Hyperbolic Geometry and Chaos in the Complex Plane

Mahan Mj,
School of Mathematics,
Tata Institute of Fundamental Research.

The best mathematics uses the whole mind, embraces human sensibility, and is not at all limited to the small portion of our brains that calculates and manipulates symbols. Through pursuing beauty we find truth, and where we find truth we discover incredible beauty.

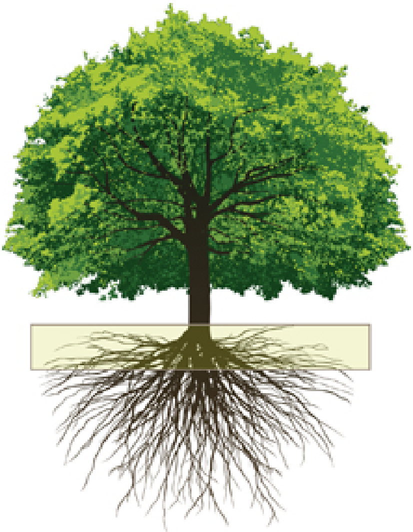
– William Thurston (October 30, 1946 – August 21, 2012)

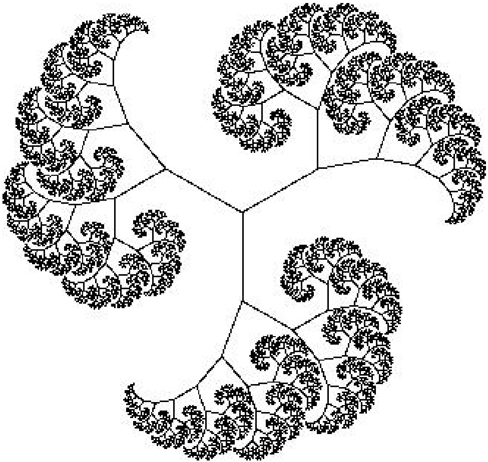
The best mathematics uses the whole mind, embraces human sensibility, and is not at all limited to the small portion of our brains that calculates and manipulates symbols. Through pursuing beauty we find truth, and where we find truth we discover incredible beauty.

– William Thurston (October 30, 1946 – August 21, 2012)

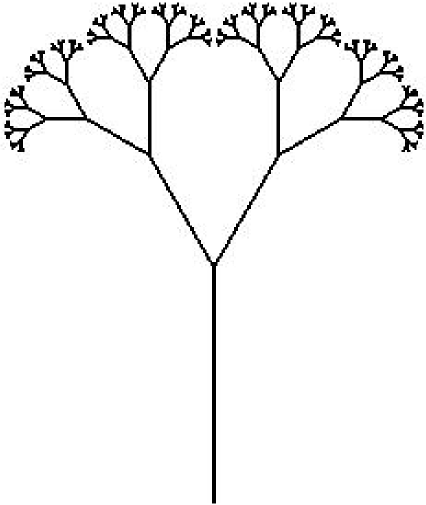
The best mathematics uses the whole mind, embraces human sensibility, and is not at all limited to the small portion of our brains that calculates and manipulates symbols. Through pursuing beauty we find truth, and where we find truth we discover incredible beauty.

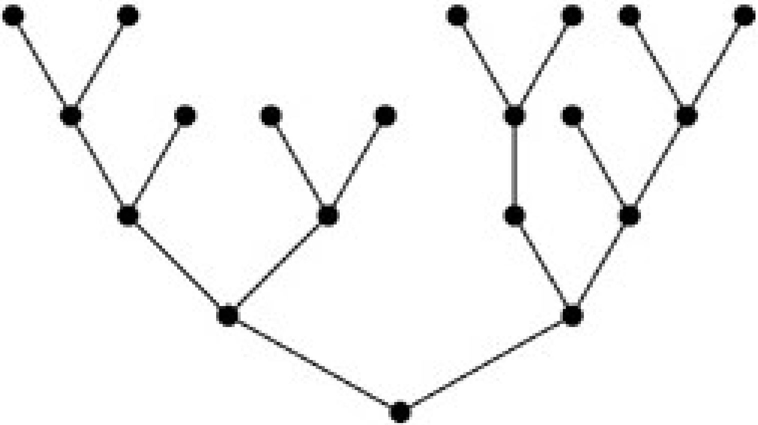
– William Thurston (October 30, 1946 – August 21, 2012)

