# SURFACE GROUP REPRESENTATIONS IN $\mathrm{SL}_2(\mathbb{C})$ WITH FINITE MAPPING CLASS ORBITS

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Abstract. Given an oriented surface of positive genus with finitely many punctures, we classify the finite orbits of the mapping class group action on the moduli space of semisimple complex special linear rank two representations of the fundamental group of the surface. For surfaces of genus at least two, such orbits correspond to homomorphisms with finite image. For genus one, they correspond to the finite or special dihedral representations. We also obtain an analogous result for bounded orbits in the moduli space.

Contents



## 1. Introduction

Let $\Sigma$ be an oriented surface of genus $g \geq 0$ with a finite set $\mathcal{F}$ of punctures. The $\mathrm{SL}_2(\mathbb{C})$-*character variety* of $\Sigma$

$$X(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C})) \mathbin{/\!\!/} \mathrm{SL}_2(\mathbb{C})$$

*Date*: June 7, 2019.
2000 *Mathematics Subject Classification.* Primary: 57M50; Secondary: 57M05, 20E36, 20F29.
*Key words and phrases.* Character variety; surface group; mapping class group.

is an affine algebraic variety whose complex points parametrize the conjugacy classes of semisimple representations $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ of the fundamental group of $\Sigma$. Let $\mathrm{Mod}(\Sigma)$ denote the pure mapping class group of $\Sigma$ fixing $\mathcal{F}$ pointwise. The group acts on the moduli space $X(\Sigma)$ by precomposition. This paper classifies the finite orbits of this action for surfaces of positive genus.

Our analysis divides into the cases of genus one and higher. For surfaces of genus at least two, we prove the following.

**Theorem A.** *Let $\Sigma$ be an oriented surface of genus $g \geq 2$ with $n \geq 0$ punctures. A semisimple representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ has finite mapping class group orbit in the character variety $X(\Sigma)$ if and only if $\rho$ is finite.*

To describe the corresponding result for surfaces of genus 1, define an irreducible representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ to be *special dihedral* if

- its image lies in the infinite dihedral group $D_\infty$ in $\mathrm{SL}_2(\mathbb{C})$ (its definition is recalled in Section 2), and
- there is a nonseparating simple closed curve $a$ in $\Sigma$ such that the restriction of $\rho$ to the complement $\Sigma \backslash a$ is diagonal.

**Theorem B.** *Let $\Sigma$ be an oriented surface of genus 1 with $n \geq 0$ punctures. A semisimple representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ has finite mapping class group orbit in $X(\Sigma)$ if and only if $\rho$ is finite or special dihedral up to conjugacy.*

Many of the technical issues in the proofs of Theorems A and B above arise while dealing with punctures. For $\Sigma$ closed (i.e. $n = 0$) the proofs become considerably simplified thanks to existing results, in particular [8, 17, 4]. To illustrate this, we have included a short proof of Theorem A in Section 5 for closed surfaces of genus at least 2 using [8, 17]. For surfaces of genus zero with more than three punctures, the description of the finite mapping class group orbits in the $\mathrm{SL}_2(\mathbb{C})$-character variety is more complicated and in general unknown. Lisovyy-Tykhyy [15] completed the case of the four-punctured sphere as part of their classification of algebraic solutions to Painlevé VI differential equations, and in this case there exist finite mapping class group orbits corresponding to representations with Zariski dense image in $\mathrm{SL}_2(\mathbb{C})$. We remark that the once-punctured torus case of Theorem B was essentially proved by Dubrovin-Mazzocco [5] (also in connection with Painlevé VI); the derivation from [5] is recorded in the Appendix. Our work gives a different proof in this case.

This paper pursues the theme of characterizing points on the character variety $X(\Sigma)$ with special dynamical properties. In this spirit, we also prove the result below. Given a complex algebraic variety $V$, we shall say that a subset of $V(\mathbb{C})$ is *bounded* if it has compact closure in $V(\mathbb{C})$ with respect to the Euclidean topology.

**Theorem C.** *Let $\Sigma$ be an oriented surface of genus $g \geq 1$ with $n \geq 0$ punctures. A semisimple representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ has bounded mapping class group orbit in the character variety $X(\Sigma)$ if and only if*

(a) *$\rho$ is unitary up to conjugacy, or*
(b) *$g = 1$ and $\rho$ is special dihedral up to conjugacy.*

Our results and methods answer some previously raised basic questions. Theorems A and B imply (Corollary 6.1) that a faithful representation of a positive-genus hyperbolic surface group into $\mathrm{SL}_2(\mathbb{C})$ (or $\mathrm{PSL}_2(\mathbb{C})$) cannot have finite mapping class group orbit in the character variety, answering a question raised by Lubotzky. Theorems A and B also verify the $G = \mathrm{SL}_2(\mathbb{C})$ case of the following conjecture of Kisin

[20, Chapter 1]: For $\pi$ the fundamental group of a closed surface or a free group of rank $r \geq 3$, the points with finite orbits for the action of the outer automorphism group $\mathrm{Out}(\pi)$ on the character variety $\mathrm{Hom}(\pi, G)/\!\!/G$, for reductive algebraic groups $G$, correspond to representations $\pi \to G$ with virtually solvable image. However, there are counter-examples to this conjecture for general $G$ [12]. Finally, we show that, given a closed hyperbolic surface $S$ of genus $\geq 2$, the energy of the harmonic map associated to a representation $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ is bounded along the mapping class group orbit in the character variety if and only if the representation fixes a point of $\mathbb{H}^3$ (Theorem 6.2). This answers a question due to Goldman.

For a surface $\Sigma$ of negative Euler characteristic with $n \geq 1$ marked punctures, the subvarieties $X_k(\Sigma)$ of $X(\Sigma)$ obtained by fixing the traces $k = (k_1, \cdots, k_n) \in \mathbb{C}^n$ of local monodromy along the punctures form a family of log Calabi-Yau varieties [25] with rich Diophantine structure [26]. Classifying the finite mapping class group orbits (and other invariant subvarieties) forms an important step in the study of strong approximation for these varieties, undertaken in the once-punctured torus case by Bourgain-Gamburd-Sarnak [3].

**Organization of the paper.** In Section 2, we record background on algebraic subgroups of $\mathrm{SL}_2(\mathbb{C})$, character varieties, and mapping class groups. We also introduce the notion of loop configurations as a tool to keep track of subsurfaces of a given surface. In Section 3, we study representations of surface groups whose images are contained in proper algebraic subgroups of $\mathrm{SL}_2(\mathbb{C})$, and give a characterization of those with finite mapping class group orbits for surfaces of positive genus.

In Sections 4 and 5, we prove our main results. One of the ingredients in the proof of Theorems A and B is a theorem of Patel–Shankar–Whang [16, Theorem 1.2], which states that a semisimple $\mathrm{SL}_2(\mathbb{C})$-representation of a positive-genus surface group with finite monodromy along every simple loop must in fact be finite. (For the proof of Theorem C which runs in parallel, there is an analogous result.) Along essential curves, the requisite finiteness of monodromy can be largely obtained by studying Dehn twists, as described in Section 4. Finiteness of local monodromy along the punctures is more involved, and is achieved in Section 5. In the case where $\Sigma$ is a closed surface, we also give another proof of Theorem A relying on works of Gallo–Kapovich–Marden [8] and Previte–Xia [17].

In Section 6, we provide applications of our work, answering earlier mentioned questions due to Lubotzky and Goldman. Finally, in the Appendix we demonstrate a derivation of the once-punctured torus case of Theorem B from Dubrovin–Mazzocco [5].

## 2. Background

2.1. **Subgroups of** $\mathrm{SL}_2(\mathbb{C})$**.** Let $G$ be a proper algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$. Up to conjugation, $G$ satisfies one of the following [22, Theorem 4.29]:

(1) $G$ is a subgroup of the *standard Borel group*

$$B = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\}.$$

(2) $G$ is a subgroup of the *infinite dihedral group*

$$D_\infty = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \mid c \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} \mid c \in \mathbb{C}^* \right\}.$$

(3) $G$ is one of the finite groups $BA_4$, $BS_4$, and $BA_5$, which are the preimages in $\mathrm{SL}_2(\mathbb{C})$ of the finite subgroups $A_4$ (tetrahedral group), $S_4$ (octahedral group), and $A_5$ (icosahedral group) of $\mathrm{PGL}_2(\mathbb{C})$, respectively.

We refer to the Appendix (after the statement of Theorem A.3) for an explicit description of the finite groups $BA_4$, $BS_4$, and $BA_5$.

**Definition 2.1.** A representation $\pi \to \mathrm{SL}_2(\mathbb{C})$ of a group $\pi$ is:

(1) *Zariski dense* if its image is Zariski dense in $\mathrm{SL}_2(\mathbb{C})$,
(2) *diagonal* if it factors through the inclusion $i : \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ of the maximal torus consisting of diagonal matrices,
(3) *dihedral* if it factors through the inclusion $j : D_\infty \to \mathrm{SL}_2(\mathbb{C})$,
(4) *finite* if its image is finite,
(5) *unitarizable* if its image is conjugate to a subgroup of $\mathrm{SU}(2)$,
(6) *reducible* if its image preserves a subspace of $\mathbb{C}^2$,
(7) *irreducible* if it is not reducible,
(8) *elementary* if it is unitary or reducible or dihedral, and
(9) *non-elementary* if it is not elementary.

2.2. **Character varieties.** Given a finitely presented group $\pi$, let us define the $\mathrm{SL}_2$-*representation variety* $\mathrm{Rep}(\pi)$ as the complex affine scheme determined by the functor

$$A \mapsto \mathrm{Hom}(\pi, \mathrm{SL}_2(A))$$

for every commutative $\mathbb{C}$-algebra $A$. Given a sequence of generators of $\pi$ with $m$ elements, we have a presentation of $\mathrm{Rep}(\pi)$ as a closed subscheme of $\mathrm{SL}_2^m$ defined by equations coming from relations among the generators. For each $a \in \pi$, let $\mathrm{tr}_a$ be the regular function on $\mathrm{Rep}(\pi)$ given by $\rho \mapsto \mathrm{tr}\,\rho(a)$. The *character variety* of $\pi$ over $\mathbb{C}$ is the affine invariant theoretic quotient

$$X(\pi) = \mathrm{Rep}(\pi) /\!\!/ \mathrm{SL}_2 = \mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}(\pi)]^{\mathrm{SL}_2(\mathbb{C})}$$

under the conjugation action of $\mathrm{SL}_2$. The complex points of $X(\pi)$ parametrize the isomorphism classes of semisimple representations $\pi \to \mathrm{SL}_2(\mathbb{C})$. For each $a \in \pi$ the regular function $\mathrm{tr}_a$ evidently descends to a regular function on $X(\pi)$. The scheme $X(\pi)$ has a natural model over $\mathbb{Z}$. We refer to [11], [18], [19] for details.

*Example* 2.2. We refer to Goldman [9] for details of the examples below. Let $F_m$ denote the free group on $m \geq 1$ generators $a_1, \cdots, a_m$.

(1) We have $\mathrm{tr}_{a_1} : X(F_1) \simeq \mathbb{A}^1$.
(2) We have $(\mathrm{tr}_{a_1}, \mathrm{tr}_{a_2}, \mathrm{tr}_{a_1 a_2}) : X(F_2) \simeq \mathbb{A}^3$ by Fricke [9, Section 2.2].
(3) The coordinate ring $\mathbb{Z}[X(F_3)]$ is the quotient of the polynomial ring

$$\mathbb{Z}[\mathrm{tr}_{a_1}, \mathrm{tr}_{a_2}, \mathrm{tr}_{a_3}, \mathrm{tr}_{a_1a_2}, \mathrm{tr}_{a_2a_3}, \mathrm{tr}_{a_1a_3}, \mathrm{tr}_{a_1a_2a_3}, \mathrm{tr}_{a_1a_3a_2}]$$

by the ideal generated by two elements

$$\mathrm{tr}_{a_1a_2a_3} + \mathrm{tr}_{a_1a_3a_2} - (\mathrm{tr}_{a_1a_2}\mathrm{tr}_{a_3} + \mathrm{tr}_{a_1a_3}\mathrm{tr}_{a_2} + \mathrm{tr}_{a_2a_3}\mathrm{tr}_{a_1} - \mathrm{tr}_{a_1}\mathrm{tr}_{a_2}\mathrm{tr}_{a_3})$$

and

$$\begin{aligned}\mathrm{tr}_{a_1a_2a_3}\,\mathrm{tr}_{a_1a_3a_2} - \{&(\mathrm{tr}^2_{a_1}+\mathrm{tr}^2_{a_2}+\mathrm{tr}^2_{a_3}) + (\mathrm{tr}^2_{a_1a_2}+\mathrm{tr}^2_{a_2a_3}+\mathrm{tr}^2_{a_1a_3})\\ &- (\mathrm{tr}_{a_1}\,\mathrm{tr}_{a_2}\,\mathrm{tr}_{a_1a_2}+\mathrm{tr}_{a_2}\,\mathrm{tr}_{a_3}\,\mathrm{tr}_{a_2a_3}+\mathrm{tr}_{a_1}\,\mathrm{tr}_{a_3}\,\mathrm{tr}_{a_1a_3})\\ &+ \mathrm{tr}_{a_1a_2}\,\mathrm{tr}_{a_2a_3}\,\mathrm{tr}_{a_1a_3} - 4\}.\end{aligned}$$

In particular, $\mathrm{tr}_{a_1a_2a_3}$ and $\mathrm{tr}_{a_1a_3a_2}$ are integral over the polynomial subring $\mathbb{Z}[\mathrm{tr}_{a_1},\mathrm{tr}_{a_2},\mathrm{tr}_{a_3},\mathrm{tr}_{a_1a_2},\mathrm{tr}_{a_2a_3}]$.

