QUASIPROJECTIVE THREE-MANIFOLD GROUPS AND COMPLEXIFICATION OF THREE-MANIFOLDS

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Abstract. We characterize the quasiprojective groups that appear as fundamental groups of compact 3-manifolds (with or without boundary). We also characterize all closed 3-manifolds that admit good complexifications. These answer questions of Friedl–Suciu, [FrSu], and Totaro [To].

Contents

1. Introduction

A group is called quasiprojective (respectively, Kähler) if it is the fundamental group of a smooth complex quasiprojective variety (respectively, compact Kähler manifold). Kähler and quasiprojective 3-manifold groups have attracted much attention of late [DiSu, Ko1, BMS, DPS, FrSu, Ko2]. In this paper we characterize quasiprojective 3-manifold groups.

We shall follow the convention that our 3-manifolds have **no spherical boundary components.** Capping such boundary components off by 3-balls does not change the fundamental group, which is really what interests us here.

Theorem 1.1 (See Theorem 3.4)**.** *Let N be a compact 3-manifold (with or without boundary). If $\pi_1(N)$ is a quasiprojective group, then N is either Seifert-fibered or $\pi_1(N)$ is one of the following type*

- *virtually free, or*

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- *virtually a surface group.*

Finer results leading to a complete characterization are given in Section 3.1 and Section 5 (see Theorem 5.6). We omit stating these here as they are slightly more complicated to do so.

This characterization of quasiprojective 3-manifold groups answers Questions 8.3 and Conjecture 8.4 of [FrSu]; see Corollary 5.7 and Corollary 5.9.

The following theorem provides an answer to Question 8.1 of [FrSu] under mild hypotheses.

Theorem 1.2 (See Theorem 5.13)**.** *Suppose A and B are groups, such that the free product $G = A * B$ is a quasiprojective group. In addition suppose that both A and B admit nontrivial finite index subgroups, and at least one of A, B has a subgroup of index greater than 2. Then each of A, B are free products of cyclic groups. In particular both A and B are quasiprojective groups.*

A good complexification of a closed smooth manifold M is defined to be a smooth affine algebraic variety U over the real numbers such that M is diffeomorphic to $U(\mathbb{R})$ (the locus of closed points defined over $\mathbb{R}$) and the inclusion $U(\mathbb{R}) \longrightarrow U(\mathbb{C})$ is a homotopy equivalence [To]. Totaro asks whether a closed smooth manifold M admits a good complexification if and only if M admits a metric of non-negative curvature [To, p. 69, 2nd para]. As an application of Theorem 1.1, we prove this in the following strong form for 3-manifolds.

Theorem 1.3 (See Theorem 4.5)**.** *A closed 3-manifold M admits a good complexification if and only if one of the following hold:*

(1) *M admits a flat metric,*
(2) *M admits a metric of constant positive curvature,*
(3) *M is covered by the (metric) product of a round S^2 and $\mathbb{R}$.*

Curiously, the proof of Theorem 1.3 is direct and there is virtually no use of the method or results of [Ku, To, DPS, FrSu]. Our main tools from recent developments in 3-manifolds are:

(1) The Geometrization Theorem and its consequences (see [AFW]).
(2) Largeness of 3-manifold groups [Ag, Wi, LoNi, CLR, La].

The basic complex geometric tool is a theorem of Bauer, [Bau], regarding existence of irrational pencils for quasiprojective varieties (the theorem of Bauer is recalled in Theorem 2.7). It is a useful existence result in the same genre as the classical Castelnuovo-de Franchis Theorem and a theorem of Gromov [Gr, ABCKT].

As a consequence of our results we deduce the restrictions on quasiprojective 3-manifold groups obtained by the authors of [DPS, FrSu, Ko2] and the restrictions on good complexifications of 3-manifolds deduced in [To] (this is done in in Section 3.1.1). We also indicate, in Remark 3.12, how to deduce the classification of (closed) 3-manifold Kähler groups [DiSu, Ko1, BMS] using the techniques of Theorem 1.1, thus providing a unified treatment of known results.

2. Preliminaries

2.1. **Three-manifold groups.** We collect together facts about 3-manifold groups that will be used here.

By a quasi-Kähler manifold we mean the complement of a closed complex analytic subset of a compact connected Kähler manifold.

Definition 2.1.

(1) A group G is **quasiprojective** (respectively, quasi-Kähler) if it can be realized as the fundamental group of a smooth quasiprojective complex variety (respectively, quasi-Kähler manifold).
(2) A group G is a **3-manifold group** if it can be realized as the fundamental group of a compact real 3-manifold (possibly with boundary).
(3) A group G is **large** if it has a finite index subgroup S that admits a surjective homomorphism onto a non-abelian free group. Such a subgroup S necessarily has a finite index subgroup that admits a surjective homomorphism onto F_3.

A **prime 3-manifold** (possibly with boundary) is a 3-manifold that cannot be decomposed as a non-trivial connected sum. **Graph manifolds** are prime 3-manifolds obtained by gluing finitely many Seifert-fibered JSJ components along boundary tori. In particular, torus bundles over a circle are graph manifolds. A 3-manifold M is **geometric** if it is a quotient of one of the following spaces (equipped with standard Riemannian metrics) by a discrete group acting freely properly discontinuously via isometries: $S^3, \mathbb{E}^3, \mathbb{H}^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}, Nil, Sol, \widetilde{Sl_2(\mathbb{R})}$. In this paper we shall mostly deal with closed 3-manifolds. If M is a compact 3-manifold *with* boundary, we say that M is geometric, if the interior of M is geometric. Note that in this case, the interior of M need not even have finite volume. Among the graph manifolds, Sol and Seifert manifolds are geometric; the rest are non-geometric. It follows that the gluing maps between the Seifert components in non-geometric manifolds do not identify circle fibers. (See [AFW, p. 59] and [He1, Ch. 3].)

The following omnibus theorem is the consequence of the Geometrization theorem of Thurston–Perelman and work of a large number of people culminating in the resolution of the virtual Haken problem by Agol and Wise. See [AFW] (especially Diagram 1, p. 36) for an excellent account.

Theorem 2.2. *If a 3-manifold M has a prime component N satisfying one of the following three conditions, then the fundamental group of M is large.*

(1) *N is a compact orientable irreducible 3-manifold with non-empty boundary such that M is* **not** *an I-bundle ("I" is a closed interval) over a surface with non-negative Euler characteristic* [CLR, La].
(2) *N is closed hyperbolic* [Ag, Wi].
(3) *N is a closed, non-geometric graph manifold* [LoNi].

If $\pi_1(M)$ is a nontrivial free product $G_1 * G_2$ (e.g., if M is not prime), where at least one G_i has order greater than 2, then the fundamental group of M is large. The exceptional case $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is realized only by the connected sum of two real projective spaces.

As an immediate corollary we have the following:

Corollary 2.3. *If the fundamental group of M is not large, then M is Seifert-fibered or a Sol manifold.*

A finitely presented group is **coherent** if any finitely generated subgroup is finitely presented.

Theorem 2.4 ([Sc])**.** *Fundamental groups of compact 3-manifolds are coherent.*

A consequence is the following [He1, Ch. 11].