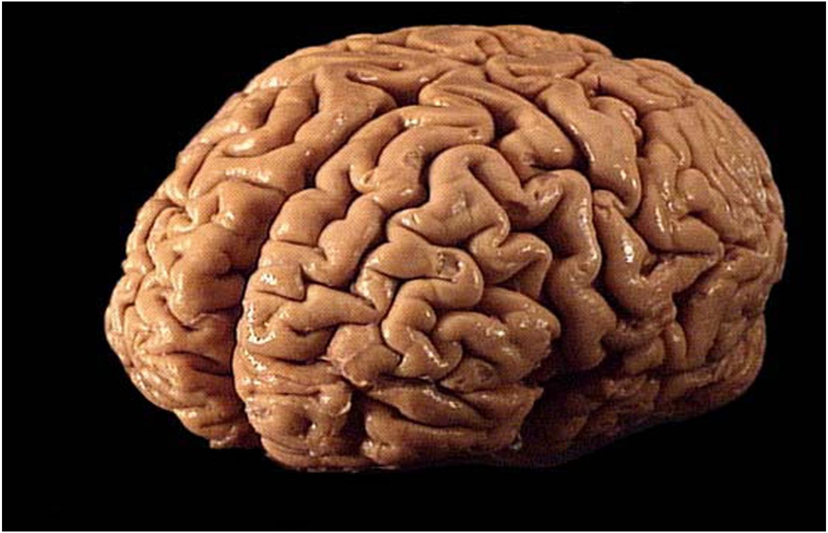


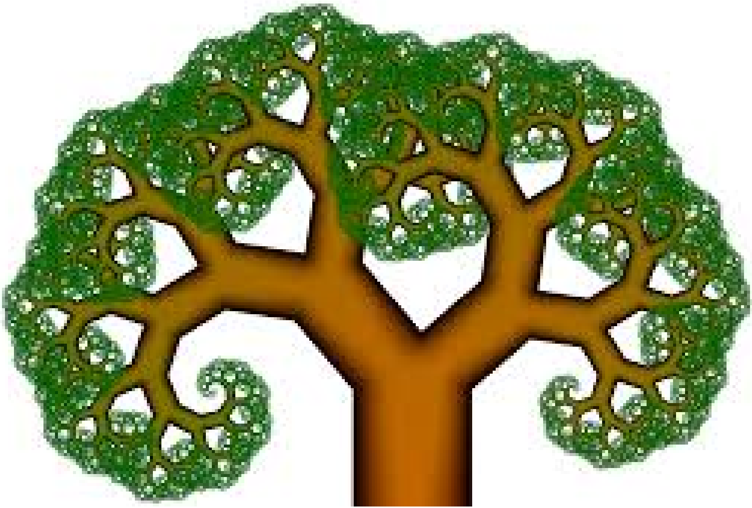


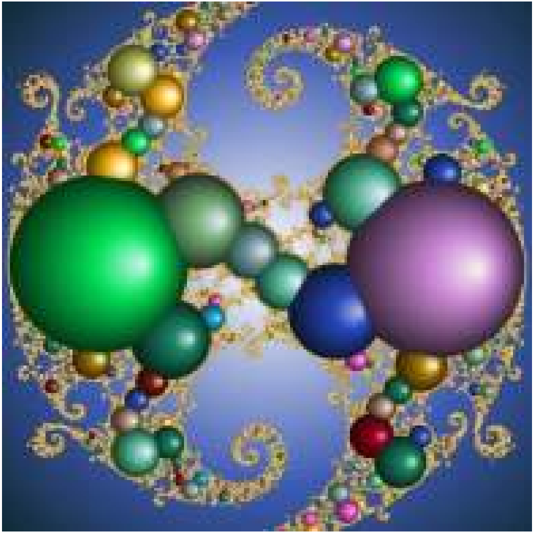
A homogeneous three-valent tree



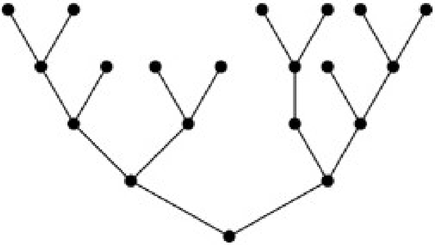






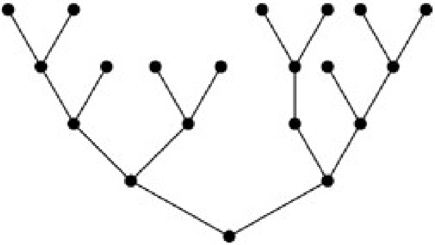


Hyperbolic metric spaces:



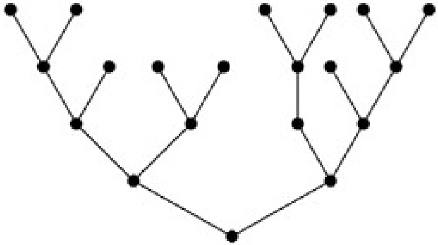
Triangles are thin: $[a, b] \subset N_\delta([a, c] \cup [b, c])$.
For a tree, $\delta = 0$.

Hyperbolic metric spaces:



Triangles are thin: $[a, b] \subset N_\delta([a, c] \cup [b, c])$.
For a tree, $\delta = 0$.

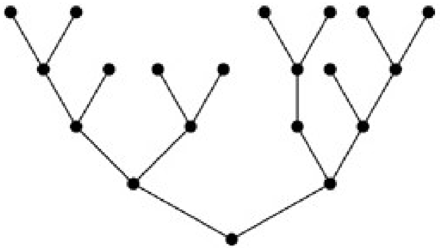
Hyperbolic metric spaces:



Triangles are thin: $[a, b] \subset N_\delta([a, c] \cup [b, c])$.