We record the following, which is attributed by Goldman, [9], to Vogt [23].

**Lemma 2.3.** *Given a finitely generated group $\pi$ and $a_1,a_2,a_3,a_4\in\pi$, the following holds:*

$$\begin{aligned}2\mathrm{tr}_{a_1a_2a_3a_4} =\ &\mathrm{tr}_{a_1}\mathrm{tr}_{a_2}\mathrm{tr}_{a_3}\mathrm{tr}_{a_4} + \mathrm{tr}_{a_1}\mathrm{tr}_{a_2a_3a_4} + \mathrm{tr}_{a_2}\mathrm{tr}_{a_3a_4a_1} + \mathrm{tr}_{a_3}\mathrm{tr}_{a_4a_1a_2}\\ &+ \mathrm{tr}_{a_4}\mathrm{tr}_{a_1a_2a_3} + \mathrm{tr}_{a_1a_2}\mathrm{tr}_{a_3a_4} + \mathrm{tr}_{a_4a_1}\mathrm{tr}_{a_2a_3} - \mathrm{tr}_{a_1a_3}\mathrm{tr}_{a_2a_4}\\ &- \mathrm{tr}_{a_1}\mathrm{tr}_{a_2}\mathrm{tr}_{a_3a_4} - \mathrm{tr}_{a_3}\mathrm{tr}_{a_4}\mathrm{tr}_{a_1a_2} - \mathrm{tr}_{a_4}\mathrm{tr}_{a_1}\mathrm{tr}_{a_2a_3} - \mathrm{tr}_{a_2}\mathrm{tr}_{a_3}\mathrm{tr}_{a_4a_1}.\end{aligned}$$

The above computations imply the following fact.

**Fact 2.4.** *If $\pi$ is a group generated by $a_1,\cdots,a_m$, then $\mathbb{Q}[X(\pi)]$ is generated as a $\mathbb{Q}$-algebra by the collection $\{\mathrm{tr}_{a_{i_1}\cdots a_{i_k}} \mid 1\le i_1<\cdots<i_k\le m\}_{1\le k\le 3}$.*

The construction of $X(\pi)$ is functorial with respect to the group $\pi$. Given a homomorphism $f:\pi\to\pi'$ of finitely presented groups, the corresponding morphism $f^*: X(\pi')\to X(\pi)$ sends a representation $\rho$ to the semisimplification of $\rho\circ f$. In particular, the automorphism group $\mathrm{Aut}(\pi)$ of $\pi$ naturally acts on $X(\pi)$. This action naturally factors through the outer automorphism group $\mathrm{Out}(\pi)$, owing to the fact that $\mathrm{tr}_{aba^{-1}}=\mathrm{tr}_b$ for every $a,b\in\pi$.

Let $i:\mathbb{C}^*\to\mathrm{SL}_2(\mathbb{C})$ and $j:D_\infty\to\mathrm{SL}_2(\mathbb{C})$ be the inclusion maps of the diagonal maximal torus and the infinite dihedral group, respectively. They induce $\mathrm{Out}(\pi)$-equivariant maps

$$i_*:\mathrm{Hom}(\pi,\mathbb{C}^*)\to X(\mathbb{C}) \quad\text{and}\quad j_*:\mathrm{Hom}(\pi,D_\infty)/D_\infty\to X(\mathbb{C}). \qquad (*)$$

**Lemma 2.5.** *The following two hold.*

1. *The map $i_*:\mathrm{Hom}(\pi,\mathbb{C}^*)\to X(\mathbb{C})$ in $(*)$ has finite fibers.*
2. *The map $j_*:\mathrm{Hom}(\pi,D_\infty)/D_\infty\to X(\mathbb{C})$ in $(*)$ has finite fibers.*

*Proof.* (1) Let $\rho_1,\rho_2:\pi\to\mathbb{C}^*$ be characters such that $gi_*(\rho_1)g^{-1}=i_*(\rho_2)$ for some $g\in\mathrm{SL}_2(\mathbb{C})$. Without loss of generality, we may assume that $\rho_1(x)\neq\pm1$ for some $x\in\pi$, since otherwise the image of $\rho_1$ is finite and we are done. Writing $g=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ and $\rho_1(x)=\lambda$ with $\lambda\in\mathbb{C}^*\setminus\{\pm1\}$, we have

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}\lambda&0\\0&\lambda^{-1}\end{bmatrix}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\begin{bmatrix}\lambda ad-\lambda^{-1}bc & (\lambda^{-1}-\lambda)ab\\(\lambda-\lambda^{-1})cd & \lambda^{-1}ad-\lambda bc\end{bmatrix}.$$

For the matrix on the right hand side to be diagonal, we must thus have $a=d=0$ or $b=c=0$ since $\lambda\neq\pm1$. If $a=d=0$, then $\rho_2=\rho_1^{-1}$. If $b=c=0$, then $\rho_1=\rho_2$. This proves (1).

(2) Let $\rho_1,\rho_2:\pi\to D_\infty$ be representations such that $gj_*(\rho_1)g^{-1}=j_*(\rho_2)$ for some $g\in\mathrm{SL}_2(\mathbb{C})$. Without loss of generality, we may assume that $\mathrm{tr}\,\rho_1(x)\notin\{0,\pm2\}$ for some $x\in\pi$, since otherwise the image of $\rho_1$ is finite and we are done. Now note that $\rho_1(x)$ must be diagonal. The equation $g\rho_1(x)g^{-1}=\rho_2(x)$ shows that, by the same computation as above, we have $g\in D_\infty$. This proves (2). □

2.3. **Surfaces.** Here, we set our notational convention and terminology for various topological notions. Throughout this paper, a *surface* is the complement of a finite collection of interior points in a compact oriented topological manifold of dimension 2, with or without boundary.

A *simple closed curve* on a surface is an embedded copy of an unoriented circle. We shall often refer to simple closed curves simply as curves (since immersed curves will not be important in this paper). Given a surface $\Sigma$, a curve in $\Sigma$ is *nondegenerate* if it does not bound a disk on $\Sigma$. A curve in the interior of $\Sigma$ is *essential* if it is nondegenerate, does not bound a punctured disk on $\Sigma$, and is not isotopic to a boundary curve of $\Sigma$. Given a surface $\Sigma$ and an essential curve $a \subset \Sigma$, we denote by $\Sigma|a$ the surface obtained by cutting $\Sigma$ along $a$. An essential curve $a \subset \Sigma$ is *separating* if the two boundary curves of $\Sigma|a$ corresponding to $a$ are in different connected components, and *nonseparating* otherwise.

Let $\Sigma$ be a surface of genus $g$ with $n$ punctures or boundary curves. We shall denote by $\mathrm{Mod}(\Sigma)$ the (pure) mapping class group of $\Sigma$. By definition, it is the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma$ fixing the punctures and boundary points individually. Given a simple closed curve $a \subset \Sigma$, we shall denote by $\mathrm{tw}_a \in \mathrm{Mod}(\Sigma)$ the associated (left) Dehn twist.

We define the character variety of $\Sigma$ (cf. Section 2.2) to be

$$X(\Sigma) = X(\pi_1(\Sigma)).$$

The complex points of $X(\Sigma)$ can be seen as parametrizing the isomorphism classes of semisimple $\mathrm{SL}_2(\mathbb{C})$-local systems on $\Sigma$. Note that a simple closed curve $a \subset \Sigma$ unambiguously defines a function $\mathrm{tr}_a$ on $X(\Sigma)$, coinciding with $\mathrm{tr}_\alpha$ for any loop $\alpha \in \pi_1(\Sigma)$ freely homotopic to a parametrization of $a$.

Given a continuous map $f : \Sigma' \to \Sigma$ of surfaces, we have an induced morphism of character varieties $f^* : X(\Sigma) \to X(\Sigma')$ depending only on the homotopy class of $f$. In particular, the mapping class group $\mathrm{Mod}(\Sigma)$ acts naturally on $X(\Sigma)$ by precomposition. If $\Sigma' \subset \Sigma$ is a subsurface, the induced morphism on character varieties is $\mathrm{Mod}(\Sigma')$-equivariant for the induced morphism $\mathrm{Mod}(\Sigma') \to \mathrm{Mod}(\Sigma)$ of mapping class groups. In particular, if a semisimple $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(\Sigma)$ has a finite $\mathrm{Mod}(\Sigma)$-orbit in $X(\Sigma)$, then its restriction to any subsurface $\Sigma' \subset \Sigma$ has a finite $\mathrm{Mod}(\Sigma')$-orbit in $X(\Sigma')$.

2.4. **Loop configurations.** Let $\Sigma$ be a surface of genus $g$ with $n$ punctures. We fix a base point in $\Sigma$. For convenience, we shall say that a sequence $\ell = (\ell_1, \cdots, \ell_m)$ of based loops on $\Sigma$ is *clean* if each loop is simple and the loops pairwise intersect only at the base point.

*Example* 2.6. Recall the standard presentation of the fundamental group

$$\pi_1(\Sigma) = \langle a_1, d_1, \cdots, a_g, d_g, c_1, \cdots, c_n \mid [a_1, d_1] \cdots [a_g, d_g] c_1 \cdots c_n \rangle.$$

We can choose (the based loops representing) the generators so that the sequence of loops $(a_1, d_1, \cdots, a_g, d_g, c_1, \cdots, c_n)$ is clean. For $i = 1, \cdots, g$, let $b_i$ be the based simple loop parametrizing the curve underlying $d_i$ with the opposite orientation. Note that $(a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)$ is a clean sequence with the property that any product of distinct elements preserving the cyclic ordering on the sequence, such as $a_1 b_g$ or $a_1 a_2 b_2 b_g$ or $b_g c_n a_1$, can be represented by a simple loop in $\Sigma$. We shall refer to $(a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)$ as *an optimal sequence of generators* of $\pi_1(\Sigma)$. See Figure 1 for an illustration of the optimal generators for $(g, n) = (2, 1)$.

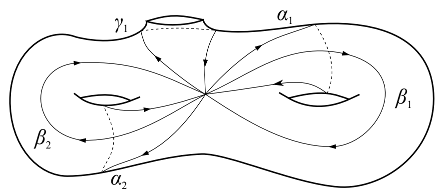


FIGURE 1. Optimal generators for $(g, n) = (2, 1)$

**Definition 2.7.** A *loop configuration* is a planar graph consisting of a single vertex $v$ and a finite cyclically ordered sequence of directed rays, equipped with a bijection between the set of rays departing from $v$ is and the set of rays arriving at $v$. We denote by $L_{g,n}$ the loop configuration whose sequence of rays is of the form

$$(a_1, b_1, \overline{a}_1, \overline{b}_1, \cdots, a_g, b_g, \overline{a}_g, \overline{b}_g, c_1, \overline{c}_1, \cdots, c_n, \overline{c}_n),$$

where $a_i, b_i, c_i$ are the rays directed away from $v$, respectively corresponding to the rays $\overline{a}_i, \overline{b}_i, \overline{c}_i$ directed towards $v$. See Figure 2 for an illustration of $L_{2,1}$.

Given a clean sequence $\ell = (\ell_1, \cdots, \ell_m)$ of loops on $\Sigma$, we have an associated loop configuration $L(\ell)$, obtained by taking a sufficiently small open neighborhood of the base point and setting the departing and arriving ends of the loops $\ell_i$ to correspond to each other. For example, if $(a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)$ is a sequence of optimal generators for $\pi_1(\Sigma)$, then

$$L(a_1, b_1, \cdots, a_g, b_g, c_1 \cdots, c_n) \simeq L_{g,n}.$$

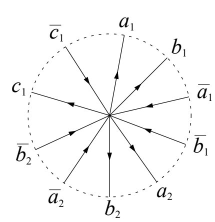


FIGURE 2. Loop configuration $L_{2,1}$

**Definition 2.8.** Let $h$ and $m$ be nonnegative integers. A sequence of based loops $\ell = (\ell_1, \cdots, \ell_{2h+m})$ on $\Sigma$ is said to be *in* $(h, m+1)$*-position* if it is homotopic termwise to a clean sequence $\ell' = (\ell'_1, \cdots, \ell'_{2h+m})$ such that $L(\ell') \simeq L_{h,m}$. We denote by $\Sigma(\ell) \subset \Sigma$ the (isotopy class of a) subsurface of genus $h$ with $m+1$ boundary curves obtained by taking a small closed tubular neighborhood of the union of the simple curves underlying $\ell'_1, \cdots, \ell'_{2h+m}$ in $\Sigma$.

## 3. Non-Zariski-dense representations

Let $\Sigma$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. The purpose of this section is to characterize representations $\pi_1(\Sigma)$ with non-Zariski-dense image in $\mathrm{SL}_2(\mathbb{C})$ that have finite mapping class group orbit in the character variety $X(\Sigma)$. An irreducible representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ will be called *special dihedral* if it factors through $D_\infty$ and there is a nonseparating essential curve $a$ in $\Sigma$ such that the restriction $\rho|(\Sigma \setminus a)$ is diagonal. The main result of this section is the following.

