  $N^h$  be a bounded geometry 3-manifold corresponding to a representation of the fundamental group of a punctured surface  $S^h$ . Excise the cusps (if any) of  $N^h$  leaving us a manifold that has one or two ends. Let  $N$  denote  $N^h$  minus cusps. Then  $N$  is quasi-isometric to the universal curve over a Lipschitz path in Teichmüller space from which cusps have been removed. This path is semi-infinite or bi-infinite according as  $N$  is one-ended or two-ended. Fix a reference finite volume hyperbolic surface  $S^h$ . Let  $S$  denote  $S^h$  minus cusps. Then  $\tilde{S}$  is quasi-isometric to the Cayley graph of  $\pi_1(S)$  which is (Gromov) hyperbolic. We fix a base surface in  $N$  and identify it with  $S$ . Now look at  $\tilde{S} \subset \tilde{N}$ . Let  $\lambda = [a, b]$  be a geodesic segment in  $\tilde{S}$ . We ‘flow’  $\lambda$  out the end(s) of  $\tilde{N}$  to generate the **hyperbolic ladder-like set**  $B_\lambda$  as in the proof of Theorem 3.1.

Theorem 3.1 ensures that there is a retraction from  $\tilde{N}$  to  $B_\lambda$  which does not increase distances much. From this it follows that  $B_\lambda$  is quasi-isometrically embedded in  $\tilde{N}$ . Recall that for the construction of  $B_\lambda$ , we only require the hyperbolicity of  $\tilde{S}$  and not that of  $\tilde{N}$ .

Now if  $\lambda$  lies outside a large ball about a fixed reference point  $p$  in  $\tilde{S}$ , then so does  $B_\lambda$  in  $\tilde{N}$ . Since  $B_\lambda$  is q.i. embedded in  $\tilde{N}$ , there exists an ambient  $\tilde{N}$ -quasigeodesic  $\mu$  lying in a bounded neighborhood of  $B_\lambda$  joining the end-points of  $\lambda$ . If  $S^h$  had no cusps, as in the case for closed surfaces, we could immediately conclude that for any geodesic segment  $\lambda$  in  $\tilde{S}^h$  lying outside large balls around  $p$ , there is a quasigeodesic in  $\tilde{N}^h$  joining its endpoints and lying outside a large ball around  $p$  in  $\tilde{N}^h$ . However, since  $S^h$  has cusps,  $\tilde{S}^h$  and  $\tilde{S}$  are different. So a little more work is necessary. Suppose as before that  $\lambda_0$  is a geodesic in  $\tilde{S}^h$  lying outside a large ball around  $p$ . For ease of exposition we assume that the end-points of  $\lambda_0$  lie outside cusps. Let  $\lambda \subset \tilde{S}$

be the geodesic in (the path-metric on)  $\tilde{S}$  joining the same pair of points. Then off horodisks,  $\lambda_0$  and  $\lambda$  track each other. Construct  $B_\lambda$  as before, and let  $\mu$  be an ambient  $\tilde{N}$ -quasigeodesic lying in a bounded neighborhood of  $B_\lambda$  joining the end-points of  $\lambda$ . Then, modulo horoballs in  $\tilde{N}^h$ ,  $\mu$  lies outside a large ball around  $p$ . Let  $\mu_0$  be the hyperbolic geodesic joining the end points of  $\mu$ . Off horoballs,  $\mu$  and  $\mu_0$  track each other. Hence, off horoballs,  $\mu_0$  lies outside large balls about  $p$ . The points at which  $\mu_0$  enters and leaves a particular horoball therefore lie outside large balls about  $p$ . But then the hyperbolic segment joining them must do the same. This shows that  $\mu_0$  must itself lie outside large balls around  $p$ . As before we conclude that there exists a continuous extension of the inclusion of  $\tilde{S}^h$  into  $\tilde{N}^h$  to the boundary. The remaining part of this section fleshes out this argument.

## 5.2 Relative Hyperbolicity

We shall be requiring certain properties of hyperbolic spaces minus horoballs. These were studied by Farb [9] under the garb of ‘electric geometry’. We combine Farb’s results with a version that is a (slight variant of) theorem due to McMullen (Theorem 8.1 of [13]).

**Definition:** A path  $\gamma : I \rightarrow Y$  to a path metric space  $Y$  is an ambient  $K$ -quasigeodesic if we have

$$L(\beta) \leq KL(A) + K$$

for any subsegment  $\beta = \gamma|[a, b]$  and any path  $A : [a, b] \rightarrow Y$  with the same endpoints.

The following definitions are adapted from [9]

**Definition:** Let  $M$  be a convex hyperbolic manifold. Let  $Y$  be the universal cover of  $M$  minus cusps and  $X = \tilde{M}$ .  $\gamma$  is said to be a  $K$ -quasigeodesic in  $X$  **without backtracking** if

- $\gamma$  is a  $K$ -quasigeodesic in  $X$
- $\gamma$  does not return to any horoball  $\mathbf{H}$  after leaving it.

