MOTIONS OF LIMIT SETS: A SURVEY

MAHAN MJ

Abstract. This is a survey article, giving an expanded version of a talk delivered at the Workshop on Grothendieck-Teichmller Theories, held on July 24-30, 2016, at Nankai University, Tianjin, China. Most of the work described is joint with either Caroline Series or Ken'ichi Ohshika.

Contents

1. Introduction: Setting up the problem 2
1.1. Algebraic and Geometric Limits 3
1.2. Dynamics: Cannon-Thurston Maps 3
1.3. Main Results 4
1.4. Plan of the paper 6
2. Criteria for convergence of CT maps: 7
2.1. Existence of Cannon-Thurston Maps 7
2.2. Uniform convergence criterion 8
2.3. Pointwise convergence criterion 11
3. Strong Convergence: Special Cases 11
3.1. Quasi Fuchsian groups converging strongly to cusped groups 12
3.2. Models of Ends 15
3.3. Quasi Fuchsian groups converging to simply degenerate groups 17
3.4. Handlebodies and Schottky groups 19
4. Strong Convergence: General Case 20
4.1. Strong Convergence with Geometrically Infinite Limit 20
4.2. From Strictly Type-Preserving to Weakly Type-Preserving 23
5. Algebraic limits and Pointwise Convergence 26
5.1. The Kerckhoff-Thurston Examples 27
5.2. Uniform and Pointwise Convergence 28
6. Algebraic limits and Pointwise Non-convergence 31

Date: December 26, 2016.
2010 *Mathematics Subject Classification.* 57M50.
Research partly supported by a DST JC Bose Fellowship.

6.1. Brock's Examples 31
6.2. The Model 31
6.3. Absence of Uniform Convergence 33
6.4. Pointwise Convergence and non-convergence 33
6.5. Generalization 40
Acknowledgments 40
References 41
Appendix 42

1. Introduction: Setting up the problem

Following [MS13, MS17] we give below an account of our study of the following problem.

Question 1.1. *Let G_n be a sequence of Kleinian groups converging to a Kleinian group G. Does the corresponding dynamics of G_n on the Riemann sphere S^2 converge to the dynamics of G on S^2?*

Question 1.1 is a paraphrasing of the second part of Problem 14 of [Thu82]. We shall motivate most of the discussion in this paper using surface groups, i.e. Kleinian groups abstractly isomorphic to $\pi_1(S)$ for a closed surface S of genus greater than one. However, we shall have occasion to deal with more general cases as well. Of course we need to first make precise what "converges" and "dynamics" mean in Question 1.1. Convergence of Kleinian groups can be taken in the sense of algebraic or geometric limits. Also by "dynamics" we shall mean the action of the groups on their limit sets. Limit sets may be regarded as the locus of chaotic dynamics. Convergence of limit sets under such limits have been studied by several authors [Mar07][p. 203]. The usual notion of convergence used in this context is Hausdorff convergence, where we look at convergence of limit sets regarded as sequences of closed subsets of the sphere equipped with the Hausdorff metric. Here we shall however look at a finer notion of convergence of limit sets, which are thought of as being parametrized by Cannon-Thurston maps [CT89] from the circle to the sphere. Existence of such maps was established in [Mj14a]. The notion of convergence of dynamics thus translates to convergence (pointwise or uniform) of Cannon-Thurston maps. Question 1.1 thus splits into the following two questions:

Question 1.2.
1. *If $G_n \to G$ strongly then do the corresponding Cannon-Thurston maps converge uniformly?*
2. *If $G_n \to G$ algebraically then do the corresponding Cannon-Thurston maps converge pointwise?*

We give the quick answer here. The rest of the paper will be an elaboration of these answers.

Answers 1.3. (1) *The answer to Question 1.2 (1) is Yes.*
(2) *The answer to Question 1.2 (2) is No, in general.*

1.1. **Algebraic and Geometric Limits.** In this subsection we explain what we mean by a sequence of Kleinian groups "converging" in Question 1.1. There are a number of notions of convergence of Kleinian groups.

Definition 1.4. *If the generators of Kleinian groups G_i converge to the generators of an isomorphic Kleinian group G as elements of $PSL_2(\mathbb{C})$, then we say that G_i converges to G* **algebraically***.*

Definition 1.5. *Let $f_j : \Gamma \to \mathbb{PSL}_{\not\geq}(\mathbb{C})$ be a sequence of discrete, faithful representations of a finitely generated, torsion-free, nonabelian group Γ, $f_j(\Gamma)$ converges as a sequence of closed subsets of $\mathbb{PSL}_{\not\geq}(\mathbb{C})$ to a torsion-free, nonabelian Kleinian group G. Then G is called the geometric limit of the sequence.*

Definition 1.6. *G_i converges* **strongly** *to G if the convergence is both geometric and algebraic.*

The best results about convergence of limit sets in the Hausdorff topology are due to Evans [Eva00], [Eva04].

Theorem 1.7. [Eva00], [Eva04] *Let $\rho_n : H \to G_n$ be a sequence of weakly type-preserving isomorphisms from a geometrically finite group H to Kleinian groups G_n with limit sets Λ_n, such that ρ_n converges algebraically to $\rho_\infty : H \to G^a_\infty$ and geometrically to G^g_∞. Let Λ_a and Λ_g denote the limit sets of G^a_∞ and G^g_∞. Then $\Lambda_n \to \Lambda_g$ in the Hausdorff metric. Further, the sequence converges strongly if and only $\Lambda_n \to \Lambda_a$ in the Hausdorff metric.*

1.2. **Dynamics: Cannon-Thurston Maps.** We shall now make precise what we mean by "dynamics" in Question 1.1.

Let (X, d_X) be a hyperbolic metric space and let H be a (Gromov-)hyperbolic group acting freely and properly discontinuously by isometries on X. By adjoining the Gromov boundaries ∂X and ∂H to X and H, one obtains their compactifications. $\widehat{X}$ and $\widehat{H}$ respectively.

Let $i : H \to X$ be an orbit map identifying $h \in H$ with $h.o \in X$ for some base-point $o \in X$.

Definition 1.8. *Let (X, d_X) be a hyperbolic metric space and let H be a (Gromov-)hyperbolic group (equipped with a finite generating set and* *word metric) acting freely and properly discontinuously by isometries on* X. $i : Y \to X$ *be an orbit map. A* **Cannon-Thurston map** $\hat{i}$ *from* $\widehat{H}$ *to* $\widehat{X}$ *is a continuous extension of* i.

We shall often have occasion to refer to only the boundary value $h = \partial i$ of the continuous extension $\hat{i}$ as the Cannon-Thurston map from ∂H to ∂X. It will be clear from the context what is meant. The Cannon-Thurston map h gives a semiconjugacy of actions, i.e. a group-equivariant surjection from the boundary of H to its limit set Λ_H contained in ∂X. Hence when a Cannon-Thurston map h exists, we can regard the limit set as being parametrized by h.

1.3. **Main Results.** We now summarize the main results (see [MS13, MS17]).

1.3.1. *Strong convergence and uniform convergence.* The following is a strengthening of the corresponding result from [MS17], where it was proven under some additional hypotheses.

Theorem 4.1 *Let* Γ *be a fixed group and* $\rho_n(\Gamma) = G_n$ *be a sequence of geometrically finite Kleinian groups converging strongly to a Kleinian group* G. *Let* M_n *be the corresponding hyperbolic manifolds. Let* K *be a fixed complex with fundamental group* Γ. *Consider embeddings* $\phi_n : K \to M_n, n = 1, \cdots, \infty$ *such that the maps* ϕ_n *are homotopic to each other by uniformly bounded homotopies. Then Cannon-Thurston maps for* $\tilde{\phi}_n$ *exist and converge uniformly to the Cannon-Thurston map for* $\tilde{\phi}_\infty$.

In [MS17], we proved Theorem 4.1 for closed surface groups. The purpose here is to indicate the changes necessary to the work in [MS17] to obtain Theorem 4.1 for arbitrary (finitely generated) Kleinian groups.

The principal special cases of Theorem 4.1 are Propositions 3.6, 3.14, 3.16 which deal respectively with the following cases:

1) Γ is a closed surface group. G_n's are quasi-Fuchsian. G is a geometrically finite surface group with accidental parabolics.
2) Γ is a closed surface group. G_n's are quasi-Fuchsian. G is a simply degenerate surface group without accidental parabolics.
3) Γ is a free group. G_n's are Schottky. G is a simply degenerate free (handlebody) group without accidental parabolics.

Our next Theorem says that for the Kerckhoff-Thurston examples [KT90] of a sequence G_m of quasi-Fuchsian groups converging geometrically to a geometrically finite group G with a rank 2 cusp the Cannon-Thurston maps converge pointwise, but not uniformly, on $\partial\mathbb{H}^2$. This theorem illustrates the more general case of geometrically finite geometric limits dealt with in [MS13].

Theorem 5.3 *Fix a closed hyperbolic surface S and a simple closed geodesic σ on it. Let tw^i denote the automorphism of S given by an i-fold Dehn twist along σ. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, tw^i(S))$. Let G_∞ denote the geometric limit of the G_m's. Let S_{i-} denote the lower boundary component of the convex core of G_i, $i = 1, \cdots, \infty$ (including ∞). Let $\phi_i : S \to S_{i-}$ be such that if $0 \in \mathbb{H}^2 = \widetilde{S}$ denotes the origin of $\mathbb{H}^2$ then $\widetilde{\phi_i}(0)$ lies in a uniformly bounded neighborhood of $0 \in \mathbb{H}^3 = \widetilde{M_m}$. We also assume (using the fact that M_∞ is a geometric limit of M_m's) that S_{i-}'s converge geometrically to $S_{\infty -}$. Then the Cannon-Thurston maps for $\widetilde{\phi_i}$ converge pointwise, but not uniformly, on $\partial\mathbb{H}^2$ to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.*

In [MS17] we obtained the following surprising non-convergence result for certain examples of geometric limits constructed by Brock in [Bro01].

Theorem 6.6 *Fix a hyperbolic surface S and a separating simple closed geodesic σ on it, cutting S up into two pieces S_- and S_+. Let ϕ denote an automorphism of S such that $\phi|_{S_-}$ is the identity and $\phi|_{S_+} = \psi$ is a pseudo-Anosov diffeomorphism of S_+ preserving the boundary. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, \phi^i(S))$. Let G_∞ denote the geometric limit of the G_m's. Let S_{i0} denote the lower boundary component of the convex core of G_i, $i = 1, \cdots, \infty$ (including ∞). Let $\phi_i : S \to S_{i0}$ be such that if $0 \in \mathbb{H}^2 = \widetilde{S}$ denotes the origin of $\mathbb{H}^2$ then $\widetilde{\phi_i}(0)$ lies in a uniformly bounded neighborhood of $0 \in \mathbb{H}^3 = \widetilde{M_m}$. We also assume (using the fact that M_∞ is a geometric limit of M_m's) that S_{i0}'s converge geometrically to $S_{\infty 0}$.*

Let Σ be a complete hyperbolic structure on S_+ such that σ is homotopic to a cusp on Σ. Let Λ consist of pairs (ξ_-, ξ) of endpoints (on $\mathbb{S}^1_\infty$ of stable leaves λ of the stable lamination of ψ acting on $\widetilde{\Sigma}$. Also let $\partial\widetilde{\mathcal{H}}$ denote the collection of basepoints of lifts (to $\widetilde{\Sigma}$) of the cusp in Σ corresponding to σ. Let

$$\Xi = \{\xi : \text{There exists } \xi_- \text{ such that } (\xi_-, \xi) \in \Lambda; \xi_- \in \partial\widetilde{\mathcal{H}}\}.$$

Let $\partial\phi_i$, $i = 1 \cdots, \infty$ denote the Cannon-Thurston maps for $\widetilde{\phi_i}$. Then

1. *$\partial\phi_i(\xi)$ does not converge to $\partial\phi_\infty(\xi)$ for $\xi \in \Xi$.*
2. *$\partial\phi_i(\xi)$ converges to $\partial\phi_\infty(\xi)$ for $\xi \notin \Xi$.*

In [MO17], we identify the exact criteria that lead to the discontinuity phenomenon of Theorem 6.6 (see Theorem 6.7).

1.4. **Plan of the paper.** In Section 2, we set out the necessary and sufficient conditions that guarantee convergence (both uniform and pointwise) of a sequence of Cannon-Thurston maps. Section 3 deals with certain special cases of strong convergence of Kleinian groups and proves uniform convergence of Cannon-Thurston maps in each of these cases:

(1) Section 3.1 deals with examples where the strong limit is a geometrically finite surface group with accidental parabolics. This illustrates the corresponding theorem from [MS13].
(2) Section 3.2 summarizes the model geometry of simply degenerate ends from [Mj14a] and relegates some of the details to the Appendix.
(3) Section 3.3 uses the material of Section 3.2 to prove uniform convergence of Cannon-Thurston maps for a sequence of quasi-Fuchsian groups converging strongly to a simply degenerate group. This case is dealt with in detail in [MS17].
(4) In Section 3.4 we deduce the analogous result for Schottky groups converging strongly to a degenerate handlebody group.

Given the illustrative examples of Section 3, we are now in a position to deal with the general case of arbitrary strongly convergent sequences of finitely generated Kleinian groups. This is done in Section 4.

We then turn to the question of whether algebraic convergence of Kleinian groups implies pointwise convergence of Cannon-Thurston maps. In Section 5, we illustrate a positive answer to this question by focusing on the Kerckhoff-Thurston examples. This is a particular instance of a *geometrically finite geometric limit*. A positive answer in the general case is dealt with in [MS13]. The main ingredients of that argument are all there in the special case of the Kerckhoff-Thurston examples; hence this expository choice.

In Section 6.1, we illustrate a negative answer to the same question by discussing in detail Brock's examples of a sequence of algebraically convergent Kleinian groups, where the *geometric limit is geometrically infinite*. This case is dealt with in detail in [MS17].

Finally in Section 6.5, we discuss work with Ohshika [MO17] where the exact conditions leading to the non-convergence phenomenon in Section 6.1 are isolated.

2. Criteria for convergence of CT maps:

In this section we list a set of criteria for existence of Cannon-Thurston maps and for convergence of a sequence of Cannon-Thurston maps.

2.1. **Existence of Cannon-Thurston Maps.** Let $i : Y \to X$ be an inclusion of hyperbolic metric spaces. The following lemma says that a Cannon-Thurston map exists if and only if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if a geodesic λ in Y lies outside an N ball around y in Y then any geodesic in X joining the end-points of λ lies outside the M ball around $i(y)$ in X (or equivalently, iff sets of small visual diameter go to sets of small visual diameter.)

Lemma 2.1. [Mit98a, Mit98b] *A Cannon-Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists if and only if the following condition is satisfied:*
Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball around $y_0 \in Y$ any geodesic segment in X joining the end-points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.

2.1.1. *Cannon-Thurston maps and Relative Hyperbolicity.* We give a criterion for the existence of Cannon-Thurston maps for relatively hyperbolic spaces. We refer the reader to [Far98] and [Bow12] for details on relative hyperbolicity. Let Y, X be hyperbolic rel. $\mathcal{Y}, \mathcal{X}$ respectively. Let $Y^h = \mathcal{G}(Y, \mathcal{Y}), \widehat{Y} = \mathcal{E}(Y, \mathcal{Y})$ denote horoballifications and electrifications of Y relative to $\mathcal{Y}$. (See [DM16] for instance for these basic tools of relative hyperbolicity) Similarly, let $X^h = \mathcal{G}(X, \mathcal{X}), \widehat{X} = \mathcal{E}(X, \mathcal{X})$ denote horoballifications and electrifications of X relative to $\mathcal{X}$. Also $B^h_R(Z) \subset X^h$ denotes the R-neighborhood of Z in (X^h, d_h). Also by the Definition of $\mathcal{G}(X, \mathcal{H})$, recall that distances in (X^h, d_h) are proper functions of distances in (X, d). Similarly for Y and Y^h.

Lemma 2.2. [MP11] *A Cannon-Thurston map for $i\colon Y \to X$ exists if and only if there exists a non-negative function $M(N)$ with $M(N) \to \infty$ as $N \to \infty$ such that the following holds:*
Suppose $y_0 \in Y$, and $\hat{\lambda}$ in $\widehat{Y}$ is an electric quasigeodesic segment starting and ending outside horospheres. If $\lambda^b = \hat{\lambda} \setminus \bigcup_{K \in \mathcal{Y}} K$ lies outside an $B_N(y_0) \subset Y$, then for any electric quasigeodesic $\hat{\beta}$ joining the end points of $\hat{i}(\hat{\lambda})$ in $\widehat{X}$, $\beta^b = \hat{\beta} \setminus \bigcup_{H \in \mathcal{X}} H$ lies outside $B_{M(N)}(i(y_0)) \subset X$.