**Proposition 3.1.** *Let $\Sigma$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. Let $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ be a semisimple representation whose image is not Zariski-dense in $\mathrm{SL}_2(\mathbb{C})$. Then $\rho$ has a finite mapping class group orbit in $X(\Sigma)$ if and only if one of the following holds:*

(1) *$\rho$ is a finite representation.*
(2) *$g = 1$ and $\rho$ is special dihedral up to conjugation.*

Proposition 3.1 is evident when the image of $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ belongs to one of the finite groups $BA_4$, $BS_4$, and $BA_5$. From the discussion in Section 2.1 and Lemma 2.5, it remains to understand the finite mapping class group orbits on $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^\times)$ and $\mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty$.

Proposition 3.1 follows by combining Lemmas 3.2, 3.3, and 3.5 below.

**Case 1: Diagonal Representations.**

**Lemma 3.2.** *Assume $\Sigma$ is a surface of genus $g \geq 1$ with $n \geq 0$ punctures. A representation $\rho : \pi_1(\Sigma) \to \mathbb{C}^*$ has finite (respectively, bounded) mapping class group orbit in $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^*)$ if and only if it has finite (respectively, bounded) image.*

*Proof.* Let $\rho : \pi_1(\Sigma) \to \mathbb{C}^*$ be a representation with finite (respectively, bounded) mapping class group orbit in $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^*)$. Let $(a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)$ be the optimal generators of $\pi_1(\Sigma)$. Assume first that $n = 0$ or 1. By considering the effect of Dehn twist along the curve underlying $a_1$ on the curve underlying $b_1$, we conclude that $\rho(a_1)$ must be torsion (respectively, have absolute value 1). Applying the same argument to the other loops in the sequence of optimal generators, we conclude that $\rho$ is finite (respectively, bounded) if $n \leq 1$, as desired.

Thus, only the case $n \geq 2$ remains. Since $(a_1, b_1, \cdots, a_g, b_g)$ is in $(g, 1)$-position, the above analysis shows that $\rho(a_i)$ and $\rho(b_i)$ are roots of unity for $i = 1, \cdots, g$. Similarly, we see that the sequence

$$L_i = (a_1, b_1, \cdots, a_{g-1}, b_{g-1}, a_g, b_g c_i)$$

is in $(g, 1)$-position for every $i = 1, \cdots, n$. Since $\rho$ restricted to the surface $\Sigma(L_i)$ must have finite (respectively, bounded) $\mathrm{Mod}(\Sigma(L_i))$-orbit, it follows that $\rho(c_i)$ is a root of unity (respectively, has absolute value 1) for $i = 1, \cdots, n$ as well. This shows that $\rho$ has finite (respectively, bounded) image, as desired. □

**Case 2: Dihedral Representations.** We first prove Proposition 3.1 for surfaces of genus $g \geq 2$ in Lemma 3.3 below.

**Lemma 3.3.** *Assume $\Sigma$ is a surface of genus $g \geq 2$ with $n \geq 0$ punctures. A representation $\rho : \pi_1(\Sigma) \to D_\infty$ has finite mapping class group orbit in $\mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty$ if and only if it has finite image.*

*Proof.* The "*if*" direction is clear, and we now prove the converse. Let $\rho : \pi_1(\Sigma) \to D_\infty$ be a representation with finite mapping class group orbit in $\mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty$. We have a short exact sequence $0 \to \mathbb{C}^* \to D_\infty \to \mathbb{Z}/2\mathbb{Z} \to 0$, where the homomorphism $D_\infty \to \mathbb{Z}/2\mathbb{Z}$ is given by

$$\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \mapsto 0 \quad \text{and} \quad \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} \mapsto 1 \quad \text{for all } c \in \mathbb{C}^*.$$

This gives us a $\mathrm{Mod}(\Sigma)$-equivariant map

$$\mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty \to \mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z}/2\mathbb{Z}).$$

The fiber of this map above the zero homomorphism consists of those points given by diagonal representations, to which Lemma 3.2 applies, noting that the map $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^*) \to \mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty$ has finite fibers. It suffices to consider the case where $\rho : \pi_1(\Sigma) \to D_\infty$ is not in the fiber over the zero homomorphism. Let

$$(a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)$$

be the optimal generators of $\pi_1(\Sigma)$ (see Section 2.4 for the definition). Note that $L = (a_1, b_1, \cdots, a_g, b_g)$ is in $(g, 1)$-position, and we have a $\mathrm{Mod}(\Sigma(L))$-equivariant homomorphism

$$\mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\pi_1(\Sigma(L)), \mathbb{Z}/2\mathbb{Z}).$$

The action of $\mathrm{Mod}(\Sigma(L))$ on $\mathrm{Hom}(\pi_1(\Sigma(L)), \mathbb{Z}/2\mathbb{Z})$ factors through the projection

$$\mathrm{Mod}(\Sigma(L)) \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z}).$$

From the transitivity of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ on $(\mathbb{Z}/2\mathbb{Z})^{2g}$ away from the origin, it follows that, up to $\mathrm{Mod}(\Sigma)$-action, we may assume that

$$\overline{\rho}(a_1) = 1 \quad \text{and} \quad \overline{\rho}(b_1) = \overline{\rho}(a_2) = \cdots = \overline{\rho}(b_g) = 0\,,$$

where $\overline{\rho}$ is the image of $\rho$ in $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z}/2\mathbb{Z})$. Up to conjugation by an element in $D_\infty$, we may moreover assume that

$$\rho(a_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It suffices to show that the entries of the matrices $\rho(b_1)$, $\rho(a_2)$, $\rho(b_2)$, $\cdots$, $\rho(a_g)$, $\rho(b_g)$, $\rho(c_1)$, $\cdots$, $\rho(c_n)$ are roots of unity. Let $i_1 < \cdots < i_q$ be precisely the indices in $\{1, \cdots, n\}$ such that $\overline{\rho}(c_{i_j}) = 0$. Since

$$L' = (a_2, b_2, \cdots, a_g, b_g, c_{i_1}, \cdots, c_{i_q}, b_1)$$

is in $(g-1, q+2)$-position, and the restriction of $\rho$ to $\Sigma(L')$ is diagonal, we see that $\rho(b_1)$, $\rho(a_2)$, $\rho(b_2)$, $\cdots$, $\rho(a_g)$, $\rho(b_g)$, $\cdots$, $\rho(c_{i_1})$, $\cdots$, $\rho(c_{i_q})$ are torsion by Lemma 3.2. Let us now take $i \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_q\}$. The restriction of $\rho$ to the subsurface $\Sigma(c_i a_1, b_1)$ of genus 1 with 1 boundary curve is diagonal. Writing

$$\rho(c_i) = \begin{bmatrix} 0 & \lambda_i \\ -\lambda^{-1} & 0 \end{bmatrix}$$

with $\lambda_i \in \mathbb{C}^*$, we have

$$\rho(c_i a_i) = \begin{bmatrix} 0 & \lambda_i \\ -\lambda_i^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\lambda_i & 0 \\ 0 & -\lambda_i^{-1} \end{bmatrix}.$$

It follows from Lemma 3.2 that $\lambda_i$ is a root of unity. This completes the proof that the entries of $\rho(b_1)$, $\rho(a_2)$, $\rho(b_2)$, $\cdots$, $\rho(a_g)$, $\rho(b_g)$, $\rho(c_1)$, $\cdots$, $\rho(c_n)$ are roots of unity, and hence the image of $\rho$ is finite, as desired. □

Before proving the case $g = 1$ of Proposition 3.1 in Lemma 3.5 below, we record the following elementary lemma.

**Lemma 3.4.** *Let $\Sigma$ be a surface of genus 1 with $n \geq 0$ punctures. Suppose that $\rho : \pi_1(\Sigma) \to D_\infty$ is special dihedral. Given a pair of loops $(a, b)$ in $(1, 1)$-position on $\Sigma$, at least one of $\rho(a)$ and $\rho(b)$ is not diagonal.*

*Proof.* Let $\rho \in \pi_1(\Sigma) \to D_\infty$ be special dihedral. We argue by contradiction. Assume that $(a_1, b_1)$ is a pair of loops in $(1,1)$-position on $\Sigma$ with both $\rho(a_1)$ and $\rho(b_1)$ diagonal. We can complete $(a_1, b_1)$ to a sequence of optimal generators $(a_1, b_1, c_1, \cdots, c_n)$ of $\pi_1(\Sigma)$. Since $\rho$ is special dihedral, the matrices $\rho(c_1), \cdots, \rho(c_n)$ must be diagonal (noting that the property of a matrix being diagonal is not changed under conjugation by an element in $D_\infty$). This implies that $\rho$ is in fact diagonal, contradicting the hypothesis that $\rho$ is irreducible. $\square$

**Lemma 3.5.** *Let $\Sigma$ be a surface of genus 1 with $n \geq 0$ punctures. A representation $\rho : \pi_1(\Sigma) \to D_\infty$ has finite mapping class group orbit in* $\mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty$ *if and only if it is finite or special dihedral.*

*Proof.* The same argument as in Lemma 3.3 shows that if $\rho : \pi_1(\Sigma) \to D_\infty$ has finite $\mathrm{Mod}(\Sigma)$-orbit in $\mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty$, then $\rho$ is finite or special dihedral. Let now $\rho : \pi_1(\Sigma) \to D_\infty$ be a special dihedral representation. We shall show that $\rho$ has finite mapping class group orbit in $\mathrm{Hom}(\pi_1(\Sigma), D_\infty)/D_\infty$, or equivalently in the character variety $X(\Sigma)$.

The coordinate ring $\mathbb{C}[X(\Sigma)]$ of the character variety is finitely generated by the trace functions $\mathrm{tr}_a$ for a finite collection of essential curves $a$ in $\Sigma$ and the boundary curves, in view of Fact 2.4 and the property of an optimal sequence of generators. Therefore, it suffices to show that the set

$$\{\mathrm{tr}\,\rho(a) \mid a \subset \Sigma \text{ essential curve}\} \subseteq \mathbb{C}$$

is finite.

Let $(a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)$ be optimal generators of $\pi_1(\Sigma)$. Since $\rho$ is special dihedral, it follows that $\rho(c_1)$, $\cdots$, $\rho(c_n)$ are diagonal matrices. Suppose $a$ is a separating simple closed curve in $\Sigma$ underlying a loop in the free homotopy class of $c_{i_1} \cdots c_{i_k}$ for some integers $i_1 < \cdots < i_k$ in $\{1, \cdots, n\}$. Since $\rho(c_i)$ are diagonal for $i = 1, \cdots, n$ it follows that $\mathrm{tr}\,\rho(a)$ lies in a finite set that only depends on the traces $\mathrm{tr}\,\rho(c_1)$, $\cdots$, $\mathrm{tr}\,\rho(c_n)$. But now, every separating simple closed curve in $\Sigma$ is sent to one of the above form by some mapping class group element. Since the mapping class group action preserves the special dihedral representations, and since it fixes the traces $\mathrm{tr}\,\rho(c_1)$, $\cdots$, $\mathrm{tr}\,\rho(c_n)$, we conclude that $\{\mathrm{tr}\,\rho(a) \mid a \subset \Sigma \text{ separating curve}\}$ is finite.

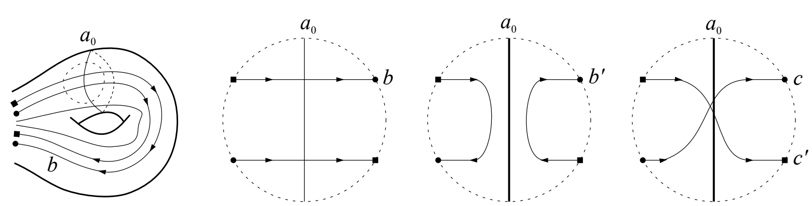


FIGURE 3. Building new loops out of old ones

It remains to show that $\{\mathrm{tr}\,\rho(a) \mid a \subset \Sigma \text{ nonseparating curve}\}$ is finite. Let $a_0$ be a nonseparating curve in $\Sigma$ such that the restriction $\rho|(\Sigma \setminus a_0)$ is diagonal; such a curve exists since $\rho$ is special dihedral. Let $b$ be a nonseparating curve in $\Sigma$. Up to isotopy, we may assume that $b$ intersects $a$ only finitely many times. If $b$ does not intersect $a_0$, then $b \subset \Sigma \setminus a_0$ and hence $\mathrm{tr}\,\rho(b)$ can take only finitely many values as $\rho|(\Sigma \setminus a_0)$ is diagonal.

