On Discreteness of Commensurators

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Summary and Motivation

Theorem

(Margulis) An irreducible lattice Γ in a semi-simple Lie group L is arithmetic iff the commensurator $\mathrm{Comm}(\Gamma)$ *is dense.*

$\mathrm{Comm}(\Gamma) = \{g \in L : g\Gamma g^{-1} \cap \Gamma$ is of finite index in both $\Gamma, g\Gamma g^{-1}\}$

Question: (Shalom) If Γ is a Zariski dense, infinite covolume, discrete subgroup of a semi-simple Lie group L, describe $\mathrm{Comm}(\Gamma)$. (i.e. Is it discrete?)

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Answer: (M–) Yes, if
a) The limit set $\Lambda_\Gamma \subset \partial_F G$ (=Furstenberg boundary) is not invariant under a simple factor, OR
b) Γ is finitely generated and $G = \mathrm{PSL}_2(\mathbb{C})$.

Theorem

(Greenberg '74) If Γ is a Zariski dense, finitely generated, infinite covolume, discrete subgroup of $G = \mathrm{PSL}_2(\mathbb{C})$, and $\Lambda_\Gamma \neq \mathbb{S}^2_\infty$ then $\mathrm{Comm}(\Gamma)$ is discrete.

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Non-full Limit sets

- Γ Zariski-dense infinite covolume subgroup of a semi-simple Lie group $L = Isom(X)$.
- X a rank one symmetric space.
- Let $\overline{\text{Comm}(\Gamma)}$ be the closure of $\text{Comm}(\Gamma)$.
- L_0 = connected component of the identity, with Lie algebra $\mathfrak{l}_0$–invariant under adjoint representation.

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- (CRUCIAL!) If L_0 is non-compact, then $\Lambda_{L_0} = \Lambda_{\overline{\mathrm{Comm}(\Gamma)}} = \Lambda_\Gamma$ is invariant under L_0.
- Zariski density implies $L_0 = L$. Hence $\Lambda_\Gamma = \partial X$.
- L_0 compact. L_0 fixes some point $x \in X$. L_0 is normal in L. Therefore L_0 fixes all $x \in X$. Therefore L_0 is trivial.

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Non-full Limit sets–Higher Rank

Theorem

(Benoist '97) Let $\Gamma \subset G = \mathrm{Isom}(X)$ be a Zariski dense subgroup. Then Λ_Γ is the unique minimal closed Γ-invariant subset of the Furstenberg boundary G/P.

This Theorem allows us to push through the crucial step in the previous page.

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Full Limit Sets–Kleinian Groups

For the rest of the talk, Γ-f.g. Kleinian group with $\Lambda_\Gamma = \mathbb{S}^2_\infty$.

Then G (as an abstract group) is hyperbolic relative to its parabolic subgroups.

For concreteness: G = surface group with or without parabolics.

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Theorem (M–) G –f.g. Kleinian group. $i : \Gamma_G \to \mathbb{H}^3$ identifies Cayley graph of G with orbit of a point in $\mathbb{H}^3$.
Then i extends continuously to a map $\hat{i} : \widehat{\Gamma_G} \to \mathbb{D}^3$, where $\widehat{\Gamma_G}$ denotes the (relative) hyperbolic compactificaton of Γ_G.
Let ∂i denote the restriction of $\hat{i}$ to the boundary $\partial\Gamma$ of Γ.
Then $\partial i(a) = \partial i(b)$ for $a \neq b \in \partial\Gamma$ if and only if a, b are either ideal end-points of a leaf of an ending lamination of G, or ideal boundary points of a complementary ideal polygon.

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Cannon-Thurston Relations

A **Cannon-Thurston map** $\hat{i}$ from $\widehat{G}$ to $\widehat{X}$ is a continuous extension of i. The restriction of $\hat{i}$ to ∂G will be denoted by ∂i.