**Definition:**  $\gamma$  is said to be an ambient  $K$ -quasigeodesic in  $Y$  **without backtracking** if

- $\gamma$  is an ambient  $K$ -quasigeodesic in  $Y$

- $\gamma$  is obtained from a  $K$ -quasigeodesic without backtracking in  $X$  by replacing each maximal subsegment with end-points on a horosphere by a quasigeodesic lying on the surface of the horosphere.

Note that in the above definition, we allow the behaviour to be quite arbitrary on horospheres (since Euclidean quasigeodesics may be quite wild); however, we do not allow wild behaviour off horoballs.

**Remark:** Our definition of *ambient quasigeodesic without backtracking* does not allow a path to follow a horosphere for a long distance without entering it. This is a point where the definition differs from the usual definition of an ambient quasigeodesic.

$B_R(Z)$  will denote the  $R$ -neighborhood of the set  $Z$ .

Let  $\mathcal{H}$  be a locally finite collection of horoballs in a convex subset  $X$  of  $\mathbb{H}^n$  (where the intersection of a horoball, which meets  $\partial X$  in a point, with  $X$  is called a horoball in  $X$ ). The following theorem is due to McMullen [13].

**Theorem 5.1** [13] *Let  $\gamma : I \rightarrow X \setminus \bigcup \mathcal{H} = \mathcal{Y}$  be an ambient  $K$ -quasigeodesic for a convex subset  $X$  of  $\mathbb{H}^n$  and let  $\mathcal{H}$  denote a collection of horoballs. Let  $\eta$  be the hyperbolic geodesic with the same endpoints as  $\gamma$ . Let  $\mathcal{H}(\eta)$  be the union of all the horoballs in  $\mathcal{H}$  meeting  $\eta$ . Then  $\eta \cup \mathcal{H}(\eta)$  is (uniformly) quasiconvex and  $\gamma(I) \subset B_R(\eta \cup \mathcal{H}(\eta))$ , where  $R$  depends only on  $K$ .*

Theorem 5.1 is similar in flavour to certain theorems about relative hyperbolicity *a la* Gromov [11], Farb [9] and Bowditch [4]. We give below a related theorem that is derived from Farb’s ‘Bounded Horosphere Penetration’ property.

Let  $\gamma_1 = \overline{pq}$  be a hyperbolic  $K$ -quasigeodesic without backtracking starting from a horoball  $\mathbf{H}_1$  and ending within (or on) a *different* horoball  $\mathbf{H}_2$ . Let  $\gamma = [a, b]$  be the hyperbolic geodesic minimising distance between  $\mathbf{H}_1$  and  $\mathbf{H}_2$ . Following [9] we put the zero metric on the horoballs that  $\gamma$  penetrates. The resultant pseudo-metric is called the electric metric. Let  $\widehat{\gamma}$  and  $\widehat{\gamma}_1$  denote the paths represented by  $\gamma$  and  $\gamma_1$  respectively in this pseudometric. It is shown in [9] that  $\gamma$ ,  $\widehat{\gamma}$  and  $\widehat{\gamma}_1$  have similar intersection patterns with horoballs, i.e. there exists  $C_0$  such that

- 1 If only one of  $\gamma$  and  $\widehat{\gamma}_1$  penetrates a horoball  $\mathbf{H}$ , then it can do so for a distance  $\leq C_0$ .
- 2 If both  $\widehat{\gamma}_1$  and  $\gamma$  enter (or leave) a horoball  $\mathbf{H}$  then their entry (or exit)

points are at a distance of at most  $C_0$  from each other. [Here by ‘entry’ (resp. ‘exit’) point of a quasigeodesic we mean a point at which the path switches from being in the complement of or ‘outside’ (resp. in the interior of or ‘inside’) a closed horoball to being inside (resp. outside) it].

The point to observe here is that quasigeodesics without backtracking in our definition gives rise to quasigeodesics without backtracking in Farb’s sense. Since this is true for arbitrary  $\gamma_1$  we give below a slight strengthening of this fact. Further, by our construction of ambient quasigeodesics without backtracking, we might just as well consider ambient quasigeodesics without backtracking in place of quasigeodesics.

**Theorem 5.2** [9] *Given  $C > 0$ , there exists  $C_0$  such that if*

- 1 *either two quasigeodesics without backtracking  $\gamma_1, \gamma_2$  in  $X$ , OR*
- 2 *two ambient quasigeodesics without backtracking  $\gamma_1, \gamma_2$  in  $Y$ , OR*
- 3  *$\gamma_1$  - an ambient quasigeodesic without backtracking in  $Y$  and  $\gamma_2$  - a quasigeodesic without backtracking in  $X$ ,*

*start and end*

- 1 *either on (or within) the same horoball OR*
- 2 *a distance  $C$  from each other*

