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## CANNON–THURSTON MAPS FOR HYPERBOLIC GROUP EXTENSIONS

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### 1. INTRODUCTION

Let  $M$  be a closed hyperbolic 3-manifold fibering over the circle with fiber  $F$ . Let  $\tilde{F}$  and  $\tilde{M}$  denote the universal covers of  $F$  and  $M$ , respectively. Then  $\tilde{F}$  and  $\tilde{M}$  are quasi-isometric to  $\mathbb{H}^2$  and  $\mathbb{H}^3$ , respectively. Now let  $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}_\infty^1$  and  $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}_\infty^2$  denote the standard compactifications. In [6] Cannon and Thurston show that the usual inclusion of  $\tilde{F}$  into  $\tilde{M}$  extends to a continuous map from  $\mathbb{D}^2$  to  $\mathbb{D}^3$ .

It would be interesting to know how far this result can be generalized. Let  $H$  be a hyperbolic subgroup of a hyperbolic group  $G$ . We choose a finite generating set of  $G$  that contains a finite generating set of  $H$ . Let  $\Gamma_G$  and  $\Gamma_H$  be the Cayley graphs of  $G, H$  with respect to these generating sets. There is a continuous proper embedding  $i$  of  $\Gamma_H$  into  $\Gamma_G$ . Now every hyperbolic group admits a compactification of its Cayley graph by adjoining the Gromov boundary consisting of asymptote-classes of geodesics [11]. Let  $\widehat{\Gamma}_H, \widehat{\Gamma}_G$  denote these compactifications.

A natural question prompted by [6] is: Does the continuous proper embedding  $i: \Gamma_H \rightarrow \Gamma_G$  extend to a continuous map  $\hat{i}: \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$ ?

Questions along this line have been raised by Bonahon [4]. Related questions in the context of Kleinian groups have been studied by Bonahon [5], Floyd [8] and Minsky [12].

Such a map, if it exists, is unique. In this paper we answer the above question affirmatively when  $H$  is normal in  $G$ :

**THEOREM:** *Let  $G$  be a hyperbolic group and let  $H$  be a hyperbolic subgroup that is normal in  $G$ . Let  $i: \Gamma_H \rightarrow \Gamma_G$  be the continuous proper embedding of  $\Gamma_H$  in  $\Gamma_G$  described above. Then  $i$  extends to a continuous map  $\hat{i}$  from  $\widehat{\Gamma}_H$  to  $\widehat{\Gamma}_G$ .*

The Gromov boundary of  $\Gamma_H$  can be regarded as an “intrinsic” (or “algebraic”) limit set of  $H$ . Since  $H$  is normal in  $G$ , its limit set in  $\widehat{\Gamma}_G$  is all of the Gromov boundary of  $\Gamma_G$ . Thus, the boundary of  $\Gamma_G$  can be regarded as an “extrinsic” or “geometric” limit set of  $H$  when  $H$  is thought of as sitting inside  $G$ . The main theorem of this paper states that there is a continuous map from the former to the latter.

When  $G$  is the fundamental group of a closed hyperbolic 3-manifold fibering over the circle with fiber  $F$  and  $F$  has fundamental group  $H$ , our main theorem reduces to the result of Cannon–Thurston [6] mentioned above. However our proof, even in this case, is different, as no use is made of either an explicit “Sol-type” metric (coming from stable and unstable singular foliations on  $\tilde{F}$ ) on  $\tilde{M}$  or the uniformization theorem of Thurston [19]. In [6] or [12], for every point  $p$  in  $\mathbb{S}_\infty^1$  a sequence of leaves of stable and unstable laminations are taken converging to  $p$ . Totally geodesic “vertical” planes through these leaves “trap”

quasi-convex sets of small “Euclidean diameter” on the side containing  $p$ . These “trapped sets” are then used to prove the continuity of  $\hat{i}$ . Such “trapped sets” are done away with and replaced here by techniques that are (literally) coarser and more elementary. The techniques of this paper can also be used to prove a similar theorem (due to Minsky [12]) regarding the existence of a topological semiconjugacy between limit sets of a Fuchsian group and certain geometrically tame Kleinian groups.

2. PRELIMINARIES

We start off with some preliminaries about hyperbolic groups in the sense of Gromov [11]. For details, see [7, 9]. Let  $G$  be a hyperbolic group with Cayley graph  $\Gamma$  equipped with a word-metric  $d$ . The *Gromov boundary* of the Cayley graph  $\Gamma$ , denoted by  $\partial\Gamma$ , is the collection of equivalence classes of geodesic rays  $r: [0, \infty) \rightarrow \Gamma$  with  $r(0) = e$ , the identity element, where rays  $r_1$  and  $r_2$  are equivalent if  $\sup\{d(r_1(t), r_2(t))\} < \infty$ . Let  $\widehat{\Gamma} = \Gamma \cup \partial\Gamma$  denote the natural compactification of  $\Gamma$  topologized the usual way (cf. [9, p. 124]).

The *Gromov inner product* of elements  $a$  and  $b$  relative to  $c$  is defined by

$$(a, b)_c = 1/2[d(a, c) + d(b, c) - d(a, b)].$$

*Definitions.* A subset  $X$  of  $\Gamma$  is said to be  $k$ -quasi-convex if any geodesic joining  $a, b \in X$  lies in a  $k$ -neighborhood of  $X$ . A subset  $X$  is *quasi-convex* if it is  $k$ -quasi-convex for some  $k$ . A map  $f$  from one metric space  $(Y, d_Y)$  into another metric space  $(Z, d_Z)$  is said to be a  $(K, \varepsilon)$ -quasi-isometric embedding if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \varepsilon \leq d_Z(f(y_1), f(y_2)) \leq Kd_Y(y_1, y_2) + \varepsilon.$$

If  $f$  is a quasi-isometric embedding, and every point of  $Z$  lies at a uniformly bounded distance from some  $f(y)$  then  $f$  is said to be a *quasi-isometry*. A  $(K, \varepsilon)$ -quasi-isometric embedding that is a quasi-isometry will be called a  $(K, \varepsilon)$ -quasi-isometry.