If $b$ intersects $a$ exactly once, then we must have $\operatorname{tr}\rho(b) = 0$ by Lemma 3.4. Let us now assume that $b$ intersects $a_0$ more than once. Let us choose a parametrization of $b$. Since $b$ is nonseparating, there must be two neighboring points of intersection of $a_0$ and $b$ where the two segments of $b$ have the same orientation, as in Figure 3. The operations as in Figure 3 produce for us a new simple closed curve $b'$ which is also nonseparating, as well as a pair $(c, c')$ of simple loops in $(1,1)$-position on $\Sigma$. We have the trace relation

$$\operatorname{tr}\rho(b) = \operatorname{tr}\rho(c)\operatorname{tr}\rho(c') - \operatorname{tr}\rho(b').$$

By Lemma 3.4, we have $\operatorname{tr}\rho(c)\operatorname{tr}\rho(c') = 0$, and therefore $\operatorname{tr}\rho(b) = -\operatorname{tr}\rho(b')$. Note furthermore that $b'$ intersects $a_0$ in a smaller number of points than $b$ does. Applying induction on the number of intersection points, we thus conclude that

$$\{\operatorname{tr}\rho(a) \mid a \subset \Sigma \text{ nonseparating curve}\}$$

is finite. This completes the proof that special dihedral representations have finite mapping class group orbits in $X$. □

## 4. Analysis of Dehn twists

Throughout this section, let $\Sigma$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. For convenience of exposition, we shall denote by $(*)$ the following condition on a representation $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$:

> $(*)$ The representation is semisimple, and moreover $\Sigma$ has genus at least 2 or the representation is not special dihedral.

Given $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ satisfying $(*)$, Theorems A and B (respectively, Theorem C) state that $\rho$ has finite (respectively, bounded) image in $\mathrm{SL}_2(\mathbb{C})$ if its mapping class group orbit in the character variety $X(\Sigma)$ is finite (respectively, bounded). Let us recall the following.

**Theorem 4.1** ([16, Theorem 1.2])**.** *Let $\Sigma$ be a surface of positive genus $g \geq 1$ with $n \geq 0$ punctures. If a semisimple representation $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ has finite monodromy along all simple loops on $\Sigma$, then it has finite image.*

**Proposition 4.2** ([16, Lemma 2.2])**.** *Let $\Sigma$ be a surface of positive genus $g \geq 1$ with $n \geq 0$ punctures. If a semisimple representation $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ has elliptic or central monodromy along all simple loops on $\Sigma$, then it is unitarizable.*

Consequently, to prove Theorems A and B (respectively, Theorem C) it suffices to prove that a representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ with finite (respectively, bounded) mapping class group orbit in the character variety has finite (respectively, elliptic or central) monodromy along all nondegenerate simple closed curves on $\Sigma$. Let us divide up the curves into four types:

- Type I. Nonseparating essential curves.
- Type II. Separating essential curves $a \subset \Sigma$ with each component of $\Sigma|a$ (defined in section 2.3) having genus at least one.
- Type III. Separating essential curves $a \subset \Sigma$ with one component of $\Sigma|a$ having genus zero.
- Type IV. Boundary curves.

The purpose of this section is to show that, if $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ is a representation satisfying the hypotheses $(*)$ mentioned above with finite (respectively, bounded) mapping class group orbit in $X(\Sigma)$, then $\rho$ must have finite (respectively, elliptic or central) monodromy along all curves of type I and II. We shall also show that $\rho$ has finite (respectively, elliptic or central) monodromy along all curves of type III provided the same holds for all curves of type IV. In particular, this is enough for us to prove the following special cases of Theorems A, B and C.

**Proposition 4.3.** *Let $\Sigma$ be a surface of genus $g \geq 1$ with $n \geq 1$ punctures. Suppose that $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ is a semisimple representation with finite (respectively, elliptic or central) local monodromy around the punctures, and finite (respectively, bounded) mapping class group orbit in the character variety $X(\Sigma)$. Then one of the following holds:*

1. *$\rho$ is finite (respectively, unitarizable);*
2. *$g = 1$ and $\rho$ is special dihedral up to conjugacy.*

The rest of this Section proves Proposition 4.3 by dealing with curves of Types I, II and III. We shall complete the proof of our main results in Section 5 by treating the curves of type IV.

**Type I.** Nonseparating essential curves.

**Lemma 4.4.** *Let $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ be a representation satisfying $(*)$, and let $a \subset \Sigma$ be a curve of type I. If $\rho$ has finite (respectively, bounded) orbit under $Mod(\Sigma)$ in $X(\Sigma)$, then $\rho$ has finite (respectively, elliptic or central) monodromy along $a$.*

*Proof.* Let us choose a base point $x_0$ on $\Sigma$ lying on $a$, and let $\alpha$ be a simple based loop parametrizing $a$. Suppose $\beta$ is another simple loop such that the pair $(\alpha, \beta)$ is in $(1,1)$-position, and let $b$ denote the underlying curve. For each integer $k \in \mathbb{Z}$, the loop $\alpha^k\beta$ is homotopic to a simple loop whose underlying curve is isotopic to $\mathrm{tw}_a^k(b)$. In particular, by our hypothesis the set $\{\mathrm{tr}\,\rho(\alpha^k\beta) \mid k \in \mathbb{Z}\}$ is a finite (respectively, bounded) subset of $\mathbb{C}$. Up to global conjugation of $\rho$ by an element of $\mathrm{SL}_2(\mathbb{C})$, we may consider two cases.

(a) Suppose first that

$$\rho(\alpha) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in \mathbb{C}^\times.$$

Let $\beta$ be a simple loop on $\Sigma$ such that $(\alpha, \beta)$ is in $(1,1)$-position, and write

$$\rho(\beta) = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

For each $k \in \mathbb{Z}$, we have

$$\mathrm{tr}\,\rho(\alpha^k\beta) = \mathrm{tr}\left( \begin{bmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right) = \lambda^k b_1 + \lambda^{-k} b_4.$$

The fact that $\{\mathrm{tr}\,\rho(\alpha^k\beta) \mid k \in \mathbb{Z}\}$ is a finite (respectively, bounded) subset of $\mathbb{C}$ then implies that $\lambda$ is a root of unity (respectively, has absolute value 1) so that $\rho(\alpha)$ is torsion (respectively, elliptic or central), or that $b_1 = b_4 = 0$. Since the argument applies to any loop $\beta$ such that $(\alpha, \beta)$ is in $(1,1)$-position, it remains only to consider the case where $\rho(\beta)$ has both diagonal entries zero for every such $\beta$. In this case, we see upon reflection that the restriction $\rho|(\Sigma|a)$ must be diagonal. This shows that $\rho$ is special dihedral. Since $\rho$ satisfies $(*)$ this means that $\Sigma$ moreover has genus at least 2, so $\Sigma|a$ has genus at least 1 and it follows from Lemma 3.2 that $\rho(\alpha)$ is torsion (respectively, elliptic or central).

(b) Suppose that

$$\rho(\alpha) = s \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

for some $s \in \{\pm 1\}$ and $x \in \mathbb{C}$. Let us assume $s = +1$; the case $s = -1$ will follow similarly. Let $\beta$ be a simple loop on $\Sigma$ such that $(\alpha, \beta)$ is in $(1,1)$-position, and let us follow the notation for $\rho(\beta)$ from the previous case. For each $k \in \mathbb{Z}$, we have

$$\operatorname{tr} \rho(\alpha^k \beta) = \operatorname{tr} \left( \begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right) = b_1 + b_4 + kxb_3.$$

The fact that $\{\operatorname{tr} \rho(\alpha^k \beta) \mid k \in \mathbb{Z}\}$ is a finite (respectively, bounded) subset of $\mathbb{C}$ then implies that $x = 0$ or $b_3 = 0$. Since the argument applies to any loops $\beta$ such that $(\alpha, \beta)$ is in $(1,1)$-position, it remains only to consider the case where $\rho(\beta)$ is upper triangular for every such $\beta$. In this case, $\rho$ is upper triangular (hence diagonal). It follows from Lemma 3.2 that $\rho(\alpha)$ is torsion (respectively, elliptic or central).

The above arguments show that $\rho(\alpha)$ is torsion (respectively, elliptic or central) unless $g = 1$ and $\rho$ is special dihedral, as desired. ☐

**Types II and III.** Separating essential curves.

**Lemma 4.5.** *Let $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ be a representation satisfying $(*)$, and let $a \subset \Sigma$ be a curve of type II. If $\rho$ has finite (respectively, bounded) orbit under $Mod(\Sigma)$ in $X(\Sigma)$, then $\rho$ has finite (respectively, elliptic or central) monodromy along $a$.*

**Lemma 4.6.** *Let $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ be a representation satisfying $(*)$, and let $a \subset \Sigma$ be a curve of type III. If $\rho$ has finite (respectively, bounded) orbit under $Mod(\Sigma)$ in $X(\Sigma)$, and moreover $\rho$ has finite (respectively, elliptic or central) monodromy along all punctures of $\Sigma$, then $\rho$ has finite (respectively, elliptic or central) monodromy along $a$.*

*Proof of Lemma 4.5 and Lemma 4.6.* Let us choose a base point $x_0$ on $\Sigma$ lying on $a$, and let $\alpha$ be a simple based loop parametrizing $a$. Let $\Sigma_1$ and $\Sigma_2$ be the connected components of $\Sigma|a$, and lift the base point on $\Sigma$ to $\Sigma_1$ and $\Sigma_2$. Suppose $\beta \in \pi_1(\Sigma_1)$ and $\gamma \in \pi_1(\Sigma_2)$ are simple loops such that their product $\beta\gamma$ on $\Sigma$ is homotopic to a simple loop (with underlying curve denoted $d$, say) transversely intersecting $a$ exactly twice. For convenience, in this paragraph we shall call such a pair $(\beta, \gamma)$ *good*. The loop $\beta\alpha^k\gamma\alpha^{-k}$ for each $k \in \mathbb{Z}$ is freely homotopic to a simple loop whose underlying curve belongs to the orbit $\langle \mathrm{tw}_a \rangle \cdot d$ (all curves considered up to isotopy). In particular, by our hypothesis the set $\{\operatorname{tr} \rho(\beta\alpha^k\gamma\alpha^{-k}) \mid k \in \mathbb{Z}\}$ is a finite (respectively, bounded) subset of $\mathbb{C}$. Up to global conjugation of $\rho$ by an element of $\mathrm{SL}_2(\mathbb{C})$, we may consider two cases.

(a) Suppose first that

$$\rho(\alpha) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in \mathbb{C}^\times.$$

Let $(\beta, \gamma)$ be a good pair, and let us write

$$\rho(\beta) = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \quad \text{and} \quad \rho(\gamma) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.$$

For each $k \in \mathbb{Z}$, we have

$$\operatorname{tr}\rho(\beta\alpha^k\gamma\alpha^{-k}) = \operatorname{tr}\left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} \lambda^{-k} & 0 \\ 0 & \lambda^k \end{bmatrix}\right)$$
$$= b_1c_1 + \lambda^{-2k}b_2c_3 + \lambda^{2k}c_2b_3 + b_4c_4.$$

The fact that $\{\operatorname{tr}\rho(\beta\alpha^k\gamma\alpha^{-k}) \mid k \in \mathbb{Z}\}$ is a finite (respectively, bounded) subset of $\mathbb{C}$ then implies that $\lambda$ is a root of unity (respectively, has absolute value 1) so that $\rho(\alpha)$ is torsion (respectively, elliptic or central), or that $b_2c_3 = c_2b_3 = 0$. If the former happens, then we are done.

Suppose that the latter happens, and that at least one of $b_2$, $b_3$, $c_2$, $c_3$ is nonzero. We shall assume $b_2 \neq 0$; the other cases will follow similarly. As $b_2c_3 = 0$, we must have $c_3 = 0$. Applying the same argument with $\gamma$ replaced by any simple loop $\gamma' \in \Sigma_2$ such that $(\beta, \gamma')$ is good, we are reduced to the case where $\rho|\Sigma_2$ is upper triangular. If $\rho|\Sigma_1$ is also upper triangular, then $\rho$ must be upper triangular (whence diagonal), and from Lemma 3.2 $\rho(\alpha)$ is torsion (respectively, elliptic or central). So suppose there is a simple loop $\beta' \in \pi_1(\Sigma)$ such that $\rho(\beta')$ is not upper triangular. By repeating the above argument with $\beta$ replaced by $\beta'$, we are reduced to the case where $\rho|\Sigma_2$ must be diagonal. If $\Sigma_2$ has genus at least 1, then from Lemma 3.2 it follows that $\rho(\alpha)$ is torsion (respectively, is elliptic or central). If $\Sigma_2$ has genus 0, then the fact that $\rho$ has finite (respectively, elliptic or central) local monodromy along the punctures implies that $\rho(\alpha)$ has finite (respectively, elliptic or central) monodromy.

It remains to consider the case where $b_2 = b_3 = c_2 = c_3 = 0$. Running through all the good pairs $(\beta, \gamma)$ and repeating the above argument, we are left with the case where $\rho$ is diagonal; Lemma 3.2 then implies that $\rho(\alpha)$ is torsion (respectively, elliptic or central).

The second case:

(b) Suppose that

$$\rho(\alpha) = s\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

for some $s \in \{\pm 1\}$ and $x \in \mathbb{C}$. We assume $s = +1$; the case $s = -1$ will follow similarly. Let $(\beta, \gamma)$ be a good pair, and let us follow the notation for $\rho(\beta)$ and $\rho(\gamma)$ from the previous case. For each $k \in \mathbb{Z}$, we have

$$\operatorname{tr}\rho(\beta\alpha^k\gamma\alpha^{-k})$$
$$= \operatorname{tr}\left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} 1 & kx \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} 1 & -kx \\ 0 & 1 \end{bmatrix}\right)$$
$$= b_1c_1 + b_2c_3 + b_2c_3 + b_4c_4 + (b_1c_3 - b_3c_1 - b_4c_3 + b_3c_4)kx - b_3c_3k^2x^2.$$

The fact that $\{\operatorname{tr}\rho(\beta\alpha^k\gamma\alpha^{-k}) \mid k \in \mathbb{Z}\}$ is a finite (respectively, bounded) subset of $\mathbb{C}$ then implies that $x = 0$ or $b_3c_3x^2 = ((b_1 - b_4)c_3 - b_3(c_1 - c_4))x = 0$. If the former happens, then we are done.