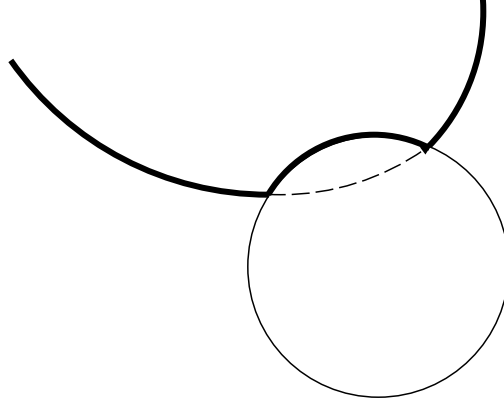
*then they have similar intersection patterns with horoballs (except possibly the first and last ones), i.e. there exists  $C_0$  such that*

- 1 *If only  $\gamma_1$  penetrates or travels along the boundary of a horoball  $\mathbf{H}$ , then it can do so for a distance  $\leq C_0$ .*
- 2 *If both  $\gamma_1$  and  $\gamma_2$  enter (or leave) a horoball  $\mathbf{H}$  then their entry (or exit) points are at a distance of at most  $C_0$  from each other.*

### 5.3 Horo-ambient Quasigeodesics

A special kind of quasigeodesic without back-tracking will be necessary. We start with a hyperbolic geodesic  $\lambda^h$  in  $\tilde{S}^h$ . Fix a neighborhood of the cusps lifting to an equivariant family of horoballs in the universal cover  $\mathbb{H}^2 = \tilde{S}^h$ . Since  $\lambda^h$  is a hyperbolic geodesic in  $\tilde{S}^h$  there are unique entry and exit points for each horoball that  $\lambda^h$  meets and hence unique Euclidean geodesics joining them on the corresponding horosphere. Replacing the segments of  $\lambda^h$  lying inside  $Z$ -horoballs by the corresponding Euclidean geodesics, we

obtain an ambient quasigeodesic  $\lambda$  in  $\widetilde{M}_0$  as a consequence of Theorem 5.1 (See Corollary 5.3 and figure below):



Horo-ambient quasigeodesics

Ambient quasigeodesics obtained by this kind of a construction will be termed **horo-ambient quasigeodesics** to distinguish them from *electro-ambient quasigeodesics* defined earlier.

The following Corollary of Theorem 5.1 justifies the terminology.

**Corollary 5.3** *There exists  $K, \epsilon$ , such that any horo-ambient quasigeodesic in  $Y$  is indeed a  $K, \epsilon$  ambient quasigeodesic in  $Y$ .*

**Proof:** Let  $[a, b]_h$  be a hyperbolic geodesic in  $X$  joining  $a, b$ , where  $a, b$  lie outside horoballs. Let  $[a, b]_{ha}$  be the horo-ambient quasigeodesic in  $Y$  joining  $a, b$ . Let  $[a, b]_a$  be the ambient geodesic in  $Y$  joining  $a, b$ . Then by Theorem 5.1, there exists  $R \geq 0$ , such that  $[a, b]_a$  lies in an  $R$ -neighborhood of  $[a, b]_h \cup \mathcal{H}(\eta)$ . Project  $[a, b]_a$  onto  $[a, b]_h \cup \mathcal{H}(\eta)$  using the nearest point projection in  $X$ . Removing the back-trackings induced, we get some  $K, \epsilon$  depending on  $R$  such that the image is an ambient quasigeodesic without backtracking in our sense. Clearly, such an ambient quasigeodesic coincides with  $[a, b]_{ha}$  off horoballs  $\mathcal{H}(\eta)$ . Since the interpolating segments in  $[a, b]_{ha}$  on any horoball in  $\mathcal{H}(\eta)$  are Euclidean *geodesics*, the result follows.  $\square$

## 5.4 Trees of Spaces

We want to first show that the universal cover of  $N^h$  minus  $Z$ -cusps is quasi-isometric to a tree of hyperbolic metric spaces.

Let  $E^h$  be a simply degenerate end of  $N^h$ . Then  $E^h$  is homeomorphic to  $S^h \times [0, \infty)$  for some surface  $S^h$  of negative Euler characteristic. Cutting off a neighborhood of the cusps of  $S^h$  we get a surface with boundary denoted as  $S$ . Let  $E$  denote  $E^h$  minus a neighborhood of the  $Z$ -cusps. We assume that each  $Z$ -cusp has the standard form coming from a quotient of a horoball in  $\mathbb{H}^3$  by  $Z$ . Also, we shall take our pleated surfaces to be such that the pair  $(S, \text{cusps})$  is mapped to the pair  $(E, \text{cusps})$  for each pleated  $S^h$ . We shall now show that each  $\tilde{E}$  is quasi-isometric to a ray of hyperbolic metric spaces satisfying the q-i embedded condition. Each edge and vertex space will be a copy of  $\tilde{S}$  and the edge to vertex space inclusions shall be quasi-isometries. Note that each  $\tilde{S}$  can be thought of as a copy of  $\mathbb{H}^2$  minus an equivariant family of horodisks. The following Lemmas are generalisations to punctured surfaces of Lemmas 4.2, 4.3, 4.4.