More informally:
If λ lies outside a large ball **modulo horoballs** *then so does any geodesic in X joining its endpoints.*

As a special case we have:

Lemma 2.3. *Let $\rho : \Gamma \to G$ be a weakly type preserving isomorphism of finitely generated Kleinian groups and suppose that Γ is geometrically finite. The Cannon-Thurston map from $\Lambda_\Gamma \to \Lambda_G$ exists if and only if given basepoints O_Γ, O_G, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ with the following property. Suppose that λ is a d_Γ-geodesic segment lying outside $B_\Gamma(O_\Gamma; N)$ in $\mathcal{G}\Gamma$. Then the hyperbolic geodesic in $\mathbb{H}^3$ joining the end-points of $i(j_\Gamma(\lambda))$ lies outside $B_{\mathbb{H}^3}(O_G; M(N))$ in $\mathbb{H}^3$.*

In the context of the present paper Lemmas 2.1 and 2.2 may be thought of as a criterion for the convergence of a sequence of Cannon-Thurston maps when the sequence of Kleinian groups is constant.

2.2. **Uniform convergence criterion.** We recall from [MS13, MS17] criteria for convergence of Cannon-Thurston maps. We shall discuss this in the context of Kleinian groups, but it is clear that the same notion goes through in the more general context of a sequence of actions of a fixed hyperbolic (or relatively hyperbolic) group H on a hyperbolic metric space X.

Let Γ be a fixed geometrically finite Kleinian group and suppose that $\rho_n : \Gamma \to PSL_2(\mathbb{C})$ is a sequence of discrete faithful representations converging (algebraically or strongly) to $\rho_\infty : G \to PSL_2(\mathbb{C})$. Let $G_n = \rho_n(\Gamma), n = 1, 2, \ldots, \infty$. Assuming they exist, we shall say that Cannon-Thurston maps $\hat{i}_n : \Lambda_\Gamma \to \Lambda_{G_n}$ converge uniformly (resp. pointwise) to $\hat{i}_\infty$ if they do so as maps from Λ_Γ to $\hat{\mathbb{C}} = \mathbb{S}^2_\infty$.

Definition 2.4. *Suppose that a sequence of abstractly isomorphic Kleinian groups $G_n \to G_\infty$ algebraically. Let $M_n = (\mathbf{H}^3/G_n), n = 1, \cdots, \infty$. A sequence of embeddings $\phi_n : K \to M_n, n = 1, \cdots, \infty$ for a 2-complex K is said to be a sequence of* **coarsely equivalent homotopy equivalences** *if*
a) ϕ_n is a homotopy equivalence for $n = 1, \cdots, \infty$.
b) There exists $L_0 \geq 1$, compact subsets $K_n \subset M_n$ and L_0-bi-Lipschitz homeomorphisms $\psi_{mn} : K_m \to K_n$ such that $\psi_n(K) \subset K_n \subset M_n$.
c) For all m and n, ϕ_n and $\psi_{mn} \circ \phi_m$ are homotopic to each other by uniformly bounded homotopies, i.e. lengths of tracks of homotopies are uniformly bounded independent of $n = 1, \cdots, \infty$. We shall say that ϕ_n's are homotopic to each other by **uniformly bounded homotopies** *if such a collection of ψ_{mn}'s exist.*

We shall say that $G_n \to G_\infty$ **semi-strongly** *if there exists such a sequence ϕ_n of coarsely equivalent homotopy equivalences.*

Note that strong convergence implies semi-strong convergence. Also, semi-strong convergence necessarily implies that the algebraic limit is contained in the geometric limit.

Remark 2.5. Relative Hyperbolic Generalization: Suppose G is (strongly) hyperbolic relative to a collection $\mathcal{H}$. Let (K, K_0) be a simplicial complex pair such that $\pi_1(K) = G$ and the fundamental groups of the components of K_0 correspond to the elements of $\mathcal{H}$. Coarsely equivalent homotopy equivalences can now be defined as above for pairs of maps $(\phi_n, \partial\phi_n) : (K, K_0) \to (M_n, \partial M_n)$, where the pair $(M_n, \partial M_n)$ denotes now a *pared manifold pair*, i.e. the quotient manifold *minus* cusps and its boundary.

Normalization:
Suppose first that a sequence of abstractly isomorphic finitely presented Kleinian groups $G_n \to G_\infty$ semi-strongly. Let $M_n = (\mathbf{H}^3/G_n), n = 1, \cdots, \infty$ and K, K_n, ϕ_n be as above.

We normalize as follows. Fix $0 \in \widetilde{K}$, $o \in \mathbf{H}^3$, $L > 0$ such that $\widetilde{\phi_n}(0) \in N_L(0)$ for all n. Let $\partial\phi_n$ be the Cannon-Thurston maps for $\widetilde{\phi_n}$ from $\partial\widetilde{K}$ to $\mathbb{S}^2_\infty$ (assuming they exist). We shall say that the Cannon-Thurston maps for $\widetilde{\phi_n}$ converge uniformly (resp. pointwise) if they do so as maps from $\partial\widetilde{K}$ to $\mathbb{S}^2_\infty$. We state below a criterion for the **uniform convergence of Cannon-Thurston maps.**

Proposition 2.6. *Let G be a fixed (relatively) hyperbolic group. Suppose that the following holds.*
$\rho_n : G \to PSL_2(\mathcal{C})$ is a sequence of discrete faithful representations converging to $\rho_\infty : G \to PSL_2(\mathcal{C})$ **semi-strongly**. *Let $G_n = \rho_n(G), n = 1, \cdots, \infty$ and $M_n = (\mathbf{H}^3/G_n)$.*
Semi-strong convergence ensures the existence of a compact 2-complex K with $\pi_1(K) = G$ and coarsely equivalent homotopy equivalences $\phi_n : K \to M_n, n = 1, \cdots, \infty$. Equip the universal cover $\widetilde{K}$ with a simplicial metric, by assigning length one to each edge. Let $\widetilde{\phi}_n : \widetilde{K} \to \mathbf{H}^3$ be lifts of ϕ_n and $o \in \widetilde{K}, 0 \in \mathbf{H}^3$ be fixed points. There exists L such that $\phi_n(o) \in N_L(0), n = 1, \cdots, \infty$.

Then
a) Cannon-Thurston maps for $\widetilde{\phi}_n : \widetilde{K} \to \mathbf{H}^3$ exist iff there exist non-negative functions $f(n, N)$, such that $f(n, N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball (modulo horoballs) around $o \in \widetilde{K}$ any geodesic segment in $\mathbf{H}^3$ joining the end-points of $\widetilde{\phi}_n(\lambda)$ lies outside the $f(n, N)$-ball around $0 \in \mathbf{H}^3$ (modulo horoballs).

b) Further, Cannon-Thurston maps for $\widetilde{\phi}_n$ converge uniformly to the Cannon-Thurston map for $\widetilde{\phi}_\infty$ iff there exists $g(N)$ such that
(i) $g(N) \to \infty$ as $N \to \infty$ and $f(n,N) \geq g(N), n = 1, \cdots, \infty$.
*(ii)***Visually small tails of sequences** *For all g lying outside an N-ball around $o \in \widetilde{K}$ and $m,n \geq N$, any geodesic segment in $\mathbf{H}^3$ joining $\widetilde{\phi}_m(g), \widetilde{\phi}_n(g)$ lies outside the $g(N)$-ball around $0 \in \mathbf{H}^3$.*

Note: The criterion in $b(ii)$ above ensures that orbits of points lying outside a large ball eventually have small visual diameter. Hence the Cannon Thurston maps ensured by a and $b(i)$ above are uniformly close.

The following Proposition ensures that the hypothesis on *visually small tails of sequences* follows from strong convergence and Conclusion (a) of Proposition 2.6.

Proposition 2.7. *Let G be a fixed (relatively) hyperbolic group. Suppose that $\rho_n : G \to PSL_2(\mathcal{C})$ is a sequence of discrete faithful representations converging to $\rho_\infty : G \to PSL_2(\mathcal{C})$* **strongly**. *Further suppose that there exists a non-negative functions $f(N)$, such that $f(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball (modulo horoballs) around $o \in \Gamma_G$ any geodesic segment in $\mathbf{H}^3$ joining the end-points of $\widetilde{\phi}_n(\lambda)$ lies outside the $f(N)$-ball around $0 \in \mathbf{H}^3$ (modulo horoballs).*
Then there exists $g(N)$ such that
(i) $g(N) \to \infty$ as $N \to \infty$
(ii) for all g lying outside an N-ball around $o \in \Gamma_G$ and $m,n \geq N$, any geodesic segment in $\mathbf{H}^3$ joining $\widetilde{\phi}_m(g), \widetilde{\phi}_n(g)$ lies outside the $g(N)$-ball around $0 \in \mathbf{H}^3$.

We recall a couple of notions from [MS13].

Definition 2.8. *Let Γ be a fixed finitely generated group and $\rho_n(\Gamma) = G_n$ be a sequence of Kleinian groups converging algebraically to $G_\infty = \rho_\infty(\Gamma)$.*

The sequence (ρ_n) is said to satisfy **UEP (Uniform Embedding of Points)** *if there exists a non-negative function $f(N)$, with $f(N) \to \infty$ as $N \to \infty$, such that for all $g \in \Gamma$ and $0 \in \mathbf{H}^3$, $d_\Gamma(1,g) \geq N$ implies $d_{\mathbf{H}}(\rho_n(g)(0), 0) \geq f(N)$ for all $n = 1, \ldots, \infty$. Here d_Γ represents the distance in the Cayley graph of $\mathcal{G}(\Gamma)$ and $d_{\mathbf{H}}$ denotes the distance in $\mathbf{H}^3$.*

The sequence (ρ_n) is said to satisfy **UEPP (Uniform Embedding of Pairs of Points)** *if there exists a non-negative function $f(N)$, with*

$f(N) \to \infty$ as $N \to \infty$, such that for all $g, h \in \Gamma$ and $0 \in \mathbf{H}^3$, $d_\Gamma(1, [g,h]_\Gamma) \geq N$ implies $d_{\mathbf{H}}([\rho_n(g)(0), \rho_n(h)(0)], 0) \geq f(N)$ for all $n = 1, \ldots, \infty$, where $[g,h]_\Gamma$ denotes a geodesic in Γ joining g, h and $[\rho_n(g)(0), \rho_n(h)(0)]$ denotes a geodesic in $\mathbf{H}^3$ joining $\rho_n(g)(0), \rho_n(h)(0)$.

Property UEP is used in [MS13] to give a criterion under which algebraic convergence is also geometric. Property UEPP is used to give the following alternate criterion for proving uniform convergence of Cannon-Thurston maps.

Proposition 2.9. [MS13] *Let Γ be a geometrically finite Kleinian group and let $\rho_n : \Gamma \to G_n$ be weakly type preserving isomorphisms to Kleinian groups. Suppose that ρ_n converges algebraically to a representation ρ_∞. Then if (ρ_n) satisfies UEPP, the* CT*-maps converge uniformly. If Γ is non-elementary, the converse also holds.*

2.3. **Pointwise convergence criterion.**

Proposition 2.10. *Let G be a fixed Gromov-hyperbolic group, Γ a Cayley graph with respect to a finite generating set, and ∂G its Gromov boundary. Let $\xi \in \partial G$ and $[1, \xi)$ be a geodesic ray in Γ from 1 to ξ. Suppose*
1) $\rho_n : G \to PSL_2(\mathcal{C})$ is a sequence of discrete faithful representations converging to $\rho_\infty : G \to PSL_2(\mathcal{C})$ algebraically. Let $G_n = \rho_n(G), n = 1, \cdots, \infty$ and $M_n = (\mathbf{H}^3/G_n)$.
2) Let $0 \in \mathbf{H}^3$ be a fixed point. Let ρ_n also denote an embedding of the (vertex set of) Cayley graph Γ given by $\rho_n(g) = \rho_n(g)(0)$.

Then Cannon-Thurston maps for $\tilde{\rho_n}$ converge to the Cannon-Thurston map for $\tilde{\rho_\infty}$ at ξ if
a) **Uniform embedding of points:** *there exists a non-negative function $f(N)$, such that $f(N) \to \infty$ as $N \to \infty$ and for all $g \in [1, \xi)$ lying outside $B_N(1)$, $\rho_n(g)$ lies outside the $f(N)$-ball around $0 \in \mathbf{H}^3$.*
b) **Embedding of pairs of points:** *there exists $g(N)$ such that $g(N) \to \infty$ as $N \to \infty$ such that for any geodesic subsegment $[a, b]$ of $[1, \xi) \subset \Gamma$ lying outside $B_N^G(1)$ (the N-ball about $1 \in \Gamma$), any geodesic in $\mathbf{H}^3$ joining $\rho_n(a), \rho_n(b)$ lies outside $B^3_{g(N)}(0)$, (the $g(N)$-ball about $0 \in \mathbf{H}^3$).*

Proof: Apply Propositions 2.6 and 2.9 to the hyperbolic metric space $\widetilde{K} = [1, \xi)$. □

3. Strong Convergence: Special Cases

In this section we shall illustrate the main strong convergence results of [MS13] and [MS17] by particular examples, postponing the general case to the next Section.

3.1. **Quasi Fuchsian groups converging strongly to cusped groups.** In [MS13], we showed that if a sequence G_n of geometrically finite Kleinian groups converges strongly to a geometrically finite Kleinian group, then the corresponding Cannon-Thurston maps converge uniformly. We shall illustrate this by means of a sequence of quasi-Fuchsian groups.

3.1.1. *Electro-ambient representatives.*

Definition 3.1. *Let (X, d_X) be hyperbolic and $\mathcal{H}$ be a collection of uniformly quasiconvex subsets. Let $\alpha = [a, b]$ be a geodesic in (X, d_X) and β be an electric P-quasigeodesic without backtracking joining a, b. Starting from the left of β, replace each maximal subsegment, (with end-points p, q, say) lying within some $H \in \mathcal{H}$ by a geodesic $[p, q]$. The resulting path β_q is called an* **electro-ambient representative** *in X.*

Lemma 3.2 ([Mj11] Lemma 3.7)**.** *Given δ, D, C, P as above, there exists C_3 such that the following holds:*
Let α, β be as above. Then α lies in a C_3 neighborhood of β_q.

We proved (a stronger version of) the following in [Mj16]:

Lemma 3.3. *Given C_0 there exists C_1 such that if $A \subset Y$ and $A \cap H$ are C_0-quasiconvex for all $H \in \mathcal{H}$, then (A, d_{el}) is C_1-quasiconvex in (X, d_{el}).*

Electrocuting curves
A specific example will illustrate the above construction. Let $\sigma_1, \cdots, \sigma_n$ be disjoint closed geodesics on a hyperbolic surface S. Let λ_e be an electric geodesic in $(\widetilde{S}, d_e)$ for $\widetilde{S}$ equipped with some electric metric obtained by electrocuting the collection of *mutually cobounded separated* geodesics given by lifts of $\sigma_1, \cdots, \sigma_n$ to $\widetilde{S}$. Then, each segment of λ_e between electrocuted geodesics is perpendicular to the electrocuted geodesics that it starts and ends at. We shall refer to these pieces of λ_e as **transverse segments**. Interpolating the segments of $\sigma_1, \cdots, \sigma_n$ that these segments cut off, we obtain the *electro-ambient representative* λ_q of λ_e, which is a (uniform) hyperbolic K-quasigeodesic by Lemma 3.2.

3.1.2. *Standard Estimates.* We shall need a couple of standard length estimates (see [Thu80][Chapter 2] or any standard text on hyperbolic geometry for a proof).

Lemma 3.4. *Let a, b be points on a horocycle σ and let $l(a, b)$ be the length of σ along a, b. Also let $d(a, b)$ be the hyperbolic distance between a, b. Then $l(a, b) = O(k^{d(a,b)})$ for some $k > 1$ independent of a, b.*

A similar estimate exists for equidistant arcs from geodesics.