Proposition 2.5. *Let $1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$ be a short exact sequence of infinite finitely generated groups with G the fundamental group of a compact orientable 3-manifold N (possibly with boundary). Then*

(1) *either H is infinite cyclic and Q is the fundamental group of a compact 2-orbifold (possibly with boundary), in which case N is Seifert-fibered;*
(2) *or H is the fundamental group of a compact surface (possibly with boundary) and Q is virtually cyclic.*

Another theorem that will be used is:

Theorem 2.6 ([Bas])**.** *A finitely generated group G is virtually free if and only if G can be represented as the fundamental group of a finite graph of groups where all vertex and edge groups are finite.*

2.2. **Logarithmic irrational pencil.** We shall require an extension, due to Bauer, of the classical Castelnuovo-de Franchis theorem on the existence of an irrational pencil on a projective variety to the more general case of quasiprojective varieties. We refer to [Bau] for details and quickly recall here the basic definitions used in this subsection (see also [Ca, Di] for related material). All varieties are defined over $\mathbb{C}$.

A surjective morphism $f : X \longrightarrow C$ between quasi-projective varieties is said to be a *fibration* if f has an irreducible (and hence connected) general fiber. If C is a curve of genus greater than zero, then f is called an *irrational pencil*.

Theorem 2.7 ([Bau, p. 442]). *Let X be a smooth complex quasiprojective variety such that $\pi_1(X)$ admits a surjective homomorphism to a group G that admits a finite presentation with n generators and m relations, where $n-m \geq 3$. Then there exists an integer $\beta \geq n-m$ and a quasiprojective curve C with first Betti number β and a logarithmic irrational pencil $f : X \longrightarrow C$ with connected fibers.*

The *proof* of Theorem 2.7 in [Bau] combined with Remark 2.3(1) in [Bau] furnishes the following:

Proposition 2.8 ([Bau]). *Let X be a smooth quasiprojective variety, and let $\overline{X}$ denote a smooth compactification such that $\overline{X} \setminus X$ is a divisor with normal crossings. Further suppose that $\pi_1(X)$ admits a surjection onto a group G that admits a finite presentation with n generators and m relations, where $n-m \geq 3$. Let C, f be the quasiprojective curve and logarithmic pencil obtained in Theorem 2.7. Let $\overline{C}$ denote the projective completion of C. Then there exists $f_1 : \overline{X} \longrightarrow \overline{C}$ such that $f_1|_X = f$. In particular, the fibers of f are quasiprojective.*

Proof. Only the last statement (which is really obvious) is not explicitly mentioned in [Bau]. However since we need it explicitly we say a couple of words here:

Note that the fibers of f are intersections of fibers of f_1 with X. All fibers of f_1 are projective varieties as f_1 is algebraic. Hence the fibers of f are quasiprojective. □

The *logarithmic genus* g^* of a curve C is defined by the equality $b_1(C) = g + g^*$, where g is the genus of a smooth completion of C.

Let X be a variety. A subspace $V \subset H^1(X, \mathbb{C})$ is called *isotropic* if the image of $\bigwedge^2 V$ in $H^2(X, \mathbb{C})$ is zero [Bau, p. 441]. A (complex linear) subspace $V \subset H^1(X, \mathbb{C})$ is called *real* if $\overline{V} = V$.

We owe the comment below to the referee:

Remark 2.9. There is a one-to-one correspondence between $\mathbb{R}-$linear subspaces of $H^1(X, \mathbb{R})$ and real subspaces of $H^1(X, \mathbb{C})$ in the above sense, that is, $\mathbb{C}-$linear subspaces V such that $\overline{V} = V$. The correspondence sends any $\mathbb{R}-$linear subspace $W \subset H^1(X, \mathbb{R})$ to

$$W \otimes_{\mathbb{R}} \mathbb{C} \subset H^1(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^1(X, \mathbb{C}).$$

This is the convention we follow.

We could have alternately defined a $\mathbb{R}$–linear subspace V of $H^1(X, \mathbb{C})$ to be *real* if $\overline{V} = V$. Now Theorem 2.10 below deals with maximal real isotropic subspaces V of $H^1(X, C)$. If V is a real isotropic subspace of $H^1(X, \mathbb{C})$ in this sense, then $V + \sqrt{-1}V \subset H^1(X, \mathbb{C})$ is also isotropic. Since V is maximal, it is equal to $V + \sqrt{-1}V$. So a maximal real isotropic subspaces V in this sense is automatically a complex linear subspace of $H^1(X, \mathbb{C})$. Thus the two definitions are essentially equivalent. However "dimension" in Theorem 2.10 below and in [Bau] means complex dimension.

The necessary and sufficient condition for C to be complete in Theorem 2.10 below is slightly misstated in [Bau].

It is a standard fact that the inclusion $X \subset \overline{X}$ induces an injective map from $H^1(\overline{X}, \mathbb{C})$ into $H^1(X, \mathbb{C})$. We identify $H^1(\overline{X}, \mathbb{C})$ with its image in $H^1(X, \mathbb{C})$ in the following:

Theorem 2.10 ([Bau, Theorem 2.1], [Ca, Theorem 2.11]). *Let X be a smooth quasiprojective variety, and let $\overline{X}$ denote a smooth compactification such that $(\overline{X} \setminus X) = D$ is a divisor with normal crossings. Every maximal real isotropic subspace $V \subset H^1(X, \mathbb{C})$ of dimension ≥ 3 determines a unique logarithmic irrational pencil $f : X \longrightarrow C$ onto a curve C with logarithmic genus $g^* \geq 2$. The curve C is complete if and only if V is a maximal isotropic real subspace of $H^1(C, \mathbb{C})$, and so $dim_{\mathbb{C}}(V)$ is equal to the genus of C. Else $V = f^*(H^1(C, \mathbb{C}))$.*

We introduce some more notation towards the final result of this subsection. For $f : X \longrightarrow Y$ be a fibration of quasiprojective varieties, $Sing(f) \subset X$ will denote the set of critical points of f. For any $y \in Y$, let $F_y := f^{-1}(y)$. Let F_b be a regular fiber of f and $\widetilde{b} \in F_b$.

Proposition 2.12 below will use the following Lemma of Nori.

Lemma 2.11 ([No, Lemma 1.5], [Sh, Proposition 3.1])**.** *Let $f : X \to Y$ be a fibration of quasiprojective varieties so that the regular fiber F_b is connected. Let $\iota : F_b \longrightarrow X$ denote the inclusion map. Let*

$$\Xi \subset Y$$

be a Zariski closed subset of codimension greater than one such that for all $y \in Y \setminus \Xi$ we have

$$F_y \setminus (F_y \bigcap Sing(f)) \neq \emptyset .$$

Then $f_* : \pi_1(X, \widetilde{b}) \longrightarrow \pi_1(Y, b)$ is surjective, and its kernel is equal to the image of

$$\iota_* : \pi_1(F_b, \widetilde{b}) \longrightarrow \pi_1(X, \widetilde{b}) .$$

A large part of the proof of the following proposition was supplied by the referee.

Proposition 2.12. *Let X, C, f be as in Theorem 2.8. Then there is an exact sequence*

$$1 \longrightarrow H \longrightarrow \pi_1(X) \longrightarrow \pi_1^{orb}(C) \longrightarrow 1$$

with H finitely generated, where $\pi_1^{orb}(C)$ denotes the orbifold fundamental group of some orbifold (with finitely many orbifold points) whose underlying topological space is C.

Proof. We apply Lemma 2.11 with $Y = C$ and $\Xi = \emptyset$. If $F_y \subset Sing(f)$ for some y, then F_y is a multiple fiber. If every singular fiber of f has an irreducible component with multiplicity one, Lemma 2.11 directly gives an exact sequence

$$1 \longrightarrow \iota_*(\pi_1(F_b, \widetilde{b})) \longrightarrow \pi_1(X, \widetilde{b}) \longrightarrow \pi_1(C) \longrightarrow 1 .$$

Since $\pi_1(F_b, \widetilde{b})$ is finitely generated by the last statement of Proposition 2.8, so is $H = \iota_*(\pi_1(F_b, \widetilde{b}))$ and we are done in this case.