**Lemma 5.4** [21] *There exists  $D_1 > 0$  such that for all  $x \in E$ , there exists a pleated surface  $g : (S^h, \sigma) \rightarrow E^h$  with  $g(S) \cap B_{D_1}(x) \neq \emptyset$ . Also  $g$  maps  $(S, \text{cusps})$  to  $(E, \text{cusps})$ .*

**Lemma 5.5** [3],[21] *There exists  $D_2 > 0$  such that if  $g : (S^h, \sigma) \rightarrow N^h$  is a pleated surface, then the diameter of the image of  $S$  is bounded, i.e.  $\text{dia}(g(S)) < D_2$ .*

The following Lemma due to Thurston (Theorems 9.2 and 9.6.1 of [21] ) and Minsky [14] follows from compactness of pleated surfaces.

**Lemma 5.6** [14] *Fix  $S^h$  and  $\epsilon > 0$ . Given  $a > 0$  there exists  $b > 0$  such that if  $g : (S^h, \sigma) \rightarrow E^h$  and  $h : (S^h, \rho) \rightarrow E^h$  are homotopic pleated surfaces which are isomorphisms on  $\pi_1$  and  $E^h$  is of bounded geometry, then*

$$d_E(g(S), h(S)) \leq a \Rightarrow d_{\text{Teich}}(\sigma, \rho) \leq b,$$

where  $d_{\text{Teich}}$  denotes Teichmuller distance.

In [14] a specialisation of this statement is proven for closed surfaces. However, the main ingredient, a Theorem due to Thurston is stated and proven in [21] (Theorems 9.2 and 9.6.1 - '*algebraic limit is geometric limit*') for finite area surfaces. The arguments given by Minsky to prove the above Lemma from Thurston's Theorems (Lemma 4.5, Corollary 4.6 and Lemma 4.7 of [14]) go through with very little change for surfaces of finite area.

**Construction of equispaced pleated surfaces exiting the end**

We next construct a sequence of **equispaced** pleated surfaces  $S^h(i) \subset E^h$  exiting the end as before. Assume that  $S^h(0), \dots, S^h(n)$  have been constructed such that:

1.  $S(i)$ , *cusps* is mapped to  $E$ , *cusps*
2. If  $E(i)$  be the component of  $E \setminus S(i)$  for which  $E(i)$  is non-compact, then  $S(i+1) \subset E(i)$ .
3. Hausdorff distance between  $S(i)$  and  $S(i+1)$  is bounded above by  $3(D_1 + D_2)$ .
4.  $d_E(S(i), S(i+1)) \geq D_1 + D_2$ .
5. From Lemma 4.4 and condition (2) above there exists  $D_3$  depending on  $D_1, D_2$  and  $S$  such that  $d_{Teich}(S(i), S(i+1)) \leq D_3$

Next choose  $x \in E(n)$ , such that  $d_E(x, S_n) = 2(D_1 + D_2)$ . Then by Lemma 5.4, there exists a pleated surface  $g : (S^h, \tau) \rightarrow E^h$  such that  $d_E(x, g(S)) \leq D_1$ . Let  $S^h(n+1) = g(S^h)$ . Then by the triangle inequality and Lemma 5.5, if  $p \in S(n)$  and  $q \in S(n+1)$ ,

$$D_1 + D_2 \leq d_E(p, q) \leq 3(D_1 + D_2).$$

This allows us to continue inductively.  $S(i)$  corresponds to a point  $x_i$  of  $Teich(S)$ . Joining the  $x_i$ 's in order, one gets a Lipschitz path in  $Teich(S)$ .

**Definition:** A sequence of pleated surfaces satisfying conditions (1-5) above will be called an **equispaced sequence of pleated surfaces**. The corresponding sequence of  $S(i) \subset E$  will be called an **equispaced sequence of truncated pleated surfaces**

Each  $S(i)$  being compact (with or without boundary),  $\widetilde{S}(i)$  is a hyperbolic metric space. We can think of the universal cover  $\widetilde{E}$  of  $E$  as being quasi-isometric to a ray  $T$  of hyperbolic metric spaces by setting  $T = [0, \infty)$ , with vertex set  $\mathcal{V} = \{ n : n \in \mathbb{N} \cup \{ 0 \} \}$ , edge set  $\mathcal{E} = \{ [n-1, n] : n \in \mathbb{N} \}$ ,  $X_n = \widetilde{S}(n) = X_{[n-1, n]}$ . Further, by Lemma 5.6 this tree of hyperbolic metric spaces satisfies the quasi-isometrically embedded condition. We have thus shown

**Lemma 5.7** *If  $E^h$  be a simply degenerate end of a hyperbolic 3 manifold  $N^h$  with bounded geometry, then there is a sequence of equispaced pleated surfaces exiting  $E^h$  and hence a sequence of truncated pleated surfaces exiting  $\widetilde{E}$ . Further,  $\widetilde{E}$  is quasi-isometric to a ray of hyperbolic metric spaces satisfying the q.i. embedded condition.*

## 5.5 Construction of $B_\lambda$ : Modifications for Punctured Surfaces

Now, let  $N^h$  be a bounded geometry 3-manifold corresponding to a representation of the fundamental group of a punctured surface  $S^h$ . Excise the cusps (if any) of  $N^h$  leaving us a manifold that has one or two ends. Let  $N$  denote  $N^h$  minus cusps. Fix a reference finite volume hyperbolic surface  $S^h$ . Let  $S$  denote  $S^h$  minus cusps. Then  $\widetilde{S}$  is quasi-isometric to the Cayley graph of  $\pi_1(S)$  which is (Gromov) hyperbolic, in fact quasi-isometric to a tree. We fix a base surface in  $N$  and identify it with  $S$ . Now look at  $\widetilde{S} \subset \widetilde{N}$ .

