

CANNON-THURSTON MAPS

MAHAN MJ

ABSTRACT. We give an overview of the theory of Cannon-Thurston maps which forms one of the links between the complex analytic and hyperbolic geometric study of Kleinian groups. We also briefly sketch connections to hyperbolic subgroups of hyperbolic groups and end with some open questions.

1. KLEINIAN GROUPS AND LIMIT SETS

The Lie group $PSL_2(\mathbb{C})$ can be viewed from three different points of view:

- (1) As a Lie group or a matrix group (**group-theoretic**).
- (2) As the isometry group of hyperbolic 3-space \mathbf{H}^3 —the upper half space $\{(x, y, z) : z > 0\}$ equipped with the metric $ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$, or equivalently the open ball $\{(x, y, z) : (x^2 + y^2 + z^2) = r^2 < 1\}$ equipped with the metric $ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - r^2)^2}$ (**geometric**).
- (3) As the group of Möbius transformations of the Riemann sphere $\hat{\mathbb{C}}$ (**complex dynamic/analytic**).

A finitely generated discrete subgroup $\Gamma \subset PSL_2(\mathbb{C})$ is called a **Kleinian group**. Depending on how we decide to look at $PSL_2(\mathbb{C})$, the group Γ can accordingly be thought of as a discrete subgroup of a Lie group, as the fundamental group of the complete hyperbolic 3-manifold \mathbf{H}^3/Γ , or in terms of its action by holomorphic automorphisms of $\hat{\mathbb{C}}$. If Γ is not virtually abelian, it is called **non-elementary**. Henceforth, unless explicitly stated otherwise, we shall assume that *all Kleinian groups in this article are non-elementary*. If Γ can be conjugated by an element of $PSL_2(\mathbb{C})$ to be contained in $PSL_2(\mathbb{R})$ it is referred to as a **Fuchsian group**¹. If Γ is abstractly isomorphic to $\pi_1(S)$, the fundamental group of a closed surface S , we shall refer to it as a **surface Kleinian group**.

In the 1960's Ahlfors and Bers studied deformations of Fuchsian surface groups in $PSL_2(\mathbb{C})$, giving rise to the theory of quasi-Fuchsian groups. Their techniques were largely complex analytic in nature. In the 1970's and 80's, the field was revolutionized by Thurston, who brought in a rich and varied set of techniques from three-dimensional hyperbolic geometry. A conjectural picture of the deep relationships between the analytic and geometric points of view was outlined by Thurston in his visionary paper [Thu82]. Perhaps the most well-known problem (predating Thurston) in this line of study was the Ahlfors' measure zero conjecture, resolved in the last decade by Brock-Canary-Minsky [BCM12]. Another (more topological) well-known problem that also predates Thurston asks if limit sets of Kleinian groups are locally connected. This is the specific problem that will concern us here. We need to fix some terminology and notation first. Identify the Riemann sphere $\hat{\mathbb{C}}$ with the sphere at infinity S^2 of \mathbf{H}^3 . Thus, S^2 encodes the 'ideal' boundary of \mathbb{H}^3 , consisting of asymptote classes of geodesics. By adjoining S^2 to \mathbf{H}^3 , we obtain the closed 3-ball \mathbf{D}^3 . The topology on S^2 is the usual one induced by the round metric given by the angle subtended at the origin $0 \in \mathbf{D}^3$. The geodesics turn out to be semicircles meeting the boundary S^2 at right angles.

Definition 1.1. *The limit set Λ_Γ of the Kleinian group Γ is the collection of accumulation points of a Γ -orbit $\Gamma \cdot z$ for some (any) $z \in \hat{\mathbb{C}}$.*

The limit set Λ_Γ is independent of z and may be regarded as the locus of chaotic dynamics of Γ on $\hat{\mathbb{C}}$. For non-elementary Γ and any $z \in \Lambda_\Gamma$, $\Gamma \cdot z$ is dense in Λ_Γ . Hence Λ_Γ is the smallest closed non-empty

Date: September 2, 2017.

2010 *Mathematics Subject Classification*. 57M50, 30F40 (Primary), 20F65, 20F67 (Secondary).

Research partly supported by a DST JC Bose Fellowship.

¹Both Fuchsian and Kleinian groups were discovered by Poincaré.

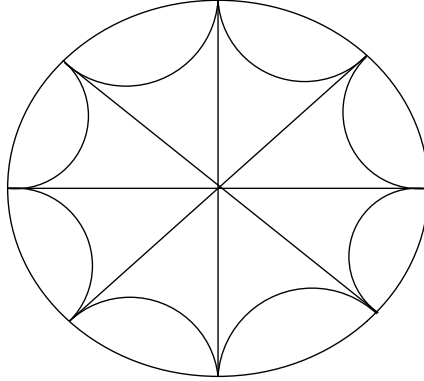
Γ -invariant subset of $\hat{\mathbb{C}}$. If we take $z \in \mathbf{H}^3$ instead, then the collection of accumulation points of any Γ -orbit $\Gamma \cdot z \subset \mathbf{D}^3$ is also Λ_Γ .

Definition 1.2. *The complement of the limit set $\hat{\mathbb{C}} \setminus \Lambda_\Gamma$ is called the domain of discontinuity Ω_Γ of Γ .*

The Kleinian group Γ acts freely and properly discontinuously on Ω_Γ and Ω_Γ/Γ is a (possibly disconnected) Riemann surface.

1.1. Fuchsian and Quasi-Fuchsian Groups. We first give an explicit example of a Fuchsian group.

An example of a Fuchsian group: Consider the standard identification space description of the genus two orientable surface Σ_2 as an octagon with edge labels $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}$. A hyperbolic metric on Σ_2 is one where each point has a small neighborhood isometric to a small disk in \mathbf{H}^2 . By the Poincaré polygon theorem, it suffices to find a regular hyperbolic octagon (with all sides equal and all angles equal) with each interior angle equal to $\frac{2\pi}{8}$. Now, the infinitesimal regular octagon at the tangent space to the origin has interior angles equal to $\frac{3\pi}{4}$. Also the ideal regular octagon in \mathbb{H}^2 has all interior angles zero. See figure below.



By the Intermediate Value Theorem, there exists an intermediate regular octagon with all interior angles equal to $\frac{\pi}{4}$. The group G that results from side-pairing transformations corresponds to a Fuchsian group, or equivalently, a discrete faithful representation ρ of $\pi_1(\Sigma_2)$ into $PSL_2(\mathbb{R})$. Equivalently we may regard ρ as a representation of $\pi_1(\Sigma_2)$ into $PSL_2(\mathbb{C})$ with image that can be conjugated to lie in $PSL_2(\mathbb{R})$. Alternately, $\rho(\pi_1(\Sigma_2))$ preserves a totally geodesic plane in \mathbf{H}^3 . The limit set of $G = \rho(\pi_1(\Sigma_2))$ is then a round circle.

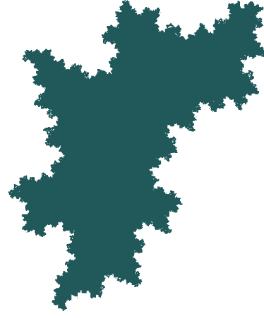
Quasi-Fuchsian groups: If we require the limit set to be only topologically a circle, i.e. a Jordan curve, then we obtain a more general class of Kleinian groups:

Definition 1.3. *Let $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ be a discrete faithful representation such that the limit set of $G = \rho(\pi_1(S))$ is a Jordan curve in S^2 . Then G is said to be **quasi-Fuchsian**. The collection of conjugacy classes of quasi-Fuchsian with the complex analytic structure inherited from $PSL_2(\mathbb{C})$ is denoted as $QF(S)$.*

The domain of discontinuity Ω of a quasi-Fuchsian G consists of two open invariant topological disks Ω_1, Ω_2 in $\hat{\mathbb{C}}$. Hence the quotient Ω/G is the disjoint union $\Omega_1/G \sqcup \Omega_2/G$ and we have a map $\tau : QF(S) \rightarrow Teich(S) \times Teich(S)$, where $Teich(S)$ denotes the Teichmüller space of S —the space of marked hyperbolic (or complex) structures on S . The **Bers simultaneous Uniformization Theorem** asserts:

Theorem 1.4. $\tau : QF(S) \rightarrow Teich(S) \times Teich(S)$ is bi-holomorphic.

Thus, given any two conformal structures T_1, T_2 on a surface, there is a unique discrete quasi-Fuchsian G (up to conjugacy) whose limit set Λ_G is topologically a circle, and the quotient of whose domain of discontinuity is $T_1 \sqcup T_2$. See figure below [Kab16], where the inside and the outside of the Jordan curve correspond to Ω_1, Ω_2 .



1.2. Degenerate groups and the Ending Lamination Theorem. Quasi-Fuchsian were studied by Ahlfors and Bers analytically as deformations of Fuchsian groups. Thurston [Thu80] introduced a new set of geometric techniques in the study.

Definition 1.5. *The convex hull CH_G of Λ_G is the smallest non-empty closed convex subset of \mathbb{H}^3 invariant under G .*

Let $M = \mathbf{H}^3/G$. The quotient of CH_G by G is called the convex core $CC(M)$ of M .

The convex hull CH_G can be constructed by joining all distinct pairs of points on Λ_G by bi-infinite geodesics, iterating this construction and finally taking the closure. It can also be described as the closure of the union of all ideal tetrahedra, whose vertices lie in Λ_G . The convex core $CC(M)$ is homeomorphic to $S \times [0, 1]$.

The distance between the boundary components $S \times \{0\}$ and $S \times \{1\}$ in the convex core $CC(M)$, measured with respect to the hyperbolic metric, is a crude geometric measure of the complexity of the quasi Fuchsian group G . We shall call it the **thickness** t_G of $CC(M)$, or of the quasi Fuchsian group G . (We note here parenthetically that the notions of convex hull CH_G and convex core $CC(M)$ go through for any Kleinian group G and the associated complete hyperbolic manifold \mathbf{H}^3/G .) For quasi-Fuchsian groups, we ask:

Question 1.6. *What happens as thickness tends to infinity?*

To address this question more precisely we need to introduce a topology on the space of representations.

Definition 1.7. *A sequence of representations $\rho_n : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ is said to converge **algebraically** to $\rho_\infty : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ if for all $g \in \pi_1(S)$, $\rho_n(g) \rightarrow \rho_\infty(g)$ in $PSL_2(\mathbb{C})$. The collection of conjugacy classes of discrete faithful representations of $\pi_1(S)$ into $PSL_2(\mathbb{C})$ equipped with the algebraic topology is denoted as $AH(S)$.*

It is not even clear a priori that, as t_G tends to infinity, limits exist in $AH(S)$. However, Thurston's **Double Limit Theorem** [Thu86, Kap01, Ota01] guarantees that if we have a sequence G_n with thickness t_{G_n} tending to infinity, subsequential limits (in the space of discrete faithful representations with a suitable topology, see Definition 5.7) do in fact exist.

Geodesic Laminations:

Definition 1.8. *A geodesic lamination on a hyperbolic surface is a foliation of a closed subset with geodesics, i.e. it is a closed set given as a disjoint union of geodesics (closed or bi-infinite).*

A geodesic lamination on a surface may further be equipped with a transverse (non-negative) measure to obtain a **measured lamination**. The space of measured (geodesic) laminations on S is then a positive cone in a vector space and is denoted as $\mathcal{ML}(S)$. It can be projectivized to obtain the space of projectivized measured laminations $\mathcal{PML}(S)$. It was shown by Thurston [FLP79] that $\mathcal{PML}(S)$ is homeomorphic to a sphere and can be adjoined to $Teich(S)$ compactifying it to a closed ball.

Definition 1.9. [Thu80, Definition 8.8.1] A **pleated surface** in a hyperbolic three-manifold N is a complete hyperbolic surface S of finite area, together with an isometric map $f : S \rightarrow N$ such that every $x \in S$ is in the interior of some geodesic segment (in S) which is mapped by f to a geodesic segment (in N). Also, f must take every cusp (corresponding to a maximal parabolic subgroup) of S to a cusp of N .

