RELATIVE HYPERBOLICITY, TREES OF SPACES AND CANNON-THURSTON MAPS

MAHAN MJ. AND ABHIJIT PAL

Abstract. We prove the existence of continuous boundary extensions (Cannon-Thurston maps) for the inclusion of a vertex space into a tree of (strongly) relatively hyperbolic spaces. The result follows for finite graphs of (strongly) relatively hyperbolic groups. This generalises a result of Bowditch for punctured surfaces in 3 manifolds.

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Contents

1. Introduction

For a closed hyperbolic 3-manifold M, fibering over the circle with fiber F, let $i : \widetilde{F} \to \widetilde{M}$ denote the inclusion of universal covers. In [5] (now published as [6]) Cannon and Thurston show that i extends to a continuous map $\hat{i} : \mathbb{D}^2 \to \mathbb{D}^3$ where $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ denote the standard compactifications. In [10], Minsky generalised Cannon and Thurston's result to bounded geometry surface Kleinian groups *without* parabolics

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In [12], one of us extended Cannon-Thurston's and Minsky's result to trees of hyperbolic metric spaces satisfying certain conditions. In the process, an alternate proof of Cannon-Thurston's original result was found.

Bowditch [3] [4] made use of some of the ideas of [12] amongst other things and proved the Cannon-Thurston property for bounded geometry surface Kleinian groups with parabolics. (It is worth remarking parenthetically that this generalisation from the case without punctures to that with punctures required essentially new ideas and a fair bit of time.)

In [13] one of us gave a different proof of Bowditch's result.

The appropriate framework for synthesizing and generalizing the above results is that of trees of (strong) relatively hyperbolic metric spaces. A combination theorem was described by Mj-Reeves in [14]. The notion of partial electrocution introduced there will be used essentially here. Relatively hyperbolic spaces (Gromov [9], Farb [7], Bowditch [2], etc.) generalize (fundamental groups of) finite volume manifolds of pinched negative curvature. We shall implicitly use the fact, due to Bowditch [2], that the (strong) relative hyperbolic boundary of a space is well-defined. Unless otherwise mentioned, relative hyperbolicity will mean strong relative hyperbolicity. Our main Theorem is:

Theorem 2.7: Let X be a tree (T) of relatively hyperbolic spaces satisfying the quasi-isometrically (qi) embedded condition. Further suppose that inclusion of edge-spaces into vertex spaces is strictly type-preserving, and that the induced tree of coned-off spaces continue to satisfy the qi-embedded condition. If X is *strongly hyperbolic* relative to the family $\mathcal{C}$ of maximal cone-subtrees of horosphere-like sets, then a Cannon-Thurston map exists for the proper embedding $i\colon X_v \to X$, where v is a vertex of T and (X_v, d_v) is the relatively hyperbolic metric space corresponding to v.

We assume that the reader is familiar with the basic notions about hyperbolic metric spaces in the sense of Gromov [9]. (See [8] for instance.) For a hyperbolic metric space X, the Gromov compactification will be denoted by $\overline{X}$.

1.1. **Relative Hyperbolicity.** Let X be a path metric space. A collection of closed subsets $\mathcal{H} = \{H_\alpha\}$ of X will be said to be **uniformly separated** if there exists $\epsilon > 0$ such that $d(H_1, H_2) \geq \epsilon$ for all distinct $H_1, H_2 \in \mathcal{H}$.

Definition 1.1. *(Farb [7]) The* **electric space** *(or coned-off space)* $\widehat{X}$ *corresponding to the pair* $(X, \mathcal{H})$ *is a metric space which consists of* X *and a collection of vertices* v_α *(one for each* $H_\alpha \in \mathcal{H}$*) such that each point of* H_α *is joined to (coned off at)* v_α *by an edge of length* $\frac{1}{2}$*. The sets* H_α *shall be referred to as* **horosphere-like sets**.
X *is said to be* **weakly hyperbolic** *relative to the collection* $\mathcal{H}$ *if* $\hat{X}$ *is a hyperbolic metric space.*

