CLASS NUMBER FOR PSEUDO-ANOSOVS

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Abstract. Given two automorphisms of a group G, one is interested in knowing whether they are conjugate in the automorphism group of G, or in the abstract commensurator of G, and how these two properties may differ. When G is the fundamental group of a closed orientable surface, we present a uniform finiteness theorem for the class of pseudo-Anosov automorphisms. We present an explicit example of a commensurably conjugate pair of pseudo-Anosov automorphisms of a genus 3 surface, that are not conjugate in the Mapping Class Group, and we also show that infinitely many independent automorphisms of hyperbolic orbifolds have class number equal to one.

1. Introduction: commensurated conjugacy

When G is a group (or any structure), a natural problem is to classify the conjugacy classes in its automorphism group $\mathrm{Aut}\,(G)$. A familiar example is $G = \mathbb{Z}^n$, for which $\mathrm{Aut}\,(G) \simeq GL_n(\mathbb{Z})$.

The *abstract commensurator* $\mathrm{Comm}\,(G)$ of G is the group of equivalence classes of isomorphisms between two finite index subgroups of G, where two such automorphisms ϕ_1, ϕ_2 are declared to be equivalent if they agree on further finite index subgroups.

There is a natural homomorphism from $\mathrm{Aut}\,(G)$ to $\mathrm{Comm}\,(G)$. It need not be injective, but in many cases of interest it is. In general, $\mathrm{Comm}\,(G)$ is much larger than the image of $\mathrm{Aut}\,(G)$. In our familiar example, $\mathrm{Comm}\,(\mathbb{Z}^n) \simeq GL_n(\mathbb{Q})$. Of course, conjugation in $GL_n(\mathbb{Q})$ is more easily understood than in $GL_n(\mathbb{Z})$.

For a group G, and $\phi \in \mathrm{Aut}\,(G)$, we define its *commensurated-conjugacy class* to be the set of automorphisms of G that are conjugate to ϕ in $\mathrm{Comm}\,(G)$, and we say that these automorphisms are commensurably conjugate to ϕ. Any commensurated-conjugacy class is a union of $\mathrm{Aut}\,(G)$-conjugacy classes.

Questions. *When are the commensurated-conjugacy class strictly larger than* $\mathrm{Aut}\,(G)$*-conjugacy classes? When do they consist of finitely many* $\mathrm{Aut}\,(G)$*-conjugacy classes?*

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We call the number of Aut (G)-conjugacy classes in the commensurated-conjugacy class of ϕ, the *class number* of ϕ. This terminology is suggested and supported by our familiar example, as we explain now.

Let $G = \mathbb{Z}^n$, so that Aut $(G) = GL_n(\mathbb{Z})$, and Comm $(G) = GL_n(\mathbb{Q})$. Choose $\phi \in GL_n(\mathbb{Z})$ such that its characteristic polynomial $\chi \in \mathbb{Z}[X]$ is irreducible over $\mathbb{Z}$. All $\psi \in GL_n(\mathbb{Z})$ with the same characteristic polynomial as ϕ are conjugate to ϕ in $GL_n(\mathbb{C})$, hence in $GL_n(\mathbb{Q})$ as well. Thus, they are commensurably conjugate. The Latimer-MacDuffee theorem [8, 14] states that the $GL_n(\mathbb{Z})$-conjugacy classes of such elements are in correspondence with the ideal classes of the ring $\mathbb{Z}[X]/(\chi)$. Their cardinality, which is the order of the ideal class group of the ring, is called the *class number* of the ring. In other words, the class number of ϕ in our sense, is the class number of the ring $\mathbb{Z}[X]/(\chi)$. In particular, by Minkowski's bound on ideal classes, it is finite for all $\phi \in GL_n(\mathbb{Z})$, and it is greater than 2 if and only if the ring is not a principal ideal domain.

It is worth mentioning that finiteness of class numbers is not true in general. A countable group in which the class number of an automorphism is not finite is the infinite direct product of copies of $\mathbb{Z}/3$: the automorphisms that are identity on finitely many copies, and flips on all the other copies, are all commensurably equal, but not conjugate.

We turn our attention to non-abelian counterparts of the groups $\mathbb{Z}^n$. There are two celebrated and important such classes: finite rank non-abelian free groups, and fundamental groups of surfaces of higher genus. In this work we will consider the later.

Consider a closed orientable surface Σ of genus $g \geqslant 2$. The extended Mapping Class group of Σ is the homeomorphism group of Σ quotiented by its component of identity $Homeo(\Sigma)/Homeo_0(\Sigma)$. By the Dehn-Nielsen-Baer theorem, it is isomorphic to the outer automorphism group of $\pi_1(\Sigma)$. However in order to make sense of commensuration, one must work at the level of automorphisms rather than outer automorphisms – in particular one needs a base point p in Σ. We have:

$$\mathrm{Aut}\,(\pi_1(\Sigma, p)) \twoheadrightarrow \mathrm{Out}\,(\pi_1(\Sigma)) \xrightarrow{\sim} MCG^*(\Sigma)$$

The Nielsen-Thurston classification of mapping classes of Σ distinguishes finite order mapping classes, reducible mapping classes, and pseudo-Anosovs, which are the correct objects for any consideration about irreducibility. We will say that an automorphism is a pseudo-Anosov if its image in the Mapping Class group under the above surjective homomorphism is a pseudo-Anosov. We establish the following uniform finiteness for class numbers of pseudo-Anosov automorphisms.