Suppose now that $x \neq 0$ and that the latter happens, and that $b_3 \neq 0$ or $c_3 \neq 0$. We shall consider the case $b_3 \neq 0$; the other case will follow similarly. Note then we must then have $c_3 = 0$ and $c_1 = c_4$. Applying the same argument with $\gamma$ replaced by any simple loop $\gamma' \in \Sigma_2$ such that $(\beta, \gamma')$ is good, we conclude that $\rho|\Sigma_2$ is upper triangular with image consisting of parabolic elements in $\mathrm{SL}_2(\mathbb{C})$. If $\Sigma_2$ has genus at least one, then by considering $\operatorname{tr}\rho(\beta\gamma')$ for good pairs $(\beta, \gamma')$ with $\gamma'$ nonseparating we see that in fact $\rho|\Sigma_2$ must have image in $\{\pm\mathbf{1}\}$. If $\Sigma_2$ has genus zero, then by our hypothesis on $\rho$ we again conclude that $\rho|\Sigma_2$ must have image in $\{\pm\mathbf{1}\}$. *A fortiori*, $\rho(\alpha)$ is central in both cases, which is a contradiction.

It only remains to consider the case where $b_3 = c_3 = 0$. Running through all the possible good pairs $(\beta, \gamma)$ and repeating the above argument, we are left with the case where $\rho$ is upper triangular (whence diagonal); then $\rho(\alpha)$ is torsion (respectively, elliptic or central). This completes the proof. □

## 5. Proof of the main results

The goal of this section is to complete the proof of Theorems A, B, and C. This section is organized as follows. In Section 5.1 we record an irreducibility criterion for $\mathrm{SL}_2(\mathbb{C})$-representations of positive genus surface groups, to be used in subsequent parts of this section. In Section 5.2, we establish Theorem C for surfaces of genus one with at most two punctures. In Sections 5.3, 5.4, and 5.5 we complete the proofs of our main theorems. Finally, in Section 5.6 we give an alternative proof of Theorem A for closed surfaces using [8, 17].

### 5.1. An irreducibility criterion.

**Lemma 5.1.** *Let $(a, b, c)$ be a sequence of loops on a surface in $(1, 2)$-position.*

(1) *The following pairs are in $(1, 1)$-position:*

$$(a, b), (a, bc), (ca, b), (ab, bc), (ca, cb), (ac, bc), (ca, ab).$$

(2) *The triple $(c^{-1}b^{-1}a, b, c)$ is in $(1, 2)$-position.*

*Proof.* This is seen by drawing the corresponding loop configurations of homotopic clean sequences. See Figure 4, noting that any segments not passing the central base point are not to be considered as part of each loop configuration. □

Let $\Sigma$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. Given a pair $(a, b)$ of based loops in $(1, 1)$-position on $\Sigma$, there is an embedding $\Sigma(a, b) \subset \Sigma$ of a surface of genus 1 with 1 boundary curve, i.e., a one-holed torus. Up to isotopy, every embedding of a one-holed torus is of the form $\Sigma(a, b)$ for some choice of $(a, b)$. The notion of loop configuration facilitates the proof of the following result, which will be used in the proof of our main theorems but may be of independent interest. (See [4] for a proof when $n = 0$.)

**Proposition 5.2.** *Let $\Sigma$ be a surface of genus $g \geq 1$ with $n \geq 0$ punctures. A representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ is irreducible if and only if there is a one-holed torus subsurface $\Sigma' \subset \Sigma$ such that the restriction $\rho|\Sigma'$ is irreducible.*

*Proof.* The *if* direction is clear. To prove the converse, let us begin by fixing a representation $\rho : \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ whose restriction to every one-holed torus subsurface is reducible. In this proof, given $a \in \pi_1(\Sigma)$ we shall also denote by $a$ the matrix $\rho(a) \in \mathrm{SL}_2(\mathbb{C})$ for simplicity. The statement that the restriction $\rho|\Sigma(a, b)$ is reducible for an embedding $\Sigma(a, b) \subset \Sigma$ associated to a pair $(a, b)$ of loops in $(1, 1)$-position is equivalent to saying that the pair $(\rho(a), \rho(b))$ of matrices in $\mathrm{SL}_2(\mathbb{C})$ has a common eigenvector in $\mathbb{C}^2$.

Throughout, we shall be using the following observation: if $a \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbf{1}\}$, and if $x, y, z \in \mathbb{C}^2$ are eigenvectors of $a$, then at least two of them are proportional; in notation, $x \sim y$, $x \sim z$, or $y \sim z$. First, we prove the following claim.

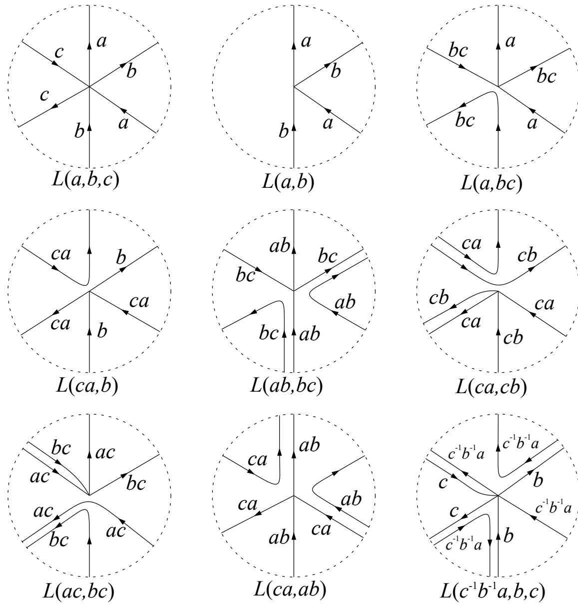


FIGURE 4. Loop configurations for Lemma 5.1

**Claim.** *Any triple $(a, b, c)$ of loops on $\Sigma$ in $(1, 2)$-position has a common eigenvector under the representation $\rho$.*

Let $(a, b, c)$ be in $(1, 2)$-position. Each of the following pairs is in $(1, 1)$-position by part (1) of Lemma 5.1, and has a common eigenvector by our hypothesis on $\rho$:

$$(a, b), (a, bc), (ca, b), (ab, bc), (ca, cb), (ac, bc), (ca, ab).$$

If $ab = \pm\mathbf{1}$, then since $(a, bc)$ has a common eigenvector we find that $(a, b, c)$ has a common eigenvector, as desired. Henceforth, assume that $ab \neq \pm\mathbf{1}$. Now $(a, b, ab)$ has a common eigenvector, say $x$; $(ab, bc)$ has a common eigenvector, say $y$; and $(ca, ab)$ has a common eigenvector, say $z$. Since $ab \neq \pm\mathbf{1}$, we must have

$$x \sim y, \quad x \sim z, \quad \text{or} \quad y \sim z$$

where the relation $\sim$ indicates that the two vectors are proportional. If the first case occurs, then $(a, b, bc)$ has a common eigenvector, and hence $(a, b, c)$ does, as required. If the second case occurs, then $(a, b, ca)$ has a common eigenvector, and hence $(a, b, c)$ does, again as required. Henceforth, assume that the third case occurs, so that $(bc, ca)$ has a common eigenvector.

Now, suppose first that $\mathrm{tr}(a) = \pm 2$. Up to conjugation, we have the following possibilities.

(1) We have $a = \pm\mathbf{1}$. Since $(b, c)$ has a common eigenvector, this implies that $(a, b, c)$ has a common eigenvector, as desired.
(2) We have

$$a = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $(a, bc)$ and $(a, b)$ each have a common eigenvector, $bc$ and $b$ must be upper triangular, and hence $c$ is upper triangular. Thus $(a, b, c)$ has a common eigenvector.

It remains to treat the case $\mathrm{tr}(a) \neq \pm 2$. As $(a, bc)$ has a common eigenvector, by Lemma A.1 we have

$$\mathrm{tr}(a)^2 + \mathrm{tr}(bc)^2 + \mathrm{tr}(abc)^2 - \mathrm{tr}(a)\,\mathrm{tr}(bc)\,\mathrm{tr}(abc) - 2 = 2$$

and this implies that we cannot have $\mathrm{tr}(bc) = \mathrm{tr}(abc) = 0$. After conjugation of $\rho$ by an element of $\mathrm{SL}_2(\mathbb{C})$, we have one of the following three cases.

(1) We have $bc = \pm \mathbf{1}$. This implies that $b = \pm c^{-1}$. Since $(a, b)$ has a common eigenvector, we are done.

(2) We have

$$bc = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $(a, bc)$ and $(bc, ca)$ each have a common eigenvector, $a$ and $ca$ are upper triangular, and hence $c$ is upper triangular. This in turn implies that $b$ is upper triangular, and we are done.

(3) We have

$$bc = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in \mathbb{C}^* \setminus \{\pm 1\}.$$

Since $(a, bc)$, $(bc, ca)$, and $(ab, bc)$ each have a common eigenvector, at least two of $a, ab, ca$ must be simultaneously upper or lower triangular. Let us assume that $a$ is upper triangular; the case where $a$ is lower triangular is dealt with similarly. If $ab$ is also upper triangular, this implies that $b$ is also upper triangular, in turn $c$ is upper triangular, and we are done. Hence, we may assume $ab$ is lower triangular. Similarly, we may assume that $ca$ is lower triangular. Since we have

$$(ab)(ca) = a(bc)a$$

with left hand side lower triangular and right hand side upper triangular, $abca$ is diagonal. Writing

$$a = \begin{bmatrix} \mu & x \\ 0 & \mu^{-1} \end{bmatrix},$$

we compute

$$a(bc)a = \begin{bmatrix} \mu^2\lambda & x(\mu\lambda + \lambda^{-1}\mu^{-1}) \\ 0 & \mu^{-2}\lambda^{-1} \end{bmatrix}$$

which implies that $x = 0$ or $\mathrm{tr}(abc) = \mu\lambda + \lambda^{-1}\mu^{-1} = 0$. If $x = 0$, that is, $a$ is diagonal, then we find that $b$ and $c$ are lower triangular, and we are done. The case $\mathrm{tr}(abc) = 0$ still remains; note the hypothesis that $\mathrm{tr}(bc) \neq \pm 2$ implies that $\mathrm{tr}(a) \neq 0$, and hence we have $\mathrm{tr}(a), \mathrm{tr}(bc) \neq 0$.

Hence, we have shown that if $(a, b, c)$ is in $(1, 2)$-position then $(a, b, c)$ has a common eigenvector, except possibly if $\mathrm{tr}_a, \mathrm{tr}(bc) \neq 0$ and $\mathrm{tr}(abc) = 0$. But in this exceptional case, since $(c^{-1}b^{-1}a, b, c)$ is in $(1, 2)$-position by part (2) of Lemma 5.1, and since

$$\begin{aligned}(\mathrm{tr}(c^{-1}b^{-1}a), \mathrm{tr}(bc), \mathrm{tr}(c^{-1}b^{-1}abc)) &= (\mathrm{tr}(a)\,\mathrm{tr}(bc) - \mathrm{tr}(abc), \mathrm{tr}(bc), \mathrm{tr}(a)) \\ &= (\mathrm{tr}(a)\,\mathrm{tr}(bc), \mathrm{tr}(bc), \mathrm{tr}(a))\end{aligned}$$

has none of the coordinates zero, running the same argument as above with $(a, b, c)$ replaced by $(c^{-1}b^{-1}a, b, c)$ we find that $(c^{-1}b^{-1}a, b, c)$ has a common eigenvector. But then $(a, b, c)$ also has a common eigenvector, as required.

Thus, we have proved our claim. To prove the proposition, we use the following inductive argument. Let $(a_1, a_2, \cdots, a_{2g-1}, a_{2g}, a_{2g+1}, \cdots, a_{2g+n})$ be an optimal sequence of generators of $\pi_1(\Sigma)$. We show that $(a_1, \cdots, a_{2g+n})$ has a common eigenvector. For simplicity of arguments we may assume that at least one element in each pair

$$(a_1, a_2), \cdots, (a_{2g-1}, a_{2g})$$

is not equal to $\pm\mathbf{1}$; for if some pair is of the form $(a, b)$ with $a, b \in \{\pm\mathbf{1}\}$ we may simply skip over that pair in the considerations below.

If $(g, n) = (1, 0)$, then we are done since every representation $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ is abelian. We thus assume $(g, n) \neq (1, 0)$. By our claim, $(a_1, a_2, a_3)$ has a common eigenvector. Assume next that $4 \leq k \leq 2g + n$ and $(a_1, \cdots, a_{k-1})$ has a common eigenvector $x \in \mathbb{C}^2$. We show that $(a_1, \cdots, a_k)$ has a common eigenvector. We consider the following cases.