Else suppose there are finitely many points $b_1, \cdots, b_k$ such that the fibers $F_i = F_{b_i}$ are the multiple fibers of the fibration. Let $Z = \{b_1, \cdots, b_k\}$ denote the critical set in C. Suppose that $F_i = F_{b_i}$ has multiplicity n_i, where we define the multiplicity of F_i to be the gcd of the multiplicities of the irreducible components of F_i. We equip C with an orbifold structure C_o with orbifold points b_i of order n_i. Since C is hyperbolic, so is C_o (since its orbifold Euler characteristic must be negative). Hence there exists a finite orbifold-cover C_1 of C_o such that C_1 has no orbifold points [Sc]. This C_1 may be thought of as a branched cover of C with n_i–fold branching at b_i. The fibration $f : X \longrightarrow C$ then lifts to a fibration $f_1 : X_1 \longrightarrow C_1$ where X_1 is a manifold cover of X (since F_i is a multiple fiber with multiplicity n_i). Further the multiplicity of each singular fiber of f_1 (in the above sense) is one. It suffices to show therefore that there is an exact sequence

$$1 \longrightarrow H \longrightarrow \pi_1(X_1, \widetilde{b}) \xrightarrow{f_{1*}} \pi_1(C_1) \longrightarrow 1$$

with H finitely generated.

Two things need to be checked now:

(1) f_{1*} is surjective.
(2) The kernel of f_{1*} is the image $H \subset \pi_1(X_1)$ of the fundamental group of a general fiber F_b.

Given that C_1 has no orbifold points, any (based) loop σ can be homotoped slightly to miss the singular set Z_1 in C_1 without changing its homotopy class. Since f_1 is a fibration away from the singular set, σ can now be lifted to a (based) loop $\sigma_1 \subset X_1$ with $f_{1*}([\sigma_1]) = [\sigma]$ and we conclude that f_{1*} is surjective.

Let $U = C_1 \setminus Z_1$ and $X_U = f_1^{-1}(U)$. Then $f_1 : X_U \longrightarrow U$ is a smooth fibration and hence $\pi_1(X_U)/\pi_1(F_b) = \pi_1(U)$. Next $\pi_1(X_1)$ is the quotient of $\pi_1(X_U)$ by the normal subgroup generated by one loop σ_K around each irreducible component K of $X \setminus X_U$. Hence $\pi_1(X_1)/H$ is the quotient of $\pi_1(U)$ by the normal subgroup generated by $f_{1*}([\sigma_K])$. If $K \subset F_i$ then $f_{1*}([\sigma_K]) = \alpha_i^{n_K}$, where α_i is a small loop around the critical point $b_i \in Z_1$ and n_K is the multiplicity of K. If $K_{i1}, \cdots, K_{il}$ are the irreducible components of F_i with multiplicities $n_{i1}, \cdots, n_{il}$ respectively, then $gcd(n_{i1}, \cdots, n_{il}) = 1$ and hence there exist integers $c_{i1}, \cdots, c_{il}$ such that $\sum_{j=1}^{l} c_{ij} n_{ij} = 1$ and hence $[\alpha_i]$ belongs to the normal subgroup generated by $f_{1*}([\sigma_K])$'s. It follows that the quotient of $\pi_1(U)$ by the normal subgroup generated by $f_{1*}([\sigma_K])$'s is precisely $\pi_1(C_1)$. This proves the proposition. □

Remark 2.13. We emphasize that in the proof of Proposition 2.12, we have actually shown the existence of a finite manifold cover X_1 of X satisfying the exact sequence

$$1 \longrightarrow H \longrightarrow \pi_1(X_1, \widetilde{b}) \longrightarrow \pi_1(C_1) \longrightarrow 1$$

with H finitely generated.

3. Quasiprojective three-manifold groups

In this section, we combine Theorem 2.2 with Theorem 2.7 to completely characterize quasiprojective 3-manifold groups.

We shall use the following restriction on quasiprojective groups due to Arapura and Nori which says that solvable quasiprojective groups are virtually nilpotent.

Theorem 3.1 ([ArNo])**.** *Let N be a closed 3-manifold such that $\pi_1(N)$ is a quasiprojective group. Then N is not a Sol manifold.*

Theorem 3.2. *Let N be a closed 3-manifold, such that $\pi_1(N)$ is a quasiprojective group. Then N is either Seifert-fibered or N is finitely covered by $\#_m S^2 \times S^1$.*

Proof. By Theorem 3.1 we can exclude the case where N is a Sol manifold. Hence it follows that if $\pi_1(N)$ is not large, then, by Corollary 2.3, the manifold N is Seifert-fibered.

Next suppose $\pi_1(N)$ is large. Then there exists a finite index subgroup G of $\pi_1(N)$ such that G admits a surjection onto the free group F_3.

Since $\pi_1(N)$ is quasiprojective, so is G. Let X be a smooth quasiprojective variety with fundamental group G. By Theorem 2.7, there exists a logarithmic pencil f (with connected fibers) of X over a quasiprojective curve C with first Betti number greater than two. By passing to a finite sheeted (orbifold) cover of the base if necessary, we can assume without loss of generality that f has no multiple fibers.

By Proposition 2.8, the generic fiber F is quasiprojective and hence has finitely generated fundamental group. Let H denote the image of $\pi_1(F)$ in $\pi_1(X)$. Now we have an exact sequence

$$1 \longrightarrow H \longrightarrow \pi_1(X) \longrightarrow \pi_1(C) \longrightarrow 1\,.$$

If C is closed, it follows from Proposition 2.5 that N is Seifert fibered. If H is infinite cyclic (or even virtually so), then also N is Seifert fibered.

Else C is quasiprojective non-compact and H is not infinite cyclic. Hence by Proposition 2.5 again, the subgroup H is finite and G is virtually free. By Grushko's theorem, [He1, p. 25, Theorem 3.4], the manifold N is finitely covered by a connected sum $\#_m S^2 \times S^1$. □

Proposition 3.3. *Let N be a 3-manifold with at least one boundary component of positive genus. Assume that $\pi_1(N)$ is an infinite quasiprojective group. Then $\pi_1(N)$ is either virtually free or virtually of the form $\mathbb{Z} \times F_n$ $(n \geq 1)$ or virtually a surface group.*

Proof. By Theorem 2.2(1), either N is an I–bundle over a surface of non-negative Euler characteristic or it is large. If N is an I–bundle over a surface of non-negative Euler characteristic, then $\pi_1(N)$ is either $\mathbb{Z}$ or virtually $\mathbb{Z} \oplus \mathbb{Z}$.

Else, by the same argument as in the proof of Theorem 3.2, we have an exact sequence

$$1 \longrightarrow H \longrightarrow \pi_1(X) \longrightarrow \pi_1(C) \longrightarrow 1$$

with H either $\mathbb{Z}$ or finite, and C a (possibly noncompact) surface. If H is finite, then $\pi_1(N)$ is either virtually free or virtually a surface group.

If H is $\mathbb{Z}$, then N is Seifert-fibered with base a compact orbifold surface with boundary. Consequently, $\pi_1(N)$ is either virtually cyclic or virtually of the form $\mathbb{Z} \times F_n$ with $n \geq 1$. □

Combining Theorem 3.2 and Proposition 3.3 we have the following:

Theorem 3.4. *Let N be a compact 3-manifold (with or without boundary) such that $\pi_1(N)$ is a quasiprojective group. One of the following is true:*

(1) *N is closed Seifert-fibered,*
(2) *$\pi_1(N)$ is virtually free,*
(3) *$\pi_1(N)$ is virtually of the form $\mathbb{Z} \times F_n$ with $n \geq 1$,*
(4) *$\pi_1(N)$ is virtually a surface group.*

3.1. **Refinements and consequences.**

Remark 3.5. The proof of Theorem 3.4 gives us a bit more. A standing assumption in this section is that N is a compact 3-manifold (with or without boundary) and $\pi_1(N)$ is quasiprojective.