Then, by Lemma 5.7  $\widetilde{N}$  is quasi-isometric to a tree  $T$  of hyperbolic metric spaces. Each of the vertex and edge spaces is a copy of  $\widetilde{S}$ . Also, the map  $\phi_i$  induced from  $\widetilde{S} \times \{i\}$  to  $\widetilde{S} \times \{i+1\}$  is a  $(K, \epsilon)$ -quasi-isometry for all  $i$  as are their quasi-isometric inverses. Further,  $T$  is a semi-infinite or bi-infinite interval in  $\mathbb{R}$  according as  $N$  is one-ended or two-ended. So far, this is exactly like the case for closed surfaces. But here, we can assume in addition that each  $\phi_i$  is the restriction of a map  $\phi_i^h$  from  $\widetilde{S}^h \times \{i\}$  to  $\widetilde{S}^h \times \{i+1\}$  which preserves horodisks. Let  $\Phi_i^h$  denote the induced map on hyperbolic geodesics and let  $\Phi_i$  denote the induced map on horo-ambient geodesics.



Let  $\lambda^h$  be a hyperbolic geodesic segment in  $\tilde{S}^h$ . Let  $\lambda$  be the horo-ambient quasigeodesic in  $\tilde{S}$  joining the end-points of  $\lambda^h$ .

Starting with a horo-ambient quasigeodesic  $\lambda \subset \tilde{S}$ , we can now proceed as in the proof of Theorem 3.1 to construct the **hyperbolic ladder-like set**  $B_\lambda$ .

There is only one difference: **Each  $\lambda_i$  in this situation is a horo-ambient quasigeodesic, and not necessarily a hyperbolic geodesic.** Thus, we set  $\lambda = \lambda_0$  to be some horo-ambient quasigeodesic in  $\tilde{S} = \tilde{S} \times \{0\}$ . Next, (for  $i \geq 0$ ), inductively, set  $\lambda_{i+1}$  to be the (unique) horo-ambient quasigeodesic in  $\tilde{S} \times \{i+1\}$  joining the end-points of  $\phi_i(\lambda_i)$ . That is to say,  $\lambda_{i+1} = \Phi_i(\lambda_i)$ . Similarly, for  $i \leq 0$ .

Note that Corollary 5.3 ensures that there exist  $K_0, \epsilon_0$  such that each  $\lambda_i$  is a  $(K_0, \epsilon_0)$ -quasigeodesic in  $\tilde{S} \times \{i\}$ .

Then Theorem 3.1 (or more precisely, the Note following it) ensures that there is a retraction from  $\tilde{N}$  to  $B_\lambda$  which does not increase distances much. From this it follows that  $B_\lambda$  is quasi-isometrically embedded in  $\tilde{N}$ . Recall that for the construction of  $B_\lambda$ , we only require the hyperbolicity of  $\tilde{S}$  and not that of  $\tilde{N}$ .

As before, (by projecting a geodesic in  $\tilde{N}$  onto  $B_\lambda$ ) we obtain an ambient quasigeodesic contained in  $B_\lambda$  joining the end-points  $a, b$  of  $\lambda$ .

Let

- $\beta^h =$  geodesic in  $\tilde{N}^h$  joining  $a, b$
- $\beta_{amb}^0 =$  horo-ambient quasigeodesic in  $\tilde{N}$  obtained from  $\beta^h$  by replacement of hyperbolic by ‘Euclidean’ geodesic segments for horoballs in  $\tilde{N}^h$
- $\beta_{amb} = \Pi_\lambda(\beta_{amb}^0) \cap B_\lambda$

## 5.6 Quasigeodesic Rays

Let  $\lambda_i^c$  denote the union of the segments of  $\lambda_i$  which lie along horocycles and let  $\lambda_i^b = \lambda_i - \lambda_i^c$ . Let

$$\begin{aligned} B_\lambda^c &= \cup_i \lambda_i^c \\ B_\lambda^b &= \cup_i \lambda_i^b \end{aligned}$$

We want to show that for all  $x \in B_\lambda^b$  there exists a  $C$ -quasigeodesic  $r_x : \{0\} \cup \mathbb{N} \rightarrow B_\lambda$  such that  $x \in r_x(\{0\} \cup \mathbb{N})$  and  $r_x(i) \in \lambda_i^b$ . Suppose

$x \in \lambda_k^b \subset B_\lambda^b$ . We define  $r_x$  by starting with  $r_x(k) = x$  and construct  $r_x(k-i)$  and  $r_x(k+i)$  inductively (of course  $(k-i)$  stops at 0 for  $T$  a semi-infinite ray but  $(k+i)$  goes on to infinity). For the sake of concreteness, we prove the existence of such a  $r_x(k+1)$ . The same argument applies to  $(k-1)$  and inductively for the rest.