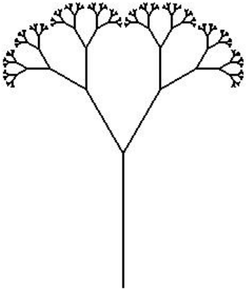
For a tree, $\delta = 0$.

Hyperbolic metric spaces:



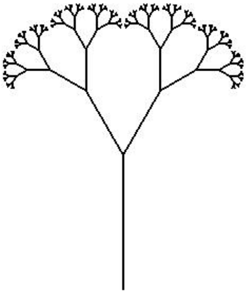
Triangles are thin: $[a, b] \subset N_\delta([a, c] \cup [b, c])$.
For a tree, $\delta = 0$.

Note that limbs get smaller and smaller in order to fit in $\mathbb{R}^2$.



Accumulates to a Cantor set.
ENTER FRACTALS.

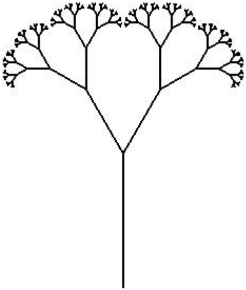
Note that limbs get smaller and smaller in order to fit in $\mathbb{R}^2$.



Accumulates to a Cantor set.

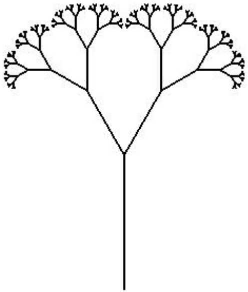
ENTER FRACTALS.

Note that limbs get smaller and smaller in order to fit in $\mathbb{R}^2$.