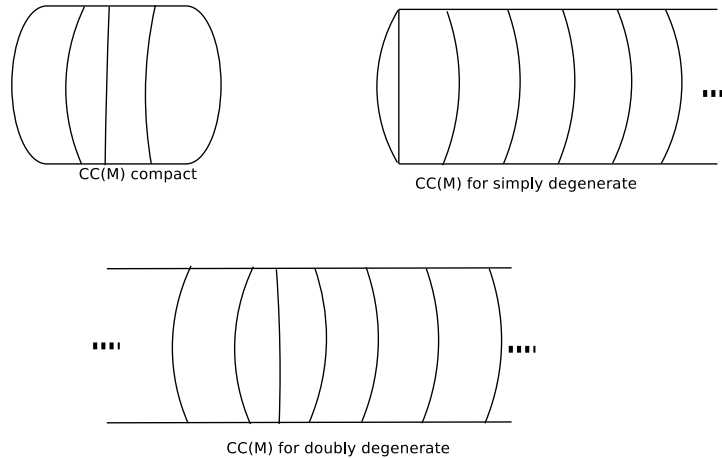
The **pleating locus** of the pleated surface $f : S \rightarrow M$ is the set $\gamma \subset S$ consisting of those points in the pleated surface which are in the interior of unique geodesic segments mapped to geodesic segments.

Proposition 1.10. [Thu80, Proposition 8.8.2] The pleating locus γ is a geodesic lamination on S . The map f is totally geodesic in the complement of γ .

Thurston further shows [Thu80, Ch. 8] that the boundary components of the convex core $CC(M)$ are pleated surfaces.

As thickness (Question 1.6) tends to infinity, the convex core may converge (up to extracting subsequences) to one of two kinds of convex manifolds $CC(M_\infty)$ (see schematic diagram below):

- (1) $CC(M_\infty)$ is homeomorphic to $S \times [0, \infty)$. Here $S \times \{0\}$ is the single boundary component of the convex core $CC(M_\infty)$. Such a manifold is called **simply degenerate** and the corresponding Kleinian group a simply degenerate surface Kleinian group. In this case the limit set Λ_G is a dendrite (topologically a tree) and the domain of discontinuity Ω_Γ is homeomorphic to an open disk with Ω_Γ/G a Riemann surface homeomorphic to $S \times \{0\}$. The end of M_∞ facing Ω_Γ/G is called a **geometrically finite end**.
- (2) $CC(M_\infty)$ is homeomorphic to $S \times (-\infty, \infty)$. Such a manifold is called **doubly degenerate** and the corresponding Kleinian group a doubly degenerate surface Kleinian group. In this case the limit set Λ_G is all of S^2 and the domain of discontinuity Ω_Γ is empty.



The ends (one for simply degenerate and two for doubly degenerate) of $CC(M_\infty)$ are called the **degenerate end(s)** of M . Thurston [Thu80] and Bonahon [Bon86] show that any such degenerate end E is **geometrically tame**, i.e. there is a sequence of simple closed curves $\{\sigma_n\}$ on S such that their geodesic realizations in $CC(M_\infty)$ exit E as n tends to ∞ . Further [Thu80, Ch. 9], the limit of any such exiting sequence (in $\mathcal{PML}(S)$; the topology on $\mathcal{PML}(S)$ is quite close to the Hausdorff topology on S) is a unique lamination λ called the **ending lamination** of E . Thus a doubly degenerate manifold has two ending laminations, one for each degenerate end, while a simply degenerate manifold has a geometrically finite end corresponding to a Riemann surface $\Omega_\Gamma/G \in \text{Teich}(S)$ at infinity and a degenerate end corresponding to an ending lamination.

These two pieces of information—Riemann surfaces at infinity and ending laminations—give the **end-invariants** of M . The ending lamination for a geometrically infinite end does not depend on the reference hyperbolic structure on S . It may be regarded as a purely topological piece of data associated to such an end.

We may thus think of the ending lamination as the analog, in the case of geometrically infinite ends, of the conformal structure at infinity for a geometrically infinite end.

The Ending Lamination Theorem of Brock-Canary-Minsky then takes the place of the Bers' simultaneous uniformization theorem and asserts:

Theorem 1.11. [Min10, BCM12] *Let M be a simply or doubly degenerate manifold. Then M is determined up to isometry by its end-invariants.*

Thus, the Ending Lamination Theorem justifies the following and may be considered an analog of Mostow Rigidity for infinite covolume Kleinian groups.

Slogan 1.12. *Topology implies Geometry.*

In order to complete the picture of the Ahlfors-Bers theory to study degenerate surface Kleinian groups G in terms of the dynamics of G on the Riemann sphere $\hat{\mathbb{C}}$, the following issue remains to be addressed:

Question 1.13. *Can the data of the ending lamination(s) be extracted from the dynamics of G on $\hat{\mathbb{C}}$?*

In more informal terms,

Question 1.14. *Is the geometric object "at infinity" of the quotient manifold \mathbf{H}^3/Γ (i.e. the ending lamination) determined by the dynamics of Γ at infinity (i.e. the action of Γ on S^2)?*

We will make these questions precise below. The attempt to make Question 1.13 precise brings us to the following.

2. CANNON-THURSTON MAPS

2.1. The main theorem for closed surface Kleinian groups. In [Thu82, Problem 14], Thurston raised the following question, which is at the heart of the work we discuss here:

Question 2.1. *Suppose Γ has the property that $(\mathbf{H}^3 \cup \Omega_\Gamma)/\Gamma$ is compact. Then is it true that the limit set of any other Kleinian group Γ' isomorphic to Γ is the continuous image of the limit set of Γ , by a continuous map taking the fixed points of an (y) element γ to the fixed points of the corresponding element γ' ?*

A special case of Question 2.1 was answered affirmatively in seminal work of Cannon and Thurston [CT85, CT07]:

Theorem 2.2. [CT07] *Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber Σ . Let $\tilde{\Sigma}$ and \tilde{M} denote the universal covers of Σ and M respectively. After identifying $\tilde{\Sigma}$ (resp. \tilde{M}) with \mathbf{H}^2 (resp. \mathbf{H}^3), we obtain the compactification $\mathbb{D}^2 = \mathbf{H}^2 \cup S^1$ (resp. $\mathbb{D}^3 = \mathbf{H}^3 \cup S^2$) by attaching the circle S^1 (resp. the sphere S^2) at infinity. Let $i : \Sigma \rightarrow M$ denote the inclusion map of the fiber and $\tilde{i} : \tilde{\Sigma} \rightarrow \tilde{M}$ the lift to the universal cover. Then \tilde{i} extends to a continuous map $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$.*

An amazing implication of Theorem 2.2 is that $\tilde{\Sigma}$ is an embedded disk in (the ball model) of \mathbf{H}^3 such that its boundary on the sphere S^2 is space-filling! See the following diagram by Thurston [Thu82, Figure 10] that illustrates 'a pattern of identification of a circle, here represented as the equator, whose quotient is topologically a sphere. This defines, topologically a sphere-filling curve.'



A version of Question 2.1 was raised by Cannon and Thurston in the context of closed surface Kleinian groups:

Question 2.3. [CT07, Section 6] *Suppose that a closed surface group $\pi_1(S)$ acts freely and properly discontinuously on \mathbb{H}^3 by isometries such that the quotient manifold has no accidental parabolics (here this just means that the image of $\pi_1(S)$ in $PSL_2(\mathbb{C})$ has no parabolics). Does the inclusion $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$ extend continuously to the boundary?*

Continuous boundary extensions as in Question 2.3, if they exist, are called **Cannon-Thurston maps**. Question 2.3 is intimately related to a much older question c.f. [Abi76] asking if limit sets are locally connected:

Question 2.4. *Let Γ be a finitely generated Kleinian group such that the limit set Λ_Γ is connected. Is Λ_Γ locally connected?*

It is shown in [CT07] that for simply degenerate surface Kleinian groups, Questions 2.3 and 2.4 are equivalent, via the Caratheodory extension Theorem.

The following Theorem of [Mj14a] [Mj14b] answers questions 2.3 and 2.4 affirmatively:

Theorem 2.5. *Let $\rho(\pi_1(S)) = G \subset PSL_2(\mathbb{C})$ be a simply or doubly degenerate (closed) surface Kleinian group. Let $M = \mathbf{H}^3/G$ and $i : S \rightarrow M$ be an embedding inducing a homotopy equivalence. Let $\tilde{i} : \tilde{S} \rightarrow \mathbf{H}^3$ denote a lift of i between universal covers. Let $\mathbb{D}^2, \mathbb{D}^3$ denote the compactifications. Then a Cannon-Thurston map $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$ exists.*

Let $\partial i : S^1 \rightarrow S^2$ denote the restriction of \hat{i} to the ideal boundaries. Then for $p \neq q$, $\partial i(p) = \partial i(q)$ if and only if p, q are the ideal end-points of a leaf of an ending lamination or ideal end-points of a complementary ideal polygon of an ending lamination.

The second part of Theorem 2.5 shows that the data of the ending lamination can be recovered from the Cannon-Thurston map and so we have an affirmative answer to Question 1.13. In conjunction with the Ending Lamination Theorem 1.11, this establishes the slogan:

Slogan 2.6. *Dynamics on the limit set determines geometry in the interior.*

A number of authors have contributed to the resolution of the above questions. Initially it was believed [Abi76] that Question 2.4 had a negative answer for simply degenerate Kleinian groups. Floyd [Flo80] proved the corresponding theorem for geometrically finite Kleinian groups. Then in the early 80's Cannon and Thurston [CT85] proved Theorem 2.2. This was extended by Minsky [Min94], Klarreich [Kla99], Alperin-Dicks-Porti [ADP99], Bowditch [Bow13, Bow07], McMullen [McM01], Miyachi [Miy02] and the author [Mit98b, Mj10, Mj11a, Mj16] for various special cases. The general surface group case was accomplished in [Mj14a] and the general Kleinian group case in [Mj17b].

2.2. Geometric Group Theory. We now turn to a generalization of Question 2.3 to a far more general context. After the introduction of hyperbolic metric spaces by Gromov [Gro85], Question 2.3 was extended by the author [Mit97b, Bes04, Mit98c] to the context of a hyperbolic group H acting freely and properly discontinuously by isometries on a hyperbolic metric space X . Any hyperbolic X has a (Gromov) boundary ∂X given by asymptote-classes of geodesics. Adjoining ∂X to X we get the Gromov compactification \widehat{X} .

There is a natural map $i : \Gamma_H \rightarrow X$, sending vertices of Γ_H to the H -orbit of a point $x \in X$, and connecting images of adjacent vertices by geodesic segments in X . Let $\widehat{\Gamma}_H, \widehat{X}$ denote the Gromov compactification of Γ_H, X respectively. The analog of Question 2.3 is the following:

Question 2.7. *Does $i : \Gamma_H \rightarrow X$ extend continuously to a map $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{X}$?*

Continuous extensions as in Question 2.7 are also referred to as **Cannon-Thurston maps** and make sense when Γ_H is replaced by an arbitrary hyperbolic metric space Y . A simple and basic criterion for the existence of Cannon-Thurston maps was established in [Mit98a, Mit98b]:

Lemma 2.8. *Let $i : (Y, y) \rightarrow (X, x)$ be a proper map between (based) Gromov-hyperbolic spaces. A continuous extension (also called a Cannon-Thurston map) $\hat{i} : \widehat{Y} \rightarrow \widehat{X}$ exists if and only if the following holds:*

There exists a non-negative proper function $M : \mathbb{N} \rightarrow \mathbb{N}$, such that if $\lambda = [a, b]_Y$ is a geodesic lying outside an N -ball around y , then any geodesic segment $[i(a), i(b)]_X$ in X joining $i(a), i(b)$ lies outside the $M(N)$ -ball around $x = i(y)$.