Case 1: N is closed prime. Then Theorem 3.2 forces N to be Seifert-fibered.

Case 2: N is closed but not prime. Then from Theorem 3.2 the fundamental group $\pi_1(N)$ is virtually free and hence by Theorem 2.6, $\pi_1(N)$ is the fundamental group of a graph of groups with edge and vertex groups finite. Hence in the prime decomposition of N, each prime component of N must have fundamental group that has virtual cohomological dimension either zero, in which case it is finite; or else virtual cohomological dimension one, in which case it is virtually cyclic. By the classification of such 3-manifold groups (see [AFW, Theorems 1.1, 1.12], [He1, Theorem 9.13]), $\pi_1(N)$ is of the form $G_1 * G_2 * \cdots * G_k$, where each G_i is either the fundamental group of a spherical 3-manifold or $\mathbb{Z}$ or $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.

Case 3: N is an I–bundle over a surface of non-negative Euler characteristic. Then $\pi_1(N)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ or the fundamental group of a Klein bottle. It turns out (see below) that all these three groups are quasiprojective.

Case 4: N has a boundary component of positive genus and $\pi_1(N)$ contains an infinite cyclic normal subgroup. Then by Proposition 3.3, the manifold N is Seifert-fibered with base a compact orbifold surface with boundary. In this case a subgroup G of index at most 2 in $\pi_1(N)$ (if N is non-orientable) or one, i.e., $\pi_1(N)$ itself (if N is orientable) contains an infinite cyclic central subgroup $\langle t \rangle$ such that the quotient $G/\langle t \rangle$ is a free product of cyclic groups (finite or infinite) [He1, p. 118].

Case 5: N has a boundary component of positive genus and $\pi_1(N)$ does not contain an infinite cyclic normal subgroup. Then by Proposition 3.3,

(1) either $\pi_1(N)$ is virtually a surface group in which case N is an I–bundle over a surface [He1, Theorem 13.6],
(2) or after compressing the boundary as far as possible, $N = M \# H$, where H is a (possibly non-orientable) handlebody and hence $\pi_1(H)$ is free, and M is a closed manifold covered by Case 2.

We now demonstrate the converse to Theorem 3.4 by describing examples of smooth quasiprojective varieties that realize the groups occurring in Remark 3.5 as their fundamental groups. To do this we shall restrict ourselves to *orientable* compact 3-manifolds with or without boundary.

We start with a lemma that is well-known to experts. We provide a proof for completeness (see [BiMj, Section 5.3] for a closely related construction).

Lemma 3.6. *Let X be a smooth complex quasiprojective variety, and let G be a finite group acting by automorphisms on X. Then the orbifold fundamental group of X/G is quasiprojective.*

Proof. Let W be a smooth simply connected projective variety admitting a *free* G–action by automorphisms. Such varieties exist by a theorem of Serre, [ABCKT, Example 1.11], which says that any finite group is realizable as the fundamental group of a smooth projective variety.

Let $Y = X \times W$. Then the diagonal action of G on Y is free and the (usual) fundamental group of the quotient Y/G coincides with the orbifold fundamental group of X/G. ☐

The next proposition addresses Cases (1) and (4) in Remark 3.5.

Proposition 3.7. *Let N be Seifert-fibered with fiber subgroup in the center of $\pi_1(N)$ such that the base surface is orientable (with or without boundary). Then $\pi_1(N)$ is quasiprojective.*

Proof. Let Q be the orientable base orbifold of N. Then Q admits the structure of an algebraic curve (projective or quasiprojective according as Q is without boundary or with boundary). Consider the quasiprojective orbifold given by Q (after we put a quasiprojective structure on it). Let $\mathcal{L}$ be an orbifold algebraic line bundle on Q such that

- for each point $x \in Q$, the action of the isotropy group for x on the fiber $\mathcal{L}_x$ is faithful, and
- the degree of $\mathcal{L}$ is the degree of the Seifert-fibration.

Let L denote the underlying variety for the orbifold $\mathcal{L}$. Let $\Sigma \subset L$ be the image of the zero-section of $\mathcal{L}$. Then the complement $L \setminus \Sigma$ is a smooth quasiprojective variety with the same fundamental group as N. □

To address Case (3), we observe first that $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are both quasiprojective. So only the fundamental group of a Klein bottle remains. Let

$$\phi : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$$

be defined by $(z_1, z_2) \longmapsto (\frac{1}{z_1}, -z_2)$. Let $Q \subset \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ be the order 2 subgroup generated by ϕ. Then Q acts freely on C, and the quotient C/Q has the same homotopy type as a Klein bottle.

In order to completely answer the question "Which 3-manifold groups are quasi-projective?", it remains to deal with virtually free groups or virtually surface groups. These will be addressed in Section 5 after developing some further tools in Section 4.

3.1.1. *Consequences.* We deduce some of the results that preceded this paper from Theorem 3.4.

Theorem 3.8 ([DPS, Theorem 1.1])**.** *Let G be the fundamental group of a closed orientable 3-manifold M. Assume M is formal. Then the following are equivalent.*

(1) *The Malcev completion of G is isomorphic to the Malcev completion of a quasi-Kähler group.*
(2) *The Malcev completion of G is isomorphic to the Malcev completion of the fundamental group of S^3, $\#_n(S^1 \times S^2)$, or $S^1 \times \Sigma_g$, where Σ_g denotes a closed orientable surface of genus g with $g \geq 1$.*

Proof. This follows from Theorem 3.2 by observing that a Seifert-fibered space is formal if and only if it is finitely covered by S^3 or a trivial circle bundle [ABCKT, Corollary 3.38]. □

Theorem 3.9 ([FrSu, Theorem 1.2])**.** *Let N be a 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ is a quasiprojective group, then all the closed prime components of N are graph manifolds.*

Proof. All the closed prime components of N are in fact Seifert-fibered by Theorem 3.2 and Remark 3.5 Case (5). □

Theorem 3.10 ([Ko2])**.** *Let N be a 3-manifold with non-empty boundary. If $\pi_1(N)$ is a projective group, then N is an I–bundle over a closed orientable surface.*

Proof. Case 3 and Case 5(1) of Remark 3.5 give that N is an I–bundle over a closed surface S. If S is non-orientable, then $\pi_1(S)$ is not projective, hence $\pi_1(N)$ is not projective.

Case 4 of Remark 3.5 forces a finite index subgroup H of $\pi_1(N)$ to be isomorphic to $F_n \times \mathbb{Z}$, with $n > 1$. The group H is not projective and hence $\pi_1(N)$ is not projective.

Case 5 (2) of Remark 3.5 along with Theorem 2.6 forces a finite index subgroup H of $\pi_1(N)$ to be isomorphic to F_n, with $n > 1$. The group H is not projective and hence $\pi_1(N)$ is not projective. □

Remark 3.11. Kotschick proves Theorem 3.10 in the context of Kähler groups. The proof we have given above works equally well in the Kähler case. The only point to be noted is that we have to replace the use of Theorem 2.7 by the analogous theorem in the Kähler context ensuring existence of irrational pencils as in [Gr] or [DeGr].