Theorem 1.1. *If ϕ is a pseudo-Anosov automorphism of the fundamental group of a closed orientable surface of genus $g \geqslant 2$, its class number is finite, bounded above by $((168(g-1))!)^{2g}$.*

We think that it is striking that our bound does not depend on ϕ. For comparison, in the case of $\mathrm{Aut}\,(\mathbb{Z}^2) \simeq GL_2(\mathbb{Z})$, by [8, 14], one encounters class numbers of certain rings of integers of real quadratic fields, which are conjectured to have interesting (but elusive) behavior: Gauss' class number problem conjectures that infinitely many such class numbers are 1, but Cohen and Lenstra's heuristics [4] suggest that these numbers are nevertheless unbounded when the field extension varies among real quadratic extensions.

Finiteness of the class number of a pseudo-Anosov is not a surprise. Here is a simple argument. Given ϕ a pseudo-Anosov automorphism on a surface Σ, the stretch factor, or entropy, $\lim\limits_{n\to\infty} \log|\phi^n(g)|/n$ does not depend on $g \neq 1$ in the fundamental group, nor on the metric up to quasi-isometry. Hence it is an invariant of the commusurated-conjugacy class of ϕ. It also equals the translation length of ϕ in the Teichmüller space of the associated surface, and there are finitely many conjugacy classes of a pseudo-Anosov that can have such translation length, hence can possibly be in the commensurated-conjugacy class of ϕ. This argument does not, however, provide any uniform bound.

Our upper bound is very likely non-optimal. It seems that an optimal bound would be given by the value of a subgroup-growth function of certain orbifold. These are difficult to estimate sharply [10].

Examples of commensurably conjugate automorphisms that are not conjugate are not immediate. Our result and methods however suggest a 'recipe' for producing examples of non-conjugate pseudo-Anosov automorphisms that are commensurably conjugate. We explain this recipe, illustrated with one explicit example.

Theorem 1.2. *Let Σ be a closed orientable surface of genus 3, with a base point p, and $G = \pi_1(\Sigma, p)$. There exist ϕ and ψ pseudo-Anosov automorphisms of G that are commensurably conjugate but whose images in $\mathrm{Aut}\,(G/[G,G])$ are not commensurably conjugate. In particular ϕ and ψ are not conjugate in $\mathrm{Aut}\,(G)$.*

Gauss famously conjectures (in his 'class number problem') that infinitely many real quadratic extensions of $\mathbb{Q}$ have class number equal to one. We observe that in our non-abelian analogue, the corresponding question can be settled.

Theorem 1.3. *There are infinitely many automorphisms of hyperbolic 2-orbifolds, that do not have any non-trivial power that are commensurably conjugate to each other, and that have class number one.*

It is tempting to conjecture that, for each hyperbolic surface, the pseudo-Anosov automorphisms of class number one are generic for random walks in the automorphism group.

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2. Mapping tori

Lemma 2.1. *If ϕ, ψ are automorphisms of G in the same commensurated-conjugacy class, there exists H, H' of finite index in G, and an isomorphism $\alpha : H \to H'$ such that, $\psi(H) = H, \phi(H') = H'$, $\psi|_H = \alpha^{-1} \circ \phi \circ \alpha|_H$, where $|_H$ denotes restriction to H .*

Proof. Let $[\alpha] \in \mathrm{Comm}\,(G)$ such that in $\mathrm{Comm}\,(G)$, $[\psi] = [\alpha]^{-1}[\phi][\alpha]$. Realize $[\alpha]$ by an isomorphism $\alpha : T_1 \to T_2$ for T_1, T_2 finite index subgroups of G.

Since $[\psi] = [\alpha]^{-1}[\phi][\alpha]$, there is a further finite index subgroup Y_1 of T_1 on which $\alpha^{-1} \circ \phi \circ \alpha = \psi$. Let $Y_2 = \alpha(Y_1)$. Observe that $\psi(Y_1)$ must be in T_1 but is perhaps not Y_1.

Let H denote the intersection of all subgroups in the $\mathrm{Aut}\,(G)$-orbit of Y_1, and H' its image under α. They continue to be of finite index. By construction, ψ preserves H (as well as every automorphism). We still have $\alpha^{-1} \circ \phi \circ \alpha = \psi$ after to restriction H. It follows that $\phi(H') = H'$.

□

Given $\phi \in \mathrm{Aut}\,(G)$, set $\langle t_\phi \rangle$ to be an abstract infinite cyclic group, and consider $\Gamma_\phi = G \rtimes_\phi \langle t_\phi \rangle$. In this semi-direct product, we say that G is the fiber.