In the generality above Question 2.7 turns out to have a negative answer. An explicit counterexample to Question 2.7 was recently found by Baker and Riley [BR13] in the context of small cancellation theory. The counterexample uses Lemma 2.8 to rule out the existence of Cannon-Thurston maps. Further, Matsuda and Oguni [MO14] further developed Baker and Riley's counterexample to show that given a(ny) non-elementary hyperbolic group H , there exists hyperbolic group G such that $H \subset G$ and there is no Cannon-Thurston map for the inclusion. We shall furnish positive answers to Question 2.7 in a number of special cases in Section 6.

3. CLOSED 3-MANIFOLDS

3.1. 3-manifolds fibering over the circle. We start by giving a sketch of the proof of Theorem 2.2. The proof is coarse-geometric in nature and follows [Mit98a, Mit98b]. We recall a couple of basic Lemmata we shall be needing from [Mit98b]. The following says that nearest point projection onto a geodesic in a hyperbolic space is coarsely Lipschitz.

Lemma 3.1. *Let (X, d) be a δ -hyperbolic metric space. Then there exists a constant $C \geq 1$ such that the following holds:*

Let $\lambda \subset X$ be a geodesic segment and let $\Pi : X \rightarrow \lambda$ be a nearest point projection. Then $d(\Pi(x), \Pi(y)) \leq Cd(x, y)$ for all $x, y \in X$.

The next Lemma says that nearest point projections and quasi-isometries almost commute.

Lemma 3.2. *Let (X, d) be a δ -hyperbolic metric space. Given (K, ϵ) , there exists C such that the following holds:*

Let $\lambda = [a, b]$ be a geodesic segment in X . Let $p \in X$ be arbitrary and let q be a nearest point projection of p onto λ . Let ϕ be a (K, ϵ) -quasi-isometry from X to itself and let $\Phi(\lambda) = [\phi(a), \phi(b)]$ be a geodesic segment in X joining $\phi(a), \phi(b)$. Let r be a nearest point projection of $\phi(p)$ onto $\Phi(\lambda)$. Then $d(r, \phi(q)) \leq C$.

Sketch of Proof: The proof of Lemma 3.2 follows from the fact that a geodesic tripod T (built from $[a, b]$ and $[p, q]$) is quasiconvex in hyperbolic space and that a quasi-isometric image $\phi(T)$ of T lies close to a geodesic tripod T' built from $[\phi(a), \phi(b)]$ and $[\phi(p), r]$. Hence the image $\phi(q)$ of the centroid q of T lies close to the centroid r of T' . \square

3.2. The key tool: hyperbolic ladder. The key idea behind the proof of Theorem 2.2 and its generalizations in [Mit98a, Mit98b] is the construction of a **hyperbolic ladder** $\mathcal{L}_\lambda \subset \widetilde{M}$ for *any* geodesic in $\widetilde{\Sigma}$. The universal cover \widetilde{M} fibers over \mathbb{R} with fibers $\widetilde{\Sigma}$. Since the context is geometric group theory, we discretize this as follows. Replace $\widetilde{\Sigma}$ and \widetilde{M} by quasi-isometric models in the form of Cayley graphs $\Gamma_{\pi_1(\Sigma)}$ and $\Gamma_{\pi_1(M)}$ respectively. Let us denote $\Gamma_{\pi_1(\Sigma)}$ by Y and $\Gamma_{\pi_1(M)}$ by X . The projection of \widetilde{M} to the base \mathbb{R} is discretized accordingly giving a model that can be thought of as (and is quasi-isometric to) a *tree T of spaces*, where

- (1) T is the simplicial tree underlying \mathbb{R} with vertices at \mathbb{Z} .
- (2) All the vertex and edge spaces are (intrinsically) isometric to Y .
- (3) The edge space to vertex space inclusions are qi-embeddings.

We summarize this by saying that X is a tree T of spaces satisfying the *qi-embedded condition* [BF92].

Given a geodesic segment $[a, b] = \lambda = \lambda_0 \subset Y$, we now sketch the promised construction of the **ladder** $\mathcal{L}_\lambda \subset X$ containing λ . Index the vertices by $n \in \mathbb{Z}$. Since the edge-to-vertex space inclusions are quasi-isometries, this induces a quasi-isometry ϕ_n from the vertex space Y_n to the vertex space Y_{n+1} for $n \geq 0$. A similar quasi-isometry ϕ_{-n} exists from Y_{-n} to the vertex space $Y_{-(n+1)}$. These quasi-isometries

are defined on the vertex sets of Y_n , $n \in \mathbb{Z}$. ϕ_n induces a map Φ_n from geodesic segments in Y_n to geodesic segments in Y_{n+1} for $n \geq 0$ by sending a geodesic in Y_n joining a, b to a geodesic in Y_{n+1} joining $\phi_n(a), \phi_n(b)$. Similarly, for $n \leq 0$. Inductively define:

- $\lambda_{j+1} = \Phi_j(\lambda_j)$ for $j \geq 0$,
- $\lambda_{-j-1} = \Phi_{-j}(\lambda_{-j})$ for $j \geq 0$,
- $\mathcal{L}_\lambda = \bigcup_j \lambda_j$.

\mathcal{L}_λ turns out to be quasiconvex in X . To prove this, we construct a coarsely Lipschitz retraction $\Pi_\lambda : \bigcup_j Y_j \rightarrow \mathcal{L}_\lambda$ as follows.

On Y_j define $\pi_j(y)$ to be a nearest-point projection of y onto λ_j and define

$$\Pi_\lambda(y) = \pi_j(y), \text{ for } y \in Y_j.$$

The following theorem asserts that Π_λ is coarsely Lipschitz.

Theorem 3.3. [Mit98a, Mit98b, Mj10] *There exists $C \geq 1$ such that for any geodesic $\lambda \subset Y$,*

$$d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_X(x, y)$$

for $x, y \in \bigcup_i Y_i$.

Sketch of Proof:

The proof requires only the hyperbolicity of Y , but not that of X . It suffices to show that for $d_X(x, y) = 1$, $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C$. Thus x, y may be thought of as

- (1) either lying in the same Y_j . This case follows directly from Lemma 3.1.
- (2) or lying vertically one just above the other. Then (up to a bounded amount of error), we can assume without loss of generality, that $y = \phi_j(x)$. This case now follows from Lemma 3.2.

Since a coarse Lipschitz retract of a hyperbolic metric space is quasiconvex, we immediately have:

Corollary 3.4. *If (X, d_X) is hyperbolic, there exists $C \geq 1$ such that for any λ , \mathcal{L}_λ is C -quasiconvex.*

Note here that we have not used any feature of Y except its hyperbolicity. In particular, we do not need the specific condition that $Y = \tilde{\Sigma}$. We are now in a position to prove a generalization of Theorem 2.2.

Theorem 3.5. [Mj10] *Let (X, d) be a hyperbolic tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition, where T is \mathbb{R} or $[0, \infty)$ with vertex and edge sets Y_j as above, $j \in \mathbb{Z}$. Assume (as above) that the edge-to-vertex inclusions are quasi-isometries. For $i : Y_0 \rightarrow X$ there is a Cannon-Thurston map $\hat{i} : \widehat{Y_0} \rightarrow \widehat{X}$.*

Proof. Fix a basepoint $y_0 \in Y_0$. By Lemma 2.8 and quasiconvexity of \mathcal{L}_λ (Corollary 3.4), it suffices to show that for all $M \geq 0$ there exists $N \geq 0$ such that if a geodesic segment λ lies outside the N -ball about $y_0 \in Y_0$, then \mathcal{L}_λ lies outside the M -ball around $y_0 \in X$. Equivalently, we need a proper function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$.

Since Y_0 is properly embedded in X , there exists a proper function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that λ lies outside the $g(N)$ -ball about $y_0 \in X$.

Let p be any point on \mathcal{L}_λ . Then $p = p_j \in Y_j$ for some j . Assume without loss of generality that $j \geq 0$. It is not hard to see that there exists C_0 , depending only on X , such that for any such p_j , there exists $p_{j-1} \in Y_{j-1}$ with $d(p_j, p_{j-1}) \leq C_0$. It follows inductively that there exists $y \in \lambda = \lambda_0$ such that $d_X(y, p) \leq C_0 j$. Hence, by the triangle inequality, $d_X(y_0, p) \geq g(N) - C_0 j$.

Next, looking at the ‘vertical direction’, $d_X(y_0, p) \geq j$ and hence

$$d_X(y_0, p) \geq \max(g(N) - C_0 j, j) \geq \frac{g(N)}{A+1}.$$

Defining $M(N) = \frac{g(N)}{A+1}$, we see that $M(N)$ is a proper function of N and we are done. \square

3.3. Quasiconvexity. The structure of Cannon-Thurston maps in Section 3.1 can be used to establish quasiconvexity of certain subgroups of $\pi_1(M)$. Let $H \subset \pi_1(\Sigma)$ be a finitely generated infinite index subgroup of the fiber group. Then, due to the LERF property for surface subgroups, a Theorem of Scott [Sco78], there is a finite sheeted cover where H is geometric, i.e. it is carried by a proper embedded subsurface of (a finite sheeted cover of) Σ . But such a proper subsurface cannot carry a leaf of the stable or unstable foliations \mathcal{F}_s or \mathcal{F}_u . This gives us the following Theorem of Scott and Swarup:

Theorem 3.6. [SS90] *Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber Σ . Let $H \subset \pi_1(\Sigma)$ be a finitely generated infinite index subgroup of the fiber group in $\pi_1(M)$. Then H is quasiconvex in $\pi_1(M)$.*

Theorem 3.6 has been generalized considerably to the context of convex cocompact subgroups of the mapping class group and $Out(F_n)$ by a number of authors [DKL14, DKT16, DT17b, DT17a, MR15].

4. KLEINIAN SURFACE GROUPS: MODEL GEOMETRIES

In this section we shall describe a sequence of models for degenerate ends of 3-manifolds following [Min01, Min94, Mj10, Mj11a, Mj16] and [Min10, BCM12, Mj14a] and indicate how to generalize the ladder construction of Section 3.2 incorporating electric geometry [Far98]. Let X be a hyperbolic metric space, e.g. \mathbf{H}^3 . Let \mathcal{H}_X be a collection of disjoint convex subsets. Roughly speaking, electrification equips each element of \mathcal{H}_X with the zero metric, while preserving the metric on $X \setminus (\bigcup_{H \in \mathcal{H}_X} H)$. The resulting electrified space $\mathcal{E}(X, \mathcal{H}_X)$ is still Gromov hyperbolic under extremely mild conditions and hyperbolic geodesics in X can be recovered from electric geodesics in the electrified space $\mathcal{E}(X, \mathcal{H}_X)$. This will allow us to establish the existence of Cannon-Thurston maps. We shall focus on closed surfaces and follow the summary in [LM16] for the exposition.

The topology of each building block is simple: it is homeomorphic to $S \times [0, 1]$, where S is a closed surface of genus greater than one. Geometrically, the top and bottom boundary components in the first three model geometries are uniformly bi-Lipschitz to a fixed hyperbolic structure on S . Assume therefore that S is equipped with such a fixed hyperbolic structure. We do so henceforth. The different types of geometries of the blocks make for different model geometries of ends.

Definition 4.1. *A model E_m is said to be built up of blocks of some prescribed geometries **glued end to end**, if*

- (1) E_m is homeomorphic to $S \times [0, \infty)$
- (2) There exists $L \geq 1$ such that $S \times [i, i+1]$ is L -bi-Lipschitz to a block of one of the three prescribed geometries: bounded, i -bounded or amalgamated (see below).

$S \times [i, i+1]$ will be called the $(i+1)$ **th block** of the model E_m .

The **thickness** of the $(i+1)$ **th block** is the length of the shortest path between $S \times \{i\}$ and $S \times \{i+1\}$ in $S \times [i, i+1] (\subset E_m)$.