A  $(K, \varepsilon)$ -quasi-geodesic is a  $(K, \varepsilon)$ -quasi-isometric embedding of a closed interval in  $\mathbb{R}$ . A  $(K, 0)$ -quasi-geodesic will also be called a  $K$ -quasi-geodesic.

Let  $G$  be a hyperbolic group and  $H$  be a subgroup that is hyperbolic. We choose a finite symmetric generating set for  $H$  and extend it to a finite symmetric generating set for  $G$ . We assume also for simplicity that the generating set for  $G$  intersects  $H$  in the generating set for  $H$ . Let  $\Gamma_H$  and  $\Gamma_G$  denote the Cayley graphs of  $H, G$ , respectively, with respect to these generating sets. By adjoining the Gromov boundaries  $\partial\Gamma_H$  and  $\partial\Gamma_G$  to  $\Gamma_H$  and  $\Gamma_G$ , one obtains their compactifications  $\widehat{\Gamma}_H$  and  $\widehat{\Gamma}_G$ , respectively.

Label the vertices of Cayley graphs by the corresponding group elements.  $G$  (resp.  $H$ ) acts on  $\Gamma_G$  (resp.  $\Gamma_H$ ) by left-translations. Denote the left action of  $g$  (resp.  $h$ ) by  $t_g$  (resp.  $t_h$ ). There is a natural embedding  $i: \Gamma_H \rightarrow \Gamma_G$  sending a vertex of  $\Gamma_H$  labeled  $h$  to the vertex of  $\Gamma_G$  labeled  $h$ .

*Definition:* A *Cannon–Thurston map*  $\hat{i}$  from  $\widehat{\Gamma}_H$  to  $\widehat{\Gamma}_G$  is a continuous extension of the natural embedding  $i: \Gamma_H \rightarrow \Gamma_G$ .

It is easy to see that such a continuous extension, if it exists, is unique. The main theorem of this paper can now be stated:

THEOREM 4.3. *Given a short exact sequence of finitely generated groups*

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1,$$

*such that  $H$  and  $G$  are hyperbolic, there exists a Cannon–Thurston map from  $\widehat{\Gamma}_H$  to  $\widehat{\Gamma}_G$ .*

When  $H$  is finite the theorem is vacuously true as  $\partial\Gamma_H = \emptyset$ . When  $H$  is virtually cyclic,  $H$  is quasi-convex in  $G$  (cf. [9, p. 155]) and the theorem is again trivial. Therefore, we shall assume henceforth that  $H$  and  $G$  are non-elementary.

The following lemma says that a Cannon–Thurston map exists if for all  $M > 0$ , there exists  $N > 0$  such that if  $\lambda$  lies outside an  $N$  ball around the identity in  $\Gamma_H$  then any geodesic in  $\Gamma_G$  joining the end-points of  $\lambda$  lies outside the  $M$  ball around the identity in  $\Gamma_G$ . For convenience of use later on, we state this somewhat differently.

LEMMA 2.1. *A Cannon–Thurston map from  $\widehat{\Gamma}_H$  to  $\widehat{\Gamma}_G$  exists if the following condition is satisfied:*

*There exists a non-negative function  $M(N)$ , such that  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  and for all geodesic segments  $\lambda$  lying outside the  $N$ -ball around the identity in  $\Gamma_H$  any geodesic segment in  $\Gamma_G$  joining the end-points of  $i(\lambda)$  lies outside the  $M(N)$ -ball around the identity in  $\Gamma_G$ .*

*Proof.* Suppose  $i: \Gamma_H \rightarrow \Gamma_G$  does not extend continuously. Since  $i$  is proper, there exist sequences  $x_m, y_m \in \Gamma_H$  and  $p \in \partial\Gamma_H$ , such that  $x_m \rightarrow p$  and  $y_m \rightarrow p$  in  $\widehat{\Gamma}_H$ , but  $i(x_m) \rightarrow u$  and  $i(y_m) \rightarrow v$  in  $\widehat{\Gamma}_G$ , where  $u, v \in \partial\Gamma_G$  and  $u \neq v$ .

Since  $x_m \rightarrow p$  and  $y_m \rightarrow p$ , any geodesic in  $\Gamma_H$  joining  $x_m$  and  $y_m$  lies outside an  $N_m$ -ball around the identity in  $\Gamma_H$ , where  $N_m \rightarrow \infty$  as  $m \rightarrow \infty$ . Any bi-infinite geodesic in  $\Gamma_G$  joining  $u, v \in \partial\Gamma_G$  has to pass through some  $M$ -ball around the identity in  $\Gamma_G$  as  $u \neq v$ . There exist constants  $c$  and  $L$  such that for all  $m > L$  any geodesic joining  $i(x_m)$  and  $i(y_m)$  in  $\Gamma_G$  passes through an  $(M + c)$ -neighborhood of the identity in  $\Gamma_G$ . Since  $(M + c)$  is a constant not depending on the index  $m$  this proves the lemma. □

*Remark.* It is easy to show that the above condition is also necessary for the existence of a Cannon–Thurston map, but this fact will not be required.

A brief outline of the plan of this paper is now given. Given a geodesic segment  $\lambda \subset \Gamma_H$  we construct a set  $B_\lambda \subset \Gamma_G$  containing  $i(\lambda)$ .  $B_\lambda$  can be roughly described as the union of the images of  $i(\lambda)$  under “right action” by elements of  $K$ . We describe this construction in the next section and show that there exists a map from  $\Gamma_G$  to  $B_\lambda$  which does not increase distances much. This shows that  $B_\lambda$ ’s are quasi-convex. In fact, the proof shows that they are uniformly quasi-convex. In the final section it is shown that if  $\lambda$  lies outside a large ball around the identity in  $\Gamma_H$ ,  $B_\lambda$  lies outside a large ball in  $\Gamma_G$ . Combining this with Lemma 2.1 the proof is completed.