Lemma 2.2. *If ϕ, ψ are automorphisms of G in the same commensurated-conjugacy class, then, Γ_ϕ and Γ_ψ are commensurable by a homomorphism commensurating the fiber.*

Proof. Let H, H' be the subgroups obtained in the previous Lemma. We make semi-direct products $H \rtimes_\psi \langle s \rangle$ and $H' \rtimes_\phi \langle q \rangle$. These two groups are isomorphic by an isomorphism restricting to α on H and sending s on q. Moreover these two groups embed as finite index subgroups of Γ_ϕ and Γ_ψ sending s and q respectively to t_ψ and t_ϕ. □

3. Surface groups and pseudo-Anosov automorphisms

Recall that in a group A, an element a is in the A-commensurator of a subgroup B if $aBa^{-1} \cap B$ has finite index in B.

Lemma 3.1. *Let G be the fundamental group of a closed orientable surface of genus $g \geqslant 2$. If ϕ and ψ are automorphisms of G in the same commensurated-conjugacy class, and if ϕ determines a pseudo-Anosov mapping class, then ψ does as well.*

Proof. By Thurston's hyperbolization of mapping tori, the group Γ_ϕ is isomorphic to a lattice in $PSL_2\mathbb{C}$, hence it is word-hyperbolic. Therefore Γ_ψ is as well, by the quasi-isometry induced by the commensuration. Hence ψ determines a pseudo-Anosov mapping class. □

Lemma 3.2. *There are faithful representations ρ_ϕ, ρ_ψ of Γ_ϕ and Γ_ψ into $PSL_2(\mathbb{C})$, such that for all $h \in H, \rho_\psi(h) = \rho_\phi(\alpha(h))$ and $\rho_\phi(t_\phi) = \rho_\psi(t_\psi)$.*

Moreover, given ρ_ϕ, the representation ρ_ψ satisfying the above is unique.

Proof. Consider the two uniform lattice representations ρ_ϕ, ρ_ψ of Γ_ϕ and Γ_ψ, respectively, into $PSL_2\mathbb{C}$.

The images contain isomorphic uniform sublattices, images of $H \rtimes \langle s\rangle$ and $H' \rtimes \langle q\rangle$ (following the notation of Lemma 2.2). Through the isomorphism between these two groups, one may see them as lattice representations of the same group $H \rtimes \langle s\rangle$. Therefore, by Mostow rigidity, one may conjugate ρ_ψ so that for all $h \in H, \rho_\psi(h) = \rho_\phi(\alpha(h))$ and also $\rho_\psi(q) = \rho_\phi(s)$.

Uniqueness follows also from Mostow rigidity: two such representations ρ_ψ, ρ'_ψ must be conjugate by a conjugator centralizing $\rho_\psi(H)$, but this later group is Zariski dense in $PSL_2(\mathbb{C})$, hence its centralizer is the center of $PSL_2(\mathbb{Z})$ which is trivial. □

Lemma 3.3. *Assume ψ_1, ψ_2 are both automorphisms of G, in the commensurated-conjugacy class of ϕ.*

If $\rho_{\psi_1}(G) = \rho_{\psi_2}(G)$, then ψ_1, ψ_2 are in the same $Aut(G)$-conjugacy class.

Proof. Since $\rho_{\psi_1}(t_{\psi_1}) = \rho_{\psi_2}(t_{\psi_2})$, therefore $\rho_{\psi_1}^{-1} \circ \rho_{\psi_2}$ is an automorphism of G that tautologically conjugates the automorphism of G given by $\rho_{\psi_1}^{-1} \circ \mathrm{ad}_{\rho_{\psi_1}(t_{\psi_1})} \circ \rho_{\psi_1}$ to $\rho_{\psi_2}^{-1} \circ \mathrm{ad}_{\rho_{\psi_1}(t_{\psi_1})} \circ \rho_{\psi_2}$. □

The following lemma is not crucial, but we record it.

Lemma 3.4. *Assume ψ_1, ψ_2 are both automorphisms of G, in the commensurated-conjugacy class of ϕ. If ψ_1, ψ_2 are in the same $Aut(G)$-conjugacy class of ϕ, then $\rho_{\psi_1}(G) = \rho_{\psi_2}(G)$.*

In particular, if ψ is conjugate to ϕ by $\alpha \in \mathrm{Aut}\,(G)$, on the entire G, then $\rho_\psi = \rho_\phi \circ \alpha$ on G, while in comparison with the previous case, the equality held only after restriction to H.