Remark 3.12. In order to recover the main Theorems of [DiSu] or [Ko1] from Theorem 3.4 with the modifications mentioned in Remark 3.11, it remains to show that fundamental groups of circle bundles N over closed surfaces of positive genus are not Kähler. If the bundle is trivial, then $b_1(N)$ is odd. If the bundle is non-trivial, then the cup product vanishes identically on H^1. Hence the maximal isotropic subspace of H^1 has dimension $2g$, which would imply that $\pi_1(N)$ would admit a surjection onto the fundamental group of a surface of genus $2g$, a contradiction.

Following [To, p. 69], define a good complexification of a closed manifold M without boundary to be a smooth affine algebraic variety U over $\mathbb{R}$ such that M is diffeomorphic to the space $U(\mathbb{R})$ of real points and the inclusion $U(\mathbb{R}) \hookrightarrow U(\mathbb{C})$ is a homotopy equivalence.

Using Theorem 3.2, we have an alternative proof of the following theorem of Totaro.

Theorem 3.13 ([To, Section 2])**.** *Let M be a closed orientable 3-manifold with a good complexification. Then either the cup product $H^1(M, \mathbb{Q}) \otimes H^1(M, \mathbb{Q}) \longrightarrow H^2(M, \mathbb{Q})$ is 0 or M is formal.*

Proof. By Theorem 3.2, M is

(1) either finitely covered by $\#_n(S^1 \times S^2)$ in which case the above cup product is 0,
(2) or M is Seifert-fibered and finitely covered by either S^3 or a trivial circle bundle over a closed orientable surface; in this case M is formal,
(3) or M is finitely covered by a non-trivial circle bundle over a closed surface of positive genus; in this case, the above cup product is zero.

This completes the proof. □

Remark 3.14. In the definition of a good complexification, if the affine variety over $\mathbb{R}$ is weakened to a Stein manifold equipped with an antiholomorphic involution, then all manifolds admit such a complexification. Indeed, given a manifold M, the total space of the cotangent bundle T^*M admits a Stein manifold structure [El, Go] such that the multiplication by -1 on T^*M is an antiholomorphic involution.

4. Classification of three-manifolds with good complexification

The definition of a good complexification was recalled prior of Theorem 3.13. In this Section we shall describe all 3-manifolds admitting a good complexification.

Lemma 4.1. *If a closed smooth manifold M admits a good complexification, and M_1 is a finite-sheeted étale cover of M, then M_1 also admits a good complexification.*

Proof. Let U be a good complexification of M. Fix a diffeomorphism of M with $U(\mathbb{R})$. Since the inclusion $U(\mathbb{R}) \hookrightarrow U(\mathbb{C})$ induces an isomorphism of fundamental groups, the covering M_1 of $M = U(\mathbb{R})$ has a unique extension to a covering U_1' of $U(\mathbb{C})$. For any point $x \in U(\mathbb{R})$, the Galois (antiholomorphic) involution σ of $U(\mathbb{C})$ for the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ induces the identity map of $\pi_1(U(\mathbb{C}), x)$ because $\sigma|_{U(\mathbb{R})} = \mathrm{Id}_{U(\mathbb{R})}$ and the inclusion $U(\mathbb{R}) \hookrightarrow U(\mathbb{C})$ induces an isomorphism of $\pi_1(U(\mathbb{C}), x)$ with $\pi_1(U(\mathbb{R}), x)$. Therefore, σ has a unique lift σ' to U_1' that fixes M_1 pointwise.

The pair (U_1', σ') defines a smooth affine variety over $\mathbb{R}$ (see [FrSu, p. 157, Lemma 4.1]). Now the variety (U_1', σ') defined over $\mathbb{R}$ is a good complexification of M_1. □

Let M be a closed 3-manifold admitting a good complexification. From Theorem 3.2 it follows that M is either closed Seifert-fibered or is finitely covered by $\#_m S^2 \times S^1$. We shall therefore consider separately the following problems:

(1) Which Seifert-fibered manifolds admit good complexifications?
(2) Does $\#_m S^2 \times S^1$, $(m > 1)$, admit a good complexification?

Seifert-fibered 3-manifolds split into three further sub-cases according to the orbifold Euler characteristic $\chi(S)$ of the orbifold base S of the fibration:

(1a) $\chi(S) > 0$,
(1b) $\chi(S) = 0$, and

(1c) $\chi(S) < 0$.

First we consider case (1a). If $\chi(S) > 0$, then M is covered by S^3 or $S^2 \times S^1$ (this follows from the Poincaré conjecture and classical 3-manifold topology [AFW, Theorem 1.12]). Further, Perelman's solution of the Geometrization conjecture also implies that M is a geometric quotient of S^3 or $S^2 \times S^1$. It is known that geometric quotients of S^3 or $S^2 \times S^1$ admit good complexification [To, Lemma 3.1], [Ku].

4.1. **Seifert-fibered manifolds with base hyperbolic.** Now we consider case (1c).

Proposition 4.2. *Let N be Seifert-fibered with hyperbolic base orbifold. Then N does not admit a good complexification.*

Proof. Seifert-fibered manifolds are finitely covered by circle bundles over surfaces. Since a finite cover of a good complexification is a good complexification (see Lemma 4.1), it suffices to rule out principal S^1–bundles N over surfaces S with $\text{genus}(S) = g > 1$ and trivial orbifold structure.

So N is now a principal S^1–bundle over a compact oriented surface S with $\text{genus}(S) = g > 1$.

Let, if possible, X be a good complexification of N. Let $X_{\mathbb{C}} = X(\mathbb{C})$ be the base change of X to $\mathbb{C}$.

If the principal S^1–bundle $N \longrightarrow S$ is nontrivial, then the fundamental group $\pi_1(N)$ admits a presentation

$$\langle a_1, \cdots, a_g, b_1, \cdots, b_g, t \mid [a_i, t], [b_i, t], \prod_{i=1}^{g} [a_i, b_i] t^n \rangle .$$

Then $\pi_1(N)$ admits a surjection onto the surface group $\pi_1(\Sigma_g) = \langle a_1, \cdots, a_g, b_1, \cdots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle$. Hence, by Theorem 2.7, there exists an irrational logarithmic pencil

$$f : X_{\mathbb{C}} \longrightarrow C \tag{4.1}$$

onto a quasiprojective curve C with $b_1(C) \geq (2g-1)$. If C is non-compact, then $\pi_1(N)$ must admit a surjection onto the free group F_{2g-1}, which is impossible as this would induce a surjection of $\pi_1(\Sigma_g)$ (the fundamental group of the closed orientable surface of genus $g > 1$) onto F_{2g-1}. Hence C is compact.

Alternatively, if N is the trivial principal S^1–bundle over S, then $\pi_1(N)$ admits a surjection onto $\pi_1(S)$. Hence by Theorem 2.7, there exists a logarithmic pencil as in (4.1) onto a quasiprojective curve C with $b_1(C) \geq (2g-1)$. If C is is non-compact, then $\pi_1(N)$ must admit a surjection onto F_{2g-1} which is impossible. Hence C is compact also in this case.

In either case the genus of C is g and $f_* : \pi_1(X(\mathbb{C})) \longrightarrow \pi_1(C)$ has exactly $\langle t \rangle$ as its kernel.