4.1. Bounded geometry. Minsky [Min01, Min94] calls an end E of a hyperbolic 3-manifold to be of bounded geometry if there are no arbitrarily short closed geodesics in E .

Definition 4.2. *Let $B_0 = S \times [0, 1]$ be given the product metric. If B is L -bi-Lipschitz homeomorphic to B_0 , it is called an L -**thick block**.*

An end E is said to have a model of bounded geometry if there exists L such that E is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing L -thick blocks end-to-end.

It follows from work of Minsky [Min92] that if E is of bounded geometry, it has a model of bounded geometry. The existence of Cannon-Thurston maps in this setup is then a replica of the proof of Theorem 3.5.

4.2. i -bounded Geometry.

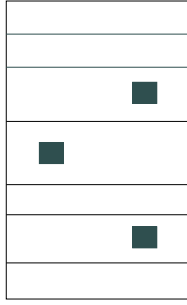
Definition 4.3. [Mj11a] *An end E of a hyperbolic 3-manifold M has **i -bounded geometry** if the boundary torus of every Margulis tube in E has bounded diameter.*

We give an alternate description. Fix a closed hyperbolic surface S . Let \mathcal{C} be a finite collection of (not necessarily disjoint) simple closed geodesics on S . Let $N_\epsilon(\sigma_i)$ denote an ϵ neighborhood of σ_i , $\sigma_i \in \mathcal{C}$, where ϵ is small enough to ensure that lifts of $N_\epsilon(\sigma_i)$ to \tilde{S} are disjoint.

Definition 4.4. Let $I = [0, 3]$. Equip $S \times I$ with the product metric. Let $B^c = (S \times I - \cup_j N_\epsilon(\sigma_j)) \times [1, 2]$, equipped with the induced path-metric. Here $\{\sigma_j\}$ is a subcollection of \mathcal{C} consisting of disjoint curves. Perform Dehn filling on some $(1, n)$ curve on each resultant torus component of the boundary of B^c (the integers n are quite arbitrary and may vary: we omit subscripts for expository ease). We call n the **twist coefficient**. Foliate the relevant torus boundary component of B^c by translates of $(1, n)$ curves. Glue in a solid torus Θ , which we refer to as a **Margulis tube**, with a hyperbolic metric foliated by totally geodesic disks bounding the $(1, n)$ curves.

The resulting copy of $S \times I$ thus obtained, equipped with the above metric is called a **thin block**.

Definition 4.5. A model manifold E_m of **i-bounded geometry** is built out of gluing L -thick and thin blocks end-to-end (for some L) (see schematic diagram below where the black squares indicate Margulis tubes and the horizontal rectangles indicate the blocks).



It follows from work in [Mj11a] that

Proposition 4.6. An end E of a hyperbolic 3-manifold M has **i-bounded geometry** if and only if it is bi-Lipschitz homeomorphic to a model manifold E_m of *i*-bounded geometry.

We give a brief indication of the construction of \mathcal{L}_λ and the proof of the existence of Cannon-Thurston maps in this case. First electrify all the Margulis tubes, i.e. equip them with a zero metric (see [Far98] for details on relative hyperbolicity and electric geometry). This ensures that in the resulting electric geometry, each block is of bounded geometry. More precisely, there is a (metric) product structure on $S \times [0, 3]$ such that each $\{x\} \times [0, 3]$ has uniformly bounded length in the electric metric.

Further, since the curves in \mathcal{C} are electrified in a block, Dehn twists are isometries from $S \times \{1\}$ to $S \times \{2\}$ in a thin block. This allows the construction of \mathcal{L}_λ to go through as before and ensures that it is quasiconvex in the resulting electric metric.

Finally given an electric geodesic lying outside large balls modulo Margulis tubes one can recover a genuine hyperbolic geodesic tracking it outside Margulis tubes. A relative version of the criterion of Lemma 2.8 can now be used to prove the existence of Cannon-Thurston maps.

4.3. Amalgamation Geometry. Again, as in Definition 4.4, start with a fixed closed hyperbolic surface S , a collection of simple closed curves \mathcal{C} and set $I = [0, 3]$. Perform Dehn surgeries on the Margulis tubes corresponding to \mathcal{C} as before. Let $K = S \times [1, 2] \subset S \times [0, 3]$ and let $K^c = (S \times I - \cup_i N_\epsilon(\sigma_i)) \times [1, 2]$. Instead of fixing the product metric on the complement K^c of Margulis tubes in K , allow these complementary components to have *arbitrary geometry* subject only to the restriction that the geometries of $S \times \{1, 2\}$ are fixed. Equip $S \times [0, 1]$ and $S \times [2, 3]$ with the product metrics. The resulting block is said to be a **block of amalgamation geometry**. After lifting to the universal cover, complements of Margulis tubes in the lifts $\tilde{S} \times [1, 2]$ are termed **amalgamation components**.

Definition 4.7. An end E of a hyperbolic 3-manifold M has **amalgamated geometry** if

- (1) it is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing L -thick and amalgamation geometry blocks end-to-end (for some L).

(2) *Amalgamation components are (uniformly) quasiconvex in \tilde{E}_m .*

To construct the ladder \mathcal{L}_λ we electrify amalgamation components as well as Margulis tubes. This ensures that in the electric metric,

- (1) Each amalgamation block has bounded geometry
- (2) The mapping class element taking $S \times \{1\}$ to $S \times \{2\}$ induces an isometry of the electrified metrics.

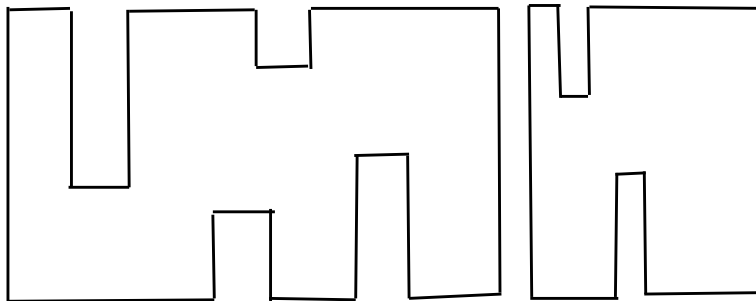
Quasiconvexity of \mathcal{L}_λ in the electric metric now follows as before. To recover the data of hyperbolic geodesics from quasigeodesics lying close to \mathcal{L}_λ , we use (uniform) quasiconvexity of amalgamation components and existence of Cannon-Thurston maps follows.

4.4. Split Geometry. We need now to relax the assumption that the boundary components of model blocks are of (uniformly) bounded geometry. Roughly speaking, split geometry is a generalization of amalgamation geometry where

- (1) A Margulis tube is allowed to travel through a uniformly bounded number of contiguous blocks and split them.
- (2) The complementary pieces, now called split components, are quasiconvex in a somewhat weaker sense (see Definition 4.8 below).

Each split component is allowed to contain Margulis tubes, called **hanging tubes** that do not go all the way across from the top to the bottom, i.e. they do not split both $\Sigma_i^s, \Sigma_{i+1}^s$.

A split component $B^s \subset B = S \times I$ is topologically a product $S^s \times I$ for some, necessarily connected $S^s \subset S$. However, the upper and lower boundaries of B^s need only be split subsurfaces of S^s to allow for hanging tubes starting or ending (but not both) within the split block.



Split Block with hanging tubes

An end E of a hyperbolic 3-manifold M has **weak split geometry** if it is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing L -thick and split blocks as above end-to-end (for some L). Electrifying split components as in Section 4.3, we obtain a new electric metric called the **graph metric** d_G on E .

Definition 4.8. *A model of weak split geometry is said to be of **split geometry** if the convex hull of each split component has uniformly bounded d_G -diameter.*

5. CANNON-THURSTON MAPS FOR KLEINIAN GROUPS AND APPLICATIONS

5.1. Cannon-Thurston maps for degenerate manifolds. Let M be a hyperbolic 3-manifold homotopy equivalent to a closed hyperbolic surface S . Once we establish that M has split geometry, the proof proceeds as in Section 4.3 by electrifying split components, constructing a hyperbolic ladder \mathcal{L}_λ and finally recovering a hyperbolic geodesic from an electric one. We shall therefore dwell in this subsection on showing that any degenerate end has split geometry. We shall do this under two simplifying assumptions, directing the reader to [Mj14a] (especially the Introduction) for a more detailed road-map.

We borrow extensively from the hierarchy and model manifold terminology and technology of Masur-Minsky [MM00] and Minsky [Min10]. The model manifold of [Min10, BCM12] furnishes a resolution, or equivalently, a sequence $\{P_m\}$ of pants decompositions of S exiting E and hence a *hierarchy path*. Let

τ_m denote the simple multicurve on S constituting P_m . The P_m in turn furnish split level surfaces $\{S_m\}$ exiting E : a *split level surface* S_m is a collection of (nearly) totally geodesic embeddings of the pairs of pants comprising P_m . Next, corresponding to the hierarchy path $\{\tau_m\}$, there is a tight geodesic in $\mathcal{C}(S)$ consisting of the bottom geodesic $\{\eta_i\}$ of the hierarchy. We proceed to extract a subsequence of the resolution τ_m using the bottom geodesic $\{\eta_i\}$ under two key simplifying assumptions:

- (1) For all i , the length of exactly one curve in η_i is sufficiently small, less than the Margulis constant in particular. Call it η_i for convenience.
- (2) Let S_i correspond to the first occurrence of the vertex η_i in the resolution τ_m . Assume further that the S_i 's are actually split surfaces and not just split level surfaces, i.e. they all have injectivity radius uniformly bounded below,

It follows that the Margulis tube η_i splits both S_i and S_{i+1} and that the tube T_i is trapped entirely between S_i and S_{i+1} . The product region B_i between S_i and S_{i+1} is therefore a split block for all i and T_i splits it. The model manifold thus obtained is one of weak split geometry. In a sense, this is a case of ‘pure split geometry’, where all blocks have a split geometry structure (no thick blocks). To prove that the model is indeed of split geometry, it remains to establish the quasiconvexity condition of Definition 4.8.

Let K be a split component and \tilde{K} an elevation to \tilde{E} . Let v be a boundary short curve for the split component and let T_v be the Margulis tube corresponding to v abutting K . Denote the hyperbolic convex hull by $CH(\tilde{K})$ and pass back to a quotient in M . A crucial observation that is needed here is the fact that any pleated surface has bounded d_G -diameter. This is because thin parts of pleated surfaces lie inside Margulis tubes that get electrified in the graph metric. It therefore suffices to show that any point in $CH(K)$ lies close to a pleated surface passing near the fixed tube T_v . This last condition follows from the Brock-Bromberg drilling theorem [BB04] and the fact that the convex core of a quasi-Fuchsian group is filled by pleated surfaces [Fan99]. This completes our sketch of a proof of the following main theorem of [Mj14a]:

Theorem 2.5: Let $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ be a simply or doubly degenerate (closed) surface Kleinian group. Then a Cannon-Thurston map exists.

It follows that the limit set of $\rho(\pi_1(S))$ is a continuous image of S^1 and is therefore locally connected. As a first application of Theorem 2.5, we shall use the following Theorem of Anderson and Maskit [AM96] to prove that connected limit sets of Kleinian groups without parabolics are locally connected.

Theorem 5.1. [AM96] *Let Γ be an analytically finite Kleinian group with connected limit set. Then the limit set $\Lambda(\Gamma)$ is locally connected if and only if every simply degenerate surface subgroup of Γ without accidental parabolics has locally connected limit set.*

Combining the remark after Theorem 2.5 with Theorem 5.1, we immediately have the following affirmative answer to Question 2.4.

Theorem 5.2. *Let Γ be a finitely generated Kleinian group without parabolics and with a connected limit set Λ . Then Λ is locally connected.*

Theorem 2.5 can be extended to punctured surfaces [Mj14a] and this allows Theorem 5.2 to be generalized to arbitrary finitely generated Kleinian groups.