Lemma 3.5. *Let α be a geodesic and σ an equidistant arc at distance r from it. Let $a, b \in \alpha$. Construct perpendiculars of length r to α, meeting σ at a_1, b_1 respectively. Let $l(a_1, b_1)$ be the length of σ along a_1, b_1. Also let $d(a, b)$ be the hyperbolic distance between a, b. Then $l(a, b) = O(k^r)d(a, b)$ for some $k > 1$ independent of α, a, b.*

3.1.3. *A model for convex cores.* Let G_n be a sequence of quasi-Fuchsian surface groups converging strongly to a geometrically finite surface group with accidental parabolics. We shall show that the Cannon-Thurston maps for G_n converge uniformly to the Cannon-Thurston map for G. We construct first a model for the convex cores K_n (resp. K) of the manifolds $M_n = \mathbf{H}^3/G_n$ (resp. $M = \mathbf{H}^3/G$).

Let β_ϵ be a hyperbolic isometry of $\mathbf{H}^2$ with fixed points $0, \infty$ and $A_\epsilon = \mathbf{H}^2/\beta_\epsilon$ be the quotient annulus with core geodesic σ_ϵ of length ϵ. Let $A^1_{\epsilon,E} = N_k(\sigma_\epsilon) \subset A_\epsilon$ be the closed sub-annulus with the two boundary components of length E each. Let $A_{\epsilon,E}$ be the closed 'half' sub-annulus of $A^1_{\epsilon,E}$ one of whose boundary curves is σ_ϵ and the other a boundary component of $A^1_{\epsilon,E}$. Let S be a hyperbolic surface and $\sigma_1, \cdots, \sigma_n$ be disjoint simple closed geodesics on S with lengths $E_1, \cdots, E_n$. Given $e_i \leq E_i$ ($i = 1, \cdots, n$), glue copies of A_{e_i,E_i} to S via isometries on the boundary curves of length E_i (of A_{e_i,E_i}) to σ_i. Let $S(\sigma_1, \cdots, \sigma_n, e_1, \cdots, e_n)$ denote the resulting 2-complex. If some $e_i = 0$, we replace $A_{\epsilon,E}$ by the quotient of a horodisk with boundary curve of length E. (Below we identify G with G_∞, M with M_∞ and so on. We also assume that the indexing set runs from 1 to ∞ and includes ∞ as an index.)

Then

1. Each convex core K_j of M_j has one boundary component that is uniformly bi-Lipschitz to some fixed hyperbolic surface S.
2. There exist disjoint simple closed geodesics $\sigma_1, \cdots, \sigma_n$ on S such that there are uniformly bi-Lipschitz embeddings ϕ_j of $S(\sigma_1, \cdots, \sigma_n, e^j_1, \cdots, e^j_n)$ in K_j, where $e^j_1, \cdots, e^j_n$ denote the lengths of the geodesic representatives of $\phi_j(\sigma_1), \cdots, \phi_j(\sigma_n)$ in K_j.
3. There exist $L > 0$ and deformation retracts $p_j : K_j \to \phi_j(S(\sigma_1, \cdots, \sigma_n, e^j_1, \cdots, e^j_n))$ such that $p_j^{-1}(x)$ has diameter less than L for all $x \in S(\sigma_1, \cdots, \sigma_n, e^j_1, \cdots, e^j_n)$ and all $j = 1, \cdots, \infty$.

Let $\tilde{S}(\sigma_1, \cdots, \sigma_n, e^j_1, \cdots, e^j_n)$ be the universal cover of $S(\sigma_1, \cdots, \sigma_n, e^j_1, \cdots, e^j_n)$ and let $(A^j_1, \cdots, A^j_n)$ be the annuli glued to S to obtain $S(\sigma_1, \cdots, \sigma_n, e^j_1, \cdots, e^j_n)$. Then for all j, $\tilde{S}(\sigma_1, \cdots, \sigma_n, e^j_1, \cdots, e^j_n)$ with lifts of A^j_i's electrocuted (for $i = 1 \cdots n$) is (uniformly) quasi-isometric to $\widetilde{S}$ with the lifts of the σ_i's electrocuted (for $i = 1 \cdots n$). In fact the natural embedding of $\widetilde{S}$

into $\tilde{S}(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j)$ induces an isometry of the electrocuted (pseudo) metrics. Let d_e be the electric metric on $\widetilde{S}$ with the lifts of the σ_i's electrocuted (for $i=1\cdots n$) and let d_e^j be the electric metric on $\tilde{S}(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j)$ with lifts of A_i^j's electrocuted (for $i=1\cdots n$). Hence, given $a,b\in\widetilde{S}$, the *transverse segments* for the electric geodesic joining a,b in $(\widetilde{S},d_e)$ are the same as the *transverse segments* for the electric geodesic joining a,b in $(\tilde{S}(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j),d_e^j)$ (via the natural embedding of $\widetilde{S}$ in $\tilde{S}(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j)$. Hence, given $a,b\in\widetilde{S}$, the *transverse segments* for the electro-ambient geodesic joining a,b in $(\widetilde{S}$ are the same as the *transverse segments* for the electro-ambient geodesic joining a,b in $\tilde{S}(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j)$.

3.1.4. *Fulfilling uniform convergence criteria.* We recall from Propositions 2.6 and 2.9 the criteria for uniform convergence of Cannon-Thurston maps. We shall state and check the criteria in Propositions 2.6 first.

(1) $\rho_n:G\rightarrow PSL_2(\mathcal{C})$ is a sequence of discrete faithful representations converging to $\rho_\infty:G\rightarrow PSL_2(\mathcal{C})$ strongly. Let $G_n=\rho_n(G),n=1,\cdots,\infty$ and $M_n=(\mathbf{H}^3/G_n)$.
In the present situation, $G=\pi_1(S)$. G_n is a sequence of quasi-Fuchsian surface groups converging strongly to a geometrically finite surface group G_∞.
(2) There exists a compact 2-complex L with $\pi_1(L)=G$ and homotopy equivalences $\phi_n:L\rightarrow M_n,n=1,\cdots,\infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies.
Here $L=S$ embedded naturally in $S(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j)$. $S(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j)$ in turn is embedded in K_j.
(3) Let $\widetilde{\phi_n}:\widetilde{K}\rightarrow\mathbf{H}^3$ be lifts of ϕ_n and $o\in\widetilde{K},0\in\mathbf{H}^3$ be fixed points. There exists L such that $\phi_n(o)\in N_L(0),n=1,\cdots,\infty$. This is a normalization condition and is satisfied by choosing the lift of a base point on $S\subset S(\sigma_1,\cdots,\sigma_n,e_1^j,\cdots,e_n^j)$ to lie in a uniformly bounded neighborhood of $0\in\mathbf{H}^3$.

The crucial criteria (4) and (5) below from Proposition 2.9 will require separate checking:
4) Cannon-Thurston maps for $\widetilde{\phi_n}:\widetilde{K}\rightarrow\mathbf{H}^3$ exist iff there exist non-negative functions $f(n,N)$, such that $f(n,N)\rightarrow\infty$ as $N\rightarrow\infty$ and for all geodesic segments λ lying outside an N-ball around $o\in\tilde{K}$ any geodesic segment in $\mathbf{H}^3$ joining the end-points of $\tilde{\phi}_n(\lambda)$ lies outside the $f(n,N)$-ball around $0\in\mathbf{H}^3$.

5) Cannon-Thurston maps for $\widetilde{\phi_n}$ converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$ iff there exists $g(N)$ such that $g(N) \to \infty$ as $N \to \infty$ and $f(n,N) \geq g(N), n = 1, \cdots, \infty$.

Checking Criteria 4 and 5

Let λ be a geodesic in $\tilde{S}$ lying outside an N-ball around $o \in \tilde{S}$. Let λ_{ea} be its electro-ambient realization. Then λ_{ea} is a (hyperbolic) quasi-geodesic in $\tilde{S}$ by Lemma 3.2. Hence, there exists C_0 (independent of λ such that λ_{ea} lies outside the $(N - C_0)$ ball about $o \in \tilde{S}$. Also, by Lemmas 3.4 and 3.5, the metric on $\tilde{S}$ is at most exponentially distorted with respect to the metrics on $\tilde{S}(\sigma_1, \cdots, \sigma_n, e_1^j, \cdots, e_n^j)$. Hence there exists $k > 1, C_1 > 0$ by Lemmas 3.4 and 3.5 and Criterion (3) above, such that λ_{ea} lies outside the $(ln_k(N - C_0) - C_1)$ ball about $0 \in \mathbf{H}^3$.

Next, let λ_{ea}^j be the electro-ambient geodesic joining the end-points of λ (and hence λ_{ea}) in $\tilde{S}(\sigma_1, \cdots, \sigma_n, e_1^j, \cdots, e_n^j)$, where the electric metric is obtained by electrocuting the lifts of the annuli A_{e_i,E_i} (which were attached to S to obtain $S(\sigma_1, \cdots, \sigma_n, e_1^j, \cdots, e_n^j)$). Again, by Lemma 3.2, λ_{ea}^j is a (uniform) quasigeodesic in $\tilde{S}(\sigma_1, \cdots, \sigma_n, e_1^j, \cdots, e_n^j)$ and hence (by the properties of K_j) in $\widetilde{K_j}$.

Finally, any path joining $0 \in \mathbf{H}^3$ to λ_{ea}^j must cross λ_{ea} as each segment of λ_{ea} lying along a lift of some σ_i separates some annulus lift from $0 \in \mathbf{H}^3$. Hence, $d_j(0, \lambda_{ea}^j) \geq d_j(0, \lambda_{ea})$, where d_j denotes metric on $\tilde{S}(\sigma_1, \cdots, \sigma_n, e_1^j, \cdots, e_n^j) \geq (ln_k(N - C_0) - C_1)$. Thus we conclude

Proposition 3.6. *Let G_n be a sequence of quasi-Fuchsian surface groups converging strongly to a geometrically finite surface group with accidental parabolics. Let M_n be the corresponding hyperbolic manifolds. Consider homotopy equivalences $\phi_n : S \to M_n, n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies. Then Cannon-Thurston maps for $\tilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\tilde{\phi_\infty}$.*

3.2. **Models of Ends.** We recall some of the material from [Mj14a, Mj14b, Mj10] that we shall need in the next subsection. Further details may be found in the Appendix. For convenience of exposition, we shall deal primarily with the case of Kleinian groups without parabolics. Modifications necessary in the presence of parabolics will be indicated from time to time. The main theorem of [Mj14a] is the following.

Theorem 3.7. [Mj14a] *Cannon-Thurston maps exist for hyperbolic 3-manifolds corresponding to simply or doubly degenerate surface Kleinian groups.*

Theorem 3.7 is generalized in [Mj10] to arbitrary finitely generated Kleinian groups. The structure of point-preimages is described in [Mj14b].

Theorem 3.8. *Let G be a finitely generated Kleinian group. Let $i : \Gamma_G \to \mathbb{H}^3$ be the natural identification of a Cayley graph of G with the orbit of a point in $\mathbb{H}^3$. Let $M = \mathbb{H}^3/G$ and assume that each degenerate end E of M admits a bi-Lipschitz Minsky model. Then i extends continuously to a map $\hat{i} : \widehat{\Gamma_G} \to \mathbb{D}^3$, where $\widehat{\Gamma_G}$ denotes the (relative) hyperbolic compactification of Γ_G. Let ∂i denote the restriction of $\hat{i}$ to the boundary $\partial\Gamma_G$ of Γ_G.*

Let E be a degenerate end of $N^h = \mathbb{H}^3/G$ and $\widetilde{E}$ a lift of E to $\widetilde{N^h}$ and let M_{gf} be an augmented Scott core of N^h. Then the ending lamination $\mathcal{L}_E$ for the end E lifts to a lamination on $\widetilde{M_{gf}} \cap \widetilde{E}$. Each such lift $\mathcal{L}$ of the ending lamination of a degenerate end defines a relation $\mathcal{R}_\mathcal{L}$ on the (Gromov) hyperbolic boundary $\partial\widetilde{M_{gf}}$ (equal to the relative hyperbolic boundary $\partial\Gamma_G$ of Γ_G), given by $a\mathcal{R}_\mathcal{L}b$ if and only if a, b are end-points of a leaf of $\mathcal{L}$. Let $\{\mathcal{R}_i\}_i$ be the entire collection of relations on $\partial\widetilde{M_{gf}}$ obtained this way. Let $\mathcal{R}$ be the transitive closure of the union $\bigcup_i \mathcal{R}_i$. Then $\partial i(a) = \partial i(b)$ if and only if $a\mathcal{R}b$.

We also proved the following stronger consequence (Theorem 3.10 below) of the *proof* of Theorem 3.7 in [Mj14a], which will be useful in dealing with sequences of manifolds. The notion of a Scott core we shall use here is slightly different from the one existing in literature and is usually referred to as the augmented Scott core (where cusps are adjoined to the compact core).

Definition 3.9. *A Scott core K of a complete hyperbolic 3-manifold M_0 with ends $E_1, \cdots, E_N$ is the convex core K of a geometrically finite manifold N and a κ-bi-Lipschitz embedding $\phi : K \to M_0$ such that $M_0 \setminus \phi(K) = \bigcup_i E_i$.*

We shall identify K with $\phi(K) \subset M_0$. We shall find it convenient to fix a common Scott core K for a collection of manifolds (denoted by a generic M for convenience).

Theorem 3.10. [Mj14a]*[Corollary 6.13] Let K be a Scott core of M_0 with ends $E_1, \cdots, E_N$ all of which are simply degenerate. Let $S_1, \cdots, S_N$ be the boundary components of K.*

Then for all compact $E_i^1 \subset E_i$ homeomorphic to $S_i \times [0, 1]$ with $E_i^1 \cap K = S_i$, $i = 1 \cdots N$, there exist
1) Compact $E_i^2 \subset E_i$ homeomorphic to $S_i \times [0, 1]$ with $E_i^2 \cap K = S_i$, $i = 1 \cdots N$ and $E_i^1 \subset E_i^2$

2) and a function $N_1(N) \to \infty$ as $N \to \infty$ such that the following holds.

Let $M^{(j)} = K \bigcup_i E_i^j$, $j = 1, 2$ glued along the boundary components $S_1, \cdots, S_N$.
Let M be any hyperbolic 3-manifold such that
a) there exists a $2-$ bi-Lipschitz embedding $i : M^{(2)} \to M$, and
b) $M \setminus i(M^{(2)})$ consists of ends (geometrically finite or infinite).
Then for all geodesic segments λ lying outside an N-ball around $o \in \widetilde{K}$ (the lift of K to the universal cover) and any geodesic segment μ in $\widetilde{M}$ joining the end-points of λ, $\mu \cap \widetilde{i(M^{(1)})}$ lies outside the $N_1(N)$-ball around $o \in \widetilde{M}$.

Remark 3.11. Theorem 3.10 above is a paraphrasing of Corollary 6.13 of [Mj14a]. Further details on the structure of ends that goes into its proof are summarized in the Appendix.

Remark 3.12. Modification in the presence of punctures: The analogous statement in the presence of parabolics is a small modification of Proposition 3.10 above. The hypothesis is exactly the same. The conclusion is modified as follows:
For all geodesic segments λ lying outside an N-ball around $o \in \widetilde{K}$ modulo horoballs, and any geodesic segment μ in $\widetilde{M}$ joining the end-points of λ, $\mu \cap \widetilde{i(M^{(1)})}$ lies outside the $N_1(N)$-ball around $o \in \widetilde{M}$ modulo horoballs.

Remark 3.13. Modification in the presence of geometrically finite ends: If some end(s) E_k of M_0 are geometrically finite in Proposition 3.10 or Remark 3.12 then the same conclusion holds provided
a) the index i runs over the ends of M_0 that are degenerate.
b) the ends of M corresponding to the geometrically finite ends of M_0 are required to be **uniformly** bi-Lipschitz to them.

3.3. **Quasi Fuchsian groups converging to simply degenerate groups.** In this subsection we shall illustrate the strong convergence result of [MS17] by looking at a sequence G_n of quasi-Fuchsian surface groups converging strongly to a simply degenerate surface group (without accidental parabolics). Let M_n be the corresponding hyperbolic manifolds. Consider homotopy equivalences $\phi_n : S \to M_n, n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies. We shall show in this subsection that the Cannon-Thurston maps for $\tilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\tilde{\phi_\infty}$.

Since G_n converges to $G = G_\infty$ strongly, we can fix a base surface $\phi_n : S \subset M_n$ for all $n = 1 \cdots \infty$. Also, by geometric convergence of M_n to $M = M_\infty$, we may identify all the copies of $\phi_n(S) \subset M_n$ with each other. Choose the lift of a base point on $S \subset M_n$ to lie in a uniformly bounded neighborhood of $0 \in \mathbf{H}^3$. Thus criteria (1)-(3) of Proposition 2.6 (by choosing the complex K to be S) are satisfied.