Let

$$\sigma : X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$$

denote the antiholomorphic involution corresponding to the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$. Fix an identification of N with $X_{\mathbb{C}}^{\sigma} = X(\mathbb{R})$. The action of σ on $H^1(X_{\mathbb{C}}, \mathbb{C})$ is trivial because the inclusion $X_{\mathbb{C}}^{\sigma} \hookrightarrow X_{\mathbb{C}}$ is a homotopy equivalence. There is a natural bijection between the irrational logarithmic pencils as in (4.1) and the maximal real isotropic subspaces of $H^1(X_{\mathbb{C}}, \mathbb{C})$ satisfying certain conditions (see the first paragraph in [Bau, p. 442]). In view of this bijective correspondence, from the fact that the action of σ on $H^1(X_{\mathbb{C}}, \mathbb{C})$ is trivial we conclude that the map f in (4.1) commutes with σ. In other words, σ descends to an antiholomorphic involution

$$\sigma_1 : C \longrightarrow C \tag{4.2}$$

of C. Note that inclusion

$$C^{\sigma_1} \supset f(X_{\mathbb{C}}^{\sigma}) \tag{4.3}$$

holds, where C^{σ_1} is the fixed point set for σ_1.

Since $f_* : \pi_1(X(\mathbb{C})) \longrightarrow \pi_1(C)$ has exactly $\langle t \rangle$ as its kernel, the same is true for $(f|_N)_* : \pi_1(N) \longrightarrow \pi_1(C)$. Since N and C are both Eilenberg–Maclane spaces, it follows that f is homotopic to the bundle projection map from N (a circle bundle over C) to C. Hence the restriction of f to $N = X_{\mathbb{C}}^{\sigma}$ is surjective. Therefore, from (4.3) it follows that $C^{\sigma_1} = C$. This is a contradiction because the identity map of C is not antiholomorphic. Hence N cannot admit a good complexification. □

4.2. **Nil manifolds.** We now consider the second case where the orbifold base of the Seifert fibration is flat (the genus of the orbifold is 1).

Non-trivial circle bundles over Euclidean orbifolds are also called *nil manifolds.*

Proposition 4.3. *Let N be a Nil manifold. Then N does not admit a good complexification.*

Proof. As before, in view of Lemma 4.1 it suffices to rule out non-trivial principal S^1–bundles N over the torus with trivial orbifold structure.

So N is a nontrivial principal S^1–bundle over a surface of genus one.

Suppose X is a good complexification of N. As before, let

$$\sigma \,:\, X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$$

denote the antiholomorphic involution corresponding to the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Let

$$Alb \,:\, X_{\mathbb{C}} \longrightarrow C \tag{4.4}$$

be the (quasi) Albanese map. Then C has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ and hence C is either an elliptic curve or the semiabelian variety $\mathbb{C}^* \times \mathbb{C}^*$ (see [NWY]).

Case 1: If C is an elliptic curve, then the same arguments as in Section 4.1 now go through as before. It leads to the conclusion that the real dimension of the fixed set of the involution σ is 4, which is a contradiction.

Case 2: Assume therefore that C is the semiabelian variety $\mathbb{C}^* \times \mathbb{C}^*$. If $\dim_{\mathbb{C}}(Alb(X)) = 1$, then $Alb(X)$ is a curve with fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ and the same argument as in the proof of Proposition 4.2 goes through.

Case 3: Hence suppose that $\dim_{\mathbb{C}}(Alb(X)) = 2$, in which case all the fibers of Alb are quasiprojective curves.

Case 3A: If some fiber of Alb is a singular curve, the same (complex Morse theoretic) arguments as in [Ka, Lemmas 4, 7] (see also [BMP, Theorem 7.9]) show that the kernel of $Alb_* \,:\, \pi_1(X) \longrightarrow \pi_1(C)$ is infinitely presented.

Case 3B: Hence the fibers of Alb must all be regular. This forces $\pi_1(F) = \mathbb{Z}$ and hence $F = \mathbb{C}^*$ (since F is a curve). Thus X is a holomorphic $\mathbb{C}^*$–bundle over $\mathbb{C}^* \times \mathbb{C}^*$.

We note that the involution σ commutes with Alb. This is because Alb is the base change to $\mathbb{C}$ of a morphism between varieties defined over $\mathbb{R}$. Therefore, σ descends to an antiholomorphic involution

$$\sigma_1 \,:\, C \longrightarrow C\,.$$

Since the fixed point set $C^{\sigma_1} \subset C$ for the involution σ_1 contains $Alb(X^{\sigma}_{\mathbb{C}})$, and $X^{\sigma}_{\mathbb{C}}$ is nonempty, we know that C^{σ_1} is nonempty. Consequently,

$$C^{\sigma_1} \,=\, S^1 \times S^1\,.$$

Therefore, $X^{\sigma}_{\mathbb{C}} = N$ is a principal S^1–bundle over $C^{\sigma_1} = S^1 \times S^1$. We will show that the first Chern class of this principal S^1–bundle on C^{σ_1} vanishes.

The first Chern class of the above principal S^1–bundle over C^{σ_1} coincides with the first Chern class of the principal $\mathbb{C}^*$–bundle $X_{\mathbb{C}}$ in (4.4) after we identify $H^2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z})$ with $H^2(C^{\sigma_1}, \mathbb{Z})$ using the inclusion of C^{σ_1} in C. Therefore, it suffices to show that the first Chern class of an algebraic line bundle over $\mathbb{C}^* \times \mathbb{C}^*$ vanishes.

Take any algebraic line bundle L over $\mathbb{C}^* \times \mathbb{C}^*$. The line bundle L extends to an algebraic line bundle over the projective surface $\mathbb{P}^1 \times \mathbb{P}^1$. To see this, take the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of any divisor in $\mathbb{C}^* \times \mathbb{C}^*$ representing L. Let $L' \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be an extension of L. Therefore, $c_1(L) = \iota^* c_1(L')$, where $\iota \,:\, \mathbb{C}^* \times \mathbb{C}^* \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is the inclusion map. But

$$\iota^*(H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})) \,=\, 0\,.$$

Therefore, $c_1(L) = 0$.

Since $X_{\mathbb{C}}^{\sigma} = N$ is the trivial S^1–bundle over $S^1 \times S^1$, we conclude that $N = S^1 \times S^1 \times S^1$. This contradicts the given condition that N is a nil manifold. □

4.3. **Connected sum of copies of $S^2 \times S^1$.** Now we consider case (2).

Proposition 4.4. *Let N be any closed 3-manifold with virtually free fundamental group and suppose that $\pi_1(N)$ is not virtually cyclic. Then N does not admit a good complexification.*

Proof. Any closed 3-manifold with virtually free fundamental group is covered by a connected sum of copies of $S^2 \times S^1$. Therefore, in view of Lemma 4.1, it is enough to rule out $N = \#_m S^2 \times S^1$, where $m > 1$.

The argument here follows that in Section 4.1. We continue with the same notation. By passing to a finite-sheeted cover, we can assume that $m \geq 3$. So Theorem 2.7 applies to give

$$f : X_{\mathbb{C}} \longrightarrow C,$$

where C is a quasiprojective curve with $b_1(C) \geq m \geq 3$. Since $\pi_1(X_{\mathbb{C}}) = \pi_1(N) = F_m$, this forces $\pi_1(C)$ to equal F_m and $f_* : \pi_1(X_{\mathbb{C}}) \longrightarrow \pi_1(C)$ to be an isomorphism. Further, C must be noncompact.

As shown in the proof of Proposition 4.2, the morphism f commutes with the antiholomorphic involution σ of $X_{\mathbb{C}}$. Therefore, σ descends to an involution σ_1 of C (as in (4.2)). The fixed point locus C^{σ_1} is a disjoint union of (real) one dimensional proper (embedded) submanifolds of C. The image $f(X_{\mathbb{C}}^{\sigma}) \subset C^{\sigma_1}$ is a connected component of of C^{σ_1}, in particular, $f(X_{\mathbb{C}}^{\sigma})$ is a connected proper (embedded) submanifold of C of dimension one.