(1) $k = 5, 7, \cdots, 2g-1$, or $k = 2g+1, 2g+2, \cdots, 2g+n$. The triple $(a_1, a_2, a_k)$ is in $(1, 2)$-position, and hence has a common eigenvector $y$ by our claim.

Assume now first that $a_1 \neq \pm\mathbf{1}$. Given any $3 \leq j < k$, note that $(a_1, a_2a_j, a_k)$ is in $(1, 2)$-position and hence has a common eigenvector by our claim, say $y_j$. Since $a_1 \neq \pm\mathbf{1}$, we must have

$$x \sim y, \quad x \sim y_j, \quad \text{or} \quad y_j \sim y.$$

If one of the first two occurs, then we conclude that $(a_1, \cdots, a_k)$ has a common eigenvector $x$, as required. Thus, we are left with the case $y_j \sim y$ for every $3 \leq j < k$. But this implies that $(a_1, a_2, a_2a_j, a_k)$ has a common eigenvector $y$, which is thus shared also by $a_j$. Thus, $y$ is a common eigenvector of $(a_1, \cdots, a_k)$, as desired.

Now, suppose that $a_1 = \pm\mathbf{1}$, and hence $a_2 \neq \pm\mathbf{1}$. Given any $3 \leq j < k$, we observe that $(a_2, a_1^{-1}a_2a_j, a_k)$ is in $(1, 2)$-position and hence has a common eigenvector, say $y_j$. Since $a_2 \neq \pm\mathbf{1}$, we must have

$$x \sim y, \quad x \sim y_j, \quad \text{or} \quad y_j \sim y.$$

If one of the first two occurs, then we conclude that $(a_1, \cdots, a_k)$ has a common eigenvector $x$, as required. Thus, we are left with the case $y_j \sim y$ for every $3 \leq j < k$. But this implies that $(a_1, a_2, a_1^{-1}a_2a_j, a_k)$ has a common eigenvector $y$, which is thus shared also by $a_j$. Thus, $y$ is a common eigenvector of $(a_1, \cdots, a_k)$, as desired.

(2) $k = 4, 6, \cdots, 2g$. First, consider the case where $a_{k-1} = \pm\mathbf{1}$. It then suffices to show that the sequence $(a_1, \cdots, a_{k-2}, a_k)$ has a common eigenvector. This can be shown by repeating the argument the previous case (1) above. Thus, we may assume that $a_{k-1} \neq \pm\mathbf{1}$. Note that, for each integer $m$ with $2m < k$, by our claim we have:
- a common eigenvector $w_m$ of $(a_{k-1}, a_k, a_{2m})$, and
- a common eigenvector $z_m$ of $(a_{k-1}, a_k, a_{2m-1})$.

Since $a_{k-1} \neq \pm\mathbf{1}$, we must have

$$x \sim w_m, \quad x \sim z_m, \quad \text{or} \quad w_m \sim z_m.$$

If one of the first two occurs, then we conclude that $(a_1, \cdots, a_k)$ has a common eigenvector $x$, and we are done. Thus, we are left with the case where $w_m \sim z_m$ for every $2m < k$. Note in this case that $w_m$ is a common eigenvector of $(a_{2m-1}, a_{2m}, a_{k-1}, a_k)$. Comparing the vectors $x, w_m$, and $w_{m'}$ for different $m, m'$, we are in turn reduced to the case $w_m \sim w_{m'}$ for all $m, m'$, in which case $(a_1, \cdots, a_k)$ has a common eigenvector $w_1$, as desired.

Thus, $(a_1, \cdots, a_k)$ has a common eigenvector. This completes the induction, and shows that $(a_1, \cdots, a_{2g+n})$ has a common eigenvector, proving the proposition. □

5.2. **Genus 1 with 1 or 2 punctures.** We now give a separate proof of Theorem C for surfaces of genus one with one or two punctures. Let $\Sigma_{1,1}$ denote a surface of genus one with one puncture. We begin with the following:

**Lemma 5.3.** *Let $\rho : \pi_1(\Sigma_{1,1}) \to \mathrm{SL}_2(\mathbb{C})$ be a semisimple representation such that each nonseparating simple loop in $\Sigma_{1,1}$ maps to an elliptic or central element of $\mathrm{SL}_2(\mathbb{C})$. Then $\rho$ is unitarizable, i.e., $\rho(\pi_1(\Sigma_{1,1}))$ has a global fixed point in $\mathbb{H}^3$.*

*Proof.* This is known in the literature; we cite two sources below.

First, the lemma can be obtained as an immediate consequence of [21, Theorem 1.2]. We sketch the connection of the setup in the current paper with that in [21]. Since every simple closed non-peripheral curve in $\Sigma_{1,1}$ maps to an elliptic element of $\mathrm{SL}_2(\mathbb{C})$, it follows that we have $\operatorname{tr}\rho(a) \in [-2, 2]$ for every $a$ in the curve complex of $\Sigma_{1,1}$. Hence, in the terminology of [21], the set of end-invariants is given by the set of all projective measured laminations on $\Sigma_{1,1}$. It now follows from [21, Theorem 1.2] that $\rho(\pi_1(\Sigma_{1,1}))$ is either unitarizable or dihedral. We finish by observing that a dihedral representation $\rho : \pi_1(\Sigma) \to D_\infty \subset \mathrm{SL}_2(\mathbb{C})$ sending every simple nonseparating simple loop to an elliptic or central element is unitarizable.

Second, a direct proof of what is essentially the contrapositive, based on an explicit presentation of the character variety $X(\Sigma_{1,1})$ (see the Appendix), is given as the Algebraic Lemma in [5, Section 1.4.2]. □

We now turn to the twice-punctured torus $\Sigma_{1,2}$. We start with the following suggestive presentation of its fundamental group:

$$\pi_1(\Sigma_{1,2}) = \langle u, x, y, p_1, p_2 \mid uxy = p_1,\ uyx = p_2 \rangle$$

where $p_1$, $p_2$ denote loops around the two different punctures of $\Sigma_{1,2}$. See Figure 5 below and section 5.3 of [9].

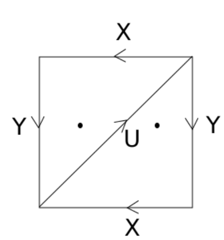


FIGURE 5. Preferred generators for $\Sigma_{1,2}$

**Proposition 5.4.** *Let $\rho : \pi_1(\Sigma_{1,2}) \to \mathrm{SL}_2(\mathbb{C})$ be a semisimple representation such that each nonseparating simple loop in $\Sigma_{1,2}$ maps to an elliptic or central element of $\mathrm{SL}_2(\mathbb{C})$. Then $\rho$ is unitarizable.*

Combining with Lemma 4.4, we note that Lemma 5.3 and Proposition 5.4 prove Theorem C for surfaces of genus one with at most two punctures. Our proof of Proposition 5.4 shall rely on the following observation. In what follows, $R_L$ denotes a hyperbolic reflection on the totally-geodesic plane $L$ in $\mathbb{H}^3$.

**Lemma 5.5.** *Let $A$ and $B$ be two non-trivial elliptic rotations with distinct but intersecting axes lying on a common plane $K$. Then $AB$ is an elliptic rotation that has axis lying on a plane $Q$ that is at equal angles from $K$ and the plane $K'$ obtained as the image of $K$ under $A$.*

*Proof.* $A$ is a composition of two hyperbolic reflections, that is, $A = R_Q \circ R_K$ where $Q$ is a totally-geodesic plane containing the axis of $A$, and is at half the rotation angle (of $A$) from $K$. Similarly, $B = R_K \circ R_S$ where $S$ is a totally-geodesic plane that is at half the rotation angle of $B$ from $K$. Hence the composition $AB = R_Q \circ R_S$, and its axis is the intersection of the planes $Q$ and $S$. In particular, the axis lies on the plane $Q$, proving the lemma. □

*Proof of Proposition 5.4.* Suppose that $\rho : \pi_1(\Sigma_{1,2}) \to \mathrm{SL}_2(\mathbb{C})$ is a representation satisfying the hypothesis of the proposition. We may assume that $\rho$ is irreducible, since otherwise the result is clear.

Moreover, we may assume that the restriction of $\rho$ to any one-holed torus subsurface of $\Sigma_{1,2}$ is semisimple. Indeed, otherwise, we may choose an optimal sequence $(a_1, b_1, c_1, c_2)$ of generators of $\pi_1(\Sigma)$ such that the restriction of $\rho$ to the one-holed torus $\Sigma' \subset \Sigma$ associated to the pair $(a_1, b_1)$ is upper triangular, with the boundary loop $c = c_1c_2$ having non-central parabolic monodromy. The irreducibility of $\rho$ then implies that $\rho(c_1)$ and $\rho(c_2)$ cannot be upper triangular. But then an argument as in part (b) of the proof of Lemma 4.6 shows that the restriction $\rho|\Sigma'$ is reducible and moreover $\rho(a_1)$ and $\rho(b_1)$ are parabolic, whence they must both be central and $\rho(c)$ is also central; a contradiction.

Thus, in what follows, $\rho : \pi_1(\Sigma_{1,2}) \to \mathrm{SL}_2(\mathbb{C})$ is an irreducible representation satisfying the hypothesis of the proposition, with the property that its restriction to every one-holed torus subsurface is semisimple. We shall also assume that none of the nonseparating simple loops have central monodromy, since otherwise (using the preferred presentation of $\pi_1(\Sigma_{1,2})$ given above) we reduce to the case of one-holed torus, treated in Lemma 5.3.

Let us write $U = \rho(u)$, $X = \rho(x)$, and $Y = \rho(y)$ for the presentation of $\pi_1(\Sigma_{1,2})$ given above. By Lemma 5.3, the elements $U, X, Y$ have coplanar, pairwise intersecting axes. We call the common plane $P$. Let $p$ be the intersection point of the axes of $X$ and $Y$. It suffices to prove that $U$ fixes $p$ as well, that is, the axis of $U$ also passes through $p$. Assume that the axis of $U$ does not pass through $p$; we shall eventually reach a contradiction.

Consider the element $UXYX = UT$, say. Let $Z = X^{-1}YX$ so that $T = X^2Z$. Note that $T$ is elliptic as it fixes the intersection of the axes of $X, Y$. Moreover, it can be easily checked using the loop configuration diagram that the curve represented by the element $uxyx$ is simple, closed, and essential. Hence $UT$ is also an elliptic element. However, we also have:

**Claim 1.** *The axis of $T = XYX$ does not lie on the plane $P$ unless $X$ is a $\pi$–rotation.*

*Proof of claim.* Note that since $Z = X^{-1}YX$, the axis of $Z$ is the image of the axis of $Y$ under $X^{-1}$. Choose the plane $K$ to be the one containing the axes of $X$ and $Z$. Then $K = X^{-1}(P)$, i.e the image of $K$ under $X$ is the plane $P$. Since $X^2$ and $Z$ have the same axes as $X$ and $Z$ respectively, Lemma 5.5 shows that the axis of $X^2Z$ will be on a plane that is at equal angles from $K$ and $P$. In particular, the axis does not lie on the plane $P$ unless $X^2$ is the identity element, that is, unless $X$ is a $\pi$–rotation. □

Since we have assumed that the axis of $U$ does not pass through $p$, the plane $P$ is the unique plane that contains both $p$ and the axis of $U$. Note that the axis of $T$ contains $p$; hence if it does not lie in the plane $P$, the axes of $U$ and $T$ cannot lie on a common plane. By Lemma 3.4.1 of [8], we have a contradiction to the fact that $UT$ is elliptic. Hence, $X$ must be a $\pi$–rotation. Repeating the same argument for other presentations of $\pi_1(\Sigma_{1,2})$, we are thus reduced to the case where the monodromy trace of every nonseparating simple loop on $\Sigma_{1,2}$ under $\rho$ is 0. It is easy to see that such $\rho$ is in fact dihedral, whence the hypothesis of the Proposition implies that $\rho$ is unitarizable, a contradiction. □

5.3. **Proof of Theorem A.** We restate and complete the proof of Theorem A.

**Theorem A.** *Let $\Sigma$ be an oriented surface of genus $g \geq 2$ with $n \geq 0$ punctures. A semisimple representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbb{C})$ has finite mapping class group orbit in the character variety $X(\Sigma)$ if and only if $\rho$ is finite.*

*Proof.* Suppose that $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ is a semisimple representation with finite mapping class group orbit in $X(\Sigma)$. By Lemma 3.2, we are done if $\rho$ is reducible, so we may assume that $\rho$ is irreducible. It suffices to show that $\rho$ has finite monodromy along every simple closed curve around a puncture of $\Sigma$, for then we reduce to Proposition 4.3. By Lemmas 4.4 and 4.5, we see that $\rho$ must have finite monodromy along any essential curve $a \subset \Sigma$ which is either:

- nonseparating, or
- separating with each component of $\Sigma|a = \Sigma_1 \sqcup \Sigma_2$ having positive genus.

By Proposition 5.2, there is a one-holed torus subsurface $\Sigma' \subset \Sigma$ such that $\rho|\Sigma'$ is irreducible. Let $c$ be a simple closed curve around a puncture of $\Sigma$. Let $\Sigma'' \subset \Sigma$ be a two-holed torus containing $\Sigma'$ and having $c$ as one of its boundary curves; let $c'$ be the other boundary curve of $\Sigma''$. (Note here that each component of $\Sigma|c'$ has positive genus, by design.) We shall prove that $\rho|\Sigma''$ is finite, so *a fortiori* $\rho$ has finite monodromy along $c$. For this, we follow the strategy of [16] below.