Proof. The groups Γ_{ψ_1} and Γ_{ψ_2} are isomorphic by an isomorphism that sends the fiber to the fiber and t_{ψ_1} to t_{ψ_2}. Therefore, there is a representation ρ'_{ψ_2} of Γ_{ψ_2} that has the same image as ρ_{ψ_1}. Further, the fibers have same image, as do t_{ψ_1}, t_{ψ_2}. Since $\rho_{\psi_1}(t_{\psi_1}) = \rho_\phi(t_\phi)$, this representation satisfies the properties required for ρ_{ψ_2} (as defined in Lemma 3.2). By uniqueness of ρ_{ψ_2} from Lemma 3.2, $\rho_{\psi_2} = \rho'_{\psi_2}$ and we have the result. □

Let ψ be in the commensurated-conjugacy class of ϕ. The representation ρ_ψ sends G to a subgroup of $PSL_2(\mathbb{C})$ that is normalized by $\rho_\phi(t_\phi)$ and intersects $\rho_\phi(G)$ in a common finite index subgroup. After passing to a further common finite index subgroup Y, we may assume that Y is normal in the group generated by $\rho_\phi(G) \cup \rho_\psi(G)$, hence the quotient $\langle \rho_\phi(G) \cup \rho_\psi(G) \rangle / Y$ is isomorphic to a quotient of the abstract free product $\rho_\phi(G)/Y * \rho_\psi(G)/Y$. In principle this could still be infinite. However, this is not the case, as we argue below.

Observe that the limit set of $\rho_\phi(G)$ in the sphere at infinity is the whole sphere $\partial\mathbb{H}^3$, since it is a normal subgroup in a lattice. Since it has infinite index in the said lattice, it has infinite co-volume in its action on $\mathbb{H}^3$. We may apply [9, Theorem 1.1], or [11, Theorem 3.5], to obtain that the $(PSL_2\mathbb{C})$-commensurator of $\rho_\phi(G)$ is a uniform lattice Λ_ϕ containing $\rho_\phi(\Gamma_\phi)$.

Proposition 3.5. *For all ψ in the commensurated-conjugacy class of ϕ, the group $\langle \rho_\phi(G) \cup \rho_\psi(G) \cup \{\rho_\phi(t_\phi)\} \rangle$ is a uniform lattice that surjects onto $\mathbb{Z}$ with kernel $\langle \rho_\phi(G) \cup \rho_\psi(G) \rangle$.*

In particular, $\langle \rho_\phi(G) \cup \rho_\psi(G) \rangle$ is virtually a hyperbolic surface group.

Proof. Every element of $\rho_\psi(G)$ is in the $(PSL_2\mathbb{C})$-commensurator of $\rho_\phi(G)$ since the later has a finite index subgroup which is a normal finite index subgroup of $\rho_\psi(G)$. Therefore, $\rho_\psi(G) \subset \Lambda_\phi$, and $\langle \rho_\phi(G) \cup \rho_\psi(G) \cup \{\rho_\phi(t_\phi)\} \rangle \subset \Lambda_\phi$. Since it contains the uniform lattice $\rho_\phi(\Gamma_\phi)$, it is also a uniform lattice.

The obvious map sending $\rho_\phi(t_\phi)$ to 1 and $\rho_\phi(G) \cup \rho_\psi(G)$ to 0 extends to a homomorphism to $\mathbb{Z}$.

By Selberg's lemma, there is a torsion free finite index subgroup F of $\langle \rho_\phi(G) \cup \rho_\psi(G) \rangle$. Observe, though it is not directly useful, that the finite index of F is uniformly bounded, if Selberg's lemma is applied to Λ_ϕ. Considering the quotient of $\mathbb{H}^3$ by F, the Tameness Theorem and Canary's Covering Theorem imply classically that F is a closed surface group. Observe that the conclusion can be also drawn from Stallings' Theorem [13] applied to a torsion-free finite index subgroup of $\langle \rho_\phi(G) \cup \rho_\psi(G) \cup \{\rho_\phi(t_\phi)\} \rangle$, which projects onto $\mathbb{Z}$ with finitely generated kernel.

It follows that $\langle \rho_\phi(G) \cup \rho_\psi(G) \rangle$ is virtually the fundamental group of a hyperbolic surface, on which ϕ and ψ extend to an automorphism (by $\mathrm{ad}_{\rho_\phi(t_\phi)}$).

□

Proposition 3.6. *Given ρ_ϕ, there is a subgroup F_0 of Λ_ϕ that contains $\rho_\phi(G)$ as a finite index subgroup, that is normalized by $\rho_\phi(t_\phi)$ and that contains all images $\rho_\psi(G)$ for ψ in the commensurated conjugacy class of ϕ.*

The subgroup F_0 is virtually a hyperbolic surface group, and it contains finitely many finite index subgroups isomorphic to G.

This number is an upper bound for the class number of ϕ.