The inclusion $f(X_{\mathbb{C}}^{\sigma}) \hookrightarrow C$ induces an isomorphism of fundamental groups. On the other hand, we have $b_1(C) \geq m \geq 3$. Therefore, there is no connected proper (embedded) submanifold of C of dimension one such that the inclusion induces an isomorphism of fundamental groups. In view of this contradiction, the proof of the proposition is complete. □

Combining Theorem 3.2 with Propositions 4.2, 4.3 and 4.4 (along with the Geometrization Theorem) we obtain:

Theorem 4.5. *If a closed 3-manifold M admits a good complexification, then one of the following is true:*

(1) *The manifold M admits the structure of a Seifert-fibered space over a spherical orbifold and is therefore covered by S^3 or $S^2 \times S^1$. Hence M either admits a metric of constant positive curvature or is covered by the (metric) product of a round S^2 and $\mathbb{R}$.*
(2) *The manifold M is finitely covered by $S^1 \times S^1 \times S^1$. Hence M admits a flat metric.*

5. Virtually free groups and virtually surface groups

The *genus* of a complex quasiprojective curve C is defined to be the genus of its smooth compactification $\overline{C}$.

Lemma 5.1. *Let X be a smooth complex quasiprojective variety and*

$$f : X \longrightarrow C$$

a nonconstant algebraic map to a quasiprojective complex curve of positive genus. Let $\iota : S \hookrightarrow X$ be a smooth curve in X such that $f \circ \iota$ is a nonconstant map. Then the dimension of the image of the pullback homomorphism

$$\iota^* : H^1(X, \mathbb{R}) \longrightarrow H^1(S, \mathbb{R})$$

is at least two.

Proof. Let $\overline{X}$ be a smooth compactification of X such that f extends to a morphism

$$\overline{f} : \overline{X} \longrightarrow \overline{C}$$

with the image of the extension

$$\overline{\iota} : \overline{S} \longrightarrow \overline{X}$$

being smooth.

We have $(\overline{f}\circ\overline{\iota})^*(H^0(\overline{C},\, \Omega_{\overline{C}})) \subset \overline{\iota}^*(H^0(\overline{X},\, \Omega_{\overline{X}}))$, and $(\overline{f}\circ\overline{\iota})^* \,:\, H^0(\overline{C},\, \Omega_{\overline{C}}) \longrightarrow H^0(\overline{S},\, \Omega_{\overline{S}})$ is injective. Therefore,

$$\dim \overline{\iota}^*(H^0(\overline{X},\, \Omega_{\overline{X}})) \,\geq\, 1\,.$$

This implies that

$$\dim_{\mathbb{R}} \overline{\iota}^*(H^1(\overline{X},\, \mathbb{R})) \,=\, 2\dim_{\mathbb{C}} \overline{\iota}^*(H^0(\overline{X},\, \Omega_{\overline{X}})) \,\geq\, 2\,. \tag{5.1}$$

The restriction homomorphism $H^1(\overline{X},\, \mathbb{R}) \longrightarrow H^1(X,\, \mathbb{R})$ is injective, and $\overline{\iota})|_S \,=\, \iota$. Therefore, from (5.1) it follows that $\dim_{\mathbb{R}} \iota^*(H^1(X,\, \mathbb{R})) \,\geq\, 2$. □

A slight modification of the techniques developed in the proofs of Propositions 4.2 , 4.3 and 4.4 yield the following general result. (This might be regarded as a (weak) "maps" version of a theorem of Catanese [Ca, Theorem A'] which provides the analogue for spaces.)

Proposition 5.2. *Let X be a smooth complex quasiprojective variety and $f \,:\, X \longrightarrow C$ an irrational logarithmic pencil over a curve C with $b_1(C) \,\geq\, 3$. Let F be any regular fiber of f and $i \,:\, F \hookrightarrow X$ the inclusion map. Suppose that the image $i_*(\pi_1(F))$ is either infinite cyclic or finite. Let A be an algebraic automorphism of X. Then $A(F)$ is a fiber of f. Hence A induces an algebraic automorphism $A_0 \,:\, C \longrightarrow C$.*

Proof. By lifting to a further Galois cover of the base C if necessary, we can assume that the smooth projective curve $\overline{C}$ has genus greater than one.

Let i denote the inclusion of $A(F)$ in X. Assume that $f\circ i$ is not a constant map. Applying Lemma 5.1 to any smooth curve $S \,\subset\, A(F)$ such that $f|_S$ is not constant, we conclude that the dimension of the image of the homomorphism

$$i^* \,:\, H^1(X,\, \mathbb{R}) \,\longrightarrow\, H^1(A(F),\, \mathbb{R}) \tag{5.2}$$

is at least two.

Since A is a homeomorphism, from the given condition on F it follows that $i_*(\pi_1(A(F))) \,\subset\, \pi_1(X)$ is either infinite cyclic or finite. Therefore, the dimension of the image of the homomorphism

$$i_* \,:\, H_1(S,\, \mathbb{R}) \,\longrightarrow\, H_1(X,\, \mathbb{R})$$

is at most one. But this contradicts the observation that the image of the homomorphism in (5.2) is at least two. Therefore, $f\circ i$ is a constant map. □

The next proposition imposes restrictions on quasiprojective groups that are virtually free groups or virtually surface groups.

Proposition 5.3. *Let G be a quasi-projective group that is virtually a non-abelian free group or virtually the fundamental group of a closed orientable surface of genus greater than one. Then there is a short exact sequence of the form*

$$1 \,\longrightarrow\, K \,\longrightarrow\, G \,\longrightarrow\, H \,\longrightarrow\, 1\,,$$

where K is finite and H is the fundamental group of an orientable orbifold surface (possibly with boundary).

Proof. Let X be a smooth quasiprojective variety with fundamental group G. Let X_1 be a finite Galois étale cover of X with fundamental group H_1 such that

- either H_1 is non-abelian free, or
- H_1 is isomorphic to the fundamental group of a closed orientable surface of genus greater than one.

Let $f \,:\, X_1 \longrightarrow C$ be a logarithmic pencil given by Theorem 2.7, and let $i \,:\, F \hookrightarrow X_1$ be a regular fiber of f. Then $i_*\pi_1(F)$ is finite. The quotient group $Q \,=\, G/H_1$ acts by algebraic automorphisms on X_1 and hence, by Proposition 5.2, on C via algebraic automorphisms. Let K be the kernel of the action of Q on C. Let H be the orbifold fundamental group of the quotient C/Q. Then we have an exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1\,.$$

Also since Q acts on C by holomorphic automorphisms, the quotient C/Q is orientable. □

Proposition 5.4. *Let G be a quasi-projective 3-manifold group that is virtually free. Then G is one of the following:*

(1) $G = \mathbb{Z}$ *or* $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$
(2) $G = *_i G_i$ *where each* G_i *is cyclic.*

Proof. If G is virtually cyclic, then by the classification of such 3-manifold groups (see [AFW, Theorems 1.1, 1.12], [He1, Theorem 9.13]), G is one of $\mathbb{Z}$ or $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

Else G is virtually a non-abelian free group. Let N be a 3-manifold with $G = \pi_1(N)$. Then we are in Case (2) or Case 5(2) of Remark 3.5. In either case, $G = *_i G_i$ where each G_i is either finite or $\mathbb{Z}$ or $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. By [ScWa, Theorem 3.11], the group G contains no finite normal subgroup. Hence by Proposition 5.3, the group G is isomorphic to the fundamental group of an orientable orbifold surface S. Since G is virtually a non-abelian free group, the orbifold surface S must have boundary. The orbifold fundamental group G of such an S is of the form $G = *_i G_i$, where each G_i is cyclic. This is because S deformation retracts onto a wedge $(\vee_i S^1) \bigvee (\vee_j D_j)$, where each D_j is a quotient of the unit disk by a finite cyclic group acting with a single fixed point at the origin. □

Proposition 5.5. *Let G be a quasi-projective 3-manifold group that is virtually the fundamental group of a closed orientable surface of genus greater than one. Then G is isomorphic to the fundamental group of a closed orientable surface of genus greater than one.*

Proof. If G is not isomorphic to the fundamental group of a closed orientable surface of genus greater than one, then by Case 5(1) of Remark 3.5, the group G contains an index 2 subgroup H that is isomorphic to the fundamental group of a closed orientable surface of genus greater than one. Also G is isomorphic to the fundamental group of a closed non-orientable surface of genus greater than one.