5.2. Finitely generated Kleinian groups. In [Mj14b], we show that the point preimages of the Cannon-Thurston map for a simply or doubly degenerate surface Kleinian group given by Theorem 2.5 corresponds to end-points of leaves of ending laminations. In particular, the ending lamination corresponding to a degenerate end can be recovered from the Cannon-Thurston map. This was extended further in [DM16] and [Mj17b] to obtain the following general version for finitely generated Kleinian groups.

Theorem 5.3. [Mj17b] *Let G be a finitely generated Kleinian group. Let $i : \Gamma_G \rightarrow \mathbb{H}^3$ be the natural identification of a Cayley graph of G with the orbit of a point in \mathbb{H}^3 . Further suppose that each degenerate*

end of \mathbf{H}^3/G can be equipped with a Minsky model [Min10].² Then i extends continuously to a Cannon-Thurston map $\hat{i} : \widehat{\Gamma}_G \rightarrow \mathbb{D}^3$, where $\widehat{\Gamma}_G$ denotes the (relative) hyperbolic compactification of Γ_G .

Let ∂i denote the restriction of \hat{i} to the boundary $\partial\Gamma_G$ of Γ_G . Let E be a degenerate end of $N^h = \mathbb{H}^3/G$ and \tilde{E} a lift of E to \tilde{N}^h and let M_{gf} be an augmented Scott core of N^h . Then the ending lamination \mathcal{L}_E for the end E lifts to a lamination on $\widetilde{M}_{gf} \cap \tilde{E}$. Each such lift \mathcal{L} of the ending lamination of a degenerate end defines a relation $\mathcal{R}_{\mathcal{L}}$ on the (Gromov) boundary $\partial\widetilde{M}_{gf}$ (or equivalently, the relative hyperbolic boundary $\partial_r\Gamma_G$ of Γ_G), given by $a\mathcal{R}_{\mathcal{L}}b$ iff a, b are end-points of a leaf of \mathcal{L} . Let $\{\mathcal{R}_i\}$ be the entire collection of relations on $\partial\widetilde{M}_{gf}$ obtained this way. Let \mathcal{R} be the transitive closure of the union $\bigcup_i \mathcal{R}_i$. Then $\partial i(a) = \partial i(b)$ iff $a\mathcal{R}b$.

5.3. Primitive Stable Representations. In [Min13] Minsky introduced an open subset of the $PSL_2(\mathbb{C})$ character variety for a free group, properly containing the Schottky representations, on which the action of the outer automorphism group is properly discontinuous. He called these **primitive stable representations**. Let F_n be a free group of rank n . An element of F_n is primitive if it is an element of a free generating set. Let $\mathcal{P} = \dots wnw \dots$ be the set of bi-infinite words with w cyclically reduced primitive. A representation $\rho : F_n \rightarrow PSL_2(\mathbb{C})$ is primitive stable if all elements of \mathcal{P} are mapped to uniform quasigeodesics in \mathbb{H}^3 .

Minsky conjectured that primitive stable representations are characterized by the feature that every component of the ending lamination is blocking.

Using Theorem 5.3, Jeon and Kim [JK10], and Jeon, Kim, Ohshika and Lecuire [JKOL14] resolved this conjecture. We briefly sketch their argument for a degenerate free Kleinian group without parabolics.

Let $\{D_1, \dots, D_n\} = \mathcal{D}$ be a finite set of essential disks on a handlebody H cutting H into a 3-ball. A free generating set of F_n is dual to \mathcal{D} . For a lamination \mathcal{L} , the **Whitehead graph** $Wh(\Lambda, \mathcal{D})$ is defined as follows. Cut ∂H along $\partial\mathcal{D}$ to obtain a sphere with $2n$ holes, labeled by D_i^\pm . The vertices of $Wh(\mathcal{L}, \mathcal{D})$ are the boundary circles of ∂H , with an edge whenever two circles are joined by an arc of $\mathcal{L} \setminus \mathcal{D}$. For the ending lamination \mathcal{L}_E of a degenerate free group without parabolics, $Wh(\mathcal{L}_E, \mathcal{D})$ is connected and has no cutpoints.

Let ρ_E be the associated representation. If ρ is not primitive stable, then there exists a sequence of primitive cyclically reduced elements w_n such that $\rho(w_n^*)$ is not an n -quasi-geodesic. After passing to a subsequence, w_n and hence w_n^* converges to a bi-infinite geodesic w_∞ in the Cayley graph with two distinct end points w_+, w_- in the Gromov boundary of F_n . The Cannon-Thurston map identifies w_+, w_- . Hence by Theorem 5.3 they are either the end points of a leaf of \mathcal{L}_E or ideal end-points of a complementary ideal polygon of \mathcal{L}_E . It follows therefore that $Wh(w_\infty, \mathcal{D})$ is connected and has no cutpoints. Since w_n 's converge to w_∞ , $Wh(w_n, \mathcal{D})$ is connected and has no cutpoints for large enough n . A Lemma due to Whitehead says that if $Wh(w_n, \mathcal{D})$ is connected and has no cutpoints, then w_n cannot be primitive, a contradiction.

5.4. Discreteness of Commensurators. In [LLR11] and [Mj11b], Theorems 2.5 and 5.3 are used to prove that commensurators of finitely generated, infinite covolume, Zariski dense Kleinian groups are discrete. The basic fact that goes into the proof is that commensurators preserve the structure of point pre-images of Cannon-Thurston maps. The point pre-image structure is known from Theorem 5.3.

5.5. Radial and horospherical limit sets.

Definition 5.4. A point $\xi \in S^2$ is a radial or conical limit point of Kleinian group Γ , if for any base-point $o \in \tilde{M}$ and any geodesic γ_ξ ending at ξ , there exist $C \geq 0$ and infinitely many translates $g.o \in N_C(\gamma_\xi)$, $g \in \Gamma$.

A point $\xi \in S^2$ is a horospherical limit point of Kleinian group Γ , if for any base-point $o \in \tilde{M}$ and any horoball B_ξ based at ξ , there exist infinitely many translates $g.o \in B_\xi$, $g \in \Gamma$.

²This hypothesis is satisfied for all ends without parabolics as well as for ends incompressible away from cusps - see [Min10, BCM12] and [Mj17b, Appendix]. That the hypothesis is satisfied in general would follow from unpublished work of Bowditch [Bow16, Bow05].

The collection of radial (resp. horospherical) limit points of Γ is called the radial (resp. horospherical) limit set of Γ and is denoted by Λ_r (resp. Λ_h).

The multiple limit set Λ_m consists of those point of S^2 that have more than one pre-image under the Cannon-Thurston map.

Several authors [Kap95, Ger12, JKLO16] worked on the relationship between Λ_m and Λ_r . They concluded that the conical limit set is strictly contained in the set of injective points of the Cannon-Thurston map, i.e. $\Lambda_r \subset \Lambda_m^c$, but the inclusion is proper.

In [LM16], we showed:

Theorem 5.5. $\Lambda_m^c = \Lambda_h$.

5.6. Motions of limit sets. We discuss the following question in this section, which paraphrases the second part of [Thu82, Problem 14]. A detailed survey appears in [Mj17a].

Question 5.6. Let G_n be a sequence of Kleinian groups converging to a Kleinian group G . Does the corresponding dynamics of G_n on the Riemann sphere S^2 converge to the dynamics of G on S^2 ?

To make Question 5.6 precise, we need to make sense of ‘convergence’ both for Kleinian groups and for their dynamics on S^2 . There are three different notions of convergence for Kleinian groups.

Definition 5.7. Let $\rho_i : H \rightarrow PSL_2(\mathbb{C})$ be a sequence of Kleinian groups. We say that ρ_i converges to ρ_∞ **algebraically** if for all $h \in H$, $\rho_i(h) \rightarrow \rho_\infty(h)$.

Let $\rho_j : H \rightarrow PSL_2(\mathbb{C})$ be a sequence of discrete, faithful representations of a finitely generated, torsion-free, nonabelian group H . If $\{\rho_j(H)\}$ converges as a sequence of closed subsets of $PSL_2(\mathbb{C})$ to a torsion-free, nonabelian Kleinian group Γ , Γ is called the **geometric limit** of the sequence.

$G_i(= \rho_i(H))$ converges **strongly** to $G(= \rho_\infty(H))$ if the convergence is both geometric and algebraic.

Question 5.6 then splits into the following three questions.

- Question 5.8.** (1) If $G_n \rightarrow G$ geometrically, then do the corresponding limit sets converge in the Hausdorff topology on S^2 ?
- (2) If $G_n \rightarrow G$ strongly then do the corresponding Cannon-Thurston maps converge uniformly?
- (3) If $G_n \rightarrow G$ algebraically then do the corresponding Cannon-Thurston maps converge pointwise?

We give the answers straight off and then proceed to elaborate.

- Answers 5.9.** (1) The answer to Question 5.8 (1) is Yes.
- (2) The answer to Question 5.8 (2) is Yes.
- (3) The answer to Question 5.8 (3) is No, in general.

The most general answer to Question 5.8 (1) is due to Evans [Eva00], [Eva04]:

Theorem 5.10. [Eva00], [Eva04] Let $\rho_n : H \rightarrow G_n$ be a sequence of weakly type-preserving isomorphisms from a geometrically finite group H to Kleinian groups G_n with limit sets Λ_n , such that ρ_n converges algebraically to $\rho_\infty : H \rightarrow G_\infty^a$ and geometrically to G_∞^g . Let Λ_a and Λ_g denote the limit sets of G_∞^a and G_∞^g . Then $\Lambda_n \rightarrow \Lambda_g$ in the Hausdorff metric. Further, the sequence converges strongly if and only if $\Lambda_n \rightarrow \Lambda_a$ in the Hausdorff metric.

The answer to Question 5.8 (2) is due to the author and Series [MS17] in the case that $H = \pi_1(S)$ for a closed surface S of genus greater than one. This can be generalized to arbitrary finitely generated Kleinian groups as in [Mj17a]:

Theorem 5.11. Let H be a fixed group and $\rho_n(H) = \Gamma_n$ be a sequence of geometrically finite Kleinian groups converging strongly to a Kleinian group Γ . Let M_n and M_∞ be the corresponding hyperbolic manifolds. Let K be a fixed complex with fundamental group H .

Consider embeddings $\phi_n : K \rightarrow M_n, n = 1, \dots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies (in the geometric limit). Then Cannon-Thurston maps for $\widetilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.

Finally we turn to Question 5.8 (3), which turns out to be the subtlest. In [MS13] we showed that the answer to Question 5.8 (3) is ‘Yes’ if the geometric limit is geometrically finite. We illustrate this with a concrete example due to Kerckhoff and Thurston [KT90]

Theorem 5.12. *Fix a closed hyperbolic surface S and a simple closed geodesic σ on it. Let tw^i denote the automorphism of S given by an i -fold Dehn twist along σ . Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, tw^i(S))$. Let G_∞ denote the geometric limit of the G_i ’s. Let S_{i-} denote the lower boundary component of the convex core of G_i , $i = 1, \dots, \infty$ (including ∞). Let $\phi_i : S \rightarrow S_{i-}$ be such that if $0 \in \mathbb{H}^2 = \tilde{S}$ denotes the origin of \mathbb{H}^2 then $\tilde{\phi}_i(0)$ lies in a uniformly bounded neighborhood of $0 \in \mathbb{H}^3 = \tilde{M}_i$. We also assume (using the fact that M_∞ is a geometric limit of M_i ’s) that S_{i-} ’s converge geometrically to $S_{\infty-}$. Then the Cannon-Thurston maps for $\tilde{\phi}_i$ converge pointwise, but not uniformly, on $\partial\mathbb{H}^2$ to the Cannon-Thurston map for $\tilde{\phi}_\infty$.*

However, if the geometric limit is geometrically infinite, then the answer to Question 5.8 (3) may be negative. We illustrate this with certain examples of geometric limits constructed by Brock in [Bro01].

