COMMUTATORS, COMMENSURATORS, AND $\mathrm{PSL}_2(\mathbb{Z})$

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ABSTRACT. Let $H < \mathrm{PSL}_2(\mathbb{Z})$ be a finite index normal subgroup which is contained in a principal congruence subgroup, and let $\Phi(H) \neq H$ denote a term of the lower central series or the derived series of H. In this paper, we prove that the commensurator of $\Phi(H)$ in $\mathrm{PSL}_2(\mathbb{R})$ is discrete. We thus obtain a natural family of thin subgroups of $\mathrm{PSL}_2(\mathbb{R})$ whose commensurators are discrete.

1. INTRODUCTION

The Commensurability Criterion for Arithmeticity due to Margulis [12] [15, 16.3.3] says that among irreducible lattices in semi-simple Lie groups, arithmetic lattices are characterized as those that have dense commensurators. During the past decade, Zariski dense discrete subgroups of infinite covolume in semi-simple Lie groups, also known as *thin subgroups* [17], have gained a lot of attention. Heuristically, thin subgroups should be regarded as non-arithmetic, even though the essential difference between a thin group and a lattice is the the former has infinite covolume in the ambient Lie group. A question attributed to Shalom makes this heuristic precise in the following way:

Question 1.1. [8] *Suppose that Γ is a thin subgroup of a semisimple Lie group G. Is the commensurator of Γ discrete?*

Let X be the symmetric space of non-compact type associated to G and ∂X denote its Furstenberg boundary. Question 1.1 has been answered affirmatively in the following cases:

(1) Let Λ_Γ denote the limit of Γ on ∂X. If $\Lambda_\Gamma \subsetneq \partial X$, then the answer to Question 1.1 is affirmative ([6] for $G = \mathrm{PSL}_2(\mathbb{C})$ and [13] in the general case).

(2) If $G = \mathrm{PSL}_2(\mathbb{C})$ and Γ is finitely generated, then the answer to Question 1.1 is affirmative [8, 13].

Question 1.1 thus remains unaddressed when

(1) $\Lambda_\Gamma = \partial X$, and further,
(2) Γ is not a finitely generated subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

Curiously, Question 1.1 remains open even for thin subgroups Γ of the simplest non-compact simple Lie group $G = \mathrm{PSL}_2(\mathbb{R})$ with Γ satisfying the condition $\Lambda_\Gamma = \partial \mathbf{H}^2 = S^1$. It is easy to see that such a Γ cannot be finitely generated. In this paper, we shall study commensurators of certain natural infinite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. If $\Gamma < G$ is an arbitrary subgroup of a group G, we define

$$\mathrm{Comm}_G(\Gamma) = \{g \in G \mid [\Gamma : \Gamma^g \cap \Gamma] \text{ and } [\Gamma^g : \Gamma^g \cap \Gamma] < \infty\}.$$

Here, we use exponentiation notation to denote conjugation, so that $\Gamma^g = g^{-1}\Gamma g$.

1.1. **The main result.**

In this paper, we will concentrate on thin subgroups of $\mathrm{PSL}_2(\mathbb{R})$, which are of particular interest because of their intimate connections to hyperbolic geometry via the identification

$$\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Isom}^+(\mathbf{H}^2).$$

For an integer $k \geqslant 2$, we will write $\Gamma(k) < \mathrm{PSL}_2(\mathbb{Z})$ for the *level k principal congruence subgroup*, which is to say the kernel of the map $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/k\mathbb{Z})$ given by reducing the entries modulo k.

As a matter of notation, we will write $\mathrm{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}}$ for the projectivization of the set of matrices in $\mathrm{SL}_2(\mathbb{R})$ which differ from a matrix in $\mathrm{GL}_2(\mathbb{Q})$ by a scalar matrix which is a square root of a rational number. That is, we have $A \in \mathrm{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}}$ if there is a representative of A in $\mathrm{SL}_2(\mathbb{R})$, a rational number $q \in \mathbb{Q}$, and a matrix $B \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$A = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q} \end{pmatrix} \cdot B.$$

Since we may also write

$$A = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q^{-1}} \end{pmatrix} \cdot B'$$

for some $B' \in \mathrm{SL}_2(\mathbb{Q})$ and since the matrix

$$\begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q^{-1}} \end{pmatrix}$$

normalizes $SL_2(\mathbb{Q})$, it is easy to see that $PSL_2(\mathbb{Q})\sqrt{\mathbb{Q}}$ forms a group which can be viewed as the join

$$PSL_2(\mathbb{Q}) \cdot \left\langle \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q^{-1}} \end{pmatrix} \mid q \in \mathbb{Q} \right\rangle < PSL_2(\mathbb{R}).$$

We need to consider $PSL_2(\mathbb{Q})\sqrt{\mathbb{Q}}$ because of the ambient group where the commensurator lives. We recall (see [15, p. 92] or [9, Ex. 6d]) that

$$\mathrm{Comm}_{PSL_2(\mathbb{R})}(PSL_2(\mathbb{Z})) = PSL_2(\mathbb{Q})\sqrt{\mathbb{Q}},$$

which is dense in $PSL_2(\mathbb{R})$. As a consequence, it is easy to see that if $\Gamma < PSL_2(\mathbb{Z})$ has finite index then

$$\mathrm{Comm}_{PSL_2(\mathbb{R})}(\Gamma) = PSL_2(\mathbb{Q})\sqrt{\mathbb{Q}}.$$

For an arbitrary group G, we recall the definition of the *lower central series* and the *derived series* of G. For the lower central series, we define $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. The derived series is defined by $D_1(G) = G$ and $D_{i+1}(G) = [D_i(G), D_i(G)]$. We will often use the notation $G' = [G, G]$ for the derived subgroup of G. Observe that

$$H' = D_2(G) = \gamma_2(G),$$

and if G is a free group then for $i \geqslant 2$ we have that $\gamma_i(G)$ and $D_i(G)$ are both properly contained in G as infinitely generated characteristic subgroups of infinite index.

The purpose of this article is to establish following result, which answers Question 1.1 in the affirmative for, perhaps, the most 'arithmetically' defined examples:

Theorem 1.2. *Let $H < PSL_2(\mathbb{Z})$ be a finite index normal subgroup, and suppose that $H < \Gamma(k)$ for some $k \geqslant 2$. Suppose furthermore that $\Phi(H) \neq H$ is a term of the lower central series or derived series of H. If*

$$g \in \mathrm{Comm}_{PSL_2(\mathbb{R})}(\Phi(H)),$$

then we have that $g^2 \in PSL_2(\mathbb{Z})$. In particular, $\mathrm{Comm}_{PSL_2(\mathbb{R})}(\Phi(H))$ is discrete.

In Theorem 1.2, it is unclear to the authors how to weaken the hypothesis that H be normal in $PSL_2(\mathbb{Z})$, as well as the assumption that H be contained in a principal congruence subgroup. See the conjectural picture below.

In our notation, we will suppress which series for H as well as which term we are considering, since this will not cause particular confusion. Observe that since H has finite index in $PSL_2(\mathbb{Z})$ and since $\Phi(H)$ is a non-elementary normal subgroup of H, we have that on the level of limit sets,

$$\Lambda(PSL_2(\mathbb{Z})) = \Lambda(H) = \Lambda(\Phi(H)) = S^1.$$

In particular, the group $\Phi(H)$ falls outside of the purview of extant results concerning the commensurators of thin subgroups.

We remark that in order to establish a perfect analogy with Margulis' Arithmeticity Theorem, one would like to show that $\Phi(H)$ has finite index in $\mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$. This, however, is simply not feasible. On the one hand, if $H = \Gamma(k)$ for some $k \geqslant 2$ then $\Phi(H)$ is normal in $\mathrm{PSL}_2(\mathbb{Z})$, whence

$$\mathrm{PSL}_2(\mathbb{Z}) < \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H)).$$

On the other hand, the index of $\Phi(H)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is always infinite under the hypotheses of Theorem 1.2.