### 3. CONSTRUCTION OF QUASICONVEX SETS

Our starting point is a lemma due to Lee Mosher [16]. Consider a surjective homomorphism  $P: G \rightarrow K$  of finitely generated groups. A *quasi-isometric section* is a subset  $\Sigma \subset G$  mapping onto  $K$  such that for any  $g, g' \in \Sigma$ ,

$$\frac{1}{\kappa} d_K(Pg, Pg') - \varepsilon_0 \leq d_G(g, g') \leq \kappa d_K(Pg, Pg') + \varepsilon_0$$

where  $d_G$  and  $d_K$  are word metrics in  $G, K$ , respectively, and  $\kappa \geq 1, \varepsilon_0 \geq 0$  are constants. A single-valued quasi-isometric section  $\sigma: K \rightarrow G$  is defined by choosing a single element of  $\Sigma$  representing each coset of the kernel of  $P$ .

LEMMA 3.1. (*Quasi-isometric section Lemma [16]*). *Given a non-elementary hyperbolic group  $H$  and a short exact sequence of finitely generated groups*

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1,$$

*the map  $P: G \rightarrow K$  has a quasi-isometric section  $\Sigma$ . In fact, choosing a generating set  $B$  for  $G$  and letting  $P(B)$  be the generating set for  $K$ , we have for all  $g, g' \in \Sigma$ ,*

$$d_K(Pg, Pg') \leq d_G(g, g') \leq \kappa d_K(Pg, Pg') + \varepsilon_0$$

*for some constants  $\kappa \geq 1$  and  $\varepsilon_0 \geq 0$ .*

When the short exact sequence splits, this is clear. The general case is proven in [16].

Given a quasi-isometric section, choose one element from each coset of  $H$  to get a single-valued quasi-isometric section. This is still a quasi-isometric section as  $Pg = Pg'$  implies  $d_G(g, g') \leq \varepsilon_0$ . Also, using a left translation  $t_h$  by an element  $h \in H \subset G$ , one can assume that  $\Sigma$  contains the identity element of  $G$ .  $t_h(\Sigma)$  is still a single-valued quasi-isometric section as  $t_h$  preserves cosets. Assume therefore that  $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$  is an exact sequence of finitely generated groups with  $H, G$  hyperbolic and  $\sigma: K \rightarrow G$  is a single-valued quasi-isometric section containing the identity element of  $G$ .

Corresponding to every element  $g \in G$  there exists an automorphism of  $H$  taking  $h$  to  $g^{-1}hg$  for  $h \in H$ . Such an automorphism induces a bijection  $\phi_g$  of the vertices of  $\Gamma_H$ . This gives rise to a map from  $\Gamma_H$  to itself, sending an edge  $[a, b]$  linearly to a shortest edge-path joining  $\phi_g(a)$  to  $\phi_g(b)$ . Abusing notation slightly, we shall also call this map  $\phi_g$ .

Given a geodesic segment  $\lambda \subset \Gamma_H$  the construction of a certain set  $B_\lambda \subset \Gamma_G$  will now be described.  $B_\lambda$  will turn out to be quasi-convex. Let  $a, b$  denote the end-points of  $\lambda$  and let  $\lambda_g \subset \Gamma_H$  denote a geodesic joining  $\phi_g(a)$  to  $\phi_g(b)$  in  $\Gamma_H$ .

Recall that  $t_g$  denotes left translation by an element  $g$  of  $G$ . Then define

$$B_\lambda = \bigcup_{g \in \sigma(K)} t_g \cdot i(\lambda_g).$$

Mark that  $B_\lambda$  contains  $i(\lambda)$  as  $e \in \sigma(K)$ . It is important to note also that  $\lambda_g$  is contained in  $\Gamma_H$  and not in  $\Gamma_G$ . It's only after acting on  $\lambda_g$  by  $t_g \cdot i$  that we obtain a subset of  $\Gamma_G$ .

There is a simple informal way to describe  $B_\lambda$ . Suppose  $\lambda$  joins vertices  $a, b \in \Gamma_H$ . Then  $ag, bg$  lie in the same (right or left) coset. Join  $ag, bg$  by the shortest edge-path lying entirely in the corresponding coset (if there are several of these choose one). The union of all these edge-paths as  $g$  ranges over  $\sigma(K)$  is  $B_\lambda$ .

The aim of the rest of this section is to prove that  $B_\lambda$  is  $C'$ -quasi-convex for some  $C'$  independent of  $\lambda$ . To do this, we shall define a map  $\Pi_\lambda: \Gamma_G \rightarrow B_\lambda$  and show that it does not increase distances much.

There is a natural identification of  $\Gamma_H$  with  $t_g(i(\Gamma_H))$  taking  $h$  to  $t_g(i(h))$ . On  $\Gamma_H$  define a map  $\pi_{g,\lambda}: \Gamma_H \rightarrow \lambda_g$  taking  $h$  to one of the points on  $\lambda_g$  closest to  $h$  in the metric  $d_H$ . Strictly speaking,  $\pi_{g,\lambda}$  is defined only on the vertex set, but this is enough for our purposes. Now, define

$$\Pi_\lambda \cdot t_g \cdot i(h) = t_g \cdot i \cdot \pi_{g,\lambda}(h) \quad \text{for } g \in \sigma(K).$$

For every  $g' \in \Gamma_G$ , there exists a unique  $g \in \sigma(K)$  such that  $g' \in t_g(i(h))$  as  $\sigma$  is a single-valued section. Hence,  $\Pi_\lambda$  is well-defined on the entire vertex set of  $\Gamma_G$ .