We need to check the following criteria of Proposition 2.9:
4) For $\tilde{\phi_n} : \tilde{S} \to \mathbf{H}^3$, there exist non-negative functions $f(n, N)$, such that $f(n, N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball around $o \in \tilde{S}$ any geodesic segment in $\mathbf{H}^3$ joining the end-points of $\tilde{\phi_n}(\lambda)$ lies outside the $f(n, N)$-ball around $0 \in \mathbf{H}^3$. For $n \in \mathbb{N}$, this follows from the fact that the groups G_n are quasi-Fuchsian. For $n = \infty$, this is the content of Theorem 3.7. Hence Cannon-Thurston maps exist for each $n = 1, \cdots, \infty$.
5) To show that Cannon-Thurston maps for $\tilde{\phi_n}$ converge uniformly to the Cannon-Thurston map for $\tilde{\phi_\infty}$ it remains to show that there exists $g(N)$ such that $g(N) \to \infty$ as $N \to \infty$ and $f(n, N) \geq g(N), n = 1, \cdots, \infty$. We formulate this criterion somewhat differently.
5′) Given $M \geq 0$, there exists $N \geq 0$ such that if λ lies outside an N-ball around $o \in \tilde{S}$, such that any geodesic segment in $\mathbf{H}^3$ joining the end-points of $\tilde{\phi_n}(\lambda)$ lies outside the M-ball around $0 \in \mathbf{H}^3$ for all $= 1, \cdots, \infty$.

The rest of this subsection is devoted to proving criterion 5′. Start with some $L > 0$.

By Theorem 3.7, there exists ϵ_1 such that each $M_n, n = 1, \cdots, \infty$ has a sequence $S_{n,m}$ of split level surfaces satisfying the conditions of Definition-Theorem 6.8 and, in particular, (condition 4 of Definition-Theorem 6.8 in the Appendix) the distance between $S_{n,i}$ and $S_{n,i+1}$ is greater than ϵ_1 for all n, i. Also by Definition-Theorem 6.9 in the Appendix, there exists D_0 such that each split block is uniformly D_0 the first graph-quasiconvex.

Let $\nu > \frac{M}{\epsilon_1} + 2D_0$ be a positive integer. By geometric convergence of M_n to M, there exists $L_0 \in \mathbb{N}$ such that for all $n \geq L_0$, the first ν split blocks of M_n are identical. Also assume that the first D_0 of these split blocks of M_n are $B_{-D_0}, \cdots, B_{-1}$ and $S = \phi_n(S)$ is the common boundary of B_{-1} and B_0.
Since each G_n is quasi-Fuchsian, there exists C_0 such that each $\phi_n(\tilde{S})$ is C_0 quasi-convex in $\tilde{M_n} = \mathbf{H}^3$ for $n \leq L_0$. Let $N_1 = L + C_0$. Then for all $n \leq L_0$, if λ lies outside an N_1-ball around $o \in \tilde{S}$, then any geodesic

segment in $\mathbf{H}^3$ joining the end-points of $\tilde{\phi}_n(\lambda)$ lies outside the M-ball around $0 \in \mathbf{H}^3$ for all $= 1, \cdots, L_0$.

Let $m = \nu - 2D_0$. As in Proposition 3.10, let $\mathcal{B}_m = \bigcup_{-D_0}^{m+D_0} B_i$ and let $\mathcal{B}_m = \bigcup_0^m B_i \subset \mathcal{B}_n^1$. Then by Proposition 3.10, for all $L \geq 0$ there exists $N_1 \geq 0$ such that the following holds.
For all geodesic segments λ lying outside an N_1-ball around $o \in \tilde{S}$ and any geodesic segment μ in $\tilde{M}_n$ joining the end-points of λ, $\mu \cap \widetilde{\mathcal{B}_n}$ lies outside the L-ball around $o \in \tilde{M}_n$. Since the intersection of μ with $\widetilde{M_n} \setminus \widetilde{\mathcal{B}_n}$ lies at a distance of at least $m\epsilon_1 (\geq M)$ from $\tilde{S}$ and hence o, we see that criterion (5) is satisfied. We have thus shown the following.

Proposition 3.14. *Let G_n be a sequence of quasi-Fuchsian surface groups converging strongly to a simply degenerate surface group without accidental parabolics. Let M_n be the corresponding hyperbolic manifolds. Consider homotopy equivalences $\phi_n : S \to M_n, n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies. Then Cannon-Thurston maps for $\tilde{\phi}_n$ exist and converge uniformly to the Cannon-Thurston map for $\tilde{\phi}_\infty$.*

3.4. **Handlebodies and Schottky groups.** Let G_n be a sequence of Schottky groups converging strongly to a simply degenerate handlebody (free) group without accidental parabolics. Let M_n be the corresponding hyperbolic manifolds. From [Mj10] we obtain the fact that the end of M_∞ has split geometry. The proof of Theorem 6.12 of [Mj14a] gives the following analogue of Proposition 3.10. Uniform graph quasiconvexity of split components

Proposition 3.15. *Let D be a positive integer. Let $B_{-D}, \cdots, B_0, \cdots, B_n, \cdots, B_{n+D}$ be a collection of split blocks, such that B_{-D} is topologically a solid handlebody and let $\mathcal{B}_n^1$ be the union of these blocks glued along the common boundary split surfaces (i.e. B_{i-1} is glued to B_i along S_i). We assume that this gluing can be done consistently (i.e. the Margulis tubes are compatible). Let $\mathcal{B}_n = \bigcup_1^n B_i \subset \mathcal{B}_n^1$. Let M be a manifold homeomorphic to a handlebody such that its end has split geometry (not necessarily simply or doubly degenerate, i.e. we allow M to have finitely many split blocks). Also suppose that each split component is D-graph quasiconvex and $\mathcal{B}_n^1 \subset M$. Then for all $L \geq 0$ there exists $N \geq 0$ such that the following holds.*
For all geodesic segments λ lying outside an N-ball around $o \in \tilde{B}_{-D}$ and any geodesic segment μ in $\tilde{M}$ joining the end-points of λ, $\mu \cap \widetilde{\mathcal{B}_n}$ lies outside the L-ball around $o \in \tilde{M}$.

To prove that Cannon-Thurston maps converge strongly in this situation, the proof is a modification of Proposition 3.14.

First, there exists ϵ_1 such that successive split level surfaces are ϵ_1-separated. Also, for any finite collection of G_n's $\tilde{B_{-D}}$'s are uniformly quasiconvex in $\tilde{M_n}$.

Second, fix ν to be a large positive integer. By geometric convergence of M_n to M, there exists $L_0 \in \mathbb{N}$ such that for all $n \geq L_0$, the first ν split blocks of M_n are identical.

Third by Proposition 3.15, for all $L \geq 0$ there exists $N_1 \geq 0$ such that the following holds.
For all geodesic segments λ lying outside an N_1-ball around $o \in \tilde{B_{-D}}$ and any geodesic segment μ in $\tilde{M_n}$ joining the end-points of λ, $\mu \cap \widetilde{\mathcal{B}_n}$ lies outside the L-ball around $o \in \tilde{M_n}$. Since the intersection of μ with $\widetilde{M_n} \setminus \widetilde{\mathcal{B}_n}$ lies at a large distance from $\tilde{B_{-D}}$ and hence o, we see that criterion (5) is satisfied. We have thus shown the following.

Proposition 3.16. *Let G_n be a sequence of Schottky groups converging strongly to a simply free (handlebody) group without accidental parabolics. Let M_n be the corresponding hyperbolic manifolds. Let B_{-D} be a fixed handlebody. Consider embeddings $\phi_n : B_{-D} \to M_n, n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies. Then Cannon-Thurston maps for $\tilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\tilde{\phi_\infty}$.*

4. Strong Convergence: General Case

In this section we show that strongly convergent Kleinian groups give rise to uniformly convergent Cannon-Thurston maps. The ideas are exactly as in the previous section; however some technical complications occur.

4.1. **Strong Convergence with Geometrically Infinite Limit.**

Let G_m be a sequence of finitely generated Kleinian groups converging strongly to a Kleinian group G. Further, assume that the limit is strongly type-preserving. Let M_n be the corresponding hyperbolic manifolds converging strongly to M. Also, let K be the convex core of a geometrically finite hyperbolic 3-manifold admitting proper homotopy equivalences $\phi_m : K \to M_m, n = 1, \cdots, \infty$ such that the maps ϕ_m are homotopic to each other by uniformly bounded homotopies. We shall now show that the Cannon-Thurston maps for $\widetilde{\phi_m}$ exist and converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.

Theorem 4.1. *Let Γ be a fixed group and $\rho_n(\Gamma) = G_n$ be a sequence of finitely generated Kleinian groups converging strongly to a Kleinian group G. Further assume that the convergence is strictly type-preserving. Let M_n be the corresponding hyperbolic manifolds converging strongly to M. Let K be a fixed geometrically finite manifold with fundamental group Γ. Consider strictly type-preserving embeddings $\phi_n : K \to M_n, n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies. Then Cannon-Thurston maps for $\widetilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.*

Proof. Since G_m converges to $G = G_\infty$, we can fix a base codimension zero submanifold $\phi_m : K \subset M_m$ for all $m = 1 \cdots \infty$. Also, by geometric convergence of M_m to $M = M_\infty$, we may identify all the copies of $\phi_m(K) \subset M_m$ with each other. Choose the lift of a base point on $K \subset M_m$ to lie in a uniformly bounded neighborhood of $0 \in \mathbf{H}^3$. Thus the hypotheses of Proposition 2.6 (by choosing the complex K of Proposition 2.6 to be the codimension zero submanifold K) are satisfied.

We need to check:

4) For $\widetilde{\phi_m} : \widetilde{K} \to \mathbf{H}^3$, there exist non-negative functions $f(m, N)$, such that $f(m, N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball around $o \in \widetilde{K}$ any geodesic segment in $\mathbf{H}^3$ joining the end-points of $\widetilde{\phi_m}(\lambda)$ lies outside the $f(m, N)$-ball around $0 \in \mathbf{H}^3$. For each $m = 1, 2, \cdots$ or ∞, this is the content of Theorems 3.7 and 3.8. Hence Cannon-Thurston maps exist for each $m = 1, \cdots, \infty$.

5) To show that Cannon-Thurston maps for $\widetilde{\phi_m}$ converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$ it remains to show that there exists $g(N)$ such that $g(N) \to \infty$ as $N \to \infty$ and $f(m, N) \geq g(N), n = 1, \cdots, \infty$.

The rest of the discussion is devoted to proving criterion 5. We shall, as before, prove the following equivalent statement:

5′) Given $L \geq 0$, there exists $N \geq 0$ such that if λ lies outside an N-ball around $o \in \widetilde{K}$, then any geodesic segment in $\mathbf{H}^3$ joining the end-points of $\widetilde{\phi_m}(\lambda)$ lies outside the L-ball around $o \in \mathbf{H}^3$ for all $= 1, \cdots, \infty$.

First assume that there are no parabolics in M. The modifications in the presence of parabolics will be indicated later.

Step 1) Let E_i, $i = 1 \cdots N$ be the ends of M. Given L, there exist compact submanifolds $E_i^1 \subset E_i$ homeomorphic to $S_i \times [0, 1]$ such that $d_M((E_i \setminus E_i^1), K) \geq 2L$.

Step 2) By Theorem 3.10 there exist compact submanifolds E_i^2 with $E_i^1 \subset E_i^2$ homeomorphic to $S_i \times [0,1]$ and an integer N_1 such that the following holds.
Let $M^{(j)} = K \bigcup_i E_i^j$, $j = 1,2$ glued along the boundary components $S_1, \cdots, S_N$.
Let W be any hyperbolic 3-manifold such that
a) there exists a $2-$ bi-Lipschitz embedding $i : M^{(2)} \to W$, and
b) $W \setminus i(M^{(2)})$ consists of ends (geometrically finite or infinite).
For all geodesic segments λ lying outside an N_1-ball around $o \in \widetilde{K}$ and any geodesic segment μ in $\widetilde{W}$ joining the end-points of λ, $\mu \cap \widetilde{i(M^{(1)})}$ lies outside the L-ball around $o \in \widetilde{W}$.

Hence by Step 1, and since i is 2-bi-Lipschitz, μ lies outside the L-ball around $o \in \widetilde{W}$.

Step 3) By strong convergence, there exists a natural number $C_0 = C_0(L)$ such that for all $k \geq C_0$, there exists a $2-$ bi-Lipschitz embedding $i : M^{(2)} \to M_k$ and hence by Step 2, $\mu(= \mu(k))$ lies outside the L-ball around $o \in \widetilde{M_k}$ whenever λ lies outside an N_1-ball around $o \in \widetilde{K}$.

Step 4) By Theorem 3.8, there exist $N_1', \cdots, N_{C_0}'$ such that if λ lies outside an N_k'-ball around $o \in \widetilde{K}$, then $\mu(= \mu(k))$ lies outside the L-ball around $o \in \widetilde{M_k}$ for $k = 1 \cdots C_0$.

Choosing $N = max(N_1, N_1', \cdots, N_{C_0}')$ we are through. $\square$

Modifications in the presence of parabolics:
We briefly explain the modifications in the special case of quasi-Fuchsian *punctured* surface groups converging strongly to a simply degenerate punctured surface group without accidental parabolics.

For a geodesic ray $[0, \xi)$ in the universal cover $\widetilde{S}$ of the punctured surface S, $\rho_n(\xi)$ converges to $\rho_\infty(\xi)$, where (abusing notation slightly) ρ_n (resp. ρ_∞) represents the Cannon-Thurston maps corresponding to the representations ρ_n (resp. ρ_∞). Let $\rho_n([0,\xi))$ (resp. $\rho_\infty([0,\xi)))$ represent the geodesics in $\mathbb{H}^3$ joining 0 to $\rho_n(\xi)$ (resp. $\rho_\infty(\xi)$). Then $\rho_n([0,\xi))$ converges to $\rho_\infty([0,\xi))$ in the compact open topology on $\mathbb{H}^3$. Now, the universal covers $\widetilde{CC(M_n)}$ (resp. $\widetilde{CC(M_\infty)}$) of the convex cores $CC(M_n)$ (resp. $CC(M_\infty)$) of the manifolds M_n (resp. M_∞) corresponding to ρ_n (resp. ρ_∞) are strongly hyperbolic relative to the equivariant collection of horoballs corresponding to lifts of the cusps [Far98]. Let $\rho_n([0,\xi))_{ea}$ (resp. $\rho_\infty([0,\xi))_{ea}$) represent the electro-ambient quasi-geodesics corresponding to $\rho_n([0,\xi))$ (resp. $\rho_\infty([0,\xi)))$ in the appropriate spaces. Then since horoballs in $\widetilde{CC(M_n)}$ converge to those in $\widetilde{CC(M_\infty)}$, we have the following.

Corollary 4.2. *Let G_n be a sequence of quasi-Fuchsian punctured surface groups converging strongly to a simply degenerate punctured surface group without accidental parabolics. Let M_n be the corresponding hyperbolic manifolds. Consider a geodesic ray $[0, \xi)$ in the universal cover $\widetilde{S}$ of the punctured surface S. Let $\rho_n([0, \xi))_{ea}$ and $\rho_\infty([0, \xi))_{ea}$ be as above. Then $\rho_n([0, \xi))_{ea}$ converges to $\rho_\infty([0, \xi))_{ea}$.*

4.2. **From Strictly Type-Preserving to Weakly Type-Preserving.** For a given Kleinian group Γ, a representation $\rho : \Gamma \to PSL_2(\mathbb{C})$ is said to be *weakly type preserving* if the image of every parabolic element of Γ is parabolic. The hypothesis that the convergence is strictly type-preserving in Theorem 4.1 can be removed by combining Theorem 4.1 with a Theorem from [MS13] that allows new parabolics to develop for geometrically finite groups converging strongly to a geometrically finite limit. In this subsection we sketch the argument.

As in Theorem 4.1, let Γ be a fixed group and $\rho_n(\Gamma) = G_n$ be a sequence of finitely generated Kleinian groups converging strongly to a Kleinian group $G = \rho_\infty(\Gamma)$. Suppose that the convergence is *not* strictly type-preserving. Then there exists a finite number $\gamma_1, \cdots, \gamma_k$ of primitive elements in Γ that are not parabolic in infinitely many G_n but are parabolic in G. Without loss of generality we assume that $\gamma_1, \cdots, \gamma_k$ are not parabolic in any G_n (else we pass to subsequences with smaller values of k and prove uniform convergence of Cannon-Thurston maps for these). We shall refer to the γ_i's as **limiting accidental parabolics**. The proof in this general situation boils down to investigating geometric limits of ends. Let $M = \mathbb{H}^3/G$ and $M_n = \mathbb{H}^3/G_n$ as before. All the manifolds M_n have a common Scott core K.