The techniques used in the proof of Theorem 1.2 differ widely from those used in [8, 13]. These papers used an action on a topological space and discreteness of the commensurator stemmed from the fact that the commensurator preserved a 'discrete geometric pattern' (in the sense of Schwartz, cf. [18, 19, 2, 14]). In this paper we use an *algebraic* action from Chevalley-Weil theory and homological algebra as a replacement in order to conclude discreteness of the commensurator. Since the proof is somewhat tricky and consists of a number of modular pieces, we outline the steps involved (using the notation of Theorem 1.2 above):

(1) We discuss some basic facts about principal congruence groups in Section 1.2, and observe in Section 2.1 that the normalizer of $\Phi(H)$ lies in $\mathrm{PSL}_2(\mathbb{Z})$.

(2) Section 2.2 introduces the first technical tool in the paper based on Chevalley-Weil theory. We prove (Lemma 2.3) that if $g \in \mathrm{PSL}_2(\mathbb{R})$ conjugates $H < \mathrm{PSL}_2(\mathbb{Z})$ to $H^g \neq H$ and both H, H^g are contained in a common free subgroup F, then g does not lie in the commensurator of $\Phi(H)$.

(3) Section 3 is devoted to proving that if $g \in \mathrm{PSL}_2(\mathbb{R})$ commensurates $\Phi(H)$, then in fact $g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$ (Lemma 3.1). We will furnish two different proofs of this fact. The first in Section 3.1 uses ideas from invariant trace fields and quaternion division algebras and is number-theoretic in flavor. The second proof will be relegated to the appendix (see Sections A.1 and A.2). It will be more elementary and quite explicit but not nearly as pithy. The two different approaches to proving Lemma 3.1 yield two generalizations, namely Propositions 3.2 and A.9. These results are quite different in flavor; hence our decision to retain both approaches for possible future applications.

(4) In Section 4 we introduce the last tool, which to the knowledge of the authors is completely novel: a partial action on homology. We call this partial action a pseudo-action and study it on homology classes carried by cusps to complete the proof of Theorem 1.2.

A conjectural picture. The proof of Theorem 1.2 draws on several different areas of mathematics, including hyperbolic geometry, homological algebra, noncommutative algebra, and Galois theory. In order to make all the arguments work, we were forced to adopt the hypothesis that H be contained as a normal subgroup of a principal congruence subgroup. However, we expect the following more general statements to hold:

Conjecture 1.3. *Let $H < \mathrm{PSL}_2(\mathbb{Z})$ be a finite index subgroup with $b_1(H) \geqslant 1$. Then $\mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$ is discrete provided that $H \neq \Phi(H)$. More generally, let K be an infinite index normal subgroup of a lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ such that $|K| = \infty$. Then $\mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(K)$ is discrete.*

In the first statement of Conjecture 1.3, the Betti number assumption $b_1(H) \geqslant 1$ is simply to guarantee that H' has infinite index in H. It would also be a reasonable alternative assumption to require H to be torsion-free. Normality of the infinite index subgroups is always assumed in order to make the limit set coincide with the full circle.

1.2. **Preliminaries on principal congruence subgroups.** In this subsection, we gather some well-known facts about principal congruence subgroups which will be useful in the sequel. We include proofs for the convenience of the reader and to keep the discussion as self-contained as possible.

Lemma 1.4. *Let $k \geqslant 2$. Then $\Gamma(k)$ is a free group.*

Proof. The quotient $\mathbf{H}^2/\mathrm{PSL}_2(\mathbb{Z})$ is the $(2, 3, \infty)$ hyperbolic orbifold. It follows that if $g \in \mathrm{PSL}_2(\mathbb{Z})$ has finite order then it is conjugate to (the image of) one of the two matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

or

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

of orders 2 and 3 respectively. Let $q\colon \mathrm{PSL}_2(\mathbb{Z}) \to Q$ be a finite quotient. If A and B do not lie in the kernel of q then $\ker q$ is torsion-free. Indeed, if $\ker q$ contains a torsion element then this element would be conjugate to either A or B, so that normality would imply that A or B lies in $\ker q$, contrary to the assumption. Since A and B are clearly nontrivial in $\mathrm{PSL}_2(\mathbb{Z})/\Gamma(k)$, we see that $\Gamma(k)$ must be torsion-free. Since $\Gamma(k)$ is the fundamental group of a real two-dimensional orientable manifold which is not closed, we see that $\Gamma(k)$ is free. □

Note that the modular surface $\mathbf{H}^2/\mathrm{PSL}_2(\mathbb{Z})$ has exactly one cusp. As an element of the orbifold fundamental group of the modular surface, the free

homotopy class of the cusp is generated by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Lemma 1.5. *Let $k \geqslant 2$. The hyperbolic manifold $\mathbf{H}^2/\Gamma(k)$ has at least three cusps.*

Proof. We may reduce to the case where k is a prime, since if $p|k$ is a prime divisor of k then $\Gamma(k) < \Gamma(p)$. The map

$$\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$$

is surjective (as can be deduced from examining generating sets of $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$), and its image has order

$$p(p+1)(p-1)/2$$

when p is odd and order 6 when $p = 2$. The order of the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ is exactly p, so that we see that there are exactly

$$(p+1)(p-1)/2$$

distinct cusps in $\mathbf{H}^2/\Gamma(p)$ when $p \geqslant 3$ and exactly three cusps when $p = 2$. The lemma now follows.   □

The following consequence is immediate from elementary surface topology:

Corollary 1.6. *If $k \geqslant 2$, then any loop about a cusp in $\mathbf{H}^2/\Gamma(k)$ represents a nontrivial integral homology class. Moreover, two loops about distinct cusps in $\mathbf{H}^2/\Gamma(k)$ represent distinct integral homology classes.*

1.3. **Powers and discreteness.** We establish the following relatively straightforward fact about discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ which will be useful at the end of the proof of our main result:

Lemma 1.7. *Let H be a Zariski dense subgroup of $\mathrm{PSL}_2(\mathbb{R})$ and suppose that there is a finitely generated discrete subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ and an $N > 0$ such that for all $h \in H$, we have $h^N \in \Gamma$. Then H is discrete.*

Proof. If H fails to be discrete then its topological closure $\overline{H}$ must have positive dimension. Since $\mathrm{PSL}_2(\mathbb{R})$ is simple and since H is Zariski dense, we have that $\overline{H}$ is necessarily equal to $\mathrm{PSL}_2(\mathbb{R})$. It follows then that H must be topologically dense in $\mathrm{PSL}_2(\mathbb{R})$. Since the condition $|\operatorname{tr} A| < 2$ is open for $A \in \mathrm{PSL}_2(\mathbb{R})$, it follows that H must contain elements whose traces form a dense subset of $(-2, 2)$. It follows that H either contains an elliptic

element of infinite order or elliptic elements of arbitrarily high orders. In the first case, we obtain that Γ also contains an elliptic element of infinite order, violating the discreteness of Γ.

In the second case, choose a finite index subgroup $\Gamma_0 < \Gamma$ which is torsion-free, which exists by Selberg's Lemma [16]. We therefore have a torsion element $A \in H$ and a power of A which is nontrivial and which lies in Γ_0, a contradiction. $\qquad\square$

1.4. **Commensurations of** $\mathrm{PSL}_2(\mathbb{Z})$ **and rational matrices.** In this section we record the following easy observation to which we have alluded already:

Lemma 1.8. *Let*

$$g \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}} = \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\mathrm{PSL}_2(\mathbb{Z})),$$

and let $A \in \mathrm{PSL}_2(\mathbb{Z})$. Then $A^g \in \mathrm{PSL}_2(\mathbb{Q})$.

What is technically meant by the statement of Lemma 1.8 is that for any representatives of A and g in $\mathrm{SL}_2(\mathbb{R})$, the corresponding matrix A^g will have rational entries.

Proof of Lemma 1.8. Choose a representative for A in $\mathrm{SL}_2(\mathbb{Z})$ and a representative $g = \sqrt{q}B$, where $q \in \mathbb{Q}$ and where $B \in \mathrm{GL}_2(\mathbb{Q})$. It is immediate that $B^{-1}AB$ has rational entries. Since the representative for g differs from B by a multiple of the identity, we see that $A^g = B^{-1}AB$ has rational entries. $\qquad\square$

The fact that the commensurator of $\mathrm{PSL}_2(\mathbb{Z})$ requires the adjunction of square roots and is not simply $\mathrm{PSL}_2(\mathbb{Q})$ is at times an annoying issue. Later on in the paper, and especially in the appendix, the natural occurrence of square roots will aid us in the more geometric parts of the proofs, however. In the remainder of this section, we note that the occurrence of square roots is fundamentally a vestige of the fact that the center of $\mathrm{SL}_2(\mathbb{R})$ is nontrivial.