Definition 1.2. • *A path* $\gamma : I \to Y$ *in a path metric space* Y *is an* **ambient K-quasigeodesic** *if we have*

$$L(\beta) \leq KL(A) + K$$

for any subsegment $\beta = \gamma|[a, b]$ *and any rectifiable path* $A : [a, b] \to Y$ *with the same endpoints.*
• γ *is said to be an* **electric** K, ϵ**-quasigeodesic in (the electric space)** $\widehat{X}$ **without backtracking** *if* γ *is an electric* K*-quasigeodesic in* $\widehat{X}$ *and* γ *does not return to the neighborhood* H_α *of any* horosphere-like set H_α *after leaving it.*

For a path $\gamma \subset X$, there is an induced path $\hat{\gamma}$ in $\hat{X}$ obtained by identifying γ with $\hat{\gamma}$ as sets. If $\hat{\gamma}$ is an electric geodesic (resp. P-quasigeodesic), γ is called a *relative geodesic* (resp. *relative* P*-quasigeodesic*). We shall usually be concerned with the case that γ is an ambient geodesic/quasigeodesic without backtracking.

Definition 1.3. *Relative geodesics (resp.* P*-quasigeodesics) in* $(X, \mathcal{H})$ *are said to satisfy* **bounded region penetration** *if for any two relative geodesics (resp.* P*-quasigeodesics without backtracking)* β*,* γ*, joining* $x, y \in X$ *there exists* $D = D(P)$ *such that*
Similar Intersection Patterns 1: *if precisely one of* $\{\beta, \gamma\}$ *meets a horosphere-like set* H_α*, then the length (measured in the intrinsic path-metric on* H_α *) from the first (entry) point to the last (exit) point (of the relevant path) is at most* D*.*
Similar Intersection Patterns 2: *if both* $\{\beta, \gamma\}$ *meet some* H_α *then the length (measured in the intrinsic path-metric on* H_α *) from the entry point of* β *to that of* γ *is at most* D*; similarly for exit points.*

Paths which enjoy the above properties shall be said to have similar intersection patterns with horospheres.

Definition 1.4. *(Farb [7])* X *is said to be hyperbolic relative to the uniformly separated collection* $\mathcal{H}$ *if*
1) X *is weakly hyperbolic relative to* $\mathcal{H}$
2) Relative P *quasigeodesics without backtracking satisfy the bounded*

penetration property

Gromov's definition of relative hyperbolicity :

Definition 1.5. *For any geodesic metric space (H, d), the* hyperbolic cone *(analog of a horoball) H_h is the metric space $H \times [0, \infty) = H_h$ equipped with the path metric d_h obtained from two pieces of data*
1) $d_{h,t}((x, t), (y, t)) = 2^{-t} d_H(x, y)$, where $d_{h,t}$ is the induced path metric on $H_t = H \times \{t\}$. Paths joining $(x, t), (y, t)$ and lying on $H_t = H \times \{t\}$ are called horizontal paths.
2) $d_h((x, t), (x, s)) = |t - s|$ for all $x \in H$ and for all $t, s \in [0, \infty)$, and the corresponding paths are called vertical paths*.*
3) for all $x, y \in H_h$, $d_h(x, y)$ is the path metric induced by the collection of horizontal and vertical paths.

Definition 1.6. *Let X be a geodesic metric space and $\mathcal{H}$ be a collection of mutually disjoint uniformly separated subsets of X. X is said to be hyperbolic relative to $\mathcal{H}$ in the sense of Gromov, if the quotient space X_h, obtained by attaching the hyperbolic cones H_h to $H \in \mathcal{H}$ via the identification $(h, 0)$ with h for all $H \in \mathcal{H}$, is a complete hyperbolic metric space.*

Here $H \in \mathcal{H}$ are thought of as horosphere-like sets and $H \times [0, \infty)$ as horoballs. For a strong relatively hyperbolic metric space $(X, \mathcal{H})$, the space $\hat{X}_h$ obtained by coning off $H \times [0, \infty)$ for all $H \in \mathcal{H}$ is the same as $\hat{X}$.

Theorem 1.7. *(Bowditch [2]) The following are equivalent:*
1) X is hyperbolic relative to the collection $\mathcal{H}$ of uniformly separated subsets of X
2) X is hyperbolic relative to the collection $\mathcal{H}$ of uniformly separated subsets of X in the sense of Gromov
3) X_h is hyperbolic relative to the collection $\mathcal{H}_\langle$

We shall identify $\widehat{X}$ with the space obtained from X_h by coning off all the H_h's.