Recall that $\rho_n \to \rho_\infty$ strongly. By [Min10] and [BCM12], the geometrically infinite ends of M are determined by their ending laminations. Thus the limiting accidental parabolics $\gamma_1, \cdots, \gamma_k$ can be identified with simple closed curves lying on the boundary of K. For ease of exposition, let $k = 1$ and $\gamma_1 = \gamma$.

Let $E_n \subset M_n$ and $E \subset M$ be the end containing γ. Two cases arise:
a) E_n's are geometrically finite and they converge geometrically to a geometrically finite $E \subset M$ containing a limiting accidental parabolic corresponding to γ.
b) E_n's converge to E where E is geometrically infinite and contains a limiting accidental parabolic. In this case E minus the rank one cusp corresponding to γ must consist of one or two degenerate ends according as γ is non-separating or separating.

To prevent tedious book-keeping, we shall focus on two representative special cases of surface Kleinian groups to illustrate what needs to be done.

Case 1: $\Gamma = \pi_1(S)$ for a closed surface S. $\rho_n(\Gamma) = G_n$ is a sequence of quasi-Fuchsian Kleinian groups converging to G where $M = \mathbb{H}^3/G$ has one simply degenerate end and one geometrically finite end containing a single limiting accidental parabolic γ.
Case 2: $\Gamma = \pi_1(S)$ for a closed surface S. $\rho_n(\Gamma) = G_n$ is a sequence of quasi-Fuchsian Kleinian groups converging to G where $M = \mathbb{H}^3/G$ has one geometrically finite end without parabolics. The other end E of M contains a single rank one cusp (corresponding to a limiting accidental parabolic γ)

Case 1:

Abusing notation slightly let M_n (resp. M) denote the convex core of $M = \mathbb{H}^3/G_n$ (resp. $M = \mathbb{H}^3/G$). Let $P \subset M$ denote the limiting rank one cusp corresponding to γ and let $P_n \subset M_n$ denote a solid torus neighborhood of the geodesic corresponding to the group element $\rho_n(\gamma)$ in M_n. We may assume that $P_n \cap \partial M_n$ is an annulus whose core curve represents the element γ. Then $M_n \setminus P_n$ converges geometrically to $M \setminus P$. Also $M_n \setminus P_n$ (resp. $M \setminus P$) are uniformly bi-Lipschitz homeomorphic to quotients of $\mathbb{H}^3$ by quasiFuchsian Kleinian groups $\rho'_n(\Gamma)$ (resp. a quotient of $\mathbb{H}^3$ by a simply degenerate Kleinian groups $\rho'(\Gamma)$) such that ρ'_n converges strongly to ρ' and $\rho'(\Gamma)$ has no accidental parabolics. It follows from Theorem 4.1 that the Cannon-Thurston maps for ρ'_n converge uniformly to the Cannon-Thurston map for ρ'.

In fact Theorem 3.10 shows that the property UEPP holds for the collection ρ'_n, i.e. there exists a non-negative function $f(N)$, with $f(N) \to \infty$ as $N \to \infty$, such that for all $g, h \in \Gamma$ and $0 \in \mathbf{H}^3$, $d_\Gamma(1, [g,h]_\Gamma) \geq N$ implies $d_{\mathbf{H}}([\rho'_n(g)(0), \rho'_n(h)(0)], 0) \geq f(N)$ for all $n = 1, \ldots, \infty$, where $[g,h]_\Gamma$ denotes a geodesic in Γ joining g, h and $[\rho'_n(g)(0), \rho'_n(h)(0)]$ denotes a geodesic in $\mathbf{H}^3$ joining $\rho'_n(g)(0), \rho'_n(h)(0)$.

Identify $\Gamma = \pi_1(S) = \pi_1(\partial(M \setminus P))$ with the fundamental group of the component of $\partial(M_n \setminus P_n)$ intersecting P_n non-trivially. In fact we can identify the component of $\partial(M_n \setminus P_n)$ intersecting P_n non-trivially with S (by uniformly bi-Lipschitz homeomorphisms).

Now electrocute $P_n \subset M_n$ and let $(\widetilde{M_n}, d_{n,el})$ be the lifted metric to the universal cover. Similarly, electrocute $P \subset M$ and let $(\widetilde{M}, d_{el})$ be the lifted metric to the universal cover. Also let $(\widetilde{S}, d_e)$ denote the electric metric on $\widetilde{S}$ with the lifts of the geodesic corresponding to γ electrocuted.

Note that electroambient quasigeodesics in each of the spaces $(\widetilde{M_n}, d_{n,el})$, $(\widetilde{M}, d_{el})$, $(\widetilde{S}, d_e)$ are uniform quasigeodesics in $\widetilde{M_n}$, $\widetilde{M}$, $\widetilde{S}$ respectively by Lemma 3.2. Hence
1) there exists $C \geq 0$ such that for all $g, h \in \Gamma$ and $o \in \widetilde{S}$, $d_\Gamma(1, [g,h]_\Gamma) \geq N$ implies $d_{\widetilde{S}}(o, [g,h]_{ea}) \geq N - C$, where $[g,h]_{ea}$ denotes the electro-ambient quasigeodesic joining $g(o), h(o) \in \widetilde{S}$.
2) $d_{\widetilde{S}}(o, [g,h]_{ea}) \geq N - C$ implies that $d_{\mathbf{H}}([\rho'_n(g)(0), \rho'_n(h)(0)], 0) \geq f(N)$ for all $n = 1, \ldots, \infty$ (by Theorem 3.10).

Since $M_n \setminus P_n$ (resp. $M \setminus P$) are uniformly bi-Lipschitz homeomorphic to quotients of $\mathbb{H}^3$ by $\rho'_n(\Gamma)$ (resp. $\rho'(\Gamma)$) it follows that $d_{\mathbf{H}}([\rho_n(g)(0), \rho_n(h)(0)]_{ea}, 0) \geq f(N) - C'$ for some C' by Lemma 3.2 since $[\rho_n(g)(0), \rho_n(h)(0)]_{ea}$ and $[\rho'_n(g)(0), \rho'_n(h)(0)]$ track each other away from the lifts of P_n. In particular, $[\rho_n(g)(0), \rho_n(h)(0)]_{ea}$ enters and leaves a lift of P_n outside a large $(=f(N) - C')$ ball about $0 \in \mathbf{H}^3$. Since $\partial P_n \cap \partial(M_n \setminus P_n)$ is an annulus on S under the identification of the component of $\partial(M_n \setminus P_n)$ intersecting P_n non-trivially with S. Hence any piece of $[\rho_n(g)(0), \rho_n(h)(0)]_{ea}$ inside a lift of P_n also lies outside a large $(=f(N) - C')$ ball about $0 \in \mathbf{H}^3$. (This part of the argument is very similar to an argument in the geometrically finite case in [MS13] where a sequence of representations of a hyperbolic element converges to a parabolic.)

Finally, by Lemma 3.2 again, $[\rho_n(g)(0), \rho_n(h)(0)]$ lies outside a large $(=f(N) - C_1)$ ball about $0 \in \mathbf{H}^3$ for all n. Hence the Cannon-Thurston maps for ρ_n converge uniformly to the Cannon-Thurston map for ρ by Proposition 2.9.

Case 2:

As before, $\Gamma = \pi_1(S)$ for a closed surface S. $\rho_n(\Gamma) = G_n$ is a sequence of quasi-Fuchsian Kleinian groups converging to G where $M = \mathbb{H}^3/G$ has one geometrically finite end without parabolics. The other end E of M contains a single rank one cusp corresponding to a limiting accidental parabolic γ. Let $M_n = \mathbb{H}^3/G_n$ and let E_n be the end of M_n such that E_n converges to E strongly.

It follows from strong convergence that E minus the rank one cusp consists of one or two degenerate ends according as γ is non-separating or separating. In either case, from the Minsky model [Min10] of the ends, there exists a Margulis tube $T_n \subset E_n \subset M_n = \mathbb{H}^3/G_n$ with core curve homotopic to γ such that the number of building blocks on either side of T_n in the Minsky model of E_n tends to infinity as $n \to \infty$. To fix notions, suppose γ is non-separating (since we shall lift everything to the universal cover the argument in the separating case is similar).

Let $[a, b]$ be an electro-ambient quasigeodesic in $\widetilde{S}$ with lifts of γ electrocuted. By Lemma 3.2 $[a, b]$ is a (uniform) quasigeodesic in the universal cover $\widetilde{S}$. We can therefore choose finitely many blocks on either side of T_n in M_n such that this number ($m = m(n)$ say) tends to infinity as $n \to \infty$.

The degenerating end E_n^c on the two sides of T_n (there are two in case γ is separating) consisting of the union of the $m(n)$ blocks above now satisfy the hypothesis of Theorem 3.10. The electro-ambient geodesic $[a, b]$ now consists of maximal segments $l_1, \cdots, l_k$ such that the segments l_{2i+1} with odd indices are geodesics in a lift of $S \setminus \{\gamma\}$ to $\widetilde{S}$ whereas the segments l_{2i} with even indices are geodesic subsegments of lifts of γ. Let r_{2i+1} be the geodesic joining the end-points of l_{2i+1} in the corresponding lift of E_n^c. Since E_n^c satisfies the hypothesis of Theorem 3.10, UEPP follows for each segment l_{2i+1} as in the case of simply degenerate ends without accidental parabolics. Also the alternating union of r_{2i+1}'s with l_{2i}'s gives an electro-ambient quasigeodesic with (lifts of) the Margulis tube T_n electrocuted. Hence, by Lemma 3.2 again UEPP is established and uniform convergence of Cannon-Thurston maps follows as in Theorem 4.1.

5. Algebraic limits and Pointwise Convergence

In [MS13], we showed:

Theorem 5.1. *Let Γ be a fixed geometrically finite group and $\rho_n : \Gamma \to G_n$ be a sequence of weakly type-preserving isomorphisms to geometrically finite Kleinian groups converging algebraically to a Kleinian group $G_\infty = \rho_\infty(\Gamma)$, and suppose that the geometric limit of the groups G_n is also geometrically finite. Then the sequence of Cannon-Thurston maps $\hat{i}_n : \Lambda_\Gamma \to \Lambda_{G_n}$ converge pointwise to $\hat{i}_\infty : \Lambda_\Gamma \to \Lambda_{G_\infty}$.*

We shall illustrate the proof by focusing on a set of examples discovered by Kerckhoff-Thurston [KT90].

We have the following basic theorem due to Jorgensen and Marden.

Theorem 5.2. (Jorgensen-Marden [JM90]**)** *Let $\rho_n : G_0 \to G_n$ be a sequence of isomorphisms onto finitely generated non-elementary Kleinian groups G_n which converge algebraically to $\rho = \rho_\infty : G_0 \to G_\infty$ and geometrically to a geometrically finite Kleinian group G. Let $\langle a_i, b_i \rangle$, $i = 1 \cdots k$, be a system of representatives of the conjugacy classes of maximal parabolic subgroups of rank 2 in G such that $a_i =$*

$\rho(\alpha_i) \in G$ is an algebraic limit of loxodromic elements $\rho_n(\alpha_i)$ ($\alpha_i \in G_0$) but $b_i \notin G_\infty$. Then G is generated by G_∞ and b_i, $i = 1 \cdots k$.

The above Theorem describes the general situation where geometrically finite groups converge to a geometrically finite group and the latter develops extra $\mathbb{Z} \times \mathbb{Z}$ cusps. Instead of discussing this general phenomenon, we shall discuss a specific set of examples discovered by Kerckhoff and Thurston.

5.1. **The Kerckhoff-Thurston Examples.** In [KT90], Kerckhoff and Thurston describe a sequence of quasi-Fuchsian surface groups converging geometrically to a geometrically finite group G^g with a $\mathbb{Z} \times \mathbb{Z}$ cusp, but algebraically to a geometrically finite surface group G^a with an accidental parabolic. We first give a brief description of these examples and then describe a bi-Lipschitz model for the sequence of manifolds and the limit. Finally, we establish that for this class of examples Cannon-Thurston maps converge pointwise, but not uniformly.

Fix a closed hyperbolic surface S and a simple closed geodesic σ on it. Let tw^i denote the automorphism of S given by an i-fold Dehn twist along σ. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, tw^i(S))$. Then Kerckhoff and Thurston [KT90] show that these groups satisfy the properties described in the previous paragraph.

5.1.1. *The Model.* We start with a closed hyperbolic surface S. The hyperbolic structure is arbitrary, but it is important that a choice be made.

Fix a *finite* collection $\mathcal{C}$ of (geodesic representatives of) simple closed curves on S. $N_\epsilon(\sigma_i)$ will denote an ϵ neighborhood of $\sigma_i \subset S$ for some $\sigma_i \in \mathcal{C}$. ϵ and the neighborhood of the cusps in S are chosen small enough so that no two lifts of $N_\epsilon(\sigma_i)$ to the universal cover $\widetilde{S^h}$ intersect.

I will denote the closed interval $[0, 3]$. Now put a product metric structure on $S \times I$, which restricts to the path-metric on S for each slice $S \times a, a \in I$ and the Euclidean metric on the I-factor. Let B_i^c denote $(S \times I - N_\epsilon(\sigma_i) \times [1, 2]$. Equip B_i^c with the path-metric.

For each resultant torus component of the boundary of B_i^c, perform Dehn filling on some $(1, n_i)$ curve, which goes n_i times around the meridian and once round the longitude. n_i will be called the **twist coefficient**. The metric on the solid torus Θ_i glued in is arranged in such a way that it is isometric to the quotient of a neighborhood of a bi-infinite hyperbolic geodesic by a hyperbolic isometry. Further, the $(1, n_i)$-curve is required to bound a totally geodesic hyperbolic disk. In fact, we might as well foliate the boundary of Θ_i by translates (under hyperbolic isometries) of the meridian, and demand that each bounds a totally geodesic disk. Since there is no canonical way to smooth out the resulting metric, we leave it as such. Θ_i equipped with this metric will be called a **Margulis tube** in keeping with the analogy from hyperbolic space.

The resulting copy of $S \times I$ obtained, equipped with the metric just described, is called a **thin building block** and is denoted by B. It is essentially established in [KT90] that B is uniformly bi-Lipschitz to the simultaneous uniformization of the pair $(S, \Pi_i tw_i^{n_i}(S))$. Here $\Pi_i tw_i^{n_i}$ denotes the composition of n_i-fold Dehn twists along the disjoint curves σ_i. Since σ_i's are disjoint, it does not matter in what order we compose these twists. For ease of exposition below, we shall consider a sequence G_m of quasi-Fuchsian groups which are given by the simultaneous uniformization of the pair $(S, \Pi_i tw_i^m(S))$, i.e. $n_i = m$ for all i in the mth group G_m.

An Estimate:
Let M_m be the hyperbolic manifold corresponding to the quasi-Fuchsian group G_m. Also we consider uniformly bi-Lipschitz embeddings of thin building blocks $B_m \subset M_m$. Note that B_m is obtained by doing $(1, m)$ hyperbolic Dehn surgeries along σ_i's in $S \times I$ for S a fixed hyperbolic surface. For the rest of the subsection we shall focus on B_m rather than M_m. For ease of exposition again, we shall assume that the collection $\mathcal{C}$ of σ_i's consists of a single curve σ. Generalization to a finite collection is straightforward. Let a be a fixed generator of $\pi_1(S)$ represented by a geodesic meeting σ transversely at a single point, which we also pick as the base point. Also, let b denote the group element represented by σ. We estimate the length of the word ab^l in B_m. Let $(\frac{l}{m})$ denote the integer obtained by rounding off $\frac{l}{m}$ to the nearest integer and let $|(\frac{l}{m})|m - l| = Rem(l, m)$. Now each word of the form ab^m can be replaced in B_m by a geodesic loop of fixed length c of $O(1)$ using the hyperbolic Dehn filling that is performed on the $(1, m)$ curve. Therefore $ab^{(\frac{l}{m})}$ can be replaced by a geodesic loop of length approximately $c(\frac{l}{m})$, i.e. of order $O((\frac{l}{m}))$. The rest of the loop in ab^l is of the form $b^{Rem(l,m)}$ and can be realized by a geodesic of length of order $O(cosh^{-\frac{1}{2}} Rem(l, m))$ (see [McM01] for more details). Hence the word ab^m has length that can be estimated by $O((\frac{l}{m})) + O(cosh^{-\frac{1}{2}} Rem(l, m))$.