We remark that it is possible to avoid the appearance of square roots by working inside of other Lie groups. For instance, one can consider $\mathrm{PSL}_2(\mathbb{Z})$ as the group of integer points of the special orthogonal group $\mathrm{SO}^+(f)$, where

$$f = xz - y^2.$$

In this case, $\mathrm{SO}(2,1)$ has no center, and a general result of Borel [3] implies that the commensurator of the integral points is simply the group of rational points $\mathrm{SO}^+(f, \mathbb{Q})$. Thus, one avoids the complications resulting from square roots.

In this paper, we retain the 2×2 (as opposed to 3×3 in the case of $\mathrm{SO}^+(f, \mathbb{Q})$) approach for the sake of an easier-to-visualize geometric description of matrices commensurating $\mathrm{PSL}_2(\mathbb{Z})$.

2. Invariance under Commensuration

Recall the notation that $\Phi(H)$ is a term of the lower central series or derived series of H. If $\Phi(H) \neq H$ then we sometimes say that $\Phi(H)$ is a *proper term*. In this section, we prove that elements

$$g \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$$

satisfying certain special properties must lie in $\mathrm{PSL}_2(\mathbb{Z})$.

2.1. **Normalization of Zariski dense subgroups.**

Lemma 2.1. *Let $\mathscr{G}$ be a simple Lie group and $\Gamma \subset \mathscr{G}$ be a Zariski dense discrete subgroup. Then the normalizer $N_{\mathscr{G}}(\Gamma)$ is discrete.*

Proof. Since $\Gamma \subset \mathscr{G}$ is Zariski dense, so is $N_{\mathscr{G}}(\Gamma)$. If $N_{\mathscr{G}}(\Gamma)$ is not discrete, then it must be all of $\mathscr{G}$, since a positive dimensional Zariski dense Lie subgroup of a simple Lie group $\mathscr{G}$ is necessarily all of $\mathscr{G}$. This then forces $\mathscr{G}$ to admit a non-trivial Zariski dense normal subgroup Γ, contradicting simplicity of $\mathscr{G}$. Hence $N_{\mathscr{G}}(\Gamma)$ is discrete. $\qquad\square$

As a consequence, we have the following:

Lemma 2.2. *Let $H < \mathrm{PSL}_2(\mathbb{Z})$ be a finite index normal subgroup. Suppose that $g \in \mathrm{PSL}_2(\mathbb{R})$ satisfies $H^g = H$. Then $g \in \mathrm{PSL}_2(\mathbb{Z})$. More generally, if $\Phi(H) = \Phi(H)^g$ then $g \in \mathrm{PSL}_2(\mathbb{Z})$.*

Proof. Let $\mathscr{G} = \mathrm{PSL}_2(\mathbb{R})$. Since $\mathrm{PSL}_2(\mathbb{R})$ is simple and since H is Zariski dense, it follows by Lemma 2.1 that $N_{\mathscr{G}}(H)$ is discrete. We have that $H_1 = \langle g, H \rangle$ satisfies $H_1 \subset N_{\mathscr{G}}(H)$ and hence H_1 is discrete.

Since H is normal in $\mathrm{PSL}_2(\mathbb{Z})$, then we have

$$\mathrm{PSL}_2(\mathbb{Z}) < N_{\mathscr{G}}(H).$$

It follows that $\langle g, \mathrm{PSL}_2(\mathbb{Z}) \rangle$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, by Lemma 2.1. It follows that $\langle g, \mathrm{PSL}_2(\mathbb{Z}) \rangle$ is a discrete group of orientation preserving isometries of $\mathbf{H}^2$. A standard result from hyperbolic geometry says that

$$\mathbf{H}^2 / \mathrm{PSL}_2(\mathbb{Z})$$

admits no further nontrivial orientation preserving isometries and is therefore a minimal orbifold. To see this fact, we may take a quotient the modular surface by an orientation reversing reflection to obtain an orbifold whose underlying surface is a triangle with reflectors on the sides. The resulting Fuchsian group is then a Coxeter triangle group. Since the fundamental domain is a triangle with no symmetry there cannot be further quotients. Consequently, we see that $g \in \mathrm{PSL}_2(\mathbb{Z})$.

The proof in the case where

$$\Phi(H) = \Phi(H)^g$$

is identical, since $\Phi(H)$ is Zariski dense in $\mathrm{PSL}_2(\mathbb{R})$ and characteristic in H, and hence normalized by $\mathrm{PSL}_2(\mathbb{Z})$. $\qquad\square$

We note that it is at this point that we use the normality of H in $\mathrm{PSL}_2(\mathbb{Z})$. If H is not normal then we could pass to a finite index subgroup of H which is normal, though then g may no longer normalize H.

2.2. **An application of Chevalley-Weil theory.** In this section, we prove the following lemma:

Lemma 2.3. *Let $g \in \mathrm{PSL}_2(\mathbb{R})$, let $H < \mathrm{PSL}_2(\mathbb{Z})$ have finite index, and suppose that there is a free group $F < \mathrm{PSL}_2(\mathbb{R})$ such that $H, H^g < F$. Suppose furthermore that H is normal in F. If $H^g \neq H$ then g does not commensurate $\Phi(H)$, provided that $\Phi(H)$ is proper.*

For Lemma 2.3, the ambient group $\mathrm{PSL}_2(\mathbb{R})$ is irrelevant. We have the following general fact, from which Lemma 2.3 will follow with some more work.

Lemma 2.4. *Let F be a finitely generated free group of rank at least two, and let $K_1, K_2 < F$ be distinct, isomorphic, finite index subgroups. Suppose furthermore that K_2 is normal in F. Then K_1' and K_2' are not commensurable. That is to say, $K_1' \cap K_2'$ has infinite index in K_1'. If K_1 is also normal in F, then $K_1' \cap K_2'$ has infinite index in K_2' as well.*

The reader may be dissatisfied with the apparent asymmetry of Lemma 2.4. In its application to commensurators of thin groups, the asymmetry disappears, however. In deducing Lemma 2.3 from Lemma 2.4, we set H to be normal subgroup of finite index in a finite index free subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, which for the sake of explicitness we assume to be $\Gamma(k)$ for some $k \geqslant 2$. We assume furthermore that $H^g < \Gamma(k)$. Of course, H^g may fail to be normal in $\Gamma(k)$. The conclusion of Lemma 2.4 will imply (via Lemma 2.6) that $\Phi(H) \cap \Phi(H^g)$ has infinite index in $\Phi(H^g)$, but says nothing about the index of $\Phi(H) \cap \Phi(H^g)$ in $\Phi(H)$. However, if

$$g \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$$

then we see that

$$g^{-1} \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$$

as well, and so we obtain that $\Phi(H) \cap \Phi(H^{g^{-1}})$ has infinite index in $\Phi(H)$, which symmetrizes the conclusion somewhat. We note briefly that $\Phi(H^g) = \Phi(H)^g$.

Before proving Lemma 2.4, we recall the following classical fact about the homology of finite index subgroups of free groups. This is also called Gaschütz's theorem [7, 5] and is a free-group version of a well-known Theorem due to Chevalley and Weil [4]. We will not reproduce a proof of this

result, though we indicate that it can easily be deduced from the Lefschetz Fixed-Point Theorem:

Theorem 2.5. *Let F_k be a free group of rank k, and let $N < F_k$ be a finite index normal subgroup with $Q = F_k/N$. Then as a $\mathbb{Q}[Q]$-module, we have an isomorphism*

$$H_1(N, \mathbb{Q}) \cong \tau^k \oplus \rho^{k-1},$$

where τ denotes the trivial representation of Q, and $\rho = \rho_{reg}/\tau$ denotes the quotient of the regular representation modulo the trivial representation.

The trivial isotypic component of $H_1(N, \mathbb{Q})$ is canonically isomorphic to $H_1(F_k, \mathbb{Q})$ via the transfer map. Note that if $z \in H_1(N, \mathbb{Q})$ is not in the image of the transfer map, then z is not fixed by some element of Q. Moreover, if $1 \neq q \in Q$, then there is an element $z \in H_1(N, \mathbb{Q})$ such that $q \cdot z \neq z$.