First, we know from above that $\rho$ has finite monodromy along every essential curve of $\Sigma''$, as well as along the boundary curve $c'$. In particular, the restriction of $\rho$ to the one-holed torus $\Sigma'$ has image that is conjugate to a subgroup of $\mathrm{SU}(2)$. Let $A \subset \mathbb{R}$ be the $\mathbb{Z}$-algebra generated by the set of traces of $\rho$ along the essential curves of $\Sigma''$. By considering the preferred generators for $\pi_1(\Sigma'')$ introduced in Section 5.2, it follows from the trace relations given in Example 2.2 (cf. [9, Section 5.3]) that $\operatorname{tr}\rho(c)$ satisfies a monic quadratic equation over the ring $A$, with the other root being $\operatorname{tr}\rho(c')$. Since $\operatorname{tr}\rho(c') \in \mathbb{R}$, it follows that $\operatorname{tr}\rho(c_1) \in \mathbb{R}$ as well. Applying Fact 2.4 to a sequence of optimal generators for $\pi_1(\Sigma'')$, we deduce that the character of $\rho|\Sigma''$ is real, and since $\rho|\Sigma''$ is semisimple its image is conjugate to a subgroup of SU(2) or $\mathrm{SL}_2(\mathbb{R})$ (see e.g. [14, Proposition III.1.1]). The latter cannot occur, since otherwise the restriction of $\rho$ to $\Sigma'$ has image conjugate to a subgroup of SO(2), contradicting the fact that $\rho|\Sigma'$ is irreducible hence nonabelian. Thus, the restriction of $\rho$ to $\Sigma''$ has image conjugate to a subgroup of SU(2).

It also follows from the above analysis that the character of $\rho|\Sigma''$ takes values in the ring of algebraic integers in $\overline{\mathbb{Q}}$. In particular, we may assume without loss of generality that the image of $\rho|\Sigma''$ lies in $\mathrm{SL}_2(\overline{\mathbb{Q}})$. By considering conjugates of $\rho|\Sigma''$ by elements of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$ and noting that the above analysis goes through for all of these conjugates, we conclude that the eigenvalues of monodromy of $\rho$ along $c$ are algebraic integers all of whose Galois conjugates have absolute value 1. By Kronecker's theorem, it follows that the eigenvalues are roots of unity, i.e., $\rho$ has finite monodromy along $c$ (note that monodromy along $\rho$ cannot be unipotent). This is the desired result. □

5.4. **Proof of Theorem B.** We restate and complete the proof of Theorem B.

**Theorem B.** *Let $\Sigma$ be an oriented surface of genus 1 with $n \geq 0$ punctures. A semisimple representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ has finite mapping class group orbit in the character variety $X(\Sigma)$ if and only if $\rho$ is finite or special dihedral.*

*Proof.* Our proof proceeds as in the proof Theorem A, with minor modifications. Suppose that $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ is a semisimple representation with finite mapping class group orbit in $X(\Sigma)$. By Lemma 3.2, we are done if $\rho$ is reducible, so we may assume that $\rho$ is irreducible. We may also assume that $\rho$ is not special dihedral. It suffices to show that $\rho$ has finite monodromy along every simple closed curve around a puncture of $\Sigma$, for then we reduce to Proposition 4.3.

By Lemma 4.4, $\rho$ has finite monodromy along any nonseparating essential curve $a \subset \Sigma$. Given any simple closed curve $c$ around a puncture of $\Sigma$, there is a two-holed torus subsurface of $\Sigma$ having $c$ as one of its boundary components, and by the trace relations in Example 2.2 (cf. [9, Section 5.3]) we see that $\mathrm{tr}\,\rho(c)$ is an algebraic integer. By considering an optimal sequence of generators for $\pi_1(\Sigma)$ and applying Fact 2.4, we see that the coordinate ring of the character variety $X(\Sigma)$ is generated by the monodromy traces $\mathrm{tr}_a$ around a finite collection of curves $a$ each of which is either a nonseparating curve or a curve around a puncture of $\Sigma$. In particular, it follows that $\mathrm{tr}\,\rho(\alpha)$ is an algebraic integer for every $\alpha \in \pi_1(\Sigma)$. We may in particular assume that $\rho$ is a representation of $\pi_1(\Sigma)$ into $\mathrm{SL}_2(\overline{\mathbb{Q}})$.

Now, $\rho$ is unitarizable by Theorem C proved below, and in particular the monodromy eigenvalues of $\rho$ along any curve $c$ around a puncture of $\Sigma$ have absolute value 1. Applying this observation to every conjugate of $\rho$ by an element of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we see that the eigenvalues of $\rho(c)$ for any curve around a puncture of $\Sigma$ are algebraic integers all of whose conjugates have absolute value 1. By Kronecker's theorem, it follows that the eigenvalues are roots of unity, i.e., $\rho$ has finite monodromy along $c$ (note that monodromy along $\rho$ cannot be unipotent). This is the desired result. □

5.5. **Proof of Theorem C.** We restate and complete the proof of Theorem C.

**Theorem C.** *Let $\Sigma$ be an oriented surface of genus $g \geq 1$ with $n \geq 0$ punctures. A semisimple representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbb{C})$ has bounded mapping class group orbit in the character variety $X(\Sigma)$ if and only if:*

(1) *$\rho$ is unitary up to conjugacy, or*

(2) *$g = 1$ and $\rho$ is special dihedral up to conjugacy.*

*Proof.* In the case where $\Sigma$ is a surface of genus $g \geq 2$, a minor modification of the argument in the proof of Theorem A proves that $\rho$ has elliptic or central local monodromy around the punctures. Proposition 4.3 then shows that $\rho$ is unitarizable, as desired.

Let us now assume that $\Sigma$ has genus 1 with $n \geq 0$ punctures. We know the case $n \leq 2$ of Theorem C by our work in Section 5.2; we shall deduce the general case from it. So suppose we have a semisimple representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ whose monodromy is central or elliptic along every nonseparating curve on $\Sigma$. It is clear that $\rho$ is unitarizable if it is moreover reducible, so let us assume that $\rho$ is irreducible in what follows.

By Proposition 5.2, there is a one-holed torus subsurface $\Sigma' \subset \Sigma$ such that $\rho|\Sigma'$ is irreducible. Let $c$ be a simple closed curve around a puncture of $\Sigma$. Let $\Sigma'' \subset \Sigma$ be a two-holed torus containing $\Sigma'$ and having $c$ as one of its boundary curves. By Proposition 5.4, our Theorem C holds for two-holed tori, and hence we see that $\rho|\Sigma''$ is unitarizable. It follows that the monodromy of $\rho$ along $c$ is central or unitary; in particular, $\mathrm{tr}\,\rho(c)$ is real. By considering an optimal sequence of generators for $\pi_1(\Sigma)$ and applying Fact 2.4, we see that the coordinate ring of the character variety $X(\Sigma)$ is generated by the monodromy traces $\mathrm{tr}_a$ around a finite collection of curves $a$ each of which is either a nonseparating curve or a curve around a puncture of $\Sigma$. In particular, it follows that $\rho$ has real character, and therefore the image of $\rho$ is conjugate to a subgroup of $\mathrm{SU}(2)$ or $\mathrm{SL}_2(\mathbb{R})$. We claim that the latter cannot occur. Indeed, by Proposition 5.2 there is a one-holed torus subsurface $\Sigma' \subset \Sigma$ such that $\rho|\Sigma'$ is irreducible, and moreover $\rho|\Sigma'$ must be unitarizable by the case $n = 2$ of our proposition. If the image of $\rho$ lies in $\mathrm{SL}_2(\mathbb{R})$, then unitarizability implies that the image of $\rho|\Sigma'$ is conjugate to a subgroup of $\mathrm{SO}(2)$, contradicting the irreducibility of $\rho|\Sigma'$. Thus, the image of $\rho$ must be conjugate to a subgroup of $\mathrm{SU}(2)$, as desired. ☐

5.6. **Alternative proof of Theorem A for closed surfaces.** We give a different short proof of Theorem A in the case where $\Sigma$ is a closed surface of genus $g \geq 2$, using the following result of Gallo–Kapovich–Marden.

**Theorem 5.6** ([8, Section 3])**.** *Let $\Sigma$ be a closed oriented surface of genus greater than one. If $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ is non-elementary, then there exist simple loops $a, b$ on $\Sigma$ such that*

(1) *the intersection number $i(a, b) = 1$,*
(2) *The images $\rho(a), \rho(b) \in \mathrm{PSL}_2(\mathbb{C})$ are loxodromic and distinct, and generate a Schottky group.*

**Theorem 5.7.** *Let $\Sigma$ be a closed orientable surface of genus $g \geq 2$. Given a semisimple representation $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$, the orbit of $\rho$ in $X(\Sigma)$, under the action of $\mathrm{Mod}(\Sigma)$, is finite if and only if the image of $\rho$ is a finite group.*

*Proof.* Suppose first that $\rho$ is non-elementary. By Theorem 5.6, there exist simple loops $a, b$ on $\Sigma$ with $i(a, b) = 1$ such that $\rho(a), \rho(b) \in \mathrm{PSL}_2(\mathbb{C})$ are loxodromic and distinct. Since $\rho(a), \rho(b) \in \mathrm{PSL}_2(\mathbb{C})$ are loxodromic, $\mathrm{tw}_a^n(b)$ gives an infinite sequence of curves in $S$ with $\rho$-images $\rho(a)^n\rho(b)$ whose translation length in $\mathbb{H}^3$ tends to infinity as $n \to \infty$, while $\mathrm{tw}_a^n(a)$ remains fixed. It follows that the $\mathrm{tw}_a^n$-orbit of $\rho$ is infinite in $X(\Sigma)$ and hence so is the $\mathrm{Mod}(\Sigma)$-orbit.

Suppose that $\rho$ is elementary. In view of the results in Section 3, it remains only to treat the case where $\rho$ is a representation whose image is a dense subgroup of $\mathrm{SU}(2)$ in the Euclidean topology. In this case, the main theorem of [17] states that the $\mathrm{Mod}(\Sigma)$-orbit of $\rho$ is dense in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(2))/\mathrm{SU}(2)$ in the Euclidean topology. Hence it is infinite. □

## 6. Applications

In this section, we collect applications of our results and methods developed in the previous sections. The following immediate corollary of Theorem A answers a question that was posed to us by Lubotzky:

**Corollary 6.1.** *Given a surface $\Sigma$ of genus $g \geq 1$ with $n \geq 0$ punctures, a faithful representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ (or $\mathrm{PSL}_2(\mathbb{C})$) cannot have finite $\mathrm{Mod}(\Sigma)$-orbit in the character variety.*

*Proof.* It follows from Theorems A and B that if $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ has a finite $\mathrm{Mod}(\Sigma)$-orbit in $X(\Sigma)$, then $\rho$ cannot be faithful. The same holds when $\mathrm{SL}_2(\mathbb{C})$ is replaced by $\mathrm{PSL}_2(\mathbb{C})$. □

Let $\Sigma$ be a closed surface of genus greater than one; we fix a a hyperbolic metric on $\Sigma$ such that $\Sigma = \mathbb{H}^2/\Gamma$ where $\Gamma$ is a Fuchsian group. Recall that for any semisimple representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$, there exists a $\rho$-equivariant harmonic map $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^3$ from the universal cover of $\Sigma$ to the symmetric space for $\mathrm{PSL}_2(\mathbb{C})$ (see, for example, [6]). The *equivariant energy* of this harmonic map is the energy of its restriction to a fundamental domain of the $\Gamma$-action on $\mathbb{H}^2$. The following answers a question due to Goldman.

**Theorem 6.2.** *Fix a Riemann surface $\Sigma$ of genus greater than one with $\Gamma = \pi_1(\Sigma)$, and let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be an semisimple representation. Suppose that the equivariant energies of the harmonic maps corresponding to the mapping class group orbit of $\rho$ in $\mathrm{Hom}(\pi_1\Sigma, \mathrm{PSL}_2(\mathbb{C}))/\!\!/\mathrm{PSL}_2(\mathbb{C})$ is uniformly bounded. Then $\rho(\Gamma)$ fixes a point of $\mathbb{H}^3$; in particular $\rho(\Gamma)$ can be conjugated to lie in $\mathrm{PSU}(2)$.*

*Proof.* If $\rho(\Gamma)$ is elementary, but not unitary, then $\rho(\Gamma)$ fixes a geodesic in $\mathbb{H}^3$, and thus the image of any equivariant harmonic map coincides with this geodesic. Let $a$ be a simple closed curve mapped to an infinite order hyperbolic element in $\rho(\Gamma)$. Let $b$ be a simple closed on $\Sigma$ with $i(a, b) > 0$. Then the translation length of $\rho(\mathrm{tw}_a^n\, b)$ increases to infinity as $n \to \infty$. Hence the $\mathrm{Mod}(\Sigma)-$orbit of $\rho$ cannot have bounded energy, since an uniform energy bound on the harmonic maps implies that they are uniformly Lipschitz.