In order to show that  $\Pi_\lambda$  does not increase distances by more than a bounded factor, one needs to first show that  $\pi_{g,\lambda}$  does not increase distances much. More precisely if  $h, h' \in \Gamma_H$ , it will be shown that  $d_H(\pi_{g,\lambda}(h), \pi_{g,\lambda}(h')) \leq C_1 d_H(h, h')$ , where  $C_1 \geq 1$  is independent of  $g$ . This follows from the following well-known lemma about hyperbolic groups.

LEMMA 3.2. *Let  $H$  be a  $\delta$ -hyperbolic group and let  $\mu \subset \Gamma_H$  be a geodesic segment. Let  $\pi: \Gamma_H \rightarrow \mu$  map  $h \in \Gamma_H$  to one of the points on  $\mu$  nearest to  $h$ . Then  $d_H(\pi(x), \pi(y)) \leq C_1 d_H(x, y)$  for all  $x, y \in H$ , where  $C_1$  depends only on  $\delta$ .*

We will need some elementary facts about hyperbolic groups which we state below without proof.

LEMMA 3.3. *Let  $H$  be a  $\delta$ -hyperbolic group. Let  $\mu$  be a geodesic segment in  $\Gamma_H$  with end-points  $a, b$  and let  $c$  be any vertex in  $\Gamma_H$ . Let  $y$  be a vertex on  $\mu$  such that  $d_H(c, y) \leq d_H(c, z)$  for any  $z \in \mu$ . Then a geodesic path from  $c$  to  $y$  followed by a geodesic path from  $y$  to  $a$  is a  $k$ -quasi-geodesic for some  $k$  dependent only on  $\delta$ .*

Recall that the Gromov inner product  $(a, b)_c = 1/2[d_H(a, c) + d_H(b, c) - d_H(a, b)]$ .

LEMMA 3.4. *Suppose  $H$  is  $\delta$ -hyperbolic. If  $\mu$  is a  $(k_0, \varepsilon_0)$ -quasi-geodesic in  $\Gamma_H$  and  $p, q, r$  are 3 points in order on  $\mu$  then  $(p, r)_q \leq k_1$  for some  $k_1$  dependent on  $k_0, \varepsilon_0$  and  $\delta$  only.*

The reader may consult [13] for proofs of the above three well-known facts. Recall that  $P: G \rightarrow K$  is the natural surjective homomorphism.

LEMMA 3.5. *Let  $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$  be as in the statement of the main theorem and  $\sigma: K \rightarrow G$  be a single-valued quasi-isometric section obtained from Lemma 3.1. Further let  $\kappa$  and  $\varepsilon_0$  be as in 3.1. Then  $\{(\sigma \cdot P(x))^{-1} (\sigma \cdot P(y))\}$  lies in a finite ball  $S$  of radius  $\kappa_1 = \kappa + \varepsilon_0$  around the identity in  $\Gamma_G$  when  $d_G(x, y) = 1$ .*

*Proof.* From Lemma 3.1,

$$d_G(x, y) = 1 \Rightarrow d_K(P(x), P(y)) \leq 1$$

$$\text{Hence } d_G(\sigma(P(x)), \sigma(P(y))) \leq \kappa + \varepsilon_0 = \kappa_1 \text{ (say).}$$

Thus,  $\{(\sigma \cdot P(x))^{-1} (\sigma \cdot P(y))\}$  lies in a finite ball  $S$  of radius  $\kappa_1$  around the identity in  $\Gamma_G$  when  $d_G(x, y) = 1$ . □

Recall that every element  $g$  of  $G$  gives an automorphism of  $H$  (by conjugation) and this in turn gives rise to a map  $\phi_g$  of  $\Gamma_H$  to itself which is a bijection on the vertex set of  $\Gamma_H$ . Each  $\phi_g$  is a quasi-isometry. Since  $S$  is finite, there exist  $K \geq 1$  and  $\varepsilon \geq 0$  such that for all  $g \in S$ ,  $\phi_g$  is a  $(K, \varepsilon)$ -quasi-isometry. Then the image of a geodesic under  $\phi_g$  is a  $(K, \varepsilon)$ -quasi-geodesic for all  $g \in S$ . The following Lemma lies at the heart of the proof of our main Theorem. It says roughly that quasi-isometries induced by automorphisms and nearest point projections “almost commute”.

LEMMA 3.6. *Suppose  $H$  is  $\delta$ -hyperbolic. Let  $\mu_1$  be some geodesic segment in  $\Gamma_H$  joining  $a, b$  and let  $p$  be any vertex of  $\Gamma_H$ . Also let  $q$  be a vertex on  $\mu_1$  such that  $d_H(p, q) \leq d_H(p, x)$  for*

$x \in \mu_1$ . Let  $\mu_2$  be a geodesic segment in  $\Gamma_H$  joining  $\phi_g(a)$  to  $\phi_g(b)$  for some  $g \in S$ . Let  $r$  be a point on  $\mu_2$  such that  $d_H(\phi_g(p), r) \leq d_H(\phi_g(p), x)$  for  $x \in \mu_2$ . Then  $d_H(r, \phi_g(q)) \leq C_2$  for some constant  $C_2$  independent of  $a, b, g, p$ .

*Proof.* From the Lemma 3.5,  $\phi_g(\mu_1)$  is a  $(K, \varepsilon)$ -quasi-geodesic joining  $\phi_g(a)$  to  $\phi_g(b)$  and hence lies in a  $K'$ -neighborhood of  $\mu_2$  where  $K'$  depends only on  $K, \varepsilon, \delta$ . Let  $u$  be a vertex in  $\phi_g(\mu_1)$  lying at a distance at most  $K'$  from  $r$ . Without loss of generality, suppose that  $u$  lies on  $\phi_g([q, b])$ , where  $[q, b]$  denotes the geodesic subsegment of  $\mu_1$  joining  $q, b$ . (See Fig. 1 below.)