In this section and throughout the rest of the paper, when we refer to *homology*, we always mean the first homology (with coefficients that will be clear from context), unless otherwise noted.

Proof of Lemma 2.4. Since K_1 and K_2 are distinct, isomorphic, and both of finite index in F, there can be no inclusion relations between K_1 and K_2. It follows that there exists an element $a \in K_1 \backslash K_2$. Let

$$b \in K_1 \cap K_2,$$

and let $x_b = [a, b]$. Observe that

$$x_b \in K_1' \cap K_2.$$

Indeed, since a and b are in K_1, we have that $x_b \in K_1'$. From the fact that $x_b = b^{-1}b^a$ and the normality of K_2, we see that $x_b \in K_2$.

We may now compute the homology class $[x_b]$ of x_b, as an element of $H_1(K_2, \mathbb{Z})$: we have

$$[x_b] = a \cdot [b] - [b],$$

where $a \cdot [b]$ denotes the image of $[b]$ under the action of a, viewed as an element of F/K_2.

Since $a \notin K_2$, we have that a represents a nontrivial element of F/K_2. It follows that there is a homology class

$$z \in H_1(K_2, \mathbb{Z})$$

such that $a \cdot z \neq z$, by Theorem 2.5. Note that replacing z by a nonzero integral multiple, we have

$$a \cdot (nz) = n(a \cdot z) \neq nz.$$

We may therefore choose an n and an element $b \in K_1 \cap K_2$ such that $[b] = nz$, since K_1 and K_2 have finite index in F.

With such a choice of b, we have that

$$[x_b] \in H_1(K_2, \mathbb{Z})$$

is a nontrivial homology class. Since K_2 is free, we see that no power of x_b represents a trivial homology class of K_2. Therefore, for all $N \neq 0$, we have $x_b^N \in K_1'$ but $x_b^N \notin K_2'$. It follows that $K_1' \cap K_2'$ has infinite index in K_1'.

If K_1 is also normal in F, then one can switch the roles of K_1 and K_2 to conclude that $K_1' \cap K_2'$ has infinite index in K_2' as well. □

Note that Lemma 2.4 establishes Lemma 2.3 for $\Phi(H) = H'$. We prove the following fact, which now immediately implies Lemma 2.3 and which will be useful in the sequel:

Lemma 2.6. *Let A and B be commensurable, nonabelian, free subgroups of finite rank in an ambient group G. Suppose there exists an element $g \in A \cap B$ such that $g \in A'$ and such that $g \in B \backslash B'$. Then $\Phi(A)$ and $\Phi(B)$ are not commensurable for any proper term of the lower central series or of the derived series.*

In Lemma 2.6, we insist that $\Phi(A)$ and $\Phi(B)$ denote the same indexed term of the same series for A and B.

Proof of Lemma 2.6. By the definition of g, we have that $g^N \in A'$ for all N and $g^N \notin B'$ for all $N \neq 0$. We deal with the two series separately, starting with the lower central series. The nth term of the lower central series of a group H will be denoted by $\gamma_n(H)$, and the nth term of the derived series of a group H will be denoted by $D_n(H)$.

Note that $g \in \gamma_2(A)$ and

$$g \in \gamma_1(B) \backslash \gamma_2(B).$$

Note also that there is an element $x \in A \cap B$ such that $[g, x] \in \gamma_3(A)$ and such that

$$[g, x] \in \gamma_2(B) \backslash \gamma_3(B).$$

Indeed, it suffices to choose an element x lying in $A \cap B$ whose integral homology class in B is nonzero, which exists since A and B are commensurable (cf. [10]). Note that $[g, x] \in A \cap B$.

By an easy induction, we can find elements $\{y_n\}_{n \geqslant 1} \subset A \cap B$ such that $y_n \in \gamma_{n+1}(A)$ and such that

$$y_n \in \gamma_n(B) \backslash \gamma_{n+1}(B).$$

Again, we can simply define $y_{n+1} = [y_n, x]$, where $x \in A \cap B$ represents a nontrivial homology class of B. For a finitely generated free group F, we have that $\gamma_n(F)/\gamma_{n+1}(F)$ is a finitely generated torsion-free abelian group

for all $n \geqslant 1$ (see again [10]). It follows that for all $N \neq 0$, we have $y_n^N \in \gamma_{n+1}(A)$ and

$$y_n^N \in \gamma_n(B) \backslash \gamma_{n+1}(B),$$

whence no proper terms of the lower central series of A and B are commensurable.

We now consider the derived series, and begin with the same element g as above. We consider the groups $H_1(A', \mathbb{Z})$ and $H_1(B', \mathbb{Z})$, both of which are infinitely generated free abelian groups. Since

$$g \in A' = D_2(A),$$

we have that g represents a (possibly trivial) element of $H_1(A', \mathbb{Z})$. Since $g \notin B'$, we have that g represents a nontrivial element of $H_1(B, \mathbb{Z})$. Thus, if $z \in H_1(B', \mathbb{Z})$ is a nontrivial homology class, then for all sufficiently large N we have that $g^N \cdot z - z$ represents a nontrivial element of $H_1(B', \mathbb{Z})$. This claim is easily checked using covering space theory, by choosing a finite wedge of circles whose fundamental group is B and taking the cover of the wedge corresponding to B'. Then one simply uses the fact that $H_1(B, \mathbb{Z})$ acts properly discontinuously on the corresponding cover.

Since A and B are commensurable, we can choose an element $x \in A' \cap B'$ such that x represents a nontrivial element of $H_1(B', \mathbb{Z})$. Indeed, we can choose any two element b_1, b_2 in a free basis for B and nonzero exponents n_1 and n_2 such that $b_i^{n_i} \in A$ for $i \in \{1, 2\}$. Then, the element $x = [b_1^{n_1}, b_2^{n_2}]$ will represent a nontrivial element of $H_1(B', \mathbb{Z})$.

Note that for all N, we have

$$y = [g^N, x] \in D_3(A).$$

On the other hand, for N sufficiently large we have that y represents a nontrivial element of $H_1(B', \mathbb{Z})$, and therefore

$$y \in D_2(B) \backslash D_3(B).$$

Since $H_1(B', \mathbb{Z})$ is torsion-free, it follows that $D_3(A)$ and $D_3(B)$ are not commensurable.

An easy induction now shows that $D_i(A)$ and $D_i(B)$ are not commensurable for $i \geqslant 2$. Indeed, suppose we have produced an element $y_i \in A \cap B$ such that for all nonzero N, we have $y_i^N \in D_i(A)$ and

$$y_i^N \in D_{i-1}(A) \backslash D_i(A).$$

As in the case $i = 2$, it is straightforward to construct an element $x \in D_i(A) \cap D_i(B)$ such that x represents a nontrivial element of $H_1(D_i(B), \mathbb{Z})$. Then for all N we have

$$y_{i+1} = [y_i^N, x] \in D_{i+1}(A).$$

For all sufficiently large N, however, we have that y_{i+1} represents a nontrivial homology class in $H_1(D_i(B), \mathbb{Z})$. Using again the fact that $H_1(D_i(B), \mathbb{Z})$ is torsion-free, no power of y_{i+1} lies in $D_{i+1}(B)$, so that $D_{i+1}(A)$ and $D_{i+1}(B)$ are not commensurable.   □

3. Integral commensurators

In this section, we establish the following fact:

Lemma 3.1. *Let $H < \mathrm{PSL}_2(\mathbb{Z})$ be a non-elementary subgroup and let $g \in$ $\mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(H)$. Then $g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$.*

We will eventually give two quite different proofs of Lemma 3.1. The first is given in Section 3.1 below and draws from the theory of invariant trace fields and quaternion division algebras. We are grateful to Alan Reid for telling us this proof.

Since the first proof of Lemma 3.1 is rather efficient, we will relegate the second proof to Appendix A. The second proof will build on a reduction Lemma given in Section A.1 and uses in addition some aspects of the topological dynamics of a natural action of H on the Furstenberg boundary $S^1 \times S^1$ of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ in Section A.2. We remark that the second proof of Lemma 3.1, though less conceptual and more computation-intensive, readily generalizes to Lemma A.9 – a statement that is not covered by Proposition 3.2. We retain it for possible future use.