5.2. Uniform and Pointwise Convergence.

5.2.1. *Absence of Uniform Convergence.* Let ab^∞ represent the infinite quasigeodesic ray with initial segment a and infinitely periodic tail obtained by repeating b. Then the subword $ab^m \subset ab^\infty$ has uniformly bounded length of order $O(1)$ in B_m and hence M_m. In particular, the quasigeodesic tail starting at ab^m and converging to the limit point $\xi \in \partial\mathbb{H}^2$ of ab^∞ has a geodesic realization starting at a uniformly bounded distance of order $O(1)$ from the origin $0 \in \mathbb{H}^3 = \widetilde{M_m}$. However, the length of ab^m in $\pi_1(S)$ is of order $O(m)$. Hence, the the quasigeodesic tail starting at ab^m and converging to the limit point $\xi \in \partial\mathbb{H}^2$ lies outside an $O(m)$ ball about $0 \in \mathbb{H}^2$. Thus condition (5) in Lemma 2.6 is violated and the Cannon-Thurston maps for the quasi-Fuchsian groups G_m cannot converge *uniformly* to the Cannon-Thurston map for the surface group as a subgroup of the geometric limit G^g.

5.2.2. *Point-wise Convergence.* In spite of the absence of uniform convergence of Cannon-Thurston maps, we shall now establish that the Cannon-Thurston maps converge pointwise by applying the sufficient criteria from Lemma 2.10 for certain rays and by a different direct argument for other rays. Consider a point $\xi \in \partial\mathbb{H}^2$ and a geodesic ray $[0, \xi) \subset \mathbb{H}^2$. Two cases arise:
1) $\xi \neq ab^\infty, ab^{-\infty}$, i.e. ξ is not a translate of a fixed point of b.
2) ξ is a translate of a fixed point of b.

Case 1: $\xi \neq ab^\infty, ab^{-\infty}$, i.e. ξ is not a translate of a fixed point of b. We apply sufficient criteria (a) and (b) from Lemma 2.10.

Criterion a) **Uniform embedding of points:** We want to show that there exists a non-negative function $f(N)$, such that $f(N) \to \infty$ as $N \to \infty$ and for all $g \in [0, \xi)$ lying outside $B_N(1)$, $\rho_n(g)$ lies outside the $f(N)$-ball around $0 \in \mathbf{H}^3$.
Proof of Criterion (a): Consider the (uniform) electro-ambient hyperbolic K-quasigeodesic starting at 0 and converging to ξ corresponding to $\tilde{S}$ with lifts of σ electrocuted. By slight abuse of notation, we shall continue to refer to this electro-ambient hyperbolic K-quasigeodesic as $[0, \xi)$. Since ξ is not a translate of a fixed point of b, there exists a proper function $f : \mathbb{N} \to \mathbb{N}$ such that if $x_N \in [0, \xi)$ is at distance at least N from 0 in $\tilde{S}$, then the electric distance $d_{el}(0, x_N) \geq f(N)$. Next, the electric metric on $\tilde{S}$ with lifts of σ electrocuted is uniformly (independent of m) quasi-isometric to $\widetilde{B_m}$ with lifts of the Margulis tube Θ_m electrocuted. Since electric distance is less than hyperbolic

distance on $\widetilde{B_m}$, Criterion a) follows.

Criterion b) **Uniform embedding of pairs of points:** There exists $g(N)$ such that $g(N) \to \infty$ as $N \to \infty$ such that for any geodesic subsegment $[a,b]$ of $[1,\xi) \subset \Gamma$ lying outside $B_N^G(1)$ (the N-ball about $1 \in \Gamma$), any geodesic in $\mathbf{H}^3$ joining $\rho_n(a), \rho_n(b)$ lies outside $B^3_{g(N)}(0)$, (the $g(N)$-ball about $0 \in \mathbf{H}^3$).
It follows from the last paragraph that $[x_N,\xi)$ is a (uniform) electric quasigeodesic in $(\widetilde{B_m}, d_{el})$ with lifts $\widetilde{\Theta_m}$ of the Margulis tube Θ_m electrocuted. Hence the corresponding electro-ambient quasigeodesic $[x_N,\xi)_{ea}$ (obtained by replacing intersections of $[x_N,\xi)$ with the lifts $\widetilde{\Theta_m}$ by hyperbolic geodesics in $\widetilde{\Theta_m}$) is a (uniform) hyperbolic quasigeodesic in $\widetilde{B_m}$ with the hyperbolic metric. Also, it follows from criterion (a) that $[x_N,\xi)_{ea}$ lies outside some $g(N)$ ball about $0 \in \mathbf{H}^3$), where $g(N) \to \infty$ as $N \to \infty$. Thus criterion (b) is satisfied. $\Box$
Case 2: ξ is a translate of a fixed point of b.
Without loss of generality let $\xi = \xi_+$ be a translate of a fixed point of b. Also let ξ_- be the other fixed point. Then the bi-infinite geodesic b^* on $\tilde{S}$ lies on the lower boundary component of $\widetilde{B_m}$ along a fixed lift $\widetilde{\Theta_m}$. However, the core geodesic of $\widetilde{\Theta_m}$ lies at a distance equal to the radius of Θ_m from b^*. The radius of Θ_m is of the order $O(ln(m))$ (see the estimate in Section 3.1. Hence the geodesic joining ξ_- and ξ_+ in $\widetilde{B_m}$ lies at a distance of the order $O(ln(m))$ from $0 \in \mathbb{H}^3 = \widetilde{M_m}$, provided we choose the base-point in $\mathbb{H}^2 = \widetilde{S}$ in such a way that the embedding $\widetilde{\phi_m}$ maps it to within a bounded neighborhood of $0 \in \mathbb{H}^3 = \widetilde{M_m}$. In particular, the geodesic joining $\widetilde{\phi_m}(\xi_-)$ and $\widetilde{\phi_m}(\xi_+)$ lies outside an $h(m)$ ball about $0 \in \mathbb{H}^3 = \widetilde{M_m}$, where $h(m) \to \infty$ as $m \to \infty$. Thus $\widetilde{\phi_m}(\xi_-)$ and $\widetilde{\phi_m}(\xi_+)$ converge to the common limit point of b^* in the limiting geometrically finite group G_∞, where b becomes an accidental parabolic. Thus the Cannon-Thurston maps for G_m converge to the Cannon-Thurston map for G_∞ at ξ. $\Box$

We summarize the conclusion of this discussion as follows.

Theorem 5.3. *Fix a hyperbolic surface S and a simple closed geodesic σ on it. Let tw^i denote the automorphism of S given by an i-fold Dehn twist along σ. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, tw^i(S))$. Let G_∞ denote the geometric limit of the G_m's. Let S_{i-} denote the lower boundary component of the convex core of G_i, $i = 1,\cdots,\infty$ (including ∞). Let $\phi_i : S \to S_{i-}$, such that if $0 \in \mathbb{H}^2 = \widetilde{S}$ denotes the origin of $\mathbb{H}^2$ then $\widetilde{\phi_i}(0)$ lies in* *a uniformly bounded neighborhood of* $0 \in \mathbb{H}^3 = \widetilde{M_m}$. *We also assume (using the fact that* M_∞ *is a geometric limit of* M_m*'s) that* S_{i-}*'s converge geometrically to* $S_{\infty -}$. *Then the Cannon-Thurston maps for* $\widetilde{\phi_i}$ *converge pointwise on* $\partial \mathbb{H}^2$ *to the Cannon-Thurston map for* $\widetilde{\phi_\infty}$.

6. Algebraic limits and Pointwise Non-convergence

6.1. **Brock's Examples.** In [Bro01], Brock describes a sequence of quasi-Fuchsian surface groups converging geometrically to a geometrically infinite group G^g which is abstractly isomorphic to the fundamental group of a book of I bundles about a solid torus. The algebraic limit is a geometrically infinite surface group G^a with an accidental parabolic. As in Section 5 we first give a brief description of these examples and then describe a bi-Lipschitz model for the sequence of manifolds and the limit. Finally, we establish, following [MS17] that for this class of examples Cannon-Thurston maps converge pointwise for most points on the boundary $\mathbb{S}^1_\infty$. However, there exist points at which the sequence of Cannon-Thurston maps *do not* converge to the limiting Cannon-Thurston map.

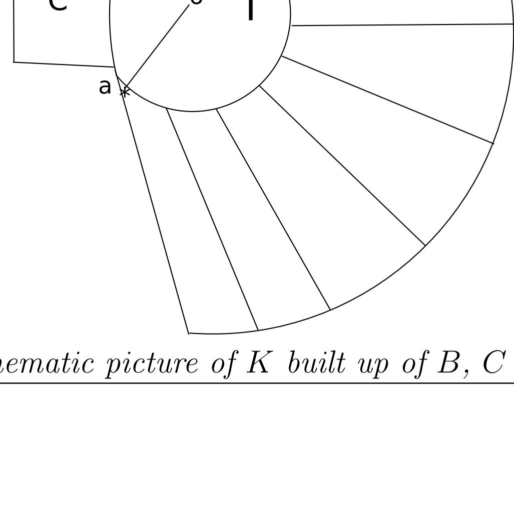
The Examples:

Fix a hyperbolic surface S and a separating simple closed geodesic σ on it, cutting S up into two pieces S_- and S_+. Let ϕ denote an automorphism of S such that $\phi|_{S_-}$ is the identity and $\phi|_{S_+} = \psi$ is a pseudo-Anosov diffeomorphism of S_+ preserving the boundary. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, \phi^i(S))$. Then Brock [Bro01] shows that the sequence of groups G_i satisfy the properties described above.

6.2. **The Model.** For brevity, let $S_+ = \Sigma$ and assume that Σ is equipped with a complete hyperbolic structure so that σ is homotopic to a cusp on Σ. Let $N(= N_\psi)$ be the hyperbolic 3-manifold with fiber Σ and monodromy ψ. Fix a fiber of the fibration and call it (by slight abuse of notation) Σ. Let Σ^c denote Σ with a small (open) neighborhood of the cusp removed so that the boundary curve has a fixed length ϵ_0.

Let N_n denote the cyclic n-fold cover of N, i.e. the kernel of the homomorphism $\pi_1(N) \to \pi_1(S^1) = \mathbb{Z} \to \mathbb{Z}_n$. Let N_n^c denote N_n with a small (open) neighborhood of the cusp removed so that the boundary curve of a fiber has length ϵ_0. Let B_n denote N_n^c cut open along a lift of Σ and completed (metrically) to a manifold (with boundary) homeomorphic to $\Sigma_c \times [0, n]$. The boundary $S^1 \times [0, n]$ has a metric which is a round metric in the S^1 direction such that the total length of the circle is ϵ_0 and the $[0,n]$ direction has the Euclidean metric on an interval of length n. To ensure this, we might need to compose with a uniformly bi-Lipschitz homeomorphism of Σ_n which we assume has been done. In any case, the model we build is only a bi-Lipschitz model. (See Minsky [Min10].) Now consider the manifold $S_- \times [0,1]$ and again assume that we have composed with a bi-Lipschitz homeomorphism to make the metric on the boundary a round metric in the S^1 direction with total length ϵ_0. Also assume that the $[0,1]$ direction has been given a Euclidean metric of length one. Glue the boundary circle (corresponding to σ) of $S_- \times \{0\}$ (resp. $S_- \times \{1\}$) to the boundary circle (corresponding to σ) of $\Sigma_c \times \{0\}$ (resp. $\Sigma_c \times \{n\}$). Let K_n^c denote the resulting space. Let η denote a circle of length $(n+1)$ obtained by moving in the Euclidean direction in K_n^c. Let K_n be the manifold (with boundary) obtained by (hyperbolic) Dehn filling η in K_n^c. The metric obtained might not be smooth at the boundary, so we smooth it out.

Let T_n denote the resulting Margulis tube. Let $\overline{B_n}$ (resp. $\overline{C_n}$) denote B_n (resp. $S_- \times [0,1]$) with T_n adjoined. See figure below, where a typical K_n is shown in 'cross-section'. Similarly let $\overline{B_\infty}$ denote B_∞ with a neighborhood of the cusp attached. B_∞ may also be thought of as 'the positive half of' the cover of N corresponding to the fiber subgroup $\pi_1(\Sigma)$.



A schematic picture of K built up of B, C and T

Brock [Bro01] (See also Minsky's model [Min10].) shows that K_n is uniformly bi-Lipschitz to the convex core of the simultaneous uniformization of the pair $(S, \phi^n(S))$. The corresponding quasiFuchsian group will be denoted by G_n.

6.3. **Absence of Uniform Convergence.** Let M_n be the hyperbolic manifold $\mathbb{H}^3/G_n$ corresponding to the quasi-Fuchsian group G_n and let ρ_n be the corresponding representation of $\pi_1(S)$ so that $G_n = \rho_n(\pi_1(S))$. Also we consider uniformly bi-Lipschitz embeddings of $K_n \subset M_n$. We think of K_n as a (bi-Lipschitz cf [Min10]) model for the convex core of M_n. For the rest of the subsection we shall focus on K_m rather than M_m.

We fix a base point $*$ on σ. Let α be a fixed generator of $\pi_1(S)$ represented by a (based) geodesic lying entirely on $T = S_+$. Also, let t_n be the path represented by $* \times [0, n]$ in K_n. Consider paths $\alpha_n = t_n \alpha \overline{t_n}$ in K_n, where $\overline{t_n}$ represents t_n with the reverse orientation. Then each α_n is homotopic to $\phi^n(\alpha) = \psi^n(\alpha) = \beta_n$ on S. Since ψ is a pseudo-Anosov, the length of $\phi^n(\alpha) = \psi^n(\alpha)$ on S increases exponentially with n. However, the length of the geodesic realization of t_n in K_n is of the order $O(1)$ since t_n is homotopic to a path which goes "the short way" around the Margulis tube, i.e. it travels $* \times [0, 1] \subset S_- \times [0, 1]$ along the boundary of $S_- \times [0, 1]$. Thus t_n is path-homotopic to a path of length one in K_n. Hence α_n has length $\leq 2 + l(\alpha)$ in K_n, which is (uniformly) O(1).

Thus the sequence of points $p_n \in \widetilde{S} \subset \widetilde{K_n}$ given by the end-point of a lift of β_n starting at a fixed lift 0 of $*$ satisfy
1) $d_{\widetilde{S}}(*, p_n) \to \infty$
2) $d_{\widetilde{K_n}}(0, p_n) \leq 2 + l(a)$

Thus the condition in conclusion (b) of Lemma 2.6 is violated and the Cannon-Thurston maps for the quasi-Fuchsian groups G_m cannot converge *uniformly* to the Cannon-Thurston map for the surface group as a subgroup of the geometric limit G^g.

6.4. **Pointwise Convergence and non-convergence.** As in Section 5, we shall now establish that the Cannon-Thurston maps converge pointwise for certain rays by applying the sufficient criteria from Lemma 2.10. Since Margulis tubes are convex, it follows that each lift $\widetilde{B_n}$ of $\overline{B_n}$ is uniformly quasiconvex in $\widetilde{K_n}$. Hence $\widetilde{K_n}$ is strongly hyperbolic relative to the collection $\mathcal{B}_n$ of lifts $\widetilde{B_n}$ of $\overline{B_n}$. Let $\widetilde{B_n^e}$ denote the resulting electric space equipped with the electric metric d_e. Consider a point $\xi \in \partial\mathbb{H}^2$ and a geodesic ray $[0, \xi) \subset \mathbb{H}^2$. Two cases arise:

1) The length of $[0,\xi)$ in the electric metric d_e is infinite, i.e. $[0,\xi)$ does not eventually lie in a fixed lift of $\widetilde{B_n}$, or equivalently, no tail of $[0,\xi)$ lies in a fixed lift of $\widetilde{B_n}$.
2) The length of $[0,\xi)$ in the electric metric d_e is finite. Translating by an appropriate element of $\pi_1(S)$ we may assume without loss of generality that the geodesic ray $[0,\xi)$ lies in a fixed lift of $\widetilde{B_n}$.