Accumulates to a Cantor set.
ENTER FRACTALS.

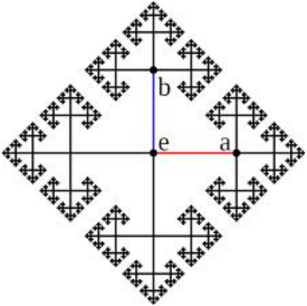
Note that limbs get smaller and smaller in order to fit in $\mathbb{R}^2$.



Accumulates to a Cantor set.
ENTER FRACTALS.

Groups = Symmetries

Cayley graph of a (discrete) group $G = \langle g_1, \cdots, g_k : r_1, \cdots, r_s \rangle$
$\mathcal{V} = \{g \in G\}$; $\mathcal{E} = \{(a, b) : a^{-1}b \in \{g_1, \cdots, g_k\}\}$.



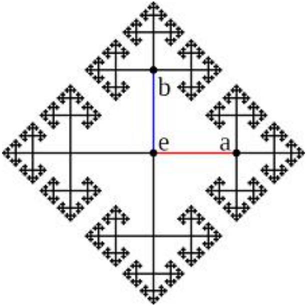
F_2

Boundaries of hyperbolic groups are fractals.

Groups = Symmetries

Cayley graph of a (discrete) group $G = \langle g_1, \cdots, g_k : r_1, \cdots, r_s \rangle$

$\mathcal{V} = \{g \in G\}$; $\mathcal{E} = \{(a, b) : a^{-1}b \in \{g_1, \cdots, g_k\}\}$.



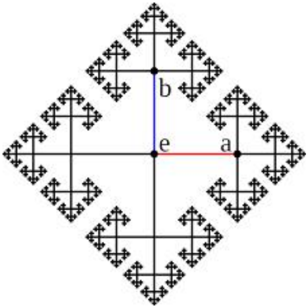
F_2

Boundaries of hyperbolic groups are fractals.

Groups = Symmetries

Cayley graph of a (discrete) group $G = \langle g_1, \cdots, g_k : r_1, \cdots, r_s \rangle$

$\mathcal{V} = \{g \in G\}$; $\mathcal{E} = \{(a, b) : a^{-1}b \in \{g_1, \cdots, g_k\}\}$.



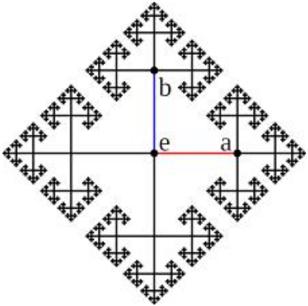
F_2

Boundaries of hyperbolic groups are fractals.

Groups = Symmetries

Cayley graph of a (discrete) group $G = \langle g_1, \cdots, g_k : r_1, \cdots, r_s \rangle$

$\mathcal{V} = \{g \in G\}$; $\mathcal{E} = \{(a, b) : a^{-1}b \in \{g_1, \cdots, g_k\}\}$.



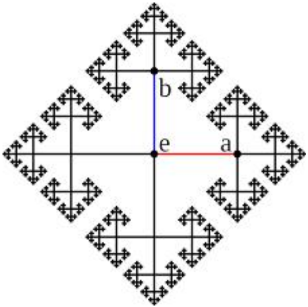
F_2

Boundaries of hyperbolic groups are fractals.

Groups = Symmetries

Cayley graph of a (discrete) group $G = \langle g_1, \cdots, g_k : r_1, \cdots, r_s \rangle$

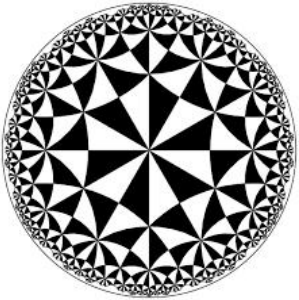
$\mathcal{V} = \{g \in G\}$; $\mathcal{E} = \{(a, b) : a^{-1}b \in \{g_1, \cdots, g_k\}\}$.