3.1. **Quaternion division algebras and commensurators.** In this section, we give our first proof of Lemma 3.1.

We refer the reader to [11] for the relevant basics on invariant trace fields and quaternion division algebras. The proof of 3.2 below is guided and informed by the argument on page 118 of [11] proving Theorem 3.3.4 there (see especially Equations 3.8 and 3.9).

Proposition 3.2. (Reid) *Let H be a non-elementary (not necessarily discrete) subgroup of $\mathrm{PSL}_2(\mathbb{C})$ such that*

$$K = \mathbb{Q}(\mathrm{tr}\, H) = \mathbb{Q}(\mathrm{tr}\, H')$$

for any subgroup H' of finite index in H. Let $B = A_0 H$ denote the quaternion algebra generated over K. That is, B is obtained by taking finite linear combinations of elements of H over K [11, Sec 3.2]. Let B^* denote the set of invertible elements of B. Suppose that H and xHx^{-1} are commensurable, i.e. $x \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{C})}(H)$. Then the following conclusions hold:

(1) $x = ta$ where $a \in B^*$ and t is a non-zero complex number.
(2) $t^2 \in K$ and so $x^2 \in B^*$.

Proof of Lemma 3.1 assuming Proposition 3.2. Note that the hypothesis

$$K = \mathbb{Q}(\operatorname{tr} H) = \mathbb{Q}(\operatorname{tr} H')$$

in Proposition 3.2 is satisfied for arbitrary subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Moreover, we have that

$$B^* = \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$$

in our notation, whence the desired conclusion follows. □

We now turn to the proof of Proposition 3.2:

Proof. Let

$$H_1 = H \cap xHx^{-1},$$

so that H_1 has finite index in H and xHx^-1. Hence,

$$\mathbb{Q}(\operatorname{tr} H_1) = \mathbb{Q}(\operatorname{tr} H) = \mathbb{Q}(\operatorname{tr}(xHx^{-1}))$$

by hypothesis. Hence the quaternion algebras $B = A_0 H$,

$$A_0(xHx^{-1}) = xA_0Hx^{-1},$$

and A_0H_1 are all defined over K and hence are all equal.

Hence by the Skolem-Noether theorem [11, 2.9.8] there exists $a \in B^*$ such that

(3.1) $$aga^{-1} = xgx^{-1}$$

for all $g \in B$. Thus

(3.2) $$a^{-1}xg = ga^{-1}x$$

for all $g \in B$.

We would like to conclude that

$$a^{-1}x \in Z(B) = K,$$

but we cannot immediately do this because $a^{-1}x$ need not be in B. However, after tensoring with $\mathbb{C}$ over K, the Equation 3.2 continues to hold:

$$a^{-1}xg = ga^{-1}x$$

for all $g \in M_2(\mathbb{C})$. Hence,

$$a^{-1}x = t \in \mathbb{C}$$

is a non-zero element and $x = ta$ as required, which proves the first conclusion.

Finally, the equation

$$\det(x) = \det(ta)$$

gives us

$$1 = t^2 \det(a),$$

and we have $\det(a) \in K$ by assumption. It follows that $t^2 \in K$, and so

$$x^2 = t^2 a^2 \in B^*,$$

which establishes the second part of the conclusion. □

4. Homological pseudo-actions and completing the proof

In this section, we complete the proof of Theorem 1.2. Let $H < \mathrm{PSL}_2(\mathbb{Z})$ be a finite index normal subgroup which is contained in the principal congruence subgroup $\Gamma(k)$. We have shown that if

$$g \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$$

then $g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$ and hence

$$g^2 \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\mathrm{PSL}_2(\mathbb{Z})).$$

Therefore, we have that $H \cap H^{g^2}$ is a finite index subgroup of both H and H^{g^2}. We wish to argue that $H^{g^2} < \Gamma(k)$ as well, so that we can apply Lemma 2.2 or Lemma 2.3, depending on whether $H = H^{g^2}$ or not.

4.1. **Building homological pseudo-actions.** Note that H and H^{g^2} both lie in $\mathrm{PSL}_2(\mathbb{Q})$, as is checked by an easy computation (cf. Lemma 1.8). Let $H = \langle x_1, \ldots, x_m \rangle$, where $\{x_1, \ldots, x_n\}$ is a free basis for H. We write $[x_i]$ for the homology class of x_i. We have that

$$\{[x_1], \ldots, [x_n]\}$$

generate the integral homology of H, and for each $N \geqslant 1$, we have that

$$\{[x_1^N], \ldots, [x_n^N]\}$$

generate a finite index subgroup of the integral homology of H. Suppressing N from the notation, we sometimes write $z_i = [x_i^N]$. For $y \in H^{g^2}$ arbitrary, we consider the homology class of the commutator $[y, x_i^N]$, when it makes sense. Since $y \in H^{g^2}$ and since H and H^{g^2} are commensurable, there exist arbitrarily large values of N for which

$$[y, x_i^N] \in (H^{g^2})',$$

since we merely choose values of N such that $x_i^N \in H \cap H^{g^2}$.

On the other hand, we have that $x_i^N \in H$ for all i and N. Similarly, there exist arbitrarily large values of N such that $(x_i^N)^y \in H$ as well, independently of i, since $y \in \mathrm{PSL}_2(\mathbb{Q})$ and hence H and H^y are commensurable. For these values of N, we may make sense of the homology class $y \cdot z_i - z_i$ for all i, which is the homology class of $[y, x_i^N]$ in H. This defines a *pseudo-action* of H^g on the integral homology of H. We call it a pseudo-action since it is not defined for all N.

Suppose that $y \cdot z_i - z_i$ is nonzero for some i. Then we obtain

$$[y, x_i^N] \in H \backslash H'.$$

Then by Lemma 2.6, we see that the group $\Phi(H)$ and the group

$$\Phi(H^{g^2}) = \Phi(H)^{g^2}$$

are not commensurable. Note that the previous discussion was symmetric in H and H^{g^2}, so that we obtain elements of $H^{g^2} \cap H'$, no powers of which lie in $(H^{g^2})'$ unless the homological pseudo-action of H on the integral homology of H^{g^2} is trivial.

Therefore, if g commensurates $\Phi(H)$ for some proper term of the lower central series or derived series of H, we must have that $y \cdot z_i - z_i$ is the trivial integral homology class of H for all i, whenever this homology class is defined. In particular, the pseudo-action of y on the integral homology of H is trivial.

4.2. Trivial homology pseudo-actions and parabolics.
Let $\gamma \in H$ be a parabolic element fixing infinity, which exists because H has finite index in $\mathrm{PSL}_2(\mathbb{Z})$. Let $y \in H^{g^2}$ be arbitrary, and suppose that the y-pseudo-action on the integral homology of H is trivial.

We see that there is a positive integer N such that the homology class of γ^N is invariant under y. That is,

$$[(\gamma^N)^y] = [\gamma^N]$$

as homology classes of $\mathbf{H}^2/H$. Since $H < \Gamma(k)$ for some $k \geqslant 2$, we see that each cusp of $\mathbf{H}^2/H$ is homologically nontrivial, and no two distinct cusps represent the same homology class, as follows from Corollary 1.6.

It follows that the element $(\gamma^N)^y$ represents a power of a free homotopy class of a cusp of $\mathbf{H}^2/H$, which is equal to the free homotopy class represented by γ^N. In particular, we have that $(\gamma^N)^y$ is a parabolic element of $\mathrm{PSL}_2(\mathbb{Q})$, and the fixed point of $(\gamma^N)^y$ is in the H-orbit of infinity.

It follows that there is an element $h \in H$ such that

$$((\gamma^N)^y)^h = (\gamma^N)^{yh}$$

stabilizes infinity. Since both H and H^{g^2} are subgroups of $\mathrm{PSL}_2(\mathbb{Q})$, it follows that yh lies in the stabilizer of infinity in $\mathrm{PSL}_2(\mathbb{Q})$, so that we have

$$yh = \begin{pmatrix} r & t \\ 0 & r^{-1} \end{pmatrix}$$

for some suitable $r, t \in \mathbb{Q}$.