6.4.1. *Point-wise Convergence.* The proof divides into a number of cases.
Case 1: *The length of* $[0,\xi)$ *in the electric metric* d_e *is infinite.*

We apply sufficient criteria in conclusions (a) and (b) from Lemma 2.10.
Criterion a) **Uniform embedding of points:** We want to show that there exists a non-negative function $f(N)$, such that $f(N) \to \infty$ as $N \to \infty$ and for all $g \in [0,\xi)$ lying outside $B_N(1)$, $\rho_n(g)$ lies outside the $f(N)$-ball around $0 \in \mathbf{H}^3$.
Proof of Criterion (a): Consider the (uniform) electro-ambient hyperbolic K-quasigeodesic starting at 0 and converging to ξ corresponding to $\tilde{S}$ with lifts of S_+ electrocuted. By slight abuse of notation, we shall continue to refer to this electro-ambient hyperbolic K-quasigeodesic as $[0,\xi)$. Since no tail of ξ lies in a lift of B_n (and hence S_+) there exists a proper function $f:\mathbb{N}\to\mathbb{N}$ such that if $x_N \in [0,\xi)$ is at distance at least N from 0 in $\tilde{S}$, then the electric distance $d_e(0,x_N) \geq f(N)$. Next, the electric metric on $\tilde{S}$ with lifts of S_+ electrocuted is uniformly (independent of n) quasi-isometric to $\widetilde{K_n}$ with lifts of the B_n electrocuted. Since electric distance is less than hyperbolic distance on $\widetilde{B_m}$, Criterion a) follows.

Criterion b) **Uniform embedding of pairs of points:** There exists $g(N)$ such that $g(N) \to \infty$ as $N \to \infty$ such that for any geodesic subsegment $[a,b]$ of $[1,\xi) \subset \Gamma$ lying outside $B_N^G(1)$ (the N-ball about $1 \in \Gamma = \Gamma_{\pi_1(S)}$), any geodesic in $\mathbf{H}^3$ joining $\rho_n(a), \rho_n(b)$ lies outside $B^3_{g(N)}(0)$, (the $g(N)$-ball about $0 \in \mathbf{H}^3$).
It follows from the last paragraph that $[x_N,\xi)$ is a (uniform) electric quasigeodesic in $(\widetilde{K_n}, d_e)$ with lifts $\widetilde{B_n}$ electrocuted. Hence the corresponding electro-ambient quasigeodesic $[x_N,\xi)_{ea}$ (obtained by replacing intersections of $[x_N,\xi)$ with the lifts $\widetilde{B_n}$ by hyperbolic geodesics in $\widetilde{B_n}$)

is a (uniform) hyperbolic quasigeodesic in $\widetilde{K_n}$ with the hyperbolic metric. Also, it follows from criterion (a) that $[x_N, \xi)_{ea}$ lies outside some $g(N)$ ball about $0 \in \mathbf{H}^3$), where $g(N) \to \infty$ as $N \to \infty$. Thus criterion (b) is satisfied. $\Box$

Case 2: *The length of* $[0, \xi)$ *in the electric metric* d_e *is finite.*

Translating by an appropriate element of $\pi_1(S)$ we may assume without loss of generality that the geodesic ray $[0, \xi)$ lies in a fixed lift of $\widetilde{B_n}$. Thus we are interested in infinite geodesic words (geodesic rays) in the group $\pi_1(S_+) = \pi_1(B_n) = \pi_1(\overline{B_n})$. Recall that $\widetilde{B_n}$ denotes a lift of $\overline{B_n}$ (and not just B_n). Now, $\widetilde{B_n}$ is strongly hyperbolic relative to the lifts of T_n. Note that $B_n \to B_\infty$ geometrically where B_∞ denotes the infinite cyclic cover of M_n^c (for any n) corresponding to the fiber subgroup $\pi_1(S_+)$ (essentially because B_n is just $S_+ \times [0, n]$ with the appropriate metric). Since Margulis tubes are convex, it follows that each $\widetilde{B_n}$ is strongly hyperbolic relative to $\mathcal{T}_n$ of lifts of T_n contained in $\overline{B_n}$. Let $(\widetilde{B_n}, d_{el})$ denote the electric metric on $\widetilde{B_n}$ obtained by electrocuting the collection of lifts $\mathcal{T}_n$ of T_n. Similarly, let $(\widetilde{B_\infty}, d_{el})$ denote the universal cover of $\overline{B_\infty}$ with *horoballs* electrocuted. We shall continue to use ρ_n (resp. ρ_∞) to denote the restrictions of ρ_n (resp. ρ_∞) to $\pi_1(\overline{B_n})$ (resp. $\pi_1(\overline{B_\infty})$). Case 2 now splits into two further subcases. $\partial\rho_n$ (resp. $\partial\rho_\infty$) will denote the respective Cannon-Thurston maps. The geodesic realization of $[0, \xi)$ in $\widetilde{B_\infty}$, denoted by $\partial\rho_\infty([0, \xi))$ is the geodesic in $\widetilde{B_\infty}$ joining $\rho_\infty(0)$ and $\partial\rho_\infty(\xi)$.

Case 2A: *The length of the geodesic realization* $\partial\rho_\infty([0, \xi))$ *of* $[0, \xi)$ *in* $\widetilde{B_\infty}$ *equipped with the electric metric* d_{el} *is infinite, i.e. no tail of* $\partial\rho_\infty([0, \xi))$ *lies in a fixed lift of* $\widetilde{H}$*, where* H *is a small neighborhood of the cusp in* B_∞*.*

A word of clarification is necessary here. The geodesic realization of $[0, \xi)$ in $\widetilde{B_n}$ is well-defined, thinking of $\overline{B_\infty}$ as the convex core of the *simply degenerate* group arising as the *algebraic* limit of simultaneous uniformizations of $(\Sigma, \psi^n(\Sigma))$. We are now considering geodesic realizations in $\widetilde{B_\infty}$. Thus, when we say that the length of the geodesic realization $[0, \xi)_r$ of $[0, \xi)$ in $\widetilde{B_\infty}$ equipped with the electric metric d_{el} is infinite, we mean that *the geodesic ray converging to the image of* ξ *under the Cannon-Thurston map from* $\widetilde{\Sigma}$ *to* $\widetilde{B_\infty}$ (see Theorem 3.8) has infinite length in $\widetilde{B_\infty}$ equipped with the electric metric d_{el}. This is equivalent to saying that $[0, \xi)_r$ does not eventually lie inside the lift of a single horoball in $\widetilde{B_\infty}$.

Let $\overline{B_n}^e$ denote $\overline{B_n}$ with Margulis tube T_n electrocuted. Also, let B_n^e denote B_n with the boundary annulus electrocuted. Then the inclusion of B_n^e into $\overline{B_n}^e$ induces an isometry as well as an isomorphism of fundamental groups. Hence the inclusion of the universal cover $UC(B_n^e)$ into $\widetilde{B_n}^e$ induces an isometry. (To distinguish between the universal covers of B_n and $\overline{B_n}$ we use the notation $UC(B_n)$ and $\widetilde{B_n}$ respectively.)

The universal covers $\widetilde{B_n}$ (resp. $\widetilde{B_\infty}$) are strongly hyperbolic relative to the equivariant collection of lifts of T_n (resp. horoballs corresponding to lifts of the cusps [Far98]). Let $\rho_n([0,\xi))_{ea}$ (resp. $\rho_\infty([0,\xi))_{ea}$) represent the electro-ambient quasi-geodesics corresponding to $\rho_n([0,\xi))$ (resp. $\rho_\infty([0,\xi))$) in the appropriate spaces. Then since lifts of T_n in $\widetilde{B_n}$ converge to horoballs in $\widetilde{B_\infty}$, by Corollary 4.2 $\rho_n([0,\xi))_{ea}$ converges to $\rho_\infty([0,\xi))_{ea}$.

Hence sufficient criteria (a) and (b) from Lemma 2.10 are satisfied. Thus the Cannon-Thurston maps for G_m converge to the Cannon-Thurston map for G_∞ at ξ.

6.4.2. *Pointwise Non-convergence.* We are left with one special case.

Case 2B: *The length of the geodesic realization $\partial\rho_\infty([0,\xi))$ of $[0,\xi)$ in $\widetilde{B_\infty}$ equipped with the electric metric d_{el} is finite, i.e. a tail of $\partial\rho_\infty([0,\xi))$ lies in a fixed lift $\widetilde{H}$, where H is a small neighborhood of the cusp (corresponding to σ) in B_∞.*

Let $\partial\widetilde{H}$ denote the basepoint (or boundary) of the horoball $\widetilde{H}$ on the boundary S^2_∞. Since a tail of $\partial\rho_\infty([0,\xi))$ lies in $\widetilde{H}$, the following is a consequence of the existence and structure of Cannon-Thurston maps (Theorem 3.8).

Observation 6.1. $\partial\rho_\infty(\xi) = \partial\widetilde{H}$.

Translating by an appropriate element of $\pi_1(S)$ we may assume without loss of generality that the geodesic ray $\partial\rho_\infty([0,\xi))$ lies in a fixed lift $\widetilde{H}$. This case now splits into two further subcases.

Subcase (i): *A tail of $[0,\xi)$ lies in a cusp of $\widetilde{\Sigma}$*, where we recall that Σ is the base surface for B_∞. Translating by an element of $\pi_1(\Sigma)$ again we may assume that $[0,\xi)$ lies in a cusp of $\widetilde{\Sigma}$. Then, since the cusp of Σ is totally geodesic, $[0,\xi) = \partial\rho_n([0,\xi)) = \partial\rho_\infty([0,\xi))$ is a geodesic in each $\widetilde{B_n}$, $n = 1\cdots\infty$. Sufficient criteria (a) and (b) from Lemma 2.10 are trivially satisfied. Thus the Cannon-Thurston maps for G_m converge to the Cannon-Thurston map for G_∞ at ξ.

The crucial subcase is the following.

Subcase (ii): *No tail of* $[0,\xi)$ *lies in a cusp of* $\widetilde{\Sigma}$.

The Cannon-Thurston map $\partial\rho_\infty$ must map ξ to $\partial\widetilde{H}$, the basepoint of $\widetilde{H}$. It follows from Theorem 3.8 that the geodesic ray $[0,\xi)$ must be part of a stable leaf (ξ_-,ξ) of the stable lamination of ψ acting on $\tilde{\Sigma}$. Also from Theorem 3.8, $\partial\rho_\infty(\xi_-) = \partial\rho_\infty(\xi)$ and ξ_- must be the basepoint of a cusp in $\widetilde{\Sigma}$.

Fix a basepoint $* = \rho_n(0)$ on $\widetilde{S_(+)}\times\{0\} \subset \widetilde{\Sigma}\times\{0\} \subset \widetilde{B_n}$ and assume that $*$ lies on the boundary of the Margulis tube T_n. For each n, we identify $[0,\xi)$ with a geodesic $[0,x_n)$ in the path-metric on $\widetilde{\Sigma}\times\{0\}$. Also identify $\psi^n([0,\xi))$ with the image of $[0,\xi)$ in $\widetilde{\Sigma}\times\{n\}$ under the action of ψ^n. See figure below.

Then $\nu_n = \{*\}\times[0,n]\cup\psi^n([0,\xi)) \subset UC(B_n)$ must be a uniform quasi-geodesic realization of $[0,\xi)$ in $UC(B_n)$ as ψ contracts lengths of stable leaves by a constant multiplicative factor greater than one. Let ζ_n denote the geodesic in $\widetilde{T_n}$ joining $*\times\times\{0\}$ to $*\times\times\{n\}$. Since $\widetilde{B_n}$ is strongly hyperbolic relative to the fixed lift $\widetilde{T_n}$ of the Margulis tube, it follows that $\zeta_n\cup\psi^n([0,\xi)) \subset \widetilde{B_n}$ is a uniform quasi-geodesic realization of $[0,\xi)$ in $\widetilde{B_n}$. Also note that 'the short way around' T_n along the boundary of C_n from $*\times\times\{0\}$ $(\subset\widetilde{C_n})$ to $*\times\times\{1\}$ $(\subset\widetilde{C_n})$ is of length one. Recall that the Margulis tube T_n is obtained by hyperbolic Dehn filling on a curve η_n (consisting of the union of $\{*\}\times[0,n]\subset\widetilde{B_n}$ and $\{*\}\times[0,1]\subset\widetilde{C_n}$) of length $(n+1)$. Let D_n denote the totally geodesic hyperbolic disk that η_n bounds in T_n. We identify D_n with a lift containing $\{*\}\times[0,n]\subset\widetilde{B_n}$.

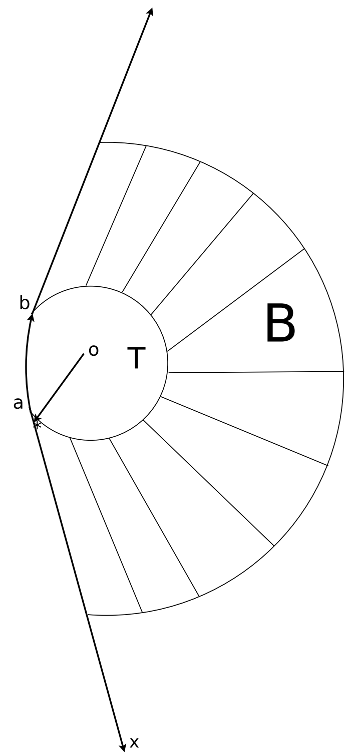


Figure 4: *Geodesics and their realization*

Now let the center of the hyperbolic disk D_n be o_n. Join o_n to $*$ by a geodesic radius on D_n, go around the short way $* \times [0,1] \subset \widetilde{C_n}$ onto the top sheet $\widetilde{S_+} \times n$, and follow $\psi^n([0,\xi))$ to infinity. Let μ_n denote this path. More formally, $\mu_n = [o_n, * \times \{0\}] \cup \{*\} \times [0,1] \cup \psi^n([0,\xi))$. In the figure we have marked $* \times \{0\} \subset \widetilde{C_n}$ as a and $* \times \{1\} \subset \widetilde{C_n}$ as b.

Claim 6.2. *The path μ_n is a uniform (independent of n) quasigeodesic in the universal cover $\widetilde{B_n}$ and hence in $\widetilde{M_n}$.*

Proof of Claim: It is clear that $[o_n, *]$ and $\psi^n([0,\xi))$ are geodesics. Further, $[o_n, * \times \{0\}] \subset \widetilde{T_n}$ and $\psi^n([0,\xi)) \cap \widetilde{T_n} = * \times \{1\}$ $(\subset \widetilde{C_n})$ $= * \times \{n\}$ $(\subset \widetilde{B_n})$. Also the path in μ_n connecting $* \times \{0\}$ to $\psi^n(0)$ is of uniformly bounded length (one). The result follows by standard coarse hyperbolic geometry (cf. the proof of the Morse Lemma or 'stability of quasigeodesics' in Ch. 5 of [GdlH90]). $\Box$

Let $\widetilde{T_n}$ denote the lift of the Margulis tube in which o_n lies. Since the lifts of Margulis tubes converge to lifts of horoballs, it follows that $\widetilde{T_n}$ converges to $\widetilde{H}$ in the geometric limit. Hence we have

Lemma 6.3. *The points o_n converge to $\partial\widetilde{H}$ as $n \to \infty$, where we recall that $\partial\widetilde{H}$ denotes the basepoint of the horoball $\widetilde{H}$.*

Since μ_n's are uniform quasigeodesics and $* \in \mu_n$ for all n, it follows that the visual angle subtended by o_n and $\partial\rho_n(\xi)$ at $*$ is bounded below by some $\epsilon_0 > 0$. Since o_n converges to $\partial\widetilde{H}$ as $n \to \infty$ (Lemma 6.3), it follows that there exists $\epsilon_1 > 0$, such that the visual angle subtended by $\partial\widetilde{H}$ and $\partial\rho_n(\xi)$ at $*$ is bounded below by some $\epsilon_0 > 0$. In particular we have the following.

Lemma 6.4. *No subsequence of $\{\partial\rho_n(\xi)\}_n$ converges to $\partial\widetilde{H}$ as $n \to \infty$.*

Combining Observation 6.1 and Lemma 6.4 we get the conclusion we desire.

Proposition 6.5. *No subsequence of $\{\partial\rho_n(\xi)\}_n$ converges to $\partial\rho_\infty(\xi)$ as $n \to \infty$. Hence the sequence of Cannon-Thurston maps $\partial\rho_n$ do not converge to the Cannon-Thurston map $\partial\rho_\infty$ at the point ξ.*

We summarize the conclusion of this section as follows.