Let  $[p, q]$  denote a geodesic joining  $p, q$ . From Lemma 3.3  $[p, q] \cup [q, b]$  is a  $k$ -quasi-geodesic, where  $k$  depends on  $\delta$  alone. Therefore  $\phi_g([p, q]) \cup \phi_g([q, b])$  is a  $(K_0, \varepsilon_0)$ -quasi-geodesic, where  $K_0, \varepsilon_0$  depend on  $K, k, \varepsilon$ . Hence, by Lemma 3.4  $(\phi_g(p), u)_{\phi_g(q)} \leq K_1$ , where  $K_1$  depends on  $K, k, \varepsilon$  and  $\delta$  alone. Therefore,

$$\begin{aligned} (\phi_g(p), r)_{\phi_g(q)} &= 1/2 [d_H(\phi_g(p), \phi_g(q)) + d_H(r, \phi_g(q)) - d_H(r, \phi_g(p))] \\ &\leq 1/2 [d_H(\phi_g(p), \phi_g(q)) + d_H(u, \phi_g(q)) + d_H(r, u) - d_H(u, \phi_g(p)) + d_H(r, u)] \\ &= (\phi_g(p), u)_{\phi_g(q)} + d_H(r, u) \\ &\leq K_1 + K'. \end{aligned}$$

There exists  $s \in \mu_2$  such that  $d_H(s, \phi_g(q)) \leq K'$ .

$$\begin{aligned} (\phi_g(p), r)_s &= 1/2 [d_H(\phi_g(p), s) + d_H(r, s) - d_H(r, \phi_g(p))] \\ &\leq 1/2 [d_H(\phi_g(p), \phi_g(q)) + d_H(r, \phi_g(q)) - d_H(r, \phi_g(p))] + K' \\ &= (\phi_g(p), r)_{\phi_g(q)} + K' \\ &\leq K_1 + K' + K' \\ &= K_1 + 2K'. \end{aligned}$$

Also,  $(\phi_g(p), s)_r \leq 2\delta$  (see [1, p. 16]):

$$\begin{aligned} d_H(r, s) &= (\phi_g(p), s)_r + (\phi_g(p), r)_s \\ &\leq K_1 + 2K' + 2\delta \\ d_H(r, \phi_g(q)) &\leq K_1 + 2K' + 2\delta + d_H(s, \phi_g(q)) \\ &\leq K_1 + 2K' + 2\delta + K'. \end{aligned}$$

Let  $C_2 = K_1 + 3K' + 2\delta$ . Then  $d_H(r, \phi_g(q)) \leq C_2$  and  $C_2$  is independent of  $a, b, g, p$ .  $\square$

We will now prove the main theorem of this section. From Lemma 3.5, if  $d_G(x, y) = 1$  then  $d_G(\sigma(P(x)), \sigma(P(y))) \leq \kappa_1$ , where  $\kappa_1$  depends only on the quasi-isometric section  $\sigma$ . Recall that  $S$  is the set of elements lying inside the  $\kappa_1$  ball around the identity. Further recall that  $\Pi_\lambda$  is defined on the vertex set of  $\Gamma_G$  as

$$\Pi_\lambda \cdot t_g \cdot i(h) = t_g \cdot i \cdot \pi_{g, \lambda}(h) \quad \text{for } g \in \sigma(K).$$

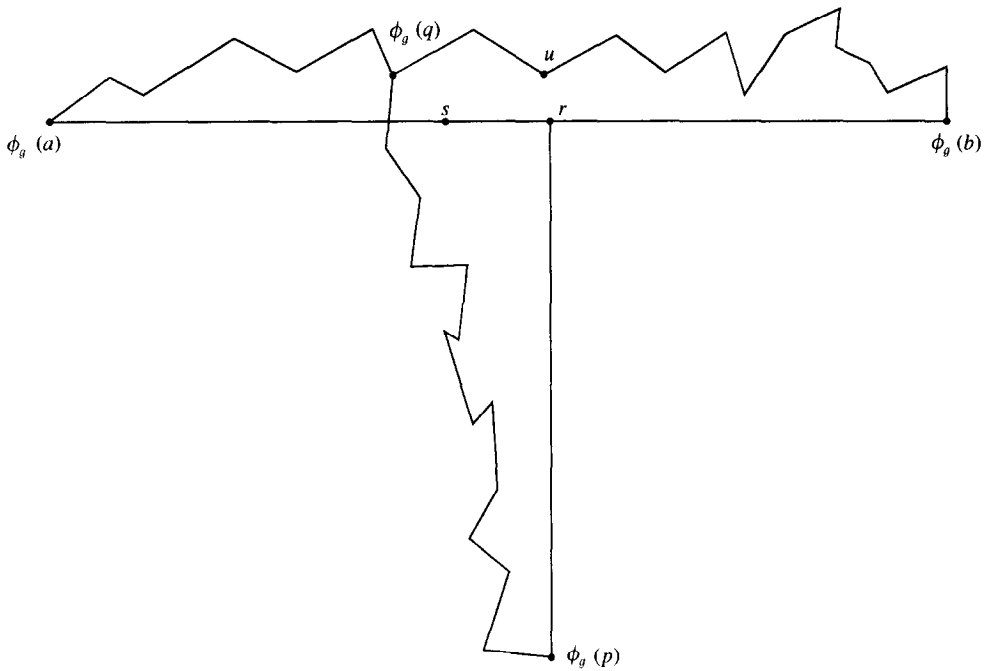


Fig. 1.