Proof. Take the collection of all $\rho_\psi(G)$ for all possible representation ψ as above commensurating $\rho_\phi(G)$. Index them over the integers, let $G_0 = \rho_\phi(G)$ and for each i let $G_i = \langle G_{i-1} \cup \rho_{\psi_i}(G)\rangle$. This sequence of subgroups of Λ_ϕ is a sequence of groups containing $\rho_\phi(G)$ as a finite index subgroup, and normalized by $\rho_\phi(t_\phi)$. The previous argument shows that, up to bounded finite index, this corresponds to a sequence of decreasing finite covers of hyperbolic surfaces, and thus must terminate, by decrease of the genus. Since the index is uniformly bounded, the sequence is stationary. Let F_0 be the union of all G_i, thus the first part of the lemma holds. Considering Euler characteristic, all finite index subgroups of F_0 that are isomorphic to G have the same index, and the second part is just counting given index subgroups. The third part is a consequence of Lemma 3.3. ◻

Lemma 3.7. *The group F_0 is a hyperbolic 2-orbifold group.*

More generally, anytime a group F_0 is a finitely generated kernel of an homomorphism to $\mathbb{Z}$ of a uniform lattice in $PSL_2(\mathbb{C})$, it is a hyperbolic 2-orbifold group.

Proof. As already mentionned, such a group F_0 contains as finite index subgroup, the fundamental group of a closed orientable surface of genus $\geqslant 2$.

Therefore there exists a finite index normal subgroup K of F_0 that is a such hyperbolic surface group. One has an exact sequence

$$1 \to K \to F_0 \to Q \to 1$$

where Q is finite. Choosing a set-theoretic section $\xi : Q \to F_0$, we obtain a map $Q \to \mathrm{Aut}\,(K)$ by realizing each $\xi(q)$ by $\mathrm{ad}_{\xi(q)}$. This is not necessarily a homomorphism (since $\xi(q)$ could be changed by any element of $\xi(q)K$), but it descends to a homomorphism $\varphi : Q \to \mathrm{Out}\,(K)$.

We first argue that this homomorphism is injective. If $q \in Q\backslash\{1\}$ is sent to the identity, then $\mathrm{ad}_{\xi(q)}$ equals conjugation by some $k \in K$, hence qk^{-1} is in the centralizer of K. However, as K is Zariski dense in $PSL_2(\mathbb{C})$ (which has trivial center), it has trivial centralizer.

By the Dehn-Nielsen-Baer theorem, $\mathrm{Out}\,(K)$ is isomorphic to the extended Mapping Class Group of the surface Σ of which K is the fundamental group. By the Nielsen realization theorem [7], there is a hyperbolic metric on Σ, a subgroup S of the finite group of (metric) symmetries of S, an identification $K \xrightarrow{\simeq} \pi_1(\Sigma, v)$ (for a base point $v \in \Sigma$ with trivial S-stabilizer), and an isomorphism $\varphi(Q) \xrightarrow{\sim} S$ in a way that the action of S by outer-automorphisms on $\pi_1(\Sigma, v)$ is the equivariant image of the action of $\varphi(Q)$ by outer-automorphisms on K.

Consider now the orbifold quotient $\overline{\Sigma}$ of Σ under the action of S. We claim that its orbifold fundamental group (with base point $\bar{v}$) is isomorphic to the extension F_0. Indeed, it is also a group extension of the form

$$1 \to \pi_1(\Sigma) \to \pi_1(\overline{\Sigma}) \to S \to 1$$

with the same representation $S \to \mathrm{Out}\,(\pi_1(\Sigma))$ as φ for F_0. Since the center of K is trivial, the classification of extensions (see for instance [3, Coro. IV(6.8)]) indicates that there is only one extension of K by Q, realizing φ, up to equivalence. Therefore $\pi_1(\overline{\Sigma}, \bar{v}) \simeq F_0$, and F_0 is a 2-orbifold group. □

Being commensurable with a hyperbolic surface group, the orbifold of which F_0 is the fundamental group is an orbifold that carries a hyperbolic metric. It must be of area greater than that of the smallest orbifold (the quotient of $\mathbb{H}^2$ by the the Coxeter triangle group $(2,3,7)$), which is $\pi/42$. On the other hand, if $\chi(G)$ is the Euler characteristic of G, it is the fundamental group of a hyperbolic surface of area $-2\pi\chi(G)$. We record this in the following.

Corollary 3.8. *The index of subgroups of F_0 that are of finite index and isomorphic to G is bounded above by $-84\chi(G)$, or $168(g-1)$. Here g is the genus of a surface Σ with fundamental group G.*

Lemma 3.9. *If g is the genus of a surface Σ with fundamental group G, then the rank of F_0 is at most $2g$.*