F_2

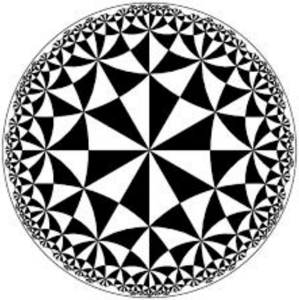
Boundaries of hyperbolic groups are fractals.

$$G = \langle a, b, c | a^2, b^2, c^2, (ab)^2, (bc)^4, (ca)^6 \rangle$$



Reflections in a hyperbolic triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}$.

$$G = \langle a, b, c | a^2, b^2, c^2, (ab)^2, (bc)^4, (ca)^6 \rangle$$



Reflections in a hyperbolic triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}$.


Circle Limit III, 1959

Relative problem:

$H \subset G$ hyperbolic subgroup of a hyperbolic group.
$i : \Gamma_H \to \Gamma_G$ inclusion of Cayley graphs.
Does i extend to a continuous map between the fractal boundaries?
Answer is "No" in this generality. (Baker-Riley 2013)

But an analogous (and much more classical) problem arises when a hyperbolic group acts by symmetries (isometries) on $\mathbf{H}^3$ – 3 dimensional hyperbolic space.
$\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ equipped with metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.
Boundary is $\mathbb{C} \cup \{\infty\}$.
If action is nice on $\mathbf{H}^3$, then geometer's instinct tells us to take a quotient.

Relative problem:

$H \subset G$ hyperbolic subgroup of a hyperbolic group.
$i : \Gamma_H \to \Gamma_G$ inclusion of Cayley graphs.
Does i extend to a continuous map between the fractal boundaries?
Answer is "No" in this generality. (Baker-Riley 2013)

But an analogous (and much more classical) problem arises when a hyperbolic group acts by symmetries (isometries) on $\mathbf{H}^3$ – 3 dimensional hyperbolic space.
$\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ equipped with metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.
Boundary is $\mathbb{C} \cup \{\infty\}$.
If action is nice on $\mathbf{H}^3$, then geometer's instinct tells us to take a quotient.

Relative problem:

$H \subset G$ hyperbolic subgroup of a hyperbolic group.
$i : \Gamma_H \to \Gamma_G$ inclusion of Cayley graphs.
Does i extend to a continuous map between the fractal boundaries?
Answer is "No" in this generality. (Baker-Riley 2013)

But an analogous (and much more classical) problem arises when a hyperbolic group acts by symmetries (isometries) on $\mathbf{H}^3$ – 3 dimensional hyperbolic space.
$\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ equipped with metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.
Boundary is $\mathbb{C} \cup \{\infty\}$.
If action is nice on $\mathbf{H}^3$, then geometer's instinct tells us to take a quotient.

Relative problem:

$H \subset G$ hyperbolic subgroup of a hyperbolic group.
$i : \Gamma_H \to \Gamma_G$ inclusion of Cayley graphs.
Does i extend to a continuous map between the fractal boundaries?
Answer is "No" in this generality. (Baker-Riley 2013)

But an analogous (and much more classical) problem arises when a hyperbolic group acts by symmetries (isometries) on $\mathbf{H}^3$ – 3 dimensional hyperbolic space.
$\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ equipped with metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.
Boundary is $\mathbb{C} \cup \{\infty\}$.
If action is nice on $\mathbf{H}^3$, then geometer's instinct tells us to take a quotient.

Relative problem:

$H \subset G$ hyperbolic subgroup of a hyperbolic group.
$i : \Gamma_H \to \Gamma_G$ inclusion of Cayley graphs.
Does i extend to a continuous map between the fractal boundaries?
Answer is "No" in this generality. (Baker-Riley 2013)

But an analogous (and much more classical) problem arises when a hyperbolic group acts by symmetries (isometries) on $\mathbf{H}^3$ – 3 dimensional hyperbolic space.