We collect together certain facts about the electric metric that Farb proves in [7]. These are proved in the context of Hadamard manifolds of pinched negative curvature, but the proofs go through in our context. $N_R(Z)$ will denote the R-neighborhood about the subset Z in the hyperbolic metric. $N^e_R(Z)$ will denote the R-neighborhood about the subset Z in the electric metric.

Lemma 1.8. *(See Lemma 4.5 and Proposition 4.6 of* [7]*)*

1. *Electric quasi-geodesics electrically track hyperbolic geodesics: Given $P > 0$, there exists $K > 0$ with the following property: Let β be any electric P-quasigeodesic without backtracking from x to y in $\widehat{X}$, and let γ be the hyperbolic geodesic from x to y in H_h. Then $\beta \subset N^e_K(\gamma)$.*

2. *Hyperbolicity: There exists K such that each H_h is uniformly quasiconvex in X_h.*
3. *electric geodesics in $\widehat{X}$, relative geodesics in X (resp. hyperbolic geodesics in X_h) joining the same pair of points in X have similar intersection patterns with H (resp. H_h), i.e. they track each other off horosphere-like (resp. horoball-like) sets.*

1.2. **Partial Electrocution.** This subsection is taken entirely from [14].

Definition 1.9. *Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple such that the following holds:*

1. *X is hyperbolic relative to a collection of subsets H_α.*
2. *For each H_α there is a uniform large-scale retraction $g_alpha : H_\alpha \to L_\alpha$ to some (uniformly) δ-hyperbolic metric space L_α, i.e. there exist $\delta, K, \epsilon > 0$ such that for all H_α there exists a δ-hyperbolic L_α and a map $g_\alpha : H_\alpha \to L_\alpha$ with $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \leq K d_{H_\alpha}(x, y) + \epsilon$ for all $x, y \in H_\alpha$. Further, we denote the collection of such g_α's as $\mathcal{G}$.*

The **partially electrocuted space** *or* partially coned off space *corresponding to $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ is obtained from X by gluing in the (metric) mapping cylinders for the maps $g_\alpha : H_\alpha \to L_\alpha$.*

In Farb's construction L_α is just a single point. The metric, geodesics and quasigeodesics in the partially electrocuted space will be referred to as the partially electrocuted metric d_{pel}, and partially electrocuted geodesics and quasigeodesics respectively. In this situation, we conclude as in Lemmma 1.8:

Lemma 1.10. *(X, d_{pel}) is a hyperbolic metric space and the sets L_α are uniformly quasiconvex.*

Note that (X, d_{pel}) is hyperbolic relative to the sets $\{L_\alpha\}$. In fact the space obtained by electrocuting the sets L_α in (X, d_{pel}) is just the space (X, d_e) obtained by (completely) electrocuting the sets $\{H_\alpha\}$ in X.

Lemma 1.11. *Given $K, \epsilon \geq 0$, there exists $C > 0$ such that the following holds:*
Let γ_{pel} and γ denote respectively a (K, ϵ) partially electrocuted quasigeodesic in (X, d_{pel}) and a hyperbolic (K, ϵ)-quasigeodesic in (Y, d) joining a, b. Then $\gamma \cap X$ lies in a (hyperbolic) C-neighborhood of (any representative of) γ_{pel}. Further, outside of a C-neighborhood of the horoballs that γ meets, γ and γ_{pel} track each other.

Definition 1.12. *We start with an electric quasi-geodesic $\hat{\lambda}$ in the electric space $\hat{X}$ without backtracking. For any H penetrated by $\hat{\lambda}$, let x_H and y_H be the first entry point and the last exit point of $\hat{\lambda}$. We join x_H and y_H by a hyperbolic geodesic segment in H_h (identifying $\widehat{X}$ with the space obtained from X_h by coning off the H_h's. This results in a path λ_h in X_h. The path λ will be called an* **electro-ambient quasigeodesic**.