Proof. The group F_0 is fundamental group of an orbifold $\mathcal{O}$ of which the surface Σ is a ramified cover. Let $\mathcal{O}_{top}$ be the underlying topological surface of $\mathcal{O}$, and $\mathcal{O}_{sing}$ the collection of singularities of $\mathcal{O}$ on $\mathcal{O}_{top}$. For p a singularity let e_p denote the order of its isotropy group. Classical presentations of 2-orbifolds show that the rank of F_0 is at most the rank of the fundamental group of $\mathcal{O}_{top}$ plus the cardinality of $\mathcal{O}_{sing}$. The surface Σ is a branch cover of $\mathcal{O}_{top}$. Let N be the degree of the cover. The Riemann-Hurwitz formula for branch covers of (topological) surfaces gives $\chi(\Sigma) = N\chi(\mathcal{O}_{top}) - \sum_{\mathcal{O}_{sing}}(e_p - 1)$. Writing the rank of the (classical) fundamental group of $\mathcal{O}_{top}$ as r_t, one has $\chi(\Sigma) = N(2 - r_t) - \sum(e_p - 1)$. The rank r of the orbifold group F_0 hence satisfies $Nr \leqslant 2N - \chi(\Sigma) - N\sum(e_p - 1) + N|\mathcal{O}_{sing}| \leqslant 2N - \chi(\Sigma)$. Finally $r \leqslant 2 + 2(g-1)/N = \frac{2g}{N} + 2 - \frac{2}{N}$. This in turn is less than $2g$ for $g \geqslant 2, N \geqslant 1$. □

Corollary 3.10. *The number of subgroups of index k in F_0 is at most $(k!)^{2g}$.*

Proof. For any index k subgroup, F_0 acts by permutation on its k cosets. Therefore, the map $Hom(F_0, \mathfrak{S}_k) \to \{H, H < F_0\}$ that assigns to every $\pi : F_0 \to \mathfrak{S}_k$ the stabilizer of $\{1\}$ of the action, surjects $Hom(F_0, \mathfrak{S}_k)$ onto the set of index $\leqslant k$ subgroups of F_0. There are at most $(k!)^{rank(F_0)}$ such homomorphisms. By the previous Lemma 3.9, we obtain the desired bound. □

A slightly better formula for surfaces was actually given by A.D. Mednykh, [10, Thm. 14.4.1].

We thus obtain a proof of Theorem 1.1 by putting together Proposition 3.6, Corollary 3.8, Corollary 3.10.

However huge, this bound is nonetheless independent of ϕ, which was perhaps unexpected. Hence, the bound obtained in Proposition 3.6 is actually uniform over the elements of $\mathrm{Aut}\,(G)$ that define pseudo-Anosov mapping classes of the surface of which G is fundamental group.

4. Recipe for commensurably-conjugate mapping classes

In this section we propose a recipe to construct examples, as suggest by the structure obtained in Proposition 3.6. We will also produce an explicit example, thus proving Theorem 1.2, in subsection 4.1.

Start with a hyperbolic surface, or 2-orbifold Σ with a base point v. Consider a single pseudo-Anosov mapping class $[\Phi]$, and realize it as an automorphism Φ of $\pi_1(\Sigma, v)$.

Consider two finite sheeted characteristic covers that are of the same degree but do not correspond to subgroups in the same orbit under the automorphism group of $\pi_1(\Sigma, v)$. For instance, one can be an abelian cover while the other is not. In fact, it is not necessary that the covers are characteristic, they only need to have fundamental groups preserved by Φ. It could thus be any cover of degree k, up to replacing Φ by $\Phi^{k!}$ (which preserves all subgroups of index k).

Let G_1, G_2 be the subgroups of $\pi_1(\Sigma, v)$ that are fundamental groups of the respective covers (Σ_1, v_1) and (Σ_2, v_2).

The mapping class Φ defines ϕ and ϕ', automorphisms of subgroups G_1, G_2.

Moving up to a common cover of Σ_1, Σ_2, corresponding the intersection of the subgroups G_1, G_2, (which is preserved by Φ), we obtain a subgroup on which ϕ, ϕ' coincide.

Turn ϕ' into an automorphism ψ of G_1 by picking an isomorphism $\alpha : G_1 \to G_2$.

Claim 4.1. *The automorphisms ϕ and ψ of the group G_1 are in the same commensurated-conjugacy class in* $\mathrm{Comm}\,(G_1)$*, but perhaps not conjugate in* $\mathrm{Aut}\,(G_1)$*.*

Write $H = G_1 \cap G_2$. On H, the restrictions of ϕ and ϕ' coincide. Since $\alpha \circ \psi \circ \alpha^{-1} = \phi'$ on G_2, we have that $\phi|_H = \alpha \circ \psi \circ \alpha^{-1}|_H$.

We thus have ϕ, ψ commensurably conjugated by α^{-1} on H.

It is not clear which condition would ensure that they are not conjugate in $\mathrm{Aut}\,(G_1)$, even though one might suspect that this happens somewhat generically.

4.1. **Explicit example.** While it appears likely that a construction as above would give examples of non-conjugated automorphisms, an actual explicit example is necessary to settle the case. Our example will be for two mapping classes on a surface of genus three.

Proposition 4.2. *Let S be a closed orientable surface of genus three, and $p \in S$. There exist two pseudo-Anosov mapping classes of S that are realized by automorphisms of the fundamental group $\pi_1(S,p)$ that are commensurably conjugate, but are not conjugate in $MCG(S)$.*

In particular they are not conjugate in $Aut(\pi_1(S,p))$.