**Lemma 5.8** *There exists  $C > 0$  such that if  $r_x(k) = x \in \lambda_k^b$  then there exists  $x' \in \lambda_{k+1}^b$  such that  $d(x, x') \leq C$ . We denote  $r_x(k+1) = x'$ .*

**Proof:** Let  $[a, b]$  be the maximal connected component of  $\lambda_k^b$  on which  $x$  lies. Then there exist two horospheres  $\mathbf{H}_1$  and  $\mathbf{H}_2$  such that  $a \in \mathbf{H}_1$  (or is the initial point of  $\lambda_k$ ) and  $b \in \mathbf{H}_2$  (or is the terminal point of  $\lambda_k$ ). Since  $\phi_k$  preserves horocycles,  $\phi_k(a)$  lies on a horocycle (or is the initial point of  $\lambda_{k+1}$ ) as does  $\phi_k(b)$  (or is the terminal point of  $\lambda_{k+1}$ ). Further, the image of  $[a, b]$  under  $\phi_k$  is a quasigeodesic in  $\tilde{S} \times \{k+1\}$  **which we now denote as**  $\phi_k([a, b])$ . Recall that  $\Phi_k([a, b])$  is the horo-ambient geodesic in  $\tilde{S} \times \{k+1\}$  joining  $\phi_k(a)$  and  $\phi_k(b)$ .

Therefore by (Gromov) hyperbolicity of  $\tilde{S}$ ,  $\Phi_k([a, b])$  lies in a bounded neighborhood of  $\phi_k([a, b])$  (which in turn lies at a bounded distance from  $\Phi_k^h([a, b])$ ) and hence by Theorem 5.2 there exists an upper bound on how much  $\Phi_k^h([a, b])$  can penetrate horoballs, i.e. there exists  $C_1 > 0$  such that for all  $z \in \Phi_k^h([a, b])$ , there exists  $z' \in \Phi_k^h([a, b])$  lying outside horoballs with  $d(z, z') \leq C_1$ . Further, since  $\phi_k$  is a quasi-isometry there exists  $C_2 > 0$  such that  $d(\phi_k(x), \Phi_k^h([a, b])) \leq C_2$ . Hence there exists  $x' \in \Phi_k^h([a, b])$  such that

- $d(\phi_k(x), x') \leq C_1 + C_2$
- $x'$  lies outside horoballs.

Again,  $\Phi_k([a, b])$  lies at a uniformly bounded distance  $\leq C_3$  from  $\lambda_{k+1}$  and so, if  $c, d \in \lambda_{k+1}$  such that  $d(a, c) \leq C_3$  and  $d(b, d) \leq C_3$  then the segment  $[c, d]$  can penetrate only a bounded distance into any horoball. Hence there exists  $C_4 > 0$  and  $x'' \in [c, d] \subset \lambda_{k+1}$  such that

- $d(x', x'') \leq C_4$
- $x''$  lies outside horoballs.

Hence,  $d(\phi_k(x), x'') \leq C_1 + C_2 + C_4$ . Since  $d(x, \phi_{k+1}(x)) = 1$ , we have, by choosing  $r_{k+1}(x) = x''$ ,

$$d(r_k(x), r_{k+1}(x)) \leq 1 + C_1 + C_2 + C_4.$$

Choosing  $C = 1 + C_1 + C_2 + C_4$ , we are through.  $\square$

Using Lemma 5.8 repeatedly (inductively replacing  $x$  with  $r_x(k+i)$ ) we obtain the values of  $r_x(i)$  for  $i \geq k$ . By an exactly similar symmetric argument, we get  $r_x(k-1)$  and proceed down to  $r_x(0)$ . Now for any  $i, z \in \tilde{S} \times \{i\}$  and  $y \in \tilde{S} \times \{i+1\}$ ,  $d(z, y) \geq 1$ . Hence, for any  $z \in \tilde{S} \times \{i\}$  and  $y \in \tilde{S} \times \{j\}$ ,  $d(z, y) \geq |i - j|$ . This gives

**Corollary 5.9** *There exist  $K, \epsilon > 0$  such that for all  $x \in \lambda_k^b \subset B_\lambda^b$  there exists a  $(K, \epsilon)$  quasigeodesic ray  $r_x$  such that  $r_x(k) = x$  and  $r_x(i) \in \lambda_i^b$  for all  $i$ .*

To fix and recall notation:

- $\lambda^h$  = hyperbolic geodesic in  $\tilde{S}^h$  joining  $a, b$
- $\lambda$  = horo-ambient quasigeodesic in  $\tilde{S}$  constructed from  $\lambda^h \subset \tilde{S}^h$
- $\beta^h$  = geodesic in  $\tilde{N}^h$  joining  $a, b$
- $\beta_{amb}^0$  = horo-ambient quasigeodesic in  $\tilde{N}$  obtained from  $\beta^h$  by replacement of hyperbolic by ‘Euclidean’ geodesic segments for horoballs in  $\tilde{N}^h$
- $\beta_{amb} = \Pi_\lambda(\beta_{amb}^0) \cap B_\lambda$