Theorem 5.13. [MS17] *Fix a closed hyperbolic surface S and a separating simple closed geodesic σ on it, cutting S up into two pieces S_- and S_+ . Let ϕ denote an automorphism of S such that $\phi|_{S_-}$ is the identity and $\phi|_{S_+} = \psi$ is a pseudo-Anosov of S_+ fixing the boundary. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, \phi^i(S))$. Let G_∞ denote the geometric limit of the G_i ’s. Let S_{i0} denote the lower boundary component of the convex core of G_i , $i = 1, \dots, \infty$ (including ∞). Let $\phi_i : S \rightarrow S_{i0}$ be such that if $0 \in \mathbb{H}^2 = \tilde{S}$ denotes the origin of \mathbb{H}^2 then $\tilde{\phi}_i(0)$ lies in a uniformly bounded neighborhood of $0 \in \mathbb{H}^3 = \tilde{M}_i$. We also assume (using the fact that M_∞ is a geometric limit of M_i ’s) that S_{i0} ’s converge geometrically to $S_{\infty 0}$.*

Let Σ be a complete hyperbolic structure on S_+ such that σ is homotopic to a cusp on Σ . Let \mathcal{L} consist of pairs (ξ_-, ξ) of ideal endpoints (on \mathbb{S}_∞^1) of stable leaves λ of the stable lamination of ψ acting on $\tilde{\Sigma}$. Also let $\partial\tilde{\mathcal{H}}$ denote the collection of ideal basepoints of horodisks given by lifts (contained in $\tilde{\Sigma}$) of the cusp in Σ corresponding to σ . Let

$$\Xi = \{\xi : \text{There exists } \xi_- \text{ such that } (\xi_-, \xi) \in \mathcal{L}; \xi_- \in \partial\tilde{\mathcal{H}}\}.$$

Let $\partial\phi_i$, $i = 1, \dots, \infty$ denote the Cannon-Thurston maps for $\tilde{\phi}_i$. Then

- (1) $\partial\phi_i(\xi)$ does not converge to $\partial\phi_\infty(\xi)$ for $\xi \in \Xi$.
- (2) $\partial\phi_i(\xi)$ converges to $\partial\phi_\infty(\xi)$ for $\xi \notin \Xi$.

In [MO17], we identify the exact criteria that lead to the discontinuity phenomenon of Theorem 5.13.

6. GROMOV-HYPERBOLIC GROUPS

6.1. Applications and Generalizations.

6.1.1. *Normal subgroups and trees.* The ladder construction of Section 3.2 has been generalized considerably. We work in the context of an exact sequence $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$, with N hyperbolic and G finitely presented. We observe that for the proof of Theorem 3.5 to go through it suffices to have a qi-section of Q into G to provide a ‘coarse transversal’ to flow. Such a qi-section was shown to exist by Mosher [Mos96]. We then obtain the following.

Theorem 6.1. [Mit98a] *Let G be a hyperbolic group and let H be a hyperbolic normal subgroup that is normal in G . Then the inclusion of Cayley graphs $i : \Gamma_H \rightarrow \Gamma_G$ gives a Cannon-Thurston map $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$.*

The ladder construction can also be generalized to the general framework of a tree of hyperbolic metric spaces.

Theorem 6.2. [Mit98b] *Let (X, d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition (i.e. edge space to vertex space inclusions are qi-embeddings). Let v be a vertex of T and*

(X_v, d_v) be the vertex space corresponding to v . If X is hyperbolic then the inclusion $i : X_v \rightarrow X$ gives a Cannon-Thurston map $\hat{i} : \widehat{X}_v \rightarrow \widehat{X}$.

Theorem 6.1 was generalized by the author and Sardar to a purely coarse geometric context, where no group action is present. The relevant notion is that of a metric bundle for which we refer the reader to [MS12]. Roughly speaking, the data of a metric bundle consists of vertex and edge spaces as in the case of a tree of spaces, with two notable changes:

- (1) The base T is replaced by an arbitrary graph B .
- (2) All edge-space to vertex space maps are quasi-isometries rather than just quasi-isometric embeddings.

With these modifications in place we have the following generalizations of Mosher's qi-section Lemma [Mos96] and Theorem 6.1:

Theorem 6.3. [MS12] *Suppose $p : X \rightarrow B$ is a metric graph bundle satisfying the following:*

- (1) B is a Gromov hyperbolic graph.
- (2) Each fiber F_b , for b a vertex of B is δ -hyperbolic (for some $\delta > 0$) with respect to the path metric induced from X .
- (3) The barycenter maps $\partial^3 F_b \rightarrow F_b$, $b \in B$, sending a triple of distinct points on the boundary ∂F_b to their centroid, are (uniformly, independent of b) coarsely surjective.
- (4) X is hyperbolic.

Then there is a qi-section $B \rightarrow X$. The inclusion map $i_b : F_b \rightarrow X$ gives a Cannon-Thurston map $\hat{i} : \widehat{F}_b \rightarrow \widehat{X}$.

6.2. Point pre-images: Laminations. In Section 3.1, it was pointed out that the Cannon-Thurston map \hat{i} identifies $p, q \in S^1$ if and only if p, q are end-points of a leaf or ideal end-points of a complementary ideal polygon of the stable or unstable lamination.

In [Mit97a] an algebraic theory of ending laminations was developed based on Thurston's theory [Thu80]. The theory was developed in the context of a normal hyperbolic subgroup of a hyperbolic group G and used to give an explicit structure for the Cannon-Thurston map in Theorem 6.1.

Definition 6.4. [BFH97, CHL07, CHL08a, CHL08b, KL10, KL15, Mit97a] *An algebraic lamination for a hyperbolic group H is an H -invariant, flip invariant, closed subset $\mathcal{L} \subseteq \partial^2 H = (\partial H \times \partial H \setminus \Delta) / \sim$, where $(x, y) \sim (y, x)$ denotes the flip and Δ the diagonal in $\partial H \times \partial H$.*

Let

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence of finitely presented groups with H, G hyperbolic. It follows by work of Mosher [Mos96] that Q is hyperbolic. In [Mit97a], we construct algebraic ending laminations naturally parametrized by points in the boundary ∂Q . We describe the construction now.

Every element $g \in G$ gives an automorphism of H sending h to $g^{-1}hg$ for all $h \in H$. Let $\phi_g : \mathcal{V}(\Gamma_H) \rightarrow \mathcal{V}(\Gamma_H)$ be the resulting bijection of the vertex set $\mathcal{V}(\Gamma_H)$ of Γ_H . This induces a map Φ_g sending an edge $[a, b] \subset \Gamma_H$ to a geodesic segment joining $\phi_g(a), \phi_g(b)$.

For some (any) $z \in \partial\Gamma_Q$ we shall describe an algebraic ending lamination Λ_z . Fix such a z and let

- (1) $[1, z] \subset \Gamma_Q$ be a geodesic ray starting at 1 and converging to $z \in \partial\Gamma_Q$.
- (2) $\sigma : Q \rightarrow G$ be a qi section.
- (3) z_n be the vertex on $[1, z)$ such that $d_Q(1, z_n) = n$.
- (4) $g_n = \sigma(z_n)$.

For $h \in H$, let S_n^h be the H -invariant collection of all free homotopy representatives (or equivalently, shortest representatives in the same conjugacy class) of $\phi_{g_n^{-1}}(h)$ in Γ_H . Identifying equivalent geodesics in S_n^h one obtains a subset \mathbb{S}_n^h of unordered pairs of points in $\widehat{\Gamma}_H$. The intersection with $\partial^2 H$ of the union of all subsequential limits (in the Hausdorff topology) of $\{\mathbb{S}_n^h\}$ is denoted by Λ_z^h .

Definition 6.5. *The set of algebraic ending laminations corresponding to $z \in \partial\Gamma_Q$ is given by*

$$\Lambda_{EL}(z) = \bigcup_{h \in H} \Lambda_z^h.$$

Definition 6.6. *The set Λ of all algebraic ending laminations is defined by*

$$\Lambda_{EL} = \bigcup_{z \in \partial\Gamma_Q} \Lambda_{EL}(z).$$

The following was shown in [Mit97a]:

Theorem 6.7. *The Cannon-Thurston map \hat{i} of Theorem 6.1 identifies the end-points of leaves of Λ_{EL} . Conversely, if $\hat{i}(p) = \hat{i}(q)$ for $p \neq q \in \partial\Gamma_H$, then some bi-infinite geodesic having p, q as its end-points is a leaf of Λ_{EL} .*

6.2.1. *Finite-to-one.* The classical Cannon-Thurston map of Theorem 2.2 is finite-to-one. Swarup asked (cf. Bestvina’s Geometric Group Theory problem list [Bes04, Prolem 1.20]) if the Cannon-Thurston maps of Theorem 6.1 are also finite-to-one. Kapovich and Lustig answered this in the affirmative in the following case.

Theorem 6.8. [KL15] *Let $\phi \in \text{Out}(F_N)$ be a fully irreducible hyperbolic automorphism. Let $G_\phi = F_N \rtimes_\phi \mathbb{Z}$ be the associated mapping torus group. Let ∂i denote the Cannon-Thurston map of Theorem 6.1 in this case. Then for every $z \in \partial G_\phi$, the cardinality of $(\partial i)^{-1}(z)$ is at most $2N$.*

Bestvina-Feighn-Handel [BFH97] define a closely related set Λ_{BFH} of algebraic laminations in the case covered by Theorem 6.8 using train-track representatives of free group automorphisms. Any algebraic lamination \mathcal{L} defines a relation $\mathcal{R}_\mathcal{L}$ on ∂F_N by $a\mathcal{R}_\mathcal{L}b$ if $(a, b) \in \mathcal{L}$. The transitive closure of \mathcal{L} will be called its diagonal closure. In [KL15], Kapovich and Lustig further show that in the case covered by Theorem 6.8, Λ_{EL} precisely equals the diagonal closure of Λ_{BFH} .

6.3. **Relative hyperbolicity.** The notion of a Cannon-Thurston map can be extended to the context of relative hyperbolicity. This was done in [MP11]. Let X and Y be relatively hyperbolic spaces, hyperbolic relative to the collections \mathcal{H}_X and \mathcal{H}_Y of ‘horosphere-like sets’ respectively. Let us denote the *horoballifications* of X and Y with respect to \mathcal{H}_X and \mathcal{H}_Y by $\mathcal{G}(X, \mathcal{H}_X), \mathcal{G}(Y, \mathcal{H}_Y)$ respectively (see [Bow12] for details). The horoballification of an H in \mathcal{H}_X or \mathcal{H}_Y is denoted as H^h . Note that $\mathcal{G}(X, \mathcal{H}_X), \mathcal{G}(Y, \mathcal{H}_Y)$ are hyperbolic. The electrifications will be denoted as $\mathcal{E}(X, \mathcal{H}_X), \mathcal{E}(Y, \mathcal{H}_Y)$.

Definition 6.9. *A map $i : Y \rightarrow X$ is strictly type-preserving if*

- (1) *for all $H_Y \in \mathcal{H}_Y$ there exists $H_X \in \mathcal{H}_X$ such that $i(H_Y) \subset H_X$, and*
- (2) *images of distinct horospheres-like sets in Y lie in distinct horosphere-like sets in X .*

Let $i : Y \rightarrow X$ be a strictly type-preserving proper embedding. Then i induces a proper embedding $i_h : \mathcal{G}(Y, \mathcal{H}_Y) \rightarrow \mathcal{G}(X, \mathcal{H}_X)$.

Definition 6.10. *A Cannon-Thurston map exists for a strictly type-preserving inclusion $i : Y \rightarrow X$ of relatively hyperbolic spaces if a Cannon-Thurston map exists for the induced map $i_h : \mathcal{G}(Y, \mathcal{H}_Y) \rightarrow \mathcal{G}(X, \mathcal{H}_X)$.*

Lemma 2.8 generalizes to the following.