Proof. Consider Σ a genus 2 surface, with base point v, and two curves that fill Σ, μ, λ, with μ separating. Writing $\pi_1(\Sigma, v) = \langle \alpha, \eta, \gamma, \delta \,|\, \alpha\beta\alpha^{-1}\beta^{-1} = \gamma^{-1}\delta\gamma\delta^{-1}\rangle$, we specify the curve μ to be the commutator and for the sake of being explicit, $\lambda = \gamma\beta\delta\gamma\beta\alpha^{-1}\delta\gamma\beta$, see Figure 1.

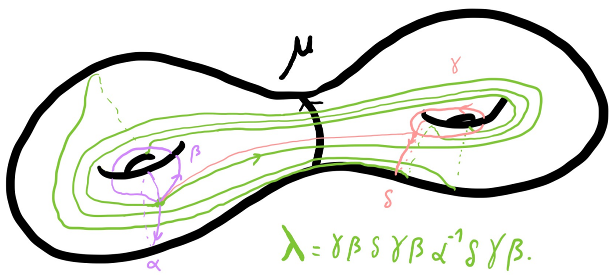

FIGURE 1. The curve λ used for twisting

Consider the mapping classes obtained by the product of Dehn twists $\phi = \tau_{\gamma_1}^{n}\tau_{\lambda}^{2m}$. The construction of Penner-Thurston [12] ensures that, for $n, m >> 1$, this defines a pseudo-Anosov mapping class on Σ.

Consider the two 2-sheeted covers Σ_1, Σ_2 of Σ whose fundamental groups are the kernels of the two homomorphisms $\pi_1(\Sigma, v) \to \mathbb{Z}/2$ given by

$$\alpha, \delta \mapsto 0, \ \beta, \gamma \mapsto 1$$

on the one hand, and

$$\alpha, \gamma, \delta \mapsto 0 \ \beta \mapsto 1$$

on the other. These are two surfaces of genus 3, pictured in Figure 2.

In both covers, the twist τ_μ lifts as a mapping class that is trivial in homology.

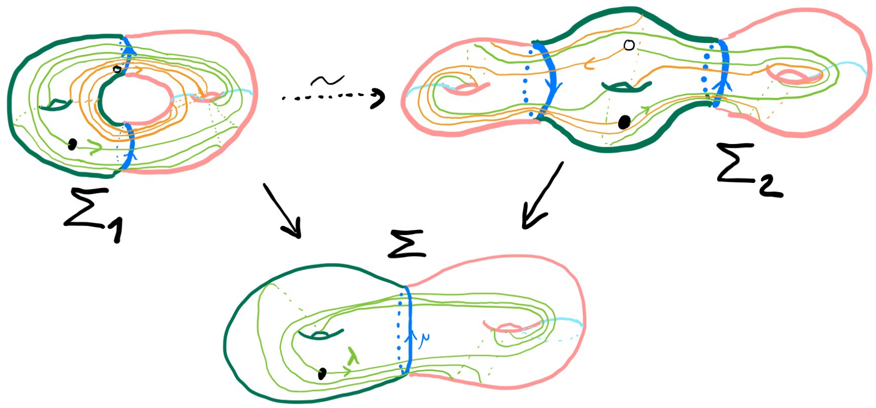

FIGURE 2. The two covers Σ_1, Σ_2 of Σ, with the lifts of λ (in green and orange) and μ (in blue). The second lift of the base point is a circled white dot.

Observe that in Σ_1 the curve λ lifts as two disjoint simple closed curves, but on Σ_2 it lifts to two arcs, that when concatenated make a simple closed curve. In other words, the squared twist τ_λ^2 lifts to both covers, as a mapping class.

In Σ_2, it lifts as a single Dehn twist over a simple (non-separating) curve. It follows that in this cover, the mapping class induced by ϕ induces, on the homology, the identity plus a rank one endomorphism.

In Σ_1, however, λ lifts as a pair of simple closed curves, and τ_λ^2 lifts as a product of two Dehn twists over the two disjoint simple closed curves. It is not automatic that this product would be different in homology as a single Dehn twist, but it turns out that, by a slightly tedious but simple computation, the action on the homology of this mapping class is indeed the identity plus a rank 2 endomorphism.

Indeed, in homology of the cover Σ_1, one can form an explicit basis, as

$$(\alpha, \beta^2, \gamma^{-1}\beta^{-1}, \delta, \gamma^2, \beta\alpha^{-1}\beta^{-1}\alpha),$$

and the matrix of the homology representation of τ_λ is the following matrix M in this basis.