$\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ equipped with metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.
Boundary is $\mathbb{C} \cup \{\infty\}$.
If action is nice on $\mathbf{H}^3$, then geometer's instinct tells us to take a quotient.

Relative problem:

$H \subset G$ hyperbolic subgroup of a hyperbolic group.
$i : \Gamma_H \to \Gamma_G$ inclusion of Cayley graphs.
Does i extend to a continuous map between the fractal boundaries?
Answer is "No" in this generality. (Baker-Riley 2013)

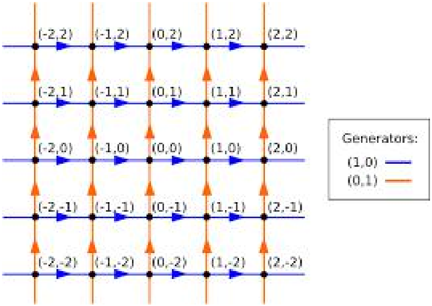
But an analogous (and much more classical) problem arises when a hyperbolic group acts by symmetries (isometries) on $\mathbf{H}^3$ – 3 dimensional hyperbolic space.
$\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ equipped with metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.
Boundary is $\mathbb{C} \cup \{\infty\}$.
If action is nice on $\mathbf{H}^3$, then geometer's instinct tells us to take a quotient.

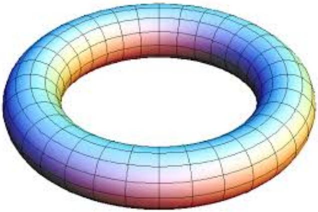
Relative problem:

$H \subset G$ hyperbolic subgroup of a hyperbolic group.
$i : \Gamma_H \to \Gamma_G$ inclusion of Cayley graphs.
Does i extend to a continuous map between the fractal boundaries?
Answer is "No" in this generality. (Baker-Riley 2013)

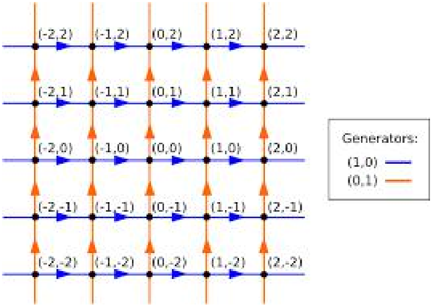
But an analogous (and much more classical) problem arises when a hyperbolic group acts by symmetries (isometries) on $\mathbf{H}^3$ – 3 dimensional hyperbolic space.
$\mathbf{H}^3 = \{(x, y, z) : z > 0\}$ equipped with metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$.
Boundary is $\mathbb{C} \cup \{\infty\}$.
If action is nice on $\mathbf{H}^3$, then geometer's instinct tells us to take a quotient.

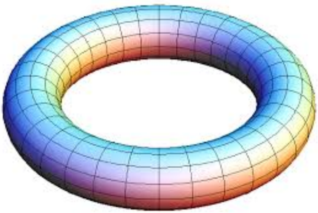
$Z \oplus Z$ acts on $\mathbb{R}^2$ to give torus $S^1 \times S^1$.

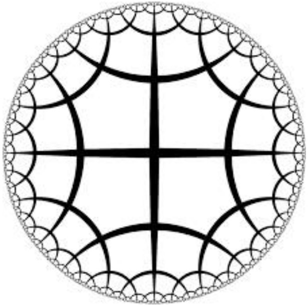


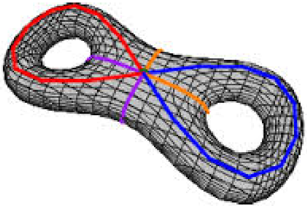


$Z \oplus Z$ acts on $\mathbb{R}^2$ to give torus $S^1 \times S^1$.









Return to 3 dimensional problem.

Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.

Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.

$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.

$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Return to 3 dimensional problem.

Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.

Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.

$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.