Lemma 6.11. *A Cannon-Thurston map for $i : Y \rightarrow X$ exists if and only if there exists a non-negative proper function $M : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds:*

Fix a base-point $y_0 \in Y$. Let $\hat{\lambda}$ in $\mathcal{E}(Y, \mathcal{H}_Y)$ be an electric geodesic segment starting and ending outside horospheres. If $\lambda^b = \hat{\lambda} \setminus \bigcup_{K \in \mathcal{H}_Y} K$ lies outside $B_N(y_0) \subset Y$, then for any electric quasigeodesic $\hat{\beta}$ joining the end points of $i(\hat{\lambda})$ in $\mathcal{E}(X, \mathcal{H}_X)$, $\beta^b = \hat{\beta} \setminus \bigcup_{H \in \mathcal{H}_X} H$ lies outside $B_{M(N)}(i(y_0)) \subset X$.

Theorem 6.2 then generalizes to:

Theorem 6.12. [MP11] *Let $P : X \rightarrow T$ be a tree of relatively hyperbolic spaces satisfying the qi-embedded condition. Assume that*

- (1) *the inclusion maps of edge-spaces into vertex spaces are strictly type-preserving*
- (2) *the induced tree of electrified (coned-off) spaces continues to satisfy the qi-embedded condition*
- (3) *X is strongly hyperbolic relative to the family \mathcal{C} of maximal cone-subtrees of horosphere-like sets.*

Then a Cannon-Thurston map exists for the inclusion of relatively hyperbolic spaces $i: X_v \rightarrow X$, where (X_v, d_{X_v}) is the relatively hyperbolic vertex space corresponding to v .

6.4. Problems. The above survey is conditioned and limited by the author’s bias on the one hand and space considerations on the other. In particular we have omitted the important work on quasigeodesic foliations by Calegari, Fenley and Frankel [Cal06, Cal00, Fen12, Fra15] and the topology of ending lamination spaces by Gabai and others [Gab14, Gab09, LMS11, HP11] as this would be beyond the scope of the present article. A more detailed survey appears in [Mj17c]. We end with some open problems.

6.4.1. *Higher dimensional Kleinian groups.* As a test-case we propose:

Question 6.13. *Let S be a closed surface of genus greater than one and let $\Gamma = \pi_1(S)$ act freely, properly discontinuously by isometries on \mathbf{H}^n , $n > 3$ (or more generally a rank one symmetric space). Does a Cannon-Thurston map exist in general?*

Can the small cancellation group of Baker-Riley in [BR13] act geometrically on a rank one symmetric space thus giving a negative answer to Question 6.13 with surface group replaced by free group? Work of Wise [Wis12, Wis04] guarantees linearity of such small cancellation groups.

The critical problem in trying to answer Question 6.13 is the absence of new examples in higher dimensions. It would be good to find new examples or prove that they do not exist. A version of a question due to Kapovich [Kap08] makes this more precise and indicates our current state of knowledge/ignorance:

Question 6.14. *Let S be a closed surface of genus greater than one and let Γ be a discrete subgroup of $SO(n, 1)$ (or more generally a rank one Lie group) abstractly isomorphic to $\pi_1(S)$ acting by isometries on \mathbf{H}^n , $n > 3$ (more generally the associated symmetric space) such that*

- (1) *orbits are not quasiconvex,*
- (2) *no element of Γ is a parabolic.*

Does ρ factor through a representation to a simply or doubly degenerate (3-dimensional) Kleinian group followed by a deformation of $SO(3, 1)$ in $SO(n, 1)$?

A closely related folklore question asks:

Question 6.15. *Can a closed higher dimensional $n > 3$ rank one manifold fiber? In particular over the circle?*

It is known, from the Chern-Gauss-Bonnet theorem that a $2n$ dimensional rank one manifold cannot fiber over the circle. Unpublished work of Kapovich [Kap98] shows that a complex hyperbolic 4-manifold cannot fiber over a 2-manifold.

6.4.2. *Surface groups in higher rank.* A topic of considerable current interest is higher dimensional Teichmüller theory and Anosov representations of surface groups [Lab06, KLP17, GGKW17]. Kapovich, Leeb and Porti give an equivalent definition of Anosov representations in purely coarse geometric terms as representations that are **asymptotic embeddings**. It will take us too far afield to define these notions precisely here. What we will say however is that if $\rho: \pi_1(S) \rightarrow \mathcal{G}$ is a discrete faithful representation into a semi-simple Lie group \mathcal{G} and $\mathcal{G}/\mathcal{P} = \mathcal{B}$ is the Furstenberg boundary, then the Anosov property implies that an orbit $\rho(\pi_1(S)) \cdot o$ ‘extends’ to a $\rho(\pi_1(S))$ -equivariant embedding of $\partial\Gamma_{\pi_1(S)} (= S^1)$ into \mathcal{B} . Thus the boundary map $\Delta: S^1 \rightarrow \mathcal{B}$ maybe thought of as a *higher rank Cannon-Thurston map*. For the representation to be Anosov, Δ is thus an embedding [KLP17].

Question 6.16. *What class of representations do we get if we require only that Δ is continuous?*

Question 6.16 is basically asking for a ‘nice’ characterization of representations that admit a higher rank Cannon-Thurston map. The core problem in addressing it again boils down to finding some rich class of examples. Question 6.14 has a natural generalization to this context where we replace $SO(n, 1)$ by \mathcal{G} .

6.4.3. *Geometric group theory.* As we have seen in Section 6.1, normal hyperbolic subgroups and trees of spaces provide examples where there is a positive answer to Question 2.7. Some sporadic new examples have also been found, e.g. hydra groups [BR12]. However no systematic theory exists. In the light of the counterexample in [BR13], the general answer to Question 2.7 is negative. Are there necessary and/or sufficient conditions beyond Lemma 2.8 to guarantee existence of Cannon-Thurston maps?

As illustrated in [Mit98b] and [BR12], distortion of subgroups [Gro93] is irrelevant. Distortion captures the relationship between $d_H(1, h)$ with $d_G(1, h)$. On the other hand Cannon-Thurston maps capture the corresponding relationship between $d_H(1, [h_1, h_2]_H)$ with $d_G(1, [h_1, h_2]_G)$, i.e. existence of Cannon-Thurston maps is equivalent to a proper embedding of ‘pairs of points’ (coding geodesic segments). The function associated with such a proper embedding is closely related to the modulus of continuity of the Cannon-Thurston map [BR12].

ACKNOWLEDGMENTS

I would like to thank Benson Farb and Thomas Koberda for several helpful comments on an earlier draft.

REFERENCES

- [Abi76] W. Abikoff. Two theorems on totally degenerate Kleinian groups. *Amer. J. Math.* 98, pages 109–118, 1976.
- [ADP99] R. C. Alperin, W. Dicks, and J. Porti. The boundary of the Gieseking tree in hyperbolic three-space. *Topology Appl.*, 93, pages 219–259, (1999).
- [AM96] J. Anderson and B. Maskit. On the local connectivity of limit sets of Kleinian groups. *Complex Variables Theory Appl.* 31, pages 177–183, 1996.
- [BB04] J. Brock and K. Bromberg. Density of Geometrically Finite Kleinian Groups. *Acta Mathematica* 192, pages 33–93, 2004.
- [BCM12] J. F. Brock, R. D. Canary, and Y. N. Minsky. The Classification of Kleinian surface groups II: The Ending Lamination Conjecture. *Ann. of Math.* 176 (1), *arXiv:math/0412006*, pages 1–149, 2012.
- [Bes04] M. Bestvina. Geometric group theory problem list. *M. Bestvina’s home page*: <http://math.utah.edu/bestvina>, 2004.
- [BF92] M. Bestvina and M. Feighn. A Combination theorem for Negatively Curved Groups. *J. Diff. Geom.*, vol 35, pages 85–101, 1992.
- [BFH97] M. Bestvina, M. Feighn, and M. Handel. Laminations, trees and irreducible automorphisms of free groups. *Geom. Funct. Anal.* vol.7 No. 2, pages 215–244, 1997.
- [Bon86] F. Bonahon. Bouts de varietes hyperboliques de dimension 3. *Ann. Math.* vol.124, pages 71–158, 1986.
- [Bow05] B. H. Bowditch. End invariants of hyperbolic manifolds. *preprint, Southampton*, 2005.
- [Bow07] B. H. Bowditch. The Cannon-Thurston map for punctured surface groups. *Math. Z.* 255, pages 35–76, (2007).
- [Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra and Computation.* 22, 1250016, 66pp, 2012.
- [Bow13] B. H. Bowditch. Stacks of hyperbolic spaces and ends of 3 manifolds. *Geometry and topology down under, Contemp. Math.*, 597, *Amer. Math. Soc., Providence, RI.*, pages 65–138, 2013.
- [Bow16] B. H. Bowditch. Ending lamination theorem. *preprint, Warwick*, 2016.
- [BR12] O. Baker and T. Riley. Cannon-Thurston maps, subgroup distortion, and hyperbolic hydra. *preprint, arXiv:1209.0815*, 2012.
- [BR13] O. Baker and T. Riley. Cannon-Thurston maps do not always exist. *Forum of Mathematics, Sigma*, 1, e3, 2013.
- [Bro01] J. Brock. Iteration of mapping classes and limits of hyperbolic 3-manifolds. *Invent. Math.* 143 no 3, pages 523–570, 2001.
- [Cal00] D. Calegari. The geometry of \mathbf{R} -covered foliations. *Geom. Topol.* 4, pages 457–515, 2000.
- [Cal06] D. Calegari. Universal circles for quasigeodesic flows. *Geom. Topol.* 10, pages 2271–2298, 2006.
- [CHL07] T. Coulbois, A. Hilion, and M. Lustig. Non-unique ergodicity, observers’ topology and the dual algebraic lamination for R-trees. *Illinois J. Math.* 51(3), 2007.
- [CHL08a] T. Coulbois, A. Hilion, and M. Lustig. R-trees and laminations for free groups. I. algebraic laminations. *J. Lond. Math. Soc.* (2), 78(3), 2008.
- [CHL08b] T. Coulbois, A. Hilion, and M. Lustig. R-trees and laminations for free groups. II. the dual lamination of an R-tree. *J. Lond. Math. Soc.* (2), 78(3), pages 737–754, 2008.
- [CT85] J. Cannon and W. P. Thurston. Group Invariant Peano Curves. *preprint, Princeton*, 1985.
- [CT07] J. Cannon and W. P. Thurston. Group Invariant Peano Curves. *Geom. Topol.* 11, pages 1315–1355, 2007.
- [DKL14] S. Dowdall, R. P. Kent, and C. J. Leininger. Pseudo-anosov subgroups of fibered 3-manifold groups. *Groups Geom. Dyn.* 8, no. 4, pages 1247–1282, 2014.
- [DKT16] S. Dowdall, I. Kapovich, and S. J. Taylor. Cannon-Thurston maps for hyperbolic free group extensions. *Israel J. Math.* 216, no. 2, pages 753–797, (2016).
- [DM16] S. Das and M. Mj. Semiconjugacies Between Relatively Hyperbolic Boundaries. *Groups Geom. Dyn.* 10, pages 733–752, 2016.