**THEOREM 3.7.** *There exists a constant  $C \geq 1$  such that for all geodesic segments  $\lambda \subset \Gamma_H$  and  $x, y \in \Gamma_G$ ,*

$$d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y).$$

*Proof.* As in Lemma 3.2 it suffices (by repeated use of the triangle inequality) to prove the theorem when  $d_G(x, y) = 1$ .

If  $x, y \in t_g(i(\Gamma_H))$  for some  $g \in \sigma(K)$  then let  $x = t_g(i(x_1))$  and  $y = t_g(i(y_1))$ . Assume for simplicity that the generating set for  $G$  intersects  $H$  in the generating set for  $H$ . So in this case,  $d_H(x_1, y_1) = 1$ :

$$\begin{aligned} d_G(\Pi_\lambda(x), \Pi_\lambda(y)) &\leq d_H(\pi_{g,\lambda}(x_1), \pi_{g,\lambda}(y_1)) \\ &\leq C_1 d_H(x_1, y_1) \quad [\text{Lemma 3.2}] \\ &= C_1. \end{aligned}$$

If  $d_G(x, y) = 1$  but  $x, y$  do not lie in the same coset, then  $d_G(\sigma(P(x)), \sigma(P(y))) \leq \kappa_1$ . Hence  $x \in t_{g_0}(i(\Gamma_H))$  for some  $g_0 \in \sigma(K)$  and  $y \in t_{g_0g}(i(\Gamma_H))$  for some  $g \in S$ . Let  $t_{g_0}(x_2) = x$  and  $t_{g_0g}(y_2) = y$  for some  $x_2, y_2 \in \Gamma_H$ .

By the triangle inequality,

$$d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq d_G(\Pi_\lambda(x), \Pi_\lambda(xg)) + d_G(\Pi_\lambda(xg), \Pi_\lambda(xg)) + d_G(\Pi_\lambda(xg), \Pi_\lambda(y)).$$

The rest of the proof is devoted to obtaining uniform bounds on the terms on the right-hand side of the above inequality.

Unraveling definitions,

$$\begin{aligned} t_{g_0g} \cdot i \cdot \phi_g \cdot \pi_{g_0,\lambda}(x_2) &= g_0 g g^{-1} (g_0^{-1} \Pi_\lambda(x)) g \\ &= \Pi_\lambda(x) g. \end{aligned}$$

And

$$\begin{aligned} t_{g_0g} \cdot i \cdot \pi_{g_0g, \lambda} \cdot \phi_g(x_2) &= \Pi_\lambda \cdot t_{g_0g} \cdot i \cdot \phi_g(x_2) \\ &= \Pi_\lambda(g_0g g^{-1} x_2 g) \\ &= \Pi_\lambda(xg). \end{aligned}$$

Now,  $\phi_g(\lambda_{g_0})$  is a  $K$ -quasi-geodesic joining the endpoints of  $\lambda_{g_0g}$ . Hence from Lemma 3.6  $d_H(\phi_g(\pi_{g_0, \lambda}(x_2)), \pi_{g_0g, \lambda}(\phi_g(x_2))) \leq C_2$ . This implies  $d_G(t_{g_0g} i \phi_g \pi_{g_0, \lambda}(x_2), t_{g_0g} i \pi_{g_0g, \lambda} \phi_g(x_2)) \leq C_2$ .

Hence  $d_G(\Pi_\lambda(xg), \Pi_\lambda(x)g) \leq C_2$ .

Now  $y, xg \in t_{g_0g} i(\Gamma_H)$ . Further,  $y = t_{g_0g} i(y_2)$  and  $xg = t_{g_0g} i \phi_g(x_2)$ . Since  $t_{g_0g}$  is an isometry

$$d_G(i(y_2), i(\phi_g(x_2))) = d_G(y, xg) \leq d_G(x, xg) + d_G(x, y) \leq \kappa_1 + 1.$$

Since  $\Gamma_H$  is properly embedded in  $\Gamma_G$  there exists a constant  $L$  independent of  $g, \lambda$  such that

$$\begin{aligned} d_G(i(y_2), i(\phi_g(x_2))) &\leq \kappa_1 + 1 \Rightarrow d_H(y_2, \phi_g(x_2)) \\ &\leq L \Rightarrow d_H(\pi_{g_0g, \lambda}(y_2), \pi_{g_0g, \lambda}(\phi_g(x_2))) \\ &\leq C_1 L \quad [\text{Lemma 3.2}]. \end{aligned}$$

Hence  $d_G(\Pi_\lambda(xg), \Pi_\lambda(y)) \leq C_1 L$ .

Therefore,

$$\begin{aligned} d_G(\Pi_\lambda(x), \Pi_\lambda(y)) &\leq d_G(\Pi_\lambda(x), \Pi_\lambda(x)g) + d_G(\Pi_\lambda(x)g, \Pi_\lambda(xg)) + d_G(\Pi_\lambda(xg), \Pi_\lambda(y)) \\ &\leq \kappa_1 + C_2 + C_1 L. \end{aligned}$$

Taking  $C = \max\{C_1, \kappa_1 + C_2 + C_1 L\}$  one sees that  $d_G(x, y) = 1 \Rightarrow d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C$  and we are through.  $\square$

*Remark.* The above theorem makes essential use of the hyperbolicity of  $H$ . In fact this theorem is false in the case of a  $Z + Z$  subgroup of the fundamental group of a closed 3-manifold with Sol-geometry. To see the idea, regard Sol as  $\mathbb{R}^3$  equipped with the metric  $ds^2 = e^t dx^2 + e^{-t} dy^2 + dt^2$ . Then the plane  $x = y$  is not quasi-isometrically embedded in Sol.