Lemma 1.13. *: An electro-ambient quasigeodesic is a quasigeodesic in the hyperbolic space X_h.*

1.3. **Trees of Spaces.**

Definition 1.14. *(Bestvina-Feighn [1]) A tree (T) of hyperbolic (resp. relatively hyperbolic) metric spaces satisfying the q(uasi) i(sometrically) embedded condition is a metric space (X, d) admitting a map $P : X \to T$ onto a simplicial tree T, such that there exist δ,ϵ and $K > 0$ satisfying the following:*
1) For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ with the induced path metric d_v is a δ-hyperbolic metric space (resp. a geodesic metric space X_v strongly hyperbolic relative to a collection $\mathcal{H}_{v\alpha}$). Further, the inclusions $i_v : X_v \to X$ are uniformly proper, i.e. for all $M > 0$, $v \in T$ and $x, y \in X_v$, there exists $N > 0$ such that $d(i_v(x), i_v(y)) \leq M$ implies $d_v(x, y) \leq N$.
2) Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point of e. Then X_e with the induced path metric is δ-hyperbolic (resp. a geodesic metric space X_e strongly hyperbolic relative to a collection $\mathcal{H}_{e\alpha}$).
3) There exist maps $f_e : X_e \times [0, 1] \to X$, such that $f_e|_{X_e\times(0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric.
4) $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ are (K, ϵ)-quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ will occassionally be

referred to as f_{v_1} *and* f_{v_2} *respectively.*
5) For a tree of relatively hyperbolic spaces, we demand in addition, that the maps f_{v_i} *above* $(i = 1, 2)$ *are* **strictly type-preserving**, *i.e.* $f_{v_i}^{-1}(H_{v_i\alpha})$, $i = 1, 2$ *(for any* $H_{v_i\alpha} \in \mathcal{H}_{v_i\alpha}$*) is either empty or some* $H_{e\alpha} \in \mathcal{H}_{e\alpha}$.
6) For a tree of strongly relatively hyperbolic spaces, we demand that the coned off spaces are uniformly δ*-hyperbolic.*

d_v *and* d_e *will denote path metrics on* X_v *and* X_e *respectively.* i_v, i_e *will denote inclusion of* X_v, X_e *respectively into* X.

For a tree of relatively hyperbolic spaces, the sets $H_{v\alpha}$ *and* $H_{e\alpha}$ *will be referred to as* **horosphere-like vertex sets and edge sets** *respectively.*

When (X, d) is a tree (T) of relatively hyperbolic metric spaces, the *strictly type-preseving condition* (Condition 5 above) ensures that we obtain an induced tree (T) (the same tree T) of *coned-off, or electric spaces*. We demand further that
• **qi-preserving electrocution condition** the induced maps of the coned-off edge spaces into the coned-off vertex spaces $\hat{f}_{v_i} : \widehat{X_e} \to \widehat{X_{v_i}}$ $(i = 1, 2)$ are uniform quasi-isometries.

The resulting tree of coned-off spaces will be called the **induced tree of coned-off spaces**. The resulting space will be denoted as $\widehat{X}$.
Definition: A finite graph of relatively hyperbolic groups is said to satisfy Condition C, if the aasociated tree of relatively hyperbolic Cayley graphs satisfies Condition C. Here C will be one of the following:

(1) the qi-embedded condition
(2) the strictly type-preserving condition
(3) the qi-preserving electrocution condition

Remark: Strictly speaking, the induced tree exists for any collection of vertex and edge spaces satisfying the *strictly type-preserving condition*. Hyperbolicity is not essential for the existence of the induced tree of spaces.

The **cone locus** of $\widehat{X}$, (the induced tree of coned-off spaces), is the graph (in fact a forest) whose vertex set $\mathcal{V}$ consists of the cone-points in the vertex set snd whose edge-set $\mathcal{E}$ consists of the cone-points in the edge set. The incidence relations are dictated by the incidence relations in T.

Note that connected components of the cone-locus can be naturally identified with sub-trees of T. Each such connected component of the cone-locus will be called a **maximal cone-subtree**. The collection of *maximal cone-subtrees* will be denoted by $\mathcal{T}$ and elements of $\mathcal{T}$ will be denoted as T_α. Further, each maximal cone-subtree T_α naturally gives rise to a tree T_α of horosphere-like subsets depending on which cone-points arise as vertices and edges of T_α. The metric space that T_α gives rise to will be denoted as C_α and will be referred to as a **maximal cone-subtree of horosphere-like spaces**. The collection of C_α's will be denoted as $\mathcal{C}$.