$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Return to 3 dimensional problem.
Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.
Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.
$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.
$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Return to 3 dimensional problem.
Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.
Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.
$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.
$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Return to 3 dimensional problem.
Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.
Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.
$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.
$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Return to 3 dimensional problem.
Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.
Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.
$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.
$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Return to 3 dimensional problem.
Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.
Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.
$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.
$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Return to 3 dimensional problem.
Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C}) = Isom(\mathbb{H}^3)$.
Quotient: Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$.
$S^2 = \widehat{\mathbb{C}}$ is the 'ideal' boundary of $\mathbb{H}^3$.
$Mob(\widehat{\mathbb{C}})$ is given by $z \to \frac{az+b}{cz+d}$.

Theorem:

There is an exact dictionary between

1) The dynamics of G on $S^2 = \widehat{\mathbb{C}}$, –Fractal.

2) The geometry of M^3 – Hyperbolic Geometry.

Theorem:

There is an exact dictionary between

1) The dynamics of G on $S^2 = \widehat{\mathbb{C}}$, –Fractal.

2) The geometry of M^3 – Hyperbolic Geometry.

Theorem:
There is an exact dictionary between
1) The dynamics of G on $S^2 = \widehat{\mathbb{C}}$, –Fractal.
2) The geometry of M^3 – Hyperbolic Geometry.

Theorem:
There is an exact dictionary between
1) The dynamics of G on $S^2 = \widehat{\mathbb{C}}$, –Fractal.
2) The geometry of M^3 – Hyperbolic Geometry.

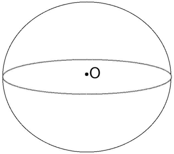
Theorem:

There is an exact dictionary between

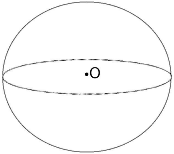
1) The dynamics of G on $S^2 = \widehat{\mathbb{C}}$, –Fractal.

2) The geometry of M^3 – Hyperbolic Geometry.

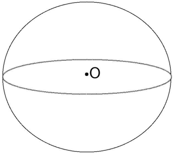
Theorem:
There is an exact dictionary between
1) The dynamics of G on $S^2 = \widehat{\mathbb{C}}$, –Fractal.
2) The geometry of M^3 – Hyperbolic Geometry.



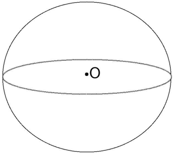
Limit set Λ_G = Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.
Hence for the $(2, 4, 6)-$group or the double torus (octagonal tiling) group, limit set = round equatorial circle.
The intrinsic boundary (a circle) embeds as a round circle in S^2.



Limit set $\Lambda_G =$ Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.
Hence for the $(2, 4, 6)-$group or the double torus (octagonal tiling) group, limit set = round equatorial circle.
The intrinsic boundary (a circle) embeds as a round circle in S^2.



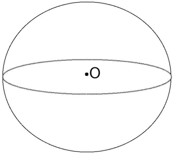
Limit set Λ_G = Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.
Hence for the $(2, 4, 6)-$group or the double torus (octagonal tiling) group, limit set = round equatorial circle.
The intrinsic boundary (a circle) embeds as a round circle in S^2.



Limit set $\Lambda_G =$ Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.

Hence for the $(2,4,6)-$group or the double torus (octagonal tiling) group, limit set = round equatorial circle.

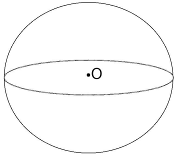
The intrinsic boundary (a circle) embeds as a round circle in S^2.



Limit set $\Lambda_G =$ Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.

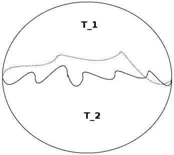
Hence for the $(2, 4, 6)-$group or the double torus (octagonal tiling) group, limit set = round equatorial circle.

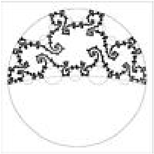
The intrinsic boundary (a circle) embeds as a round circle in S^2.



Limit set $\Lambda_G =$ Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.
Hence for the $(2, 4, 6)-$group or the double torus (octagonal tiling) group, limit set = round equatorial circle.
The intrinsic boundary (a circle) embeds as a round circle in S^2.