## 5.7 Proof of Theorem for Surfaces with Punctures

By construction, the hyperbolic geodesic  $\beta^h$  and the ambient quasigeodesic  $\beta_{amb}^0$  agree exactly off horoballs.  $\beta_{amb}$  is constructed from  $\beta_{amb}^0$  by projecting it onto  $B_\lambda$  and so by Theorem 3.1, it is an ambient quasigeodesic. But it might ‘backtrack’. Hence, we need to modify it such that it satisfies the no backtracking condition. First, observe by Theorem 5.1 that all three  $\beta^h$ ,  $\beta_{amb}^0$ ,  $\beta_{amb}$  track each other off some  $K$ -neighborhood of horoballs.

The advantage of working with  $\beta_{amb}$  is that it lies on  $B_\lambda$ . However, it might backtrack.

**Lemma 5.10** *There exists  $C > 0$  such that for all  $x \in \lambda_i^b \subset B_\lambda^b \subset B_\lambda$  if  $\lambda^h$  lies outside  $B_n(p)$  for a fixed reference point  $p \in \tilde{S}^h$  and also assuming that in fact the reference point lies in  $\tilde{S}$ , then  $x$  lies outside an  $\frac{n-C}{C+1}$  ball about  $p$  in  $\tilde{N}$ .*

**Proof:** Since  $\lambda^b$  is a part of  $\lambda^h$ , therefore  $r_x(0)$  lies outside  $B_n(p)$ . By Corollary 5.9, there exists  $C > 0$  such that for all  $i, j \in \{0, 1, 2, \dots\}$ ,

$$|i - j| \leq d(r_x(i), r_x(j)) \leq C|i - j|$$

Also,  $d(x, p) \geq i$  since  $x \in \mu_i^b$ . (Here distances are all measured in  $\widetilde{N}$ .) Hence,

$$\begin{aligned} d(x, p) &\geq \min\{i, n - C(i + 1)\} \\ &\geq \frac{n - C}{C + 1} \end{aligned}$$

This proves the result.  $\square$

If  $x \in B_\lambda$ , then  $x \in B_\lambda^b$  or  $x \in B_\lambda^c$  for some  $\mu$ . Hence  $x \in B_\lambda$  implies that either  $x$  lies on some horosphere bounding some  $\mathbf{H} \in \mathcal{H}$  or, from Lemma 5.10 above,  $d(x, p) \geq \frac{n-C}{C+1}$ . Since  $\beta_{amb}$  lies on  $B_\lambda$ , we conclude that  $\beta_{amb}$  is an ambient quasigeodesic in  $\widetilde{N}$  such that every point  $x$  on  $\beta_{amb}$  either lies on some horosphere bounding some  $\mathbf{H} \in \mathcal{H}$  or, from Lemma 5.10 above,  $d(x, p) \geq \frac{n-C}{C+1}$ .

McMullen [13] shows (cf Theorem 5.1) that in  $\widetilde{N}^h$ , any such ambient quasigeodesic  $\beta_{amb}$  lies in a bounded neighborhood of  $\beta^h \cup \mathcal{H}(\beta^h)$ . We do not as yet know that  $\beta_{amb}$  does not backtrack, but we can convert it into one without much effort. Let  $\Pi$  denote nearest point projection of  $\widetilde{N}^h$  onto  $\beta^h \cup \mathcal{H}(\beta^h)$ . Then  $\Pi(\beta_{amb}) = \beta_1$  is again an ambient quasigeodesic in  $\widetilde{N}_0$ . Further,  $\beta_1$  tracks  $\beta_{amb}$  throughout its length, since  $\Pi$  moves points through a uniformly bounded distance. Now  $\beta_1$  might backtrack, but it can do so in a trivial way, i.e. if  $\beta_1$  re-enters a horoball after leaving it, it must do so at exactly the point where it leaves it. Removing these ‘trivial backtracks’, we obtain an **ambient quasigeodesic without backtracking**  $\beta$  which tracks  $\beta_{amb}$  throughout its length.

**Note:** On the one hand  $\beta$  is an ambient quasigeodesic without backtracking. Hence, it reflects the intersection pattern of  $\beta^h$  with horoballs. On the other hand, it **tracks**  $\beta_{amb}$  whose properties we already know from Corollary 5.9 above.

Since, of  $\beta$  and  $\beta^h$ , one is an ambient quasigeodesic without backtracking, and the other a hyperbolic geodesic joining the same pair of points, we

conclude from Theorem 5.2 that they have similar intersection patterns with horoballs, i.e. there exists  $C_0$  such that

- If only of  $\beta$  and  $\beta^h$  penetrates or travels along the boundary of a horoball  $\mathbf{H}$ , then it can do so for a distance  $\leq C_0$ .
- If both  $\beta$  and  $\beta^h$  enter (or leave) a horoball  $\mathbf{H}$  then their entry (or exit) points are at a distance of at most  $C_0$  from each other.