#### 4. PROOF OF MAIN THEOREM

The hypothesis that  $G$  is hyperbolic has not yet been used. Theorem 3.7 above shows that  $d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y)$  where  $C$  is independent of  $\lambda$ . Suppose  $\mu$  is a geodesic in  $\Gamma_G$  starting and ending in  $B_\lambda$ . Then  $\Pi_\lambda(\mu)$  is a  $C$ -quasi-geodesic. Since quasi-geodesics lie in a bounded neighborhood of geodesics in any hyperbolic metric space, this proves:

**LEMMA 4.1.** *There exists  $C' \geq 0$  such that for all geodesic segments  $\lambda \subset \Gamma_H$ ,  $B_\lambda$  is  $C'$ -quasi-convex.*

Thus, there exists  $C'$  independent of  $\lambda$  such that every geodesic with end points on  $B_\lambda$  lies in a  $C'$ -neighborhood of  $B_\lambda$ . In particular, any geodesic joining the end-points of  $i(\lambda)$  lies in a  $C'$ -neighborhood of  $B_\lambda$ .

We need one final Lemma.



LEMMA 4.2. *There exists  $A \geq 1$  such that for all  $g \in \sigma(K)$  and  $x \in t_g \cdot i(\lambda_g)$  there exists  $y \in i(\lambda)$  such that  $d_G(x, y) \leq Ad_K(e, Px)$ .*

*Proof.* Let  $\mu$  be a geodesic path in  $\Gamma_K$  joining  $e \in \Gamma_K$  to  $Px \in \Gamma_K$ . Order the vertices on  $\mu$  so that we have a finite sequence  $e = y_0, y_1, \dots, y_n = Px = Pg$  such that  $d_K(y_i, y_{i+1}) = 1$ . and  $d_K(e, Px) = n$ . Since  $\sigma$  is a quasi-isometric section, this gives a sequence  $\sigma(y_i) = g_i$  such that  $d_G(g_i, g_{i+1}) \leq \kappa + \varepsilon_0 \leq \kappa_1$ . Observe that  $g_n = g$ .

From Lemma 3.5  $g_{i+1}^{-1}g_i = s_i \in S$ . Hence  $\phi_{s_i}$  is a  $(K, \varepsilon)$ -quasi-isometry. Note that  $K, \varepsilon$  are independent of the index  $i$ .

Let  $z = t_{g_{i+1}} i(u)$  where  $u \in \lambda_{g_{i+1}}$ .

$$\begin{aligned} t_{g_i} i \phi_{s_i}(u) &= g_i(g_{i+1}^{-1}g_i)^{-1}(g_{i+1}^{-1}z)(g_{i+1}^{-1}g_i) \\ &= z g_{i+1}^{-1}g_i \\ &= z s_i. \end{aligned}$$

Since  $\phi_{s_i}$  is a  $(K, \varepsilon)$ -quasi-isometry,  $\phi_{s_i}(\lambda_{g_{i+1}})$  is a  $(K, \varepsilon)$ -quasi-geodesic in  $\Gamma_H$  joining the end-points of  $\lambda_{g_i}$ . Therefore there exists  $v$  on  $\lambda_{g_i}$  such that  $d_H(v, \phi_{s_i}(u)) \leq K'$ , where  $K'$  depends on  $K, \varepsilon$  and  $\delta$ . Hence,

$$\begin{aligned} d_G(t_{g_i} i(v), t_{g_i} i \phi_{s_i}(u)) &\leq K'. \text{ Let } t_{g_i} i(v) = w. \text{ Then } d_G(w, z s_i) \leq K' \text{ and} \\ d_G(w, z) &\leq d_G(w, z s_i) + d_G(z s_i, z) \leq K' + \kappa_1 = A(\text{say}). \end{aligned}$$

Thus, we have shown that given  $z \in t_{g_{i+1}} i(\lambda_{g_{i+1}})$  there exists  $w \in t_{g_i} i(\lambda_{g_i})$  such that  $d_G(z, w) \leq A$  where  $A$  is independent of  $\lambda$ .

$x \in t_g i(\lambda_g)$ . Let  $x = x_n$ . Then there exist  $x_i \in t_{g_i} i(\lambda_{g_i})$  for  $i = 0 \dots n$  such that  $d_G(x_i, x_{i+1}) \leq A$ . Choosing  $y = x_0$  we have  $y \in i(\lambda)$  and  $d_G(x, y) \leq An = Ad_K(e, Px)$ .  $\square$

We are now in a position to prove our main theorem.

THEOREM 4.3. *Given a short exact sequence of finitely generated groups*

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

*such that  $H, G$  are hyperbolic, there exists a Cannon–Thurston map from  $\widehat{\Gamma}_H$  to  $\widehat{\Gamma}_G$ .*

*Proof.* When  $H$  is elementary this is trivial. So we assume  $H$  non-elementary. It suffices by Lemma 2.1 to show that if  $\lambda$  is a geodesic segment lying outside an  $N$ -ball around the identity in  $\Gamma_H$ , then any geodesic joining the end-points of  $i(\lambda)$  in  $\Gamma_G$  lies outside an  $M(N)$ -ball around the identity in  $\Gamma_G$  and  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$ .

Since  $\Gamma_H$  is properly embedded in  $\Gamma_G$  there exists  $f(N)$  such that  $i(\lambda)$  lies outside the  $f(N)$ -ball in  $\Gamma_G$  and  $f(N) \rightarrow \infty$  as  $N \rightarrow \infty$ .