The map ∂i induces a relation $\mathcal{R}_{CT}$ on ∂G where $x \sim y$ if $\partial i(x) = \partial i(y)$ for $x, y \in \partial G$.

Distinct pairs of points identified by ∂i will be denoted as $\mathcal{R}^2_{CT}$, which is a subset of $\partial^2(G)$.

$\mathcal{R}_{CT}$ is a closed relation on ∂G

Lemma

Suppose G acts on X without accidental parabolics If $(x, y) \in \mathcal{R}_{CT}$ and $x \neq y$, then x cannot be a pole of G.

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Cannon-Thurston Relations (Contd.)

Density of Orbits of cosets of $\mathcal{R}_{CT}$ in the Hausdorff metric:

Let $K \subset \mathcal{R}_{CT}$ be a coset (equivalence class) of the relation.

Let $C_c(\partial G)$ denote the space of closed subsets of ∂G with the Hausdorff metric.

Then for all $x \in \partial G$, the singleton set $\{x\}$ is an accumulation point of $\{g.K : g \in G\}$.

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$\overline{f} \in \mathrm{Comm}(G)$ implies $f \in \mathrm{Homeo}(\partial G)$.

"Non-proof": Pull $\mathrm{Comm}(G)$*-action back to* ∂G*. Then* $\overline{\mathrm{Comm}(G)}$ *preserves closed totally disconnected relation. Hence its a closed totally disconnected subgroup of L–discrete.*

Let f_n be a sequence of homeomorphisms of $(\partial G, d)$ that preserves the cosets of $\mathcal{R}_{CT}$, where d denotes some visual metric.

Let $\overline{f_n}$ denote the induced homeomorphisms of Λ_G.

If $f_n \to id$ in the uniform topology on $Homeo(\partial G)$ then $\overline{f_n} \to id$ in the uniform topology on $Homeo(\Lambda_G)$.

Conversely, if $\overline{f_n} \to id$ in the uniform topology on $Homeo(\Lambda_G)$ then for every pole $p \in \partial G$, $d(p, f_n(p)) \to 0$.

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Totally Degenerate Surface Groups

Lemma

Let $\overline{f_n} \in Comm(H)$ be a sequence of commensurators converging to the identity in $\mathrm{Isom}(\mathbb{H}^3)$ and let f_n be the induced homeomorphisms on the (relative) hyperbolic boundary $\partial\pi_1(S)(= S^1)$ of the group $\pi_1(S)$. Then $f_n \rightarrow Id \in Homeo(S^1)$.

Proof Idea Follows.

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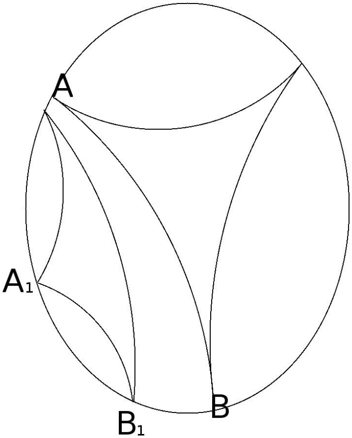

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Let H be a totally degenerate surface Kleinian group. Then the commensurator $\mathrm{Comm}(H)$ *of H is discrete in* $\mathrm{PSL}_2(\mathbb{C})$.

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By previous Lemma, for any ideal polygon Δ with boundary in the ending lamination there exists $N = N(\Delta)$ such that f_n fixes all the vertices of Δ for all $n \geq N$.

Let $z_\Delta \in S^2_\infty$ be the common image of the end-points of Δ under the Cannon-Thurston map.

Choose ideal polygons $\Delta_1, \cdots, \Delta_k$ such that the common images $\{z_1, \cdots, z_k\}$ is Zariski dense in S^2_∞.

Hence for all $n \geq max_{i=1\cdots k}\{N(\Delta_i)\}$, $\overline{f_n} = Id$. $\Box$

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