- [DT17a] S. Dowdall and S. J. Taylor. The co-surface graph and the geometry of hyperbolic free group extensions. *J. Topol.* 10, no. 2, pages 447–482, (2017).
- [DT17b] S. Dowdall and S. J. Taylor. Hyperbolic extensions of free groups. *to appear in Geom. Topol.*, *arXiv:1406.2567*, 2017.
- [Eva00] R. Evans. Deformation spaces of hyperbolic 3-manifolds: strong convergence and tameness. *Ph.D. Thesis, University of Michigan*, 2000.
- [Eva04] R. Evans. Weakly Type-Preserving Sequences and Strong Convergence. *Geometriae Dedicata, Volume 108, Number 1*, pages 71–92, October, 2004.
- [Fan99] Carol E. Fan. Injectivity radius bounds in hyperbolic convex cores I. *preprint, arXiv:math.GT/9907058*, 1999.
- [Far98] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.* 8, pages 810–840, 1998.
- [Fen12] S. Fenley. Ideal boundaries of pseudo-Anosov flows and uniform convergence groups, with connections and applications to large scale geometry. *Geom. Topol.* 16, pages 1–110, 2012.
- [Flo80] W. J. Floyd. Group Completions and Limit Sets of Kleinian Groups. *Invent. Math. vol.57*, pages 205–218, 1980.
- [FLP79] A. Fathi, M. Laudenbach, and V. Poenaru. Travaux de Thurston sur les surfaces. *Asterisque 66-67*, pages 1–284, 1979.
- [Fra15] S. Frankel. Quasigeodesic flows and sphere-filling curves. *Geom. Topol.* 19, pages 1249–1262, 2015.
- [Gab09] D. Gabai. Almost filling laminations and the connectivity of ending lamination space. *Geom. Topol.* 13, no. 2, pages 1017–1041, (2009).
- [Gab14] D. Gabai. On the topology of ending lamination space. *Geom. Topol.* 18, no. 5, pages 2683–2745, 2014.
- [Ger12] V. Gerasimov. Floyd maps for relatively hyperbolic groups. *Geom. Funct. Anal.*, 22, no. 5, pages 1361–1399, 2012.
- [GGKW17] F. Gueritaud, O. Guichard, F. Kassell, and A. Wienhard. Anosov representations and proper actions. *Geom. Topol.* 21, pages 485–584, (2017).
- [Gro85] M. Gromov. Hyperbolic Groups. in *Essays in Group Theory, ed. Gersten, MSRI Publ., vol.8, Springer Verlag*, pages 75–263, 1985.
- [Gro93] M. Gromov. Asymptotic Invariants of Infinite Groups. in *Geometric Group Theory, vol.2; Lond. Math. Soc. Lecture Notes 182, Cambridge University Press*, 1993.
- [HP11] S. Hensel and P. Przytycki. The ending lamination space of the five-punctured sphere is the Nobeling curve. *J. Lond. Math. Soc. (2)* 84 no. 1, pages 103–119, 2011.
- [JK10] W. Jeon and I. Kim. Primitive stable representations of geometrically infinite handlebody hyperbolic 3-manifolds. *C. R. Math. Acad. Sci. Paris* 348, no. 15-16, 2010.
- [JKLO16] W. Jeon, I. Kapovich, C. Leininger, and K. Ohshika. Conical limit points and the Cannon-Thurston map. *Conform. Geom. Dyn.* 20, pages 58–80, 2016.
- [JKOL14] W. Jeon, I. Kim, K. Ohshika, and C. Lecuire. Primitive stable representations of free Kleinian groups. *Israel J. Math.* 199, no. 2, pages 841–866, 2014.
- [Kab16] Y. Kabaya. Deformation of quasi-Fuchsian groups. <https://www.math.kyoto-u.ac.jp/~kabaya/galleries/quasi-fuchsian-deformation/quasi-fuchsian-deformation.html>, 2016.
- [Kap95] M. Kapovich. On absence of Ahlfors’ finiteness theorem for Kleinian groups in dimension 3. *Algebra and analysis (Irkutsk, 1989)*, (L A Bokut, M Hazewinkel, Y G Reshetnyak, editors), *Amer. Math. Soc. Transl. Ser. 2* 163, 1995.
- [Kap98] M. Kapovich. On the normal subgroups in the fundamental groups of complex surfaces. *Preprint*, 1998.
- [Kap01] M. Kapovich. Hyperbolic manifolds and discrete groups. *Progress in Mathematics, 183. Birkhuser Boston, Inc., Boston, MA. xxvi+467 pp.*, 2001.
- [Kap08] M. Kapovich. Kleinian groups in higher dimensions. geometry and dynamics of groups and spaces. *Progr. Math.*, 265, *Birkhuser, Basel*, pages 487–564, 2008.
- [KL10] I. Kapovich and M. Lustig. Intersection form, laminations and currents on free groups. *Geom. Funct. Anal.* 19, no. 5, pages 1426–1467, 2010.
- [KL15] I. Kapovich and M. Lustig. Cannon-Thurston fibers for iwip automorphisms of F_N . *J. Lond. Math. Soc. (2)* 91, no. 1, pages 203–224, 2015.
- [Kla99] E. Klarreich. Semiconjugacies between Kleinian group actions on the Riemann sphere. *Amer. J. Math* 121, pages 1031–1078, 1999.
- [KLP17] M. Kapovich, B. Leeb, and J. Porti. Anosov subgroups: Dynamical and geometric characterizations. *preprint arXiv:1703.01647*, (2017).
- [KT90] S. Kerckhoff and W. Thurston. Non-continuity of the action of the modular group at the Bers’ boundary of Teichmuller Space. *Invent. Math.* 100, pages 25–48, 1990.
- [Lab06] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math. Vol. 165, no. 1*, pages 51–114, (2006).
- [LLR11] C. J. Leininger, D. D. Long, and A. W. Reid. Commensurators of non-free finitely generated Kleinian groups. *Algebraic and Geometric Topology* 11, pages 605–624, 2011.
- [LM16] C. Lecuire and M. Mj. Horospheres in degenerate 3-manifolds. *preprint, arXiv:1606.07185, to appear in Int. Math. Res. Not. IMRN*, 2016.
- [LMS11] C. Leininger, M. Mj, and S. Schleimer. The universal cannon-thurston map and the boundary of the curve complex. *Comment. Math. Helv.* 86 no. 4, pages 769–816, 2011.
- [McM01] C. T. McMullen. Local connectivity, Kleinian groups and geodesics on the blow-up of the torus. *Invent. math.*, 97:95–127, 2001.

- [Min92] Y. N. Minsky. Teichmüller Geodesics and Ends of 3-Manifolds. *Topology*, pages 1–25, 1992.
- [Min94] Y. N. Minsky. On Rigidity, Limit Sets, and End Invariants of Hyperbolic 3-Manifolds. *J. Amer. Math. Soc.*, vol. 7, pages 539–588, 1994.
- [Min01] Y. N. Minsky. Bounded geometry in Kleinian groups. *Invent. Math.* 146, pages 143–192, 2001.
- [Min10] Y. N. Minsky. The Classification of Kleinian surface groups I: Models and Bounds. *Ann. of Math.* 171(1), *math.GT/0302208*, pages 1–107, 2010.
- [Min13] Y. N. Minsky. On dynamics of $Out(F_n)$ on $PSL_2(C)$ characters. *Israel J. Math.* 193(1), pages 47–70, 2013.
- [Mit97a] M. Mitra. Ending Laminations for Hyperbolic Group Extensions. *Geom. Funct. Anal.* vol. 7 No. 2, pages 379–402, 1997.
- [Mit97b] M. Mitra. Maps on Boundaries of Hyperbolic Metric Spaces, PhD Thesis. *UC Berkeley*, 1997.
- [Mit98a] M. Mitra. Cannon-Thurston Maps for Hyperbolic Group Extensions. *Topology* 37, pages 527–538, 1998.
- [Mit98b] M. Mitra. Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces. *Jour. Diff. Geom.* 48, pages 135–164, 1998.
- [Mit98c] M. Mitra. Coarse extrinsic geometry: a survey. *Geometry and Topology monographs, The Epstein birthday Schrift, paper no. 17*, pages 341–364, 1998.
- [Miy02] H. Miyachi. Semiconjugacies between actions of topologically tame Kleinian groups. *preprint*, 2002.
- [Mj10] M. J. Cannon-Thurston Maps and Bounded Geometry. *Teichmüller theory and moduli problem, Ramanujan Math. Soc. Lect. Notes Ser., 10, Ramanujan Math. Soc., Mysore, arXiv:math.GT/0603729*, pages 489–511, 2010.
- [Mj11a] M. J. Cannon-Thurston Maps, i-bounded Geometry and a Theorem of McMullen. *Actes du séminaire Théorie spectrale et géométrie, Grenoble, vol 28, 2009-10, arXiv:math.GT/0511104*, pages 63–108, 2011.
- [Mj11b] M. J. On Discreteness of Commensurators. *Geom. Topol.* 15, pages 331–350, 2011.
- [Mj14a] M. J. Cannon-Thurston Maps for Surface Groups. *Ann. of Math.*, 179(1), pages 1–80, 2014.
- [Mj14b] M. J. Ending Laminations and Cannon-Thurston Maps, with an appendix by S. Das and M. J. *Geom. Funct. Anal.* 24, pages 297–321, 2014.
- [Mj16] M. J. Cannon-Thurston Maps for Surface Groups: An Exposition of Amalgamation Geometry and Split Geometry. *Geometry, Topology, and Dynamics in Negative Curvature, London Mathematical Society Lecture Note Series volume 425, arXiv:math.GT/0512539*, pages 221–271, 2016.
- [Mj17a] M. J. Motions of limit sets: A survey. *Proceedings of Workshop on Grothendieck-Teichmüller theories, Chern Institute, Tianjin, July 2016*, 2017.
- [Mj17b] M. J. Cannon-Thurston Maps for Kleinian Groups. *Forum Math. Pi* 5, e1, 49 pp., 2017.
- [Mj17c] M. J. Cannon-Thurston maps in Kleinian groups and geometric group theory. *submitted to Handbook on Teichmüller Theory, ed. A. Papadopoulos*, 2017.
- [MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hierarchical structure. *Geom. Funct. Anal.* 10, pages 902–974, 2000.
- [MO14] Y. Matsuda and S. Oguni. On Cannon-Thurston maps for relatively hyperbolic groups. *Journal of Group Theory* 17(1), pages 41–47, 2014.
- [MO17] M. J. and K. Ohshika. Discontinuous Motions of Limit Sets. *preprint, arxiv:1704.00269*, 2017.
- [Mos96] L. Mosher. Hyperbolic Extensions of Groups. *J. of Pure and Applied Algebra* vol. 110 No. 3, pages 305–314, 1996.
- [MP11] M. J. and A. Pal. Relative Hyperbolicity, Trees of Spaces and Cannon-Thurston Maps. *Geom. Dedicata* 151, pages 59–78, 2011.
- [MR15] M. J. and K. Rafi. Algebraic Ending Laminations and Quasiconvexity. *preprint, arXiv:1506.08036*, 2015.
- [MS12] M. J. and P. Sardar. A combination theorem for metric bundles. *Geom. Funct. Anal.* 22, no. 6, pages 1636–1707, 2012.
- [MS13] M. J. and C. Series. Limits of Limit Sets I. *Geom. Dedicata* 167, pages 35–67, 2013.
- [MS17] M. J. and C. Series. Limits of Limit Sets II: geometrically infinite groups. *Geom. Topol.* 21, no. 2, pages 647–692, 2017.
- [Ota01] J. P. Otal. The hyperbolization theorem for fibered 3-manifolds. *Translated from the 1996 French original by Leslie D. Kay. SMF/AMS Texts and Monographs, 7. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris. xiv+126 pp.*, 2001.
- [Sco78] P. Scott. Subgroups of surface groups are almost geometric. *Journal L.M.S.* 17, pages 555–65, 1978.
- [SS90] P. Scott and G. Swarup. Geometric Finiteness of Certain Kleinian Groups. *Proc. Amer. Math. Soc.* 109, pages 765–768, 1990.
- [Thu80] W. P. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes*, 1980.
- [Thu82] W. P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.*, pages 357–382, 1982.
- [Thu86] W. P. Thurston. Hyperbolic Structures on 3-Manifolds, II: Surface Groups and 3-Manifolds which Fiber over the Circle. *preprint, arXiv:math.GT/9801045*, 1986.
- [Wis04] D. T. Wise. Cubulating small cancellation groups. *Geom. Funct. Anal.* 14, no. 1, pages 150–214, (2004).
- [Wis12] D. T. Wise. The structure of groups with a quasi-convex hierarchy. *preprint*, page 189, (2012).

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH. 1, HOMI BHABHA ROAD, MUMBAI-400005,
INDIA

E-mail address: `mahan@math.tifr.res.in`