Let  $x$  be any point on  $B_\lambda$ . There exists  $y \in i(\lambda)$  such that  $d_G(y, x) \leq Ad_K(e, Px)$  by Lemma 4.2. Therefore,

$$\begin{aligned} d_G(e, x) &\geq d_G(e, y) - Ad_K(e, Px) \\ &\geq f(N) - Ad_K(e, Px). \end{aligned}$$

Recall that in Lemma 3.1, a finite generating set  $B$  was chosen for  $G$  and  $P(B)$  was chosen to be a generating set for  $K$  so that

$$d_G(e, x) \geq d_K(Pe, Px) = d_K(e, Px).$$

Hence,

$$\begin{aligned} d_G(e, x) &\geq \max(f(N) - Ad_K(e, Px), d_K(e, Px)) \\ &\geq \frac{f(N)}{A + 1}. \end{aligned}$$

Since (Lemma 4.1)  $B_\lambda$  is a  $C'$ -quasi-convex set containing  $i(\lambda)$ , any geodesic joining the end-points of  $i(\lambda)$  lies in a  $C'$ -neighborhood of  $B_\lambda$ .

Hence any geodesic joining end-points of  $i(\lambda)$  lies outside a ball of radius  $M(N)$ , where

$$M(N) = \frac{f(N)}{A + 1} - C'.$$

Since  $f(N) \rightarrow \infty$  as  $N \rightarrow \infty$  so does  $M(N)$ . □

*Note:*  $f(N)$  can be regarded as the inverse of a “distortion function” (cf. [10]).

There is a certain analogy between the discussion in this paper and the discussion of [6]. Let  $M$  be a 3-manifold fibering over the circle with fiber  $F$ . Then the universal covers of  $F, M$  correspond to  $H, G$  in Theorem 4.3.  $M$  admits a foliation by flow-lines transverse to the fiber  $F$ . These lift to quasi-geodesics in the universal cover of  $M$ . It might be helpful to think of a quasi-isometric section as a single ‘flow-line’. The translates of a quasi-isometric section under the elements of  $i(H)$  exhaust  $\Gamma_G$  and are mutually disjoint. Extending the analogy between [6] and this paper, one can regard the translates of  $\sigma(K)$  as “flow-lines” foliating  $\Gamma_G$ .

**APPLICATIONS.** The techniques of this paper can be generalized to certain geometrically tame Kleinian groups to prove a similar theorem for these groups (see [18] for definitions). The existence of a quasi-isometric section has been used in an essential way here. We have also used the fact that  $t_g(i(\Gamma_H))$  equipped with the path metric is isometric to  $\Gamma_H$ , and is therefore  $\delta$ -hyperbolic for some uniform  $\delta$  independent of  $g$ . These two properties can be abstracted and translated into certain regularity properties of pleated surfaces exiting the end as follows.

Let  $M$  be a geometrically tame hyperbolic 3 manifold with fundamental group equal to that of a closed surface. Then the condition required to prove an analog of Theorem 4.3 is that  $M$  is quasi-isometric to a manifold that ‘fibers over a Lipschitz path in Teichmuller space’ — the so-called universal curve over a Lipschitz path. This condition is satisfied when the hyperbolic 3-manifold in question has injectivity radius bounded below. Thus, the following Theorem of Minsky ([12, Theorem B]) follows from the techniques of this paper:

**THEOREM (Minsky [12]).** Let  $\Gamma = \rho(\pi_1(S))$  be a Kleinian group, such that there is a uniform lower bound on the injectivity radius of  $M = \mathbb{H}^3/\Gamma$ . Let  $\rho_0$  be a Fuchsian representative of  $\pi_1(S)$ , with limit set  $\Lambda_0$  equal to  $S^1$ . Let  $\Lambda_\Gamma$  be the limit set of  $\Gamma$ . Then there is a continuous map  $\phi: S^1 \rightarrow \Lambda_\Gamma$ , which takes the action by  $\rho_0$  on  $S^1$  to the action of  $\rho$  on  $\Lambda_\Gamma$ .

Details will appear in [13].

There are three known classes of examples to which the main theorem of this paper, Theorem (4.3) applies:

(1) (due to Bestvina and Feighn [12]) Extensions of  $Z$  by hyperbolic groups  $H$  where the corresponding automorphism of  $H$  is hyperbolic. (See [2] for definitions and proof).

(2) (due to Mosher [15]) Extensions of finitely generated free groups by (closed) surface groups where the free group acts by pseudo-Anosov automorphisms satisfying some additional constraints.

(3) (due to Bestvina, Feighn and Handel [3]) Extensions of finitely generated free groups by finitely generated free groups where the action is by hyperbolic automorphisms satisfying some additional constraints.

Further examples of non-quasi-convex subgroups of hyperbolic groups (due to Bestvina and Feighn [2]) are given by vertex and edge groups of graphs of hyperbolic groups, satisfying certain conditions. In [13], the results of this paper are extended to include these examples:

**THEOREM ([13]):** *Let  $G$  be a hyperbolic group acting cocompactly on a simplicial tree  $T$  such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let  $H$  be the stabilizer of a vertex or edge of  $T$ . Then there exists a Cannon–Thurston map from  $\widehat{\Gamma}_H$  to  $\widehat{\Gamma}_G$ .*

This essentially completes all known examples of non-quasi-convex hyperbolic subgroups of hyperbolic groups.

We have proven in this paper that  $i: \Gamma_H \rightarrow \Gamma_G$  extends continuously to a map  $\hat{i}: \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$ . An explicit description of  $\hat{i}$  is given in [14], where some aspects of Thurston's theory of ending laminations are generalized to the context of a hyperbolic normal subgroup of a hyperbolic group.

We end with some questions. Consider an exact sequence of groups as in the statement of Theorem 4.3. It is known that a hyperbolic group with infinite outer automorphism group admits a small action on an  $\mathbb{R}$ -tree [17]. This imposes restrictions on the nature of  $H$  (by work of Rips and Sela). In [16] Mosher shows that  $K$  must be hyperbolic. Not many examples are known where  $K \neq Z$ . Mosher [15] has found examples of hyperbolic groups  $G$  with  $H$  a closed surface group and  $K$  a free group of rank bigger than one. Of course in this situation the exact sequence splits. It would be interesting to have examples (or prove that they do not exist) where the exact sequence does not split or at least where  $K$  is not free.

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