Since such a G contains no finite normal subgroup, by Proposition 5.3, the group G is isomorphic to the fundamental group of an orientable orbifold surface S. No orientable orbifold surface S has the same fundamental group as a closed non-orientable surface. Therefore, the proposition follows. □

Combining the observations in Section 3.1 with those of this section, we have the following classification result for quasiprojective 3-manifold groups.

Theorem 5.6. *Let G be a quasiprojective group that can be realized as the fundamental of a compact 3-manifold N with or without boundary. Then either N is Seifert-fibered, or G satisfies one of the following:*

(a) *G is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ or the fundamental group of a Klein bottle or the fundamental group of a closed orientable surface of genus greater than one.*
(b) $G = *_i G_i$ *where each* G_i *is cyclic.*

Each of the groups appearing in above alternatives (a) and (b) are quasiprojective. If N is closed Seifert-fibered, and N is spherical, flat or covered by $S^2 \times \mathbb{R}$, then $\pi_1(N)$ is quasiprojective. If N is an orientable closed Seifert-fibered with hyperbolic base orbifold B, then $\pi_1(N)$ is quasiprojective if and only if B is an orientable orbifold.

Proof. All the statements except for the last two are contained in Remark 3.5, the examples constructed in Section 3.1 or in Proposition 5.4 and Proposition 5.5. The penultimate statement is a consequence of the fact that such manifolds admit good complexifications [To].

It remains to deal with N an *orientable*, Seifert-fibered with hyperbolic base orbifold B. That an orientable, Seifert-fibered space N with orientable hyperbolic base orbifold B has quasiprojective fundamental group follows from Proposition 3.7 and the last statement in the first paragraph of [He1, p. 118]. We will prove the converse statement.

Let X be a smooth quasiprojective variety with $\pi_1(X) = \pi_1(N)$. Let B' be an orientable hyperbolic surface (without orbifold points) that (Galois) covers B and with $b_1(B') > 2$. There is a corresponding finite (Galois) cover N' of N which is a circle bundle over B'. Let X' be the Galois étale cover of X corresponding to the subgroup $\pi_1(N')$. By Theorem 2.7 (or more precisely by Theorem A' of [Ca] which is its generalization to the quasi-Kähler context), there is a pencil $f : X' \longrightarrow C$ with C a closed curve (as N is closed). We are now in the situation of Proposition 5.2; the deck transformation group Q induces an algebraic action on C forcing the quotient orientable orbifold C/Q to be orientable. □

The following immediate Corollary of Theorem 5.6 answers Question 8.3 of [FrSu, p. 166].

Corollary 5.7. *Let G be a quasiprojective group that can be realized as the fundamental of a closed graph manifold M. Then M is Seifert-fibered.*

Friedl and Suciu conjecture the following in [FrSu]:

Conjecture 5.8 ([FrSu, p. 166, Conjecture 8.4])**.** *Let N be a compact 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ is a quasiprojective group and N is not prime, then N is the connected sum of spherical 3-manifolds and manifolds which are either diffeomorphic to $S^1 \times D^2$, $S^1 \times S^1 \times [0,1]$, or the 3-torus.*

Following is a strong positive answer to it.

Corollary 5.9. *Let N be a compact 3-manifold with empty or toroidal boundary such that $\pi_1(N)$ is a quasiprojective group and N is not prime. Then N is the connected sum of lens spaces, $S^1 \times S^2$ and manifolds which are diffeomorphic to disk bundles over the circle.*

Proof. We are in Case (b) of Theorem 5.6. Then by the prime decomposition theorem for 3-manifolds [He1, Ch. 3], the manifold M is a connected sum of manifolds with cyclic fundamental group. A complete list of such manifolds is: lens spaces, $S^1 \times S^2$ and manifolds which are diffeomorphic to disk bundles over the circle. □

From Theorem 5.6 it follows that a closed **non-orientable** Seifert-fibered manifold N with hyperbolic base orbifold such that its orientable double cover N' is a Seifert-fibered manifold with **non-orientable** hyperbolic base orbifold cannot have quasiprojective fundamental group, because otherwise $\pi_1(N')$ is quasiprojective contradicting Theorem 5.6. The only case that thus remains unanswered by Theorem 5.6 is the following:

Question 5.10. Let N be a closed **non-orientable** Seifert-fibered space with hyperbolic base orbifold such that its orientable double cover is a Seifert-fibered space with **orientable** hyperbolic base orbifold. Is $\pi_1(N)$ quasiprojective?

5.1. **Quasiprojective free products.** In [FrSu], Friedl and Suciu ask the following:

Question 5.11 ([FrSu, p. 165, Question 8.1])**.** Suppose A and B are groups, such that the free product $A * B$ is a quasiprojective group. Does it follow that A and B are already quasiprojective groups?

Lemma 5.12. *Suppose A and B are groups, such that the free product $A * B$ is a quasiprojective group. In addition suppose that both A, B admit nontrivial finite index subgroups and at least one of A, B has a subgroup of index greater than 2. Then $A * B$ is virtually free.*

Proof. Since A, B admit nontrivial finite index subgroups, they also admit finite index normal subgroups. By the hypothesis, there exist finite quotients A_1 and B_1 (of A and B respectively) of which at least one has order more than 2. So $A * B$ admits a surjection onto $A_1 * B_1$, and hence a finite index subgroup G of $A * B$ admits a surjection onto a non-abelian free group with greater than 2 generators.

Let X be a smooth quasiprojective variety with fundamental group G. By Proposition 2.12, there exists an exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow F_n \longrightarrow 1$$

with $n \geq 3$ and H finitely generated. Hence H is trivial [ScWa, Theorem 3.11]. It follows that $A * B$ is virtually free. □

Following is a positive answer to Question 5.11 under mild hypotheses.

Theorem 5.13. *Suppose A and B are groups, such that the free product $G = A * B$ is a quasiprojective group. In addition suppose that both A, B admit nontrivial finite index subgroups and at least one of A, B has a subgroup of index greater than 2. Then each of A, B are free products of cyclic groups. In particular both A and B are quasiprojective.*

Proof. By Lemma 5.12 and Proposition 5.3, there is a short exact sequence of the form

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1\,,$$

where K is finite and H is the fundamental group of an orientable orbifold surface. The subgroup K is trivial by [ScWa, Theorem 3.11], and H is virtually free. Hence as in the proof of Proposition 5.4, we have $G = *_i G_i$, where each G_i is cyclic. Therefore, since both A and B are free factors of G, they are free product of cyclic groups. Hence A and B are fundamental groups of orientable orbifold surface. In particular, both A and B are quasiprojective. □

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