Theorem 6.6. *Fix a hyperbolic surface S and a separating simple closed geodesic σ on it, cutting S up into two pieces S_- and S_+. Let ϕ denote an automorphism of S such that $\phi|_{S_-}$ is the identity and $\phi|_{S_+} = \psi$ is a pseudo-Anosov diffeomorphism of S_+ preserving the boundary. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, \phi^i(S))$. Let G_∞ denote the geometric limit of the G_m's. Let S_{i0} denote the lower boundary component of the convex core of G_i, $i = 1, \cdots, \infty$ (including ∞). Let $\phi_i : S \to S_{i0}$ be such that if $0 \in \mathbb{H}^2 = \widetilde{S}$ denotes the origin of $\mathbb{H}^2$ then $\widetilde{\phi_i}(0)$ lies in a uniformly bounded neighborhood of $0 \in \mathbb{H}^3 = \widetilde{M_m}$. We also assume (using the fact that M_∞ is a geometric limit of M_m's) that S_{i0}'s converge geometrically to $S_{\infty 0}$.*

Let Σ be a complete hyperbolic structure on S_+ such that σ is homotopic to a cusp on Σ. Let Λ consist of pairs (ξ_-, ξ) of endpoints (on $\mathbb{S}^1_\infty$ of stable leaves λ of the stable lamination of ψ acting on $\widetilde{\Sigma}$. Also let $\partial\widetilde{\mathcal{H}}$ denote the collection of basepoints of lifts (to $\widetilde{\Sigma}$) of the cusp in Σ corresponding to σ. Let

$$\Xi = \{\xi : \text{There exists } \xi_- \text{ such that } (\xi_-, \xi) \in \Lambda; \xi_- \in \partial\widetilde{\mathcal{H}}\}.$$

Let $\partial\phi_i$, $i = 1 \cdots, \infty$ denote the Cannon-Thurston maps for $\widetilde{\phi_i}$. Then

1. *$\partial\phi_i(\xi)$ does not converge to $\partial\phi_\infty(\xi)$ for $\xi \in \Xi$.*
2. *$\partial\phi_i(\xi)$ converges to $\partial\phi_\infty(\xi)$ for $\xi \notin \Xi$.*

6.5. **Generalization.** In [MO17], we isolate the exact conditions that give rise to the discontinuity phenomenon for sequences of Cannon-Thurston maps that occur in Brock's examples. We restrict ourselves to a sequence of surface groups $(\phi_n : \pi_1(S) \to PSL_2(\mathbb{C}))$ converging algebraically to $\phi_\infty : \pi_n(S) \to PSL_2(\mathbb{C})$ and geometrically to Γ. A detailed analysis of possible geometric limits was carried out by Ohshika and Soma in [OS10]. They prove in particular that any such geometric limit admits an embedding Φ into $S \times (-1, 1)$. We identify certain features of the Brock examples dealt with earlier in this Section:

(1) First, the geometric limit has a **coupled pair of ends**, i.e. there exists two simply degenerate ends 'facing' each other in the geometric limit. More precisely there exists a proper essential subsurface Σ of S and $t_0 \in (-1, 1)$ such that the image under Φ of the geometric limit into $S \times (-1, 1)$ has two ends of the form $\Sigma \times [t_0 - \epsilon, t_0)$ and $\Sigma \times (t_0, t_0 + \epsilon]$ for some small $\epsilon > 0$. Note that $\Sigma \times \{t_0\}$ is missing.
(2) There exists a boundary curve σ of Σ that separates Σ from a geometrically finite end.
(3) The annulus corresponding to σ in the geometric limit is **untwisted**, i.e. the Dehn twist parameters along σ are bounded uniformly all along the sequence ϕ_n.
(4) Finally, the ending lamination of the end $\Sigma \times [t_0 - \epsilon, t_0)$ is **realized** (after remarking) in the coupled end $\Sigma \times (t_0, t_0 + \epsilon]$.

With these conditions enunciated, we can state the main Theorem of [MO17]:

Theorem 6.7. *Let $(\phi_n : \pi_1(S) \to PSL_2(\mathbb{C}))$ be a sequence of quasi-Fuchsian groups which converges algebraically to $\phi_\infty : \pi_n(S) \to PSL_2(\mathbb{C})$. We set $G_n = \phi_i(\pi_1(S))$ and $G_\infty = \phi_\infty(\pi_1(S))$. Suppose that (G_n) converges geometrically to Γ. Then the Cannon-Thurston maps $c_n : S^1 (= \Lambda_{\pi_1(S)}) \to \Lambda_{\phi_n(\pi_1(S))}$ for the representations ϕ_n do not converge pointwise to the Cannon-Thurston map $c_\infty : S^1 \to \Lambda_{\phi_\infty(\pi_1(S))}$ for ϕ_∞ if and only if Conditions (1)-(4) above are satisfied.*

The points of discontinuity turn out to be exactly the same as in Brock's examples, viz. the tips of crown domains described in Theorem 6.6.

Acknowledgments

The author was supported in part by the National Science Foundation under Grant No. DMS-1440140 at the Mathematical Sciences

Research Institute in Berkeley during Fall 2016, where this article was started.

References

[BCM12] J. F. Brock, R. D. Canary, and Y. N. Minsky. The Classification of Kleinian surface groups II: The Ending Lamination Conjecture. *Ann. of Math. 176 (1), arXiv:math/0412006*, pages 1–149, 2012.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra and Computation. 22, 1250016, 66pp*, 2012.

[Bro01] J. Brock. Iteration of mapping classes and limits of hyperbolic 3-manifolds. *Invent. Math. 143 no 3*, pages 523–570, 2001.

[CT89] J. Cannon and W. P. Thurston. Group Invariant Peano Curves. *preprint, Princeton*, 1989.

[DM16] F. Dahmani and M. Mj. Height, graded relative hyperbolicity and quasi-convexity. *preprint, arXiv:1602.00834*, 2016.

[Eva00] R. Evans. Deformation spaces of hyperbolic 3-manifolds: strong convergence and tameness. *Ph.D. Thesis, Unversity of Michigan*, 2000.

[Eva04] R. Evans. Weakly Type-Preserving Sequences and Strong Convergence. *Geometriae Dedicata, Volume 108, Number 1*, pages 71–92, October, 2004.

[Far98] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal. 8*, pages 810–840, 1998.

[GdlH90] E. Ghys and P. de la Harpe(eds.). Sur les groupes hyperboliques d'apres Mikhael Gromov. *Progress in Math. vol 83, Birkhauser, Boston Ma.*, 1990.

[JM90] T. Jorgensen and A. Marden. Algebraic and geometric convergence of Kleinian groups. *Math. Scand.*, 66:47–72, 1990.

[KT90] S. Kerckhoff and W. Thurston. Non-continuity of the action of the modular group at the Bers' boundary of Teichmuller Space. *Invent. Math. 100*, pages 25–48, 1990.

[Mar07] A. Marden. Outer Circles: An introduction to hyperbolic 3-manifolds. *Cambridge University Press*, 2007.

[McM01] C. T. McMullen. Local connectivity, Kleinian groups and geodesics on the blow-up of the torus. *Invent. math.*, 97:95–127, 2001.

[Min10] Y. N. Minsky. The Classification of Kleinian surface groups I: Models and Bounds. *Ann. of Math. 171(1), math.GT/0302208*, pages 1–107, 2010.

[Mit98a] Mahan Mitra. Cannon-Thurston Maps for Hyperbolic Group Extensions. *Topology 37*, pages 527–538, 1998.

[Mit98b] Mahan Mitra. Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces. *Jour. Diff. Geom.48*, pages 135–164, 1998.

[Mj10] M. Mj. Cannon-Thurston Maps for Kleinian Groups. *preprint, arXiv:math 1002.0996*, 2010.

[Mj11] M. Mj. Cannon-Thurston Maps, i-bounded Geometry and a Theorem of McMullen. *Actes du séminaire Théorie spectrale et géométrie, Grenoble, vol 28, 2009-10, arXiv:math.GT/0511104*, pages 63–108, 2011.

[Mj14a] M. Mj. Cannon-Thurston Maps for Surface Groups. *Ann. of Math., 179(1)*, pages 1–80, 2014.

[Mj14b] M. Mj. Ending Laminations and Cannon-Thurston Maps, with an appendix by S. Das and M. Mj. *Geom. Funct. Anal. 24*, pages 297–321, 2014.

[Mj16] M. Mj. Cannon-Thurston Maps for Surface Groups: An Exposition of Amalgamation Geometry and Split Geometry. *Geometry, Topology, and Dynamics in Negative Curvature, London Mathematical Society Lecture Note Series volume 425, arXiv:math.GT/0512539*, pages 221–271, 2016.

[MO17] M. Mj and K. Ohshika. Discontinuous Motions of Limit Sets. *in preparation*, 2017.

[MP11] M. Mj and A. Pal. Relative Hyperbolicity, Trees of Spaces and Cannon-Thurston Maps. *Geom. Dedicata 151*, pages 59–78, 2011.

[MS13] M. Mj and C. Series. Limits of Limit Sets I. *Geom. Dedicata 167*, pages 35–67, 2013.

[MS17] M. Mj and C. Series. Limits of Limit Sets II: geometrically infinite groups. *To appear in Geometry & Topology*, 2017.

[OS10] K. Ohshika and T. Soma. Geometry and topology of geometric limits. *arXiv:1002.4266*, 2010.

[Thu80] W. P. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes*, 1980.

[Thu82] W. P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.*, pages 357–382, 1982.

Appendix

We shall briefly summarize here the construction of models of *split geometry* from [Mj14a]. Topologically, a **split subsurface** S^s of a surface S is a (possibly disconnected, proper) subsurface with boundary such that
1) each component of S^s is an essential subsurface of S.
2) no component of S^s is an annulus.
3) $S - S^s$ consists of a non-empty family of non-homotopic essential annuli, none of which are homotopic into the boundary of S^s.

Geometrically, we assume that S is given some finite volume hyperbolic structure. A split subsurface S^s of S has bounded geometry, i.e.
1) each boundary component of S^s is of length ϵ_0, and is in fact a component of the boundary of $N_k(\gamma)$, where γ is a hyperbolic geodesic on S, and $N_k(\gamma)$ denotes its k-neighborhood.
2) For any closed geodesic β on S, either $\beta \subset S - S^s$, or, the length of any component of $\beta \cap (S - S^s)$ is greater than ϵ_0.

Definition-Theorem 6.8. (WEAK SPLIT GEOMETRY) *We constructed in* [Mj14a] *the following from the Minsky model for a simply or totally degenerate surface group:*
1) A sequence of split surfaces S^s_i *exiting the end(s) of* M*, where* M *is*

marked with a homeomorphism to $S \times J$ *(*J *is* $\mathbb{R}$ *or* $[0, \infty)$ *according as* M *is totally or simply degenerate).* $S_i^s \subset S \times \{i\}$.
2) A collection of Margulis tubes $\mathcal{T}$.
3) For each complementary annulus of S_i^s *with core* σ*, there is a Margulis tube* T *whose core is freely homotopic to* σ *and such that* T *intersects the level* i*. (What this roughly means is that there is a* T *that contains the complementary annulus.) We say that* T *splits* S_i^s.
4) There exist constants $\epsilon_0 > 0, K_0 > 1$ *such that for all* i*, either there exists a Margulis tube splitting both* S_i^s *and* S_{i+1}^s*, or else* $S_i (= S_i^s)$ *and* $S_{i+1} (= S_{i+1}^s)$ *have injectivity radius bounded below by* ϵ_0 *and bound a* **thick block** B_i*, where a thick block is defined to be a* K_0 *bi-Lipschitz homeomorphic image of* $S \times I$.
5) $T \cap S_i^s$ *is either empty or consists of a pair of boundary components of* S_i^s *that are parallel in* S_i.
6) There is a uniform upper bound $n = n(M)$ *on the number of surfaces that* T *splits.*
A model manifold satisfying conditions (1)-(6) above is said to have **weak split geometry**.

Topologically, a **split block** $B^s \subset B = S \times I$ is a topological product $S^s \times I$ for some *not necessarily connected* S^s. However, its upper and lower boundaries need not be $S^s \times 1$ and $S^s \times 0$. We only require that the upper and lower boundaries be *split subsurfaces* of S^s. This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**. Connected components of split blocks are called **split components**. We demand that there is a *non-empty* collection of Margulis tubes splitting a split block. However we *do not* require that the upper (or lower) horizontal boundary of a split component K be connected. This happens due to the presence of *hanging tubes*. See figure below, where the left split component has four hanging tubes and the right split component has two hanging tubes. The vertical space between the components is the place where a Margulis tube splits the split block into two split components.

Note that the whole manifold M is then the union of
a) Thick blocks (homeomorphic to $S \times I$)
b) Split blocks (homeomorphic to $S^s \times I$ for some split surfaces)
c) Margulis tubes.

The union of thick blocks and split blocks give rise to the complement (in $M = S \times J$) of a special collection of Margulis tubes. Each of these

Margulis tubes splits a uniformly bounded number of split blocks and might end in a hanging tube.

We define a **welded split block** to be a split block with identifications as follows: Components of $\partial S^s \times 0$ are glued together if and only if they correspond to the same geodesic in $S - S^s$. The same is done for components of $\partial S^s \times 1$. A simple closed curve that results from such an identification shall be called a **weld curve**. For hanging tubes, we also glue the boundary circles of their *lower or upper boundaries* by simply collapsing $S^1 \times [-\eta, \eta]$ to $S^1 \times \{0\}$. The same construction is repeated for all $i \geq 0$ by replacing $0, 1$ by $i, i+1$ respectively. For doubly degenerate groups we need to proceed in the negative direction too, from 0 to -1 and then inductively from $-i$ to $-(i+1)$.

Let the metric product $S^1 \times [0, 1]$ be called the **standard annulus** if each horizontal S^1 has length ϵ_0. For hanging tubes the standard annulus will be taken to be $S^1 \times [0, 1/2]$.

Next, we require another pseudometric on B which we shall term the **tube-electrocuted metric**. We first define a map from each boundary annulus $S^1 \times I$ (or $S^1 \times [0, 1/2]$ for hanging annuli) to the corresponding standard annulus that is affine on the second factor and an isometry on the first. Now glue the mapping cylinder of this map to the boundary component. The resulting 'split block' has a number of standard annuli as its boundary components. Call the split block B^s with the above mapping cylinders attached, the *stabilized split block* B^{st}.

Glue boundary components of B^{st} corresponding to the same geodesic together to get the **tube electrocuted metric** on B as follows. Suppose that two boundary components of B^{st} correspond to the same geodesic γ. In this case, these boundary components are both of the form $S^1 \times I$ or $S^1 \times [0, \frac{1}{2}]$ where there is a projection onto the horizontal S^1 factor corresponding to γ. Let $S^1_l \times J$ and $S^1_r \times J$ denote these two boundary components (where J denotes I or $[0, \frac{1}{2}]$). Then each $S^1 \times \{x\}$ has length ϵ_0. Glue $S^1_l \times J$ to $S^1_r \times J$ by the natural 'identity map'. Finally, on each resulting $S^1 \times \{x\}$ put the zero metric. Thus the annulus $S^1 \times J$ obtained via this identification has the zero metric in the *horizontal direction* $S^1 \times \{x\}$ and the Euclidean metric in the *vertical direction* J. The resulting block will be called the **tube-electrocuted block** B_{tel} and the pseudometric on it will be denoted as d_{tel}. Note that B_{tel} is homeomorphic to $S \times I$. The operation of obtaining a *tube electrocuted block and metric* (B_{tel}, d_{tel}) from a split block B^s shall be called *tube electrocution*. Note that a tube electrocuted block is homeomorphic to $S \times I$.

A lift of a split component to the universal cover of the block $B = S \times I$ or $B_{tel} = S \times I$ shall be termed a **split component** $\widetilde{K}$ of $\widetilde{B}$ or $\widetilde{B_{tel}}$.

Also, let d_G be the (pseudo)-metric obtained by electrocuting the collection of split components $\widetilde{K}$ in $\widetilde{B_{tel}}$. d_G will be called the the **graph metric**.

In [Mj14a] we constructed a sequence of split surfaces that satisfy the following two conditions in addition to Conditions (1)-(6) of Definition-Theorem 6.8 for the Minsky model of a simply or totally degenerate surface group:

Definition-Theorem 6.9. (SPLIT GEOMETRY) *7) Each split component $\widetilde{K} \subset \widetilde{B_i} \subset \widetilde{M}$ is (not necessarily uniformly) quasiconvex in the hyperbolic metric on $\widetilde{M}$.*
8) Equip $\widetilde{M}$ with the graph-metric d_G obtained by electrocuting each split component $\widetilde{K}$. Then the convex hull $CH(\widetilde{K})$ of any split component $\widetilde{K}$ has uniformly bounded diameter in the metric d_G. We say that the components $\widetilde{K}$ are **uniformly graph-quasiconvex**. *It follows that $(\widetilde{M}, d_G)$ is a hyperbolic metric space.*
A model manifold satisfying conditions (1)-(8) above is said to have **split geometry**.

School of Mathematics, Tata Institute of Fundamental Research,, 1, Homi Bhabha Road, Mumbai-400005, India
E-mail address: mahan@math.tifr.res.in