On the right, $M - I_6$ is of rank 2.

$$M = \begin{pmatrix} -2 & -2 & -1 & -3 & 4 & 0 \\ 9 & 4 & 0 & 9 & -6 & 9 \\ 9 & 0 & -2 & 9 & 0 & 18 \\ 6 & 4 & 2 & 7 & -8 & 0 \\ 9 & 3 & 0 & 9 & -5 & 9 \\ 0 & -1 & -1 & 0 & 2 & 4 \end{pmatrix}, \qquad M - I_6 = \begin{pmatrix} -3 & -2 & -1 & -3 & 4 & 0 \\ 9 & 3 & 0 & 9 & -6 & 9 \\ 9 & 0 & -3 & 9 & 0 & 18 \\ 6 & 4 & 2 & 6 & -8 & 0 \\ 9 & 3 & 0 & 9 & -6 & 9 \\ 0 & -1 & -1 & 0 & 2 & 3 \end{pmatrix}$$

Therefore, the two lifts of ϕ to the two surfaces of genus three are not conjugate by any isomorphism between the surfaces groups.

To conclude, our explicit example of a pair of pseudo-Anosov automorphisms on the fundamental group of a surface of genus 3, that are commensurably conjugate, but not conjugate, are these two lifts of ϕ. See Figure 2. These are the mapping classes obtained, in each picture of genus 3 surface, by twisting k times over the pair of blue curves, and then twisting $2m$ times along the pair of green and orange curves, in Σ_1, and only m times along the single green-and-orange curve in Σ_2. □

5. Infinitely many class number one pseudo-Anosovs

We start this section by a lemma which is well known to specialists.

Lemma 5.1. *There exists $V_0 > 0$, and infinitely many conjugacy classes of maximal non-arithmetic uniform lattices in $PSL_2(\mathbb{C})$, with co-volume $\leqslant V_0$.*

Proof. Take M_0 a non-compact, finite volume hyperbolic 3-manifold with one cusp (a complement of a hyperbolic knot for instance), and consider a sequence M_n of compact hyperbolic manifolds obtained by deeper and deeper hyperbolic Dehn filling on M_0. Their volumes V_n are bounded above by the volume V_0 of the initial manifold, and accumulate on V_0. It follows from a theorem of Borel [2, Theorem 8.2] that once extracted a subsequence, they have non-arithmetic fundamental group Γ_n in $PSL_2(\mathbb{C})$. By another theorem of Borel, on commensurability [2, Main Theorem], for all these n, there exists a unique biggest uniform lattice Γ_n^+ containing Γ_n, necessarily as a finite index subgroup. For each n, the co-volume of Γ_n^+ in $\mathbb{H}^3$ divides V_n, and is larger than a positive constant [5, 6]. Therefore, since V_n accumulates on V_0, the Γ_n^+ are eventually all of different co-volume, hence non-conjugate.

□

We finally prove Theorem 1.3.

Proof. Let us denote $\Lambda_i^0, i \in \mathbb{N}$ a sequence of representatives of different conjugacy classes of maximal non-arithmetic uniform lattices in $PSL_2(\mathbb{C})$.

In each Λ_i^0, by Agol virtual fibration theorem [1], there is a finite index subgroup Λ_i^1 that maps onto $\mathbb{Z}$ with finitely generated kernel. Since the index of Λ_i^1 in Λ_i^0 is finite, the chains of strict inclusion of subgroups from Λ_i^0 to Λ_i^1 are finite. Therefore, there exists a maximal subgroup Λ_i of Λ_i^0, containing Λ_i^1, and surjecting on $\mathbb{Z}$ (by an homomorphism $\pi_i : \Lambda_i \to \mathbb{Z}$). The index of Λ_i^1 in Λ_i is necessarily finite, less than $[\Lambda_i^1, \Lambda_i^0]$, and the kernel of π_i has therefore finite index in the fiber of Λ_i^1. In particular it is finitely generated. By Lemma 3.7, it is therefore the fundamental group of a hyperbolic 2-orbifold. Let G_i be this subgroup, and $t_i \in \Lambda_i$ such that $\Lambda_i = G_i \rtimes \langle t_i \rangle$.

Denote by ϕ_i the automorphism of G_i given by the conjugation by t_i.

Let us compute the class number of ϕ_i. Let ψ_i an commensurably conjugated automorphism, we are to prove that it is conjugate to ϕ_i in $\mathrm{Aut}\,(G_i)$. By Proposition 3.5, the lattice Λ_i lives in a larger lattice containing a copy M_i of the mapping torus of ψ_i and surjecting on $\mathbb{Z}$ with the fibers of M_i and Λ_i in the kernel. By Borel Theorem on commensurability [2], since Λ_i is non-arithmetic, there is a unique biggest lattice containing Λ_i, and it is therefore Λ_i^0. Therefore, by maximality of Λ_i, $\langle \mathrm{M}_i \cup \Lambda_i \rangle = \Lambda_i$, and it follows from Lemma 3.3 that ϕ_i and ψ_i and conjugated in $\mathrm{Aut}\,(G_i)$. Therefore, the class number of ϕ_i is one.

The same argument also reveals that for $i \neq j$, ϕ_i and ϕ_j do not have any non-trivial power that are commensurably conjugate: if they did, the mapping tori would be represented in $PLS_2(\mathbb{C})$ in a common lattice that fibers with finitely generated kernel, and Borel Theorem again ensures that the maximal lattice containing them is unique, and therefore $i = j$. ☐

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