Writing

$$\gamma^N = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix}$$

for some suitable $M \in \mathbb{Z}$, then we see that

$$(\gamma^N)^{yh} = \begin{pmatrix} 1 & r^{-2}M \\ 0 & 1 \end{pmatrix}.$$

Since

$$[(\gamma^N)^{yh}] = [(\gamma^N)^y] = [\gamma^N],$$

it follows that we must have $r = 1$, so that

$$yh = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Repeating the same argument for a parabolic element of H stabilizing 0, we find an element $q \in H$ such that

$$yq = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

for some suitable $s \in \mathbb{Q}$. We now multiply $(yh)^{-1}$ and (yq) to get

$$\begin{pmatrix} 1 - ts & -t \\ s & 1 \end{pmatrix} \in H.$$

Since $H < \Gamma(k)$, we see that $t, s \in k\mathbb{Z}$. It follows that $y \in \Gamma(k)$. Since $y \in H^{g^2}$ was chosen arbitrarily, we see that $H^{g^2} < \Gamma(k)$.

We summarize the discussion of the last two subsections as follows:

Lemma 4.1. *Let $H < \mathrm{PSL}_2(\mathbb{Z})$ be a finite index normal subgroup which is contained in $\Gamma(k)$ for some $k \geqslant 2$, and let $g \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$ commensurate $\Phi(H)$ for some proper term. Then $g \in \mathrm{PSL}_2(\mathbb{Z})$.*

Proof. The discussion of Sections 4.1 and 4.2 implies that $H^g < \Gamma(k)$. Note here that we conjugate by g and not by g^2 since we already assume that $g \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$.

Note that H is normal in $\mathrm{PSL}_2(\mathbb{Z})$ and therefore is normal in $\Gamma(k)$ as well. Since $\Gamma(k)$ is a free group and since H and H^g are isomorphic, if $H \neq H^g$ then we can apply Lemma 2.3 to conclude that $\Phi(H)$ and $\Phi(H)^g$ are not commensurable. If $H = H^g$ then Lemma 2.2 implies that $g \in \mathrm{PSL}_2(\mathbb{Z})$. $\square$

Proof of Theorem 1.2. Let H be as in the statement of the theorem and let $g \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$. By Lemma 3.1, we see that

$$g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}.$$

By Lemma 4.1, we have that $g^2 \in \mathrm{PSL}_2(\mathbb{Z})$. It follows that the square of every element in

$$\mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$$

lies in $\mathrm{PSL}_2(\mathbb{Z})$, so that the group

$$\mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H))$$

is discrete by Lemma 1.7.   □

APPENDIX A. AN ALTERNATIVE PROOF OF LEMMA 3.1

In this appendix, we will provide an alternative approach to the characterization of elements which commensurate Zariski dense groups of integral matrices inside of $PSL_2(\mathbb{R})$. The advantage of such the approach given here is that it is more or less from first principles and computational, whereas the proof of Proposition 3.2 (from which Lemma 3.1 follows easily) is much slicker and uses more machinery.

For the purposes of this appendix, H will satisfy the following assumption unless otherwise noted:

Assumption A.1. *We have $H < PSL_2(\mathbb{Z})$ and the limit set $\Lambda_H \subset S^1$ is all of S^1. In particular, $H < PSL_2(\mathbb{Z})$ is Zariski dense in $PSL_2(\mathbb{R})$.*

This assumption will become essential in Subsection A.2 below. Note that under Assumption A.1, H plainly need not be a finite index subgroup $PSL_2(\mathbb{Z})$.

A.1. **Reducing to quadratic extensions.** In this section, we consider general elements $g \in PSL_2(\mathbb{R})$ which can commensurate Zariski dense groups of integral matrices. The following lemma is purely algebraic:

Lemma A.2. *Let $g \in PSL_2(\mathbb{R})$ commensurate a Zariski dense subgroup $H < PSL_2(\mathbb{Z})$. Then $g \in PSL_2(K)$ or $g \in PSL_2(L)$, where K is a totally real quadratic extension of $\mathbb{Q}$ and where L is a real quadratic extension of K.*

More generally, let $\mathcal{O}$ be the ring of integers in a totally real number field F and let $x \in PSL_2(\mathbb{R})$ commensurate a Zariski dense subgroup $H < PSL_2(\mathcal{O})$. Then $g \in PSL_2(K)$ or $g \in PSL_2(L)$, where K is a totally real quadratic extension of F and where L is a real quadratic extension of K.

Proof. We prove the lemma for $F = \mathbb{Q}$ and $\mathcal{O} = \mathbb{Z}$ for concreteness. The proof in the general case is identical.

We write

$$x = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ x^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

We suppose that there are integral two-by-two matrices $A, B \in H$ such that $Ax = xB$. We write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

A straightforward calculation shows that

$$Ax = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

and

$$xB = \begin{pmatrix} e\alpha + g\beta & f\alpha + h\beta \\ e\gamma + g\delta & f\gamma + h\delta \end{pmatrix}.$$

From the equality $x^{-1}Ax = B$, we obtain

$$\begin{pmatrix} \alpha(a\delta - c\beta) + \gamma(b\delta - d\beta) & \beta(a\delta - c\beta) + \delta(b\delta - d\beta) \\ \alpha(c\alpha - a\gamma) + \gamma(d\alpha - b\gamma) & \beta(c\alpha - a\gamma) + \delta(d\alpha - b\gamma) \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

We first suppose $\beta = \gamma = 0$. In this case, we have $\delta = \alpha^{-1}$. Since H is Zariski dense, we may assume that there is an element $A \in H$ for which $b \neq 0$. We thus obtain $\delta^2 b = e$, whence δ is the square root of a rational number. In this case, $x \in \mathrm{PSL}_2(K)$ for a totally real quadratic extension K of $\mathbb{Q}$ (or of F in the general case), as desired. Thus, we may henceforth assume that at least one of β and γ is nonzero.

From the equality $Ax = xB$, we conclude that

$$\alpha(a - e) + b\gamma = g\beta.$$

Similarly, we see that

$$\delta(d - h) + c\beta = f\gamma.$$

Each of these equations expresses either of linear dependence among the coefficients of x or degenerates with equalities among some of the coefficients of A and B. If $a \neq e$ then we obtain

$$\alpha = \frac{g\beta - b\gamma}{a - e},$$

and similarly if $d \neq h$ then

$$\delta = \frac{f\gamma - c\beta}{d - h}.$$

If $a = e$ then we obtain the linear relation

$$\alpha(a\delta - c\beta) + \gamma(b\delta - d\beta) = a$$

among the coefficients of A. By the invertibility of x, none of the pairs

$$\{(\alpha, \gamma), (\delta, \beta), (\alpha, \beta), (\gamma, \delta)\}$$

can be simultaneously zero.

One may then examine the degenerate cases of the equation

$$\alpha(a\delta - c\beta) + \gamma(b\delta - d\beta) = a$$

when one of the entries of x is zero. From this it is straightforward to verify that the linear relation defined by $a = e$ imposes a nontrivial algebraic

condition on the coefficients of A and therefore defines a proper subvariety of $\mathrm{PSL}_2(\mathbb{R})$, except in the special case $\beta = \gamma = 0$. Thus, unless $\beta = \gamma = 0$, for a fixed x there is a choice of A such that $a \neq e$, since H is Zariski dense. We note that since $\mathrm{PSL}_2(\mathbb{R})$ is a simple Lie group, it is irreducible as a variety and therefore H is not contained in any finite collection of proper subvarieties of $\mathrm{PSL}_2(\mathbb{R})$, since any such collection would be Zariski closed and proper.

Therefore, an identical analysis shows that we may choose A so that $d \neq h$ as well, unless $\beta = \gamma = 0$. It follows then that α and δ are rational linear combinations of β and γ.

We next consider the special case when exactly one of β or γ is zero. We assume $\gamma = 0$, with the case $\beta = 0$ being analogous. Then we see that α and δ are both rational multiples of β. Since $\alpha = \delta^{-1}$, we have that β^2 is a rational number, so that β is a square root of a rational number and consequently $x \in \mathrm{PSL}_2(K)$ for a totally real quadratic extension K of $\mathbb{Q}$, as desired. We may therefore assume $\beta, \gamma \neq 0$.