Otherwise, suppose $\rho(\Gamma)$ is non-elementary. By Theorem 5.6, there exist simple closed curves $a, b$ on $S$ such that $i(a, b) = 1$ and $\rho(a), \rho(b) \in \mathrm{PSL}_2(\mathbb{C})$ are loxodromic and distinct. It follows as in the proof of Theorem 5.7 that the translation lengths in $\mathbb{H}^3$ of $\rho(\mathrm{tw}_a^n(b))$ tend to infinity, while $\mathrm{tw}_a^n(a)$ remains fixed. Hence the $\mathbb{Z}-$orbit $\rho \circ \mathrm{tw}_a^n$ of $\rho$ under $\langle \mathrm{tw} \rangle$ cannot have bounded energy. Thus the $\mathrm{Mod}(\Sigma)-$orbit of $\rho$ cannot have bounded energy, as before.

The possibility that remains is that $\rho(\Gamma)$ is unitary. □

## Appendix A. Case of once-punctured torus

In this appendix, we describe how the work of Dubrovin–Mazzocco [5] can be used to prove the once-punctured torus case of our Theorem B. We begin with the following well-known observation.

**Lemma A.1.** *A pair $(a, b)$ of elements in $\mathrm{SL}_2(\mathbb{C})$ has a common eigenvector in $\mathbb{C}^2$, or in other words lies in the standard Borel $B$ up to simultaneous conjugation, if and only if $\mathrm{tr}([a, b]) = 2$, where $[a, b] = aba^{-1}b^{-1}$.*

Let $\Sigma$ be a surface of genus one with one puncture. Let $(a, b, c)$ be an optimal sequence of generators for $\pi_1(\Sigma)$, as defined in Example 2.6. Let $X = X(\Sigma)$ be the character variety of $\Sigma$. Note that $\pi_1(\Sigma)$ is freely generated by $a$ and $b$. The trace functions on $X(\Sigma)$ furnish an isomorphism

$$(x_1, x_2, x_3) = (\mathrm{tr}_a, \mathrm{tr}_b, \mathrm{tr}_{ab}) : X(\Sigma) \xrightarrow{\sim} \mathbb{A}^3$$

by Fricke (see [9] for details). Observing that $\mathrm{tr}(\mathbf{1}) = 2$ for the identity $\mathbf{1} \in \mathrm{SL}_2(\mathbb{C})$ and that $\mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$ for any $A, B \in \mathrm{SL}_2(\mathbb{C})$, we have

$$\begin{aligned}
\mathrm{tr}_c &= \mathrm{tr}_{aba^{-1}b^{-1}} = \mathrm{tr}_{aba^{-1}}\,\mathrm{tr}_{b^{-1}} - \mathrm{tr}_{aba^{-1}b} \\
&= \mathrm{tr}_b^2 - \mathrm{tr}_{ab}\,\mathrm{tr}_{a^{-1}b} + \mathrm{tr}_{aa} = \mathrm{tr}_b^2 - \mathrm{tr}_{ab}(\mathrm{tr}_{a^{-1}}\,\mathrm{tr}_b - \mathrm{tr}_{ab}) + \mathrm{tr}_a^2 - \mathrm{tr}_{\mathbf{1}} \\
&= \mathrm{tr}_a^2 + \mathrm{tr}_b^2 + \mathrm{tr}_{ab}^2 - \mathrm{tr}_a\,\mathrm{tr}_b\,\mathrm{tr}_{ab} - 2.
\end{aligned}$$

In particular, under the identification $(x_1, x_2, x_3) = (\mathrm{tr}_a, \mathrm{tr}_b, \mathrm{tr}_{ab})$ of the coordinate functions above and Lemma A.1, we see that the locus of reducible representations in $X(\Sigma) = \mathbb{A}^3$ is the cubic algebraic surface cut out by the equation

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 - 2 = 2.$$

The mapping class group $\mathrm{Mod}(\Sigma)$ acts on $X(\Sigma)$ via polynomial transformations. For convenience, we shall denote the isotopy classes of simple closed curves lying in the free homotopy classes of $a$, $b$, and $ab$ by the same letters. We have the following descriptions of the associated Dehn twist actions.

**Lemma A.2.** *The Dehn twist actions $\mathrm{tw}_a$, $\mathrm{tw}_b$, and $\mathrm{tw}_{ab}$ on $X(\Sigma)$ are given by*

$$\begin{aligned}
\mathrm{tw}_a^* &: (x_1, x_2, x_3) \mapsto (x_1, x_3, x_1x_3 - x_2), \\
\mathrm{tw}_b^* &: (x_1, x_2, x_3) \mapsto (x_1x_2 - x_3, x_2, x_1), \\
\mathrm{tw}_{ab}^* &: (x_1, x_2, x_3) \mapsto (x_2, x_2x_3 - x_1, x_3).
\end{aligned}$$

*in terms of the above coordinates.*

*Proof.* Note that $\mathrm{tw}_a(a)$ has the homotopy class of $\alpha$, $\mathrm{tw}_a(b)$ has the homotopy class of $\alpha\beta$, and $\mathrm{tw}_a(ab)$ has the homotopy class of $\alpha\alpha\beta$. Noting that $\mathrm{tr}_{\alpha\alpha\beta} = \mathrm{tr}_\alpha\,\mathrm{tr}_{\alpha\beta} - \mathrm{tr}_\beta$, we obtain the desired expression for $\mathrm{tw}_a^*$. The other Dehn twists are similar. □

Let $\Pi$ be the group of polynomial automorphisms of $\mathbb{A}^3$ generated by $\mathrm{tw}_a^*$, $\mathrm{tw}_b^*$, and $\mathrm{tw}_{ab}^*$. It is precisely the image of the mapping class group $\mathrm{Mod}(\Sigma)$ in the group of polynomial automorphisms of $X(\Sigma) = \mathbb{A}^3$. Let $\Pi'$ be the group generated by $\Pi$ together with the following transformations:

$$\begin{aligned}
\sigma_{12} &: (x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3), \\
\sigma_{23} &: (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3), \\
\sigma_{13} &: (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3).
\end{aligned}$$

It is easy to see that $[\Pi' : \Pi] < \infty$. Hence, a point in $\mathbb{A}^3$ has finite $\Pi$-orbit if and only if it has finite $\Pi'$-orbit. Now, the group $\Pi'$ contains a group generated by transformations

$$\beta_1 = \sigma_{12}\,\mathrm{tw}^*_{ab}(\mathrm{tw}^*_b\,\mathrm{tw}^*_a)^{-1} : (x_1, x_2, x_3) \mapsto (-x_1, x_3 - x_1x_2, x_2),$$
$$\beta_2 = \sigma_{23}\,\mathrm{tw}^*_{a}(\mathrm{tw}^*_b\,\mathrm{tw}^*_a)^{-1} : (x_1, x_2, x_3) \mapsto (x_3, -x_2, x_1 - x_2x_3).$$

whose finite orbits in $\mathbb{A}^3$ were studied by Dubrovin-Mazzocco [5, Theorem 1.6] in connection with algebraic solutions of special Painlevé VI equations. They defined a triple $(x_1, x_2, x_3) \in \mathbb{A}^3(\mathbb{C})$ to be *admissible* if it has at most one coordinate zero and $x_1^2 + x_2^2 + x_2^2 - x_1x_2x_3 - 2 \neq 2$. It is easy to verify that the admissible points are precisely those which do not correspond to reducible or special dihedral representations. The result of [5] we shall use is the following.

**Theorem A.3** (Dubrovin-Mazzocco)**.** *The following is a complete set of representatives for the finite $\langle\beta_1, \beta_2\rangle$-orbits of admissible triples in $\mathbb{A}^3$:*

$$(0, -1, -1), (0, -1, -\sqrt{2}), (0, -1, -\varphi), (0, -1, -\varphi^{-1}), (0, -\varphi, -\varphi^{-1})$$

*where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.*

To deduce Theorem B in the once-punctured torus case from the above, we recall the following explicit description of the finite subgroups $BA_4$, $BS_4$, $BA_5$ of $\mathrm{SL}_2(\mathbb{C})$. First, let us identify the group of unit quaternions

$$\mathrm{Sp}(1) = \{z = (a, b, c, d) = a + bi + cj + dk \in \mathbb{H} : |z| = a^2 + b^2 + c^2 + d^2 = 1\}$$

as a subgroup of $\mathrm{SL}_2(\mathbb{C})$ by the map

$$z = (a, b, c, d) \mapsto \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}.$$

Under the identification, the *binary tetrahedral group* $BA_4$ is given by

$$BA_4 = \{\pm 1, \pm i, \pm j, \pm k, (\pm 1 \pm i \pm j \pm k)/2\}$$

with all sign combinations taken in the above. The *binary octahedral group* $BS_4$ is the union of $BA_4$ with all quaternions obtained from $(\pm 1, \pm 1, 0, 0)/\sqrt{2}$ by all permutations of coordinates and all sign combinations. The *binary icosahedral group* $BA_5$ is the union of $BA_4$ with all quaternions obtained from $(0, \pm 1, \pm\varphi^{-1}, \pm\varphi)/2$ by an even permutation of coordinates and all possible sign combinations, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

**Corollary A.4.** *If $(x_1, x_2, x_3) \in \mathbb{A}^3(\mathbb{C})$ is an admissible triple with finite* $\mathrm{Mod}(\Sigma)$*-orbit, then it corresponds to a representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ with finite image.*

*Proof of the corollary.* Replacing $(x_1, x_2, x_3)$ by another triple within its $\mathrm{Mod}(\Sigma)$-orbit if necessary, we may assume that $(x_1, x_2, x_3)$ is one of the triples in Theorem A.3 or its image under one of the transformations $\sigma_{12}$, $\sigma_{23}$, or $\sigma_{13}$. We shall show that

$$(x_1, x_2, x_3) = (\mathrm{tr}\, A, \mathrm{tr}\, B, \mathrm{tr}(AB))$$

where $A, B \in \mathrm{SL}_2(\mathbb{C})$ are elements that together lie in one of the finite subgroups $BA_4$, $BS_4$, or $BA_5$ of $\mathrm{SL}_2(\mathbb{C})$. Since the matrix $-\mathbf{1}$ is contained in every one of these groups, it suffices to treat the case where $(x_1, x_2, x_3)$ is one of the triples in

Theorem A.3. By explicit computation, we find that the triples in Theorem A.3 respectively correspond to traces of the triples of matrices

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}(1+i) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ \frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \end{bmatrix}\right),$$

$$\left(\begin{bmatrix} 0 & \frac{1}{\sqrt{2}}(1-i) \\ \frac{1}{\sqrt{2}}(1+i) & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}(1+i) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}} \end{bmatrix}\right),$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}(\varphi+\varphi^{-1}i) \\ -\frac{1}{2}(\varphi-\varphi^{-1}i) & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}(\varphi-\varphi^{-1}i) & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}(\varphi+\varphi^{-1}i) \end{bmatrix}\right),$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2}(1-\varphi i) & \frac{1}{2}\varphi^{-1} \\ -\frac{1}{2}\varphi^{-1} & -\frac{1}{2}(1+\varphi i) \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\varphi^{-1} & -\frac{1}{2}(1+\varphi i) \\ -\frac{1}{2}(1-\varphi i) & -\frac{1}{2}\varphi^{-1} \end{bmatrix}\right),$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{\varphi}{2} & \frac{\varphi^{-1}}{2}+\frac{1}{2}i \\ -\frac{\varphi^{-1}}{2}+\frac{1}{2}i & -\frac{\varphi}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\varphi^{-1}}{2}+\frac{1}{2}i & -\frac{\varphi}{2} \\ \frac{\varphi}{2} & -\frac{\varphi^{-1}}{2}-\frac{1}{2}i \end{bmatrix}\right),$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The matrices for the first triple all lie in the binary tetrahedral group $BA_4$, the matrices for the second triple all lie in the binary octahedral group $BS_4$, and the matrices for the remaining three triples all lie in the binary icosahedral group $BA_5$. In each triple, the third matrix is the product of the first two. Thus, each of the triples in Theorem A.3 correspond to representations $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ with finite image, proving the corollary. □

*Remark.* In [5], two proofs of Theorem A.3 are given. The first proof is based on an explicit analysis of certain relevant trigonometric Diophantine equations. General equations of this type are effectively solvable by Lang's $\mathbb{G}_m$ conjecture (proved by Laurent [13]), as noted in [3]. The second proof in [5], based on a suggestion of Vinberg, uses consideration of certain representations of Coxeter groups of reflections associated to admissible triples. Both methods use special features present in the once-punctured torus case which do not seem to generalize easily to the case of general surfaces treated in our work.

## Acknowledgments

Theorems A and B along with parts of this paper originally appeared in Chapter 6 of JPW's Ph.D. thesis [24], but the proof had a gap which has been filled in this paper. JPW thanks Peter Sarnak and Phillip Griffiths for encouragement, and Anand Patel and Ananth Shankar for collaborative work in [16] which led to one of the key ingredients in this paper. IB and MM thank Bill Goldman for useful email correspondence, in particular for alerting us to dihedral representations. Research of IB partly supported by a DST JC Bose Fellowship. SG acknowledges the SERB, DST (Grant no. MT/2017/000706) and the Infosys Foundation for their support. Research of MM partly supported by a DST JC Bose Fellowship, Matrics research project grant MTR/2017/000005 and CEFIPRA project No. 5801-1. MM was also partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund.

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