Again, since  $\beta$  tracks  $\beta_{amb}$ , we conclude that there exists  $C > 0$  such that  $\beta$  lies in a  $C$ -neighborhood of  $\beta_{amb}$  and hence from Lemma 5.10

- Every point  $x$  on  $\beta$  either lies on some horosphere bounding some  $\mathbf{H} \in \mathcal{H}$  or,  $d(x, p) \geq \frac{n-C}{C+1} - C$

The above three conditions on  $\beta$  and  $\beta^h$  allow us to deduce the following condition for  $\beta^h$ .

**Proposition 5.11** *Every point  $x$  on  $\beta^h$  either lies inside some horoball  $\mathbf{H} \in \mathcal{H}$  or,  $d(x, p) \geq \frac{n-C}{C+1} - C = m(n)$*

We split  $\beta^h$  into two parts.  $\beta^c$  consists of those points of  $\beta^h$  which lie within horoballs. We set  $\beta^b$  to be the closure of  $\beta^h - \beta^c$ .

We have denoted  $\frac{n-C}{C+1} - C$  by  $m(n)$ , so that  $m(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . The above Proposition asserts that the geodesic  $\beta^h$  lies outside large balls about  $p$  modulo horoballs. By Lemma 2.1 this is almost enough to guarantee the existence of a Cannon-Thurston map. The rest of the necessary work is given below.

**Theorem 5.12** *Suppose  $S^h$  is a hyperbolic surface of finite volume. Suppose that  $N^h$  is a hyperbolic manifold corresponding to a representation of  $\pi_1(S^h)$  without accidental parabolics. Let  $i : S^h \rightarrow N^h$  be a proper homotopy equivalence. Then  $\tilde{i} : \tilde{S}^h \rightarrow \tilde{N}^h$  extends continuously to the boundary  $\hat{i} : \widehat{S}^h \rightarrow \widehat{N}^h$ . If  $\Lambda$  denotes the limit set of  $\tilde{M}$ , then  $\Lambda$  is locally connected.*

**Proof:** Let  $\lambda^h$  be a geodesic segment in  $\widetilde{M}_{gf}$  lying outside  $B_n(p)$  for some fixed reference point  $p$ . Fix neighborhoods of the cusps and lift them to the universal cover. Let  $\mathcal{H}$  denote the set of horoballs. Assume without loss of generality that  $p$  lies outside horoballs. Let  $\beta^h$  be the hyperbolic geodesic in  $\widetilde{N}^h$  joining the endpoints of  $\lambda^h$ . Further, let  $\beta^h = \beta^b \cup \beta^c$  as above. Then by Proposition 5.11,  $\beta^b$  lies outside an  $m(n)$  ball about  $p$ , with  $m(n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Next, let  $\mathbf{H}_1$  be some horoball that  $\beta^h$  meets. Then the entry and exit points  $u$  and  $v$  of  $\beta^h$  into and out of  $\mathbf{H}_1$  lie outside an  $m(n)$  ball about  $p$ . Let  $z$  be the point on the boundary sphere that  $\mathbf{H}_1$  is based at. Then for any sequence  $x_i \in \mathbf{H}_1$  with  $d(p, x_i) \rightarrow \infty$ ,  $x_i \rightarrow z$ . If  $\{x_i\}$  and  $\{y_i\}$  denote two such sequences, then the visual diameter of the set  $\{x_i, y_i\}$  must go to zero. Hence, if  $[x_i, y_i]$  denotes the geodesic joining  $x_i, y_i$  then  $d(p, [x_i, y_i]) \rightarrow \infty$ . Since,  $u, v$  lie outside an  $m(n)$  ball, there exists some function  $\psi$ , such that the geodesic  $[u, v]$  lies outside a  $\psi(m(n))$  ball around  $p$ , where  $\psi(k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

Since the choice of this function does not depend on  $\mathbf{H}_1$ , which is chosen at random, we conclude that there exists such a function for all of  $\beta^c$ . We have thus established:

- $\beta^b$  lies outside an  $m(n)$  ball about  $p$ .
- $\beta^c$  lies outside a  $\psi(m(n))$  ball about  $p$ .
- $m(n)$  and  $\psi(m(n))$  tend to infinity as  $n \rightarrow \infty$

Define  $f(n) = \min(m(n), \psi(m(n)))$ . Then  $\beta^h$  lies outside an  $f(n)$  ball about  $p$  and  $f(n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

By Lemma 2.1  $i : \widetilde{S}^h \rightarrow \widetilde{N}^h$  extends continuously to the boundary  $\widehat{i} : \widehat{S}^h \rightarrow \widehat{N}^h$ . This proves the first statement of the theorem.

Now, the limit set of  $\widetilde{S}^h$  is the circle at infinity, which is locally connected. Further, the continuous image of a compact locally connected set is locally connected [12]. Hence, if  $\Lambda$  denotes the limit set of  $\widehat{N}^h$ , then  $\Lambda$  is locally connected. This proves the theorem.  $\square$

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