We now check the special cases where at least one of α and δ is zero. From the fact that α and δ are rational combinations of β and γ, we see then that γ is a rational multiple of β, and so that both α and δ are rational multiples of β. From

$$\alpha\delta - \beta\gamma = 1,$$

we again obtain that β is a square root of a rational number and so that again $x \in \mathrm{PSL}_2(K)$ for a totally real quadratic extension K of $\mathbb{Q}$, as desired.

We may therefore assume that all entries of x are nonzero. Consider x as an element of $\mathrm{SL}_2(\mathbb{R}) < \mathrm{GL}_2(\mathbb{R})$. Observe that if $x^{-1}Ax = B$ and $y \in \mathrm{GL}_2(\mathbb{R})$ differs from x by a nonzero real multiple of the identity matrix, then $y^{-1}Ay = B$ as well. So, let y be given by rescaling x by β, so that the upper right entry of y is 1. We write (ϵ, ζ, η) for $(\alpha/\beta, \gamma/\beta, \delta/\beta)$, so that

$$y = \begin{pmatrix} \epsilon & 1 \\ \zeta & \eta \end{pmatrix}.$$

Multiplying out $y^{-1}Ay$ and examining the upper right entry, we obtain the equation

$$b\eta^2 + (a - d)\eta - (c + f) = 0.$$

The Zariski density of H allows us to assume that $b \neq 0$, so that η lies in a totally real quadratic extension K of $\mathbb{Q}$. From the equation

$$\eta\beta = \frac{c\beta - f\gamma}{a - e},$$

we see that $\gamma/\beta = \zeta \in K$. Similarly, we see that $\epsilon \in K$. From the equation $\det x = 1$, we see that

$$\beta^2(\epsilon\eta - \zeta) = 1,$$

so that $\beta^2 \in K$. In particular, β is a square root of an element K, so that x has entries in an extension $L \subset \mathbb{R}$ of $\mathbb{Q}$ which has degree either one, two, or four. $\qquad\square$

Note that in general the field L may not be a quadratic extension of $\mathbb{Q}$ (or of F in the general case), and it might not be totally real. However, there is an order two Galois automorphism of L/K for a totally real quadratic extension $K/\mathbb{Q}$ (or K/F in the general case) which interchanges a positive and negative square root of a positive element of K, and hence two distinct embeddings of L into $\mathbb{R}$ which restrict to the identity on K.

A.2. **Limit sets and commensurators.** In Section A.1, we proved that if

$$g \in \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(H)$$

then either $g \in \mathrm{PSL}_2(K)$, where K is a totally real quadratic extension of $\mathbb{Q}$, or $g \in \mathrm{PSL}_2(L)$, where L is a real quadratic extension of K.

To prove Lemma 3.1, it suffices therefore to show the following:

Lemma A.3. *Let H satisfy Assumption A.1, and let*

$$g \in \mathrm{PSL}_2(L) \cap \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(H),$$

where L is a real quadratic extension of a totally real quadratic extension of $\mathbb{Q}$. Then $g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$.

Proof. Note that $\mathrm{PSL}_2(L)$ embeds as a subgroup of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, defined by

$$A \mapsto (A, \sigma(A)),$$

where σ is the unique nontrivial Galois automorphism of L/K. Restricting to $\mathrm{PSL}_2(K)$, we have that the image is simply the diagonal embedding of $\mathrm{PSL}_2(K)$ into $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, which extends to the diagonal embedding of $\mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. We will show that if g commensurates H then g differs from an element of $\mathrm{PSL}_2(K)$ by a matrix of the form

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}.$$

The scalar $\zeta \in L$ can be chosen to satisfy $\zeta^2 \in K$. Then, repeating the same argument with the roles of L and K replaced by K and $\mathbb{Q}$ respectively will show that

$$g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}},$$

which will complete the proof.

The group $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ acts naturally on the associated symmetric space of non-compact type, which is given by $\mathbf{H}^2 \times \mathbf{H}^2$, and whose Furstenberg boundary is naturally identified with the torus $S^1 \times S^1$. We equip the

torus with coordinates (θ, ϕ), coming from this direct product decomposition. The maximal compact subgroup of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ is identified with $\mathrm{SO}(2) \times \mathrm{SO}(2)$, which acts transitively on $S^1 \times S^1$. If $P < \mathrm{PSL}_2(\mathbb{R})$ is a Borel subgroup then

$$P \times P < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$$

becomes a Borel subgroup. Thus, we have identifications

$$(\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}))/(\mathrm{SO}(2) \times \mathrm{SO}(2)) \cong \mathbf{H}^2 \times \mathbf{H}^2$$

and

$$(\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}))/(P \times P) \cong \partial \mathbf{H}^2 \times \partial \mathbf{H}^2 \cong S^1 \times S^1.$$

The limit set of $\mathrm{PSL}_2(\mathbb{Z})$ as also that of H inside of $S^1 \times S^1$ under the diagonal embedding thus coincides with the circle $\theta = \phi$. The action of $\mathrm{PSL}_2(\mathbb{Z})$ preserves leaf-wise the foliation $\mathscr{F}$ of $S^1 \times S^1$ consisting of all circles of the form $(\theta_0 + \theta, \theta)$, for θ_0 fixed. If $g \in \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ commensurates H, then g need not preserve individual leaves of the foliation, but it preserves the foliation itself (see [1] for instance). Indeed, if λ is a leaf of $\mathscr{F}$ then λ coincides with the closure of the H-orbit of any point $p \in \lambda$. So, if g commensurates H then the closure of the $H \cap H^g$-orbit of $g \cdot p$ will coincide with the leaf of $\mathscr{F}$ meeting $g \cdot p$.

Now, suppose that

$$g \in \mathrm{PSL}_2(L) \backslash \mathrm{PSL}_2(K),$$

and suppose that g is represented by a matrix A whose image in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ is given by $(A, \sigma A)$. We show that if $g \in \mathrm{PSL}_2(L)$ commensurates H then $A = \pm \sigma A$.

To prove this, we first show that the diagonal embedding

$$H \to \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$$

together with $(A, \sigma A)$ generate a Zariski dense subgroup

$$\Gamma < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}),$$

except in the case $A = \pm \sigma A$. This will complete the proof of the lemma (after analyzing the ramifications of $A = \pm \sigma A$), by the following claim which we shall establish as Lemma A.6 below:

Claim A.4. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ be a Zariski dense subgroup. Then Γ does not preserve $\mathscr{F}$. In particular, there exists an element of Γ which does not commensurate H.*

We complete the proof assuming Claim A.4. Let

$$G < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$$

be the Zariski closure of Γ. Note that G contains the diagonal copy

$$\mathrm{PSL}_2(\mathbb{R}) \cong \Delta < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}).$$

Let $d \in \mathrm{PSL}_2(\mathbb{R})$ be arbitrary. Then d gives rise to the element $(d, d) \in \Delta$. We have

$$(A, \sigma A)(d, d) = (Ad, \sigma Ad) \in G.$$

Multiplying on the right by $(d^{-1}A^{-1}, d^{-1}A^{-1}) \in \Delta$, we see that

$$(1, (\sigma A)(A^{-1})) \in G.$$

Similarly, we obtain $((\sigma A)A^{-1}, 1) \in G$.

Now we compute the conjugate

$$(A, \sigma A)(d, d)(A^{-1}, \sigma A^{-1}) \in G,$$

which is given by $(AdA^{-1}, (\sigma A)d(\sigma A^{-1}))$. Again, multiplying by a suitable element of Δ, we see that

$$(AdA^{-1}(\sigma A)d^{-1}(\sigma A^{-1}), 1) \in G.$$

Using the fact that $((\sigma A)A^{-1}, 1) \in G$, we get

$$(AdA^{-1}(\sigma A)d^{-1}A^{-1}, 1) \in G$$

and

$$((\sigma A)dA^{-1}(\sigma A)d^{-1}(\sigma A^{-1}), 1) \in G.$$

Conjugating the element $(AdA^{-1}(\sigma A)d^{-1}A^{-1}, 1)$ by (A, A) and by (Ad, Ad), we obtain

$$(dA^{-1}(\sigma A)d^{-1}, 1) \in G$$

and

$$(A^{-1}(\sigma A), 1) \in G.$$

Identical computations show that

$$(1, dA^{-1}(\sigma A)d^{-1}) \in G$$

and

$$(1, A^{-1}(\sigma A)) \in G.$$

Therefore, in order for G to be a proper Lie subgroup of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, we must have that

$$\langle A^{-1}(\sigma A), dA^{-1}(\sigma A)d^{-1} \rangle < \mathrm{PSL}_2(\mathbb{R})$$

is contained in a fixed solvable subgroup of $\mathrm{PSL}_2(\mathbb{R})$, independently of d. Note that since $d \in \mathrm{PSL}_2(\mathbb{R})$ is arbitrary, we must have that $A^{-1}\sigma A$ is the identity in $\mathrm{PSL}_2(\mathbb{R})$.