Note: Each T_α thus appears in two guises:

(1) as a subset of $\widehat{X}$
(2) as the underlying tree of C_α

We shall have need for both these interpretations.

In [14] conditions on trees of relatively hyperbolic metric spaces were found ensuring the hyperbolicity of X relative to the collection $\mathcal{C}$.

Remarks:

1) If X is a tree (T) of relatively hyperbolic spaces and $\mathcal{C}$ be the collection of maximal cone subtree of horosphere like spaces C_α, then the tree (T) of coned-off spaces $\hat{X}$, can be thought of as obtained from X by partial electrocuting each C_α to the cone subtree T_α.

2) If X is hyperbolic relative to the collection $\mathcal{C}$, then from lemma 1.10, $(\hat{X}, d_{\hat{X}})$ is a hyperbolic metric space. Thus we can treat the tree (T) of coned spaces as a partially electrocuted space $(X, \mathcal{C}, \mathcal{G}, \mathcal{T})$, where $\mathcal{G}$ is the collection of maps $g_\alpha\colon C_\alpha \to T_\alpha$ collapsing C_α, the tree of horosphere-like spaces to the underlying tree T_α.

1.4. Preliminaries on Cannon-Thurston Maps.

Definition 1.15. *Let X and Y be hyperbolic metric spaces and $i : Y \to X$ be an embedding. A* **Cannon-Thurston map** *$\bar{i}$ from $\overline{Y}$ to $\overline{X}$ is a continuous extension of i to the Gromov compactifications $\overline{X}$ and $\overline{Y}$.*

The following lemma, given in [11], gives a necessary and sufficient condition for the existence of Cannon-Thurston maps.

Lemma 1.16. [11] *A Cannon-Thurston map $\bar{i}$ from $\overline{Y}$ to $\overline{X}$ exists for the proper embedding $i\colon Y \to X$ if and only if there exists a non-negative function $M(N)$ with $M(N) \to \infty$ as $N \to \infty$ such that the following holds:*
Given $y_0 \in Y$, for all geodesic segments λ in Y lying outside an N-ball around $y_0 \in Y$ any geodesic segment in X joining the end points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.

Note that due to stability of quasigeodesics, the above statement is also true if geodesics are replaced by quasigeodesics.

Let X and Y be hyperbolic relative to the collections $\mathcal{H}_{\mathcal{X}}$ and $\mathcal{H}_{\mathcal{Y}}$ respectively. Let $i\colon Y \to X$ be a strictly type-preserving proper embedding, i.e.for $H_Y \in \mathcal{H}_{\mathcal{Y}}$ there exists $H_X \in \mathcal{H}_{\mathcal{X}}$ such that $i(H_Y) \subset H_X$ and images of distinct horospheres-like sets in Y lie in distinct horosphere-like sets in X. It follows easily from the fact that the inclusion of H into H_h is uniformly proper for all $H \in \mathcal{H}_X$ or $\mathcal{H}_Y$ that the proper embedding $i\colon Y \to X$ induces a proper embedding $i_h\colon Y_h \to X_h$.

Definition 1.17. *A Cannon-Thurston map is said to exist for the pair X, Y of relatively hyperbolic metric spaces and a strictly type-preserving inclusion $i : Y \to X$ if a Cannon-Thurston map exists for the induced map $i_h\colon Y_h \to X_h$ between the respective hyperbolic cones.*

We now give a criterion for the existence of Cannon-Thurston maps for relatively hyperbolic spaces. The argument we give here is heuristic. By Lemma 1.16, we note that a Cannon-Thurston map exists for the pair (X_h, Y_h) if and only if for any geodesic $\lambda_h \subset Y_h$ lying outside a large ball around $y_0 \in Y_h$, a geodesic $\beta_h \subset X_h$ joining the end-points of $i(\lambda_h)$ also lies outside a large ball about $i(y_0) = x_0$ in X_h. Now, from Lemma 1.8, it follows that $\lambda_h \subset Y_h$ (resp. $\beta_h \subset X_h$) lies outside a large ball about y_0 (resp. x_0) if and only if $\lambda^b = \hat{\lambda} \setminus \mathcal{H}_{\mathcal{Y}}$ (resp. $\beta^b = \hat{\beta} \setminus \mathcal{H}_{\mathcal{X}}$) lies outside a large ball about y_0 (resp. x_0), where $\hat{