Deform:





Limit set is the *locus of chaotic dynamics of the G−action on* S^2.

$i : \Gamma_G \to \mathbf{H}^3$ sending $g \in G$ to $g.o \in \mathbf{H}^3$.

Does i extend to a continuous map between the circle boundary of G and its limit set?

A continuous map as above (if it exists) is called a *Cannon-Thurston map.*

Limit set is the *locus of chaotic dynamics of the $G-$action on S^2.*

$i : \Gamma_G \rightarrow \mathbf{H}^3$ sending $g \in G$ to $g.o \in \mathbf{H}^3$.

Does i extend to a continuous map between the circle boundary of G and its limit set?

A continuous map as above (if it exists) is called a *Cannon-Thurston map.*

Limit set is the *locus of chaotic dynamics of the $G-$action on S^2.*

$i : \Gamma_G \to \mathbf{H}^3$ sending $g \in G$ to $g.o \in \mathbf{H}^3$.

Does i extend to a continuous map between the circle boundary of G and its limit set?

A continuous map as above (if it exists) is called a *Cannon-Thurston map.*

Limit set is the *locus of chaotic dynamics of the $G-$action on* S^2.

$i : \Gamma_G \to \mathbf{H}^3$ sending $g \in G$ to $g.o \in \mathbf{H}^3$.

Does i extend to a continuous map between the circle boundary of G and its limit set?

A continuous map as above (if it exists) is called a *Cannon-Thurston map.*

Limit set is the *locus of chaotic dynamics of the $G-$action on S^2.*

$i : \Gamma_G \to \mathbf{H}^3$ sending $g \in G$ to $g.o \in \mathbf{H}^3$.

Does i extend to a continuous map between the circle boundary of G and its limit set?

A continuous map as above (if it exists) is called a *Cannon-Thurston map.*

Limit set is the *locus of chaotic dynamics of the* $G-$*action on* S^2.

$i : \Gamma_G \to \mathbf{H}^3$ sending $g \in G$ to $g.o \in \mathbf{H}^3$.

Does i extend to a continuous map between the circle boundary of G and its limit set?

A continuous map as above (if it exists) is called a *Cannon-Thurston map.*

Theorem

(M-) There exist Cannon-Thurston maps for finitely generated (3d) Kleinian groups.

Theorem

(M-) Connected limit sets of f.g. (3d) Kleinian groups are locally connected.

Second follows from first using a result of Anderson-Maskit.

Theorem

(M-) There exist Cannon-Thurston maps for finitely generated (3d) Kleinian groups.

Theorem

(M-) Connected limit sets of f.g. (3d) Kleinian groups are locally connected.

Second follows from first using a result of Anderson-Maskit.

Theorem

(M-) There exist Cannon-Thurston maps for finitely generated (3d) Kleinian groups.

Theorem

(M-) Connected limit sets of f.g. (3d) Kleinian groups are locally connected.

Second follows from first using a result of Anderson-Maskit.

Theorem

(M-) There exist Cannon-Thurston maps for finitely generated (3d) Kleinian groups.

Theorem

(M-) Connected limit sets of f.g. (3d) Kleinian groups are locally connected.

Second follows from first using a result of Anderson-Maskit.

Theorem

(M-) There exist Cannon-Thurston maps for finitely generated (3d) Kleinian groups.

Theorem

(M-) Connected limit sets of f.g. (3d) Kleinian groups are locally connected.

Second follows from first using a result of Anderson-Maskit.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky) Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky) Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky)* *Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky)* *Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky)* *Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky) Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky) Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky) Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.

Theorem

Ending Lamination Theorem: *(Brock-Canary-Minsky) Asymptotic topology (at infinity) of M determines geometry of M.*

Theorem

(M-) The asymptotic topology is determined by the Cannon-Thurston map.

Theorem

Chaotic dynamics on boundary determines and is determined by the geometry of M.