To analyze this case in more detail, we consider $A^{-1}\sigma A$ as an element of $\mathrm{SL}_2(L)$. By Claim A.4, we may conclude that if g commensurates H then Γ cannot be Zariski dense, so that

$$A^{-1}(\sigma A) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular $A = \pm\sigma(A)$ in $\mathrm{SL}_2(L)$.

We can now complete the proof of the lemma. If α is an arbitrary entry of A, we may write

$$\alpha = \alpha_1 + \alpha_2\zeta,$$

where $\{1, \zeta\}$ form a K–basis for L. Since we are free to choose any K–basis we like, we choose ζ so that $\zeta^2 \in K$. We quickly see that if $A = \sigma A$ then $\alpha_2 = 0$, and if $A = -\sigma A$ then $\alpha_1 = 0$. In the first case, we obtain $A \in \mathrm{PSL}_2(K)$ (after projectivizing), and in the second we obtain

$$A = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \cdot B$$

for some $B \in \mathrm{GL}_2(K)$.

In the first of these cases where $A \in \mathrm{PSL}_2(K)$, we may run the same argument as before, using the nontrivial Galois automorphism of $K/\mathbb{Q}$, to conclude that either $A \in \mathrm{PSL}_2(\mathbb{Q})$ or

$$A = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \cdot J$$

for some $J \in \mathrm{GL}_2(\mathbb{Q})$ and μ such that $\mu^2 \in \mathbb{Q}$, so that $A \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$.

In the second case, we have that

$$A \in \mathrm{PSL}_2(L) \backslash \mathrm{PSL}_2(K).$$

We obtain $A^2 \in \mathrm{PSL}_2(K)$ since $\zeta^2 \in K$. So, running the same argument as before and replacing A by A^2, we see that if g commensurates H then

$$A^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}.$$

In particular, g^2 commensurates $\mathrm{PSL}_2(\mathbb{Z})$.   □

To complete the proof of Lemma A.3, it remains to justify Claim A.4. For

$$\mathscr{G} = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}),$$

let Π_1, Π_2 denote the projections onto the first and second factors. We first observe:

Lemma A.5. *Let* $\Gamma < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ *be a Zariski dense subgroup. Then there exists* $(A, B) \in \Gamma$ *such that* $\mathrm{tr}(A) \neq \mathrm{tr}(B)$.

Proof. This follows from the fact that

$$\text{tr} \circ \Pi_1(g) = \text{tr} \circ \Pi_2(g)$$

is an algebraic condition on $g \in \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$. □

We are now in a position to prove Claim A.4 which we restate:

Lemma A.6. *Let* $\Gamma < \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$ *be a Zariski dense subgroup. Then* Γ *does not preserve the foliation* $\mathscr{F}$ *of* $S^1 \times S^1$.

Proof. Let $g = (A, B)$ be an element of Γ where each of A, B are hyperbolic elements of $\text{PSL}_2(\mathbb{R})$. For the action of A (resp. B) we denote the attracting fixed point A^∞ (resp. B^∞) on $S^1 = \partial \mathbf{H}^2$ by p (resp. q). Then $(p, q) \in S^1 \times S^1$ is a fixed point of g on the Furstenberg boundary $S^1 \times S^1$ of $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$.

We say a few words about the dynamics of g on $S^1 \times S^1$. Through (p, q) there are two natural circles $p \times S^1$ and $S^1 \times q$ with respect to the product structure of $S^1 \times S^1$. Denote $p \times S^1$ by S_p and $S^1 \times q$ by S_q. Both these circles S_p, S_q are preserved under the g−action.

There is (up to sign) exactly one other tangent direction $v \in T_{(p,q)}(S^1 \times S^1)$ in the tangent space $T_{(p,q)}(S^1 \times S^1)$ at (p, q) on $S^1 \times S^1$ that is also preserved under the g−action. This direction v is given by a slope $m_A(p)/m_B(q)$ where $m_A(p)$ (resp. $m_B(q)$) denotes the multiplier (i.e. the derivative) of A (resp. B) at p (resp. q). We shall call this tangent direction v the *neutral* tangent direction.

Identifying S^1 with the unit circle in the complex plane $\mathbb{C}$ and acting by a suitable Möbius transformation if necessary, we can assume without loss of generality that $A^{\pm\infty} = \pm 1$ and similarly, $B^{\pm\infty} = \pm 1$. Then

$$(A^\infty, B^\infty) = (p, q) = (1, 1)$$

and

$$(A^{-\infty}, B^{-\infty}) = (-1, -1).$$

Further, with these coordinates, the foliation $\mathscr{F}$ would be preserved by g if and only if the tangent vectors to the leaves of $\mathscr{F}$ are parallel to the neutral tangent direction v (of the previous paragraph). Hence the multipliers $m_A(p)$ and $m_B(q)$ must be equal. Hence $\text{tr}(A) = \text{tr}(B)$.

Therefore, to establish the lemma, it suffices to find $g = (A, B) \in \Gamma$ such that:

    (1) A, B are both hyperbolic, i.e. $\text{tr}(A) > 2$, $\text{tr}(B) > 2$;
    (2) $\text{tr}(A) \neq \text{tr}(B)$.

The first condition is Zariski open and nonempty and so is the second condition by Lemma A.5. Hence there exists

$$g = (A, B) \in \Gamma$$

satisfying the two above conditions completing the proof of the Lemma.   □

The upshot of this section is the following:

Corollary A.7. *Let H satisfy Assumption A.1 and let $\Phi(H)$ be a (not necessarily proper) term in the lower central series or derived series of H. Let*

$$g \in \mathrm{PSL}_2(L) \cap \mathrm{Comm}_{\mathrm{PSL}_2(\mathbb{R})}(\Phi(H)),$$

where L is a real quadratic extension of a totally real quadratic extension of $\mathbb{Q}$. Then $g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$.

In particular, this conclusion is true if H is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$.

Proof. Since $\Phi(H)$ is a nontrivial (and hence non-elementary) normal subgroup of H, we have that the limit set $\Lambda_{\Phi(H)} \subset \partial \mathbf{H}^2$ of $\Phi(H)$ coincides with that of H, which is to say it is the whole circle. Lemma A.3 now furnishes the conclusion.   □

In fact, Lemma A.2 and Lemma A.3 together furnish us the following (a specialization of Lemma 3.1) to $\mathrm{PSL}_2(\mathbb{R})$:

Proposition A.8. *Let $g \in \mathrm{PSL}_2(\mathbb{R})$ commensurate a Zariski dense subgroup $H < \mathrm{PSL}_2(\mathbb{Z})$ such that $\Lambda_H = S^1$. Then $g^2 \in \mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$.*

We record here for possible future use a generalization of Proposition A.8 to more general number fields. The proof is the same as that of Lemma A.3, by using the second statement of Lemma A.2.

Proposition A.9. *Let F be a totally real number field F, let $\mathcal{O}$ be its ring of integers, and let $g \in \mathrm{PSL}_2(\mathbb{R})$ commensurate a Zariski dense subgroup $H < \mathrm{PSL}_2(\mathcal{O})$ such that $\Lambda_H = S^1$. Then $g^2 \in \mathrm{PSL}_2(F) \sqrt{F}$.*

Here, the definition of the group

$$\mathrm{PSL}_2(F) \sqrt{F}$$

is identical to that of $\mathrm{PSL}_2(\mathbb{Q}) \sqrt{\mathbb{Q}}$ as considered before.

insightful comments, in particular for telling us of Proposition 3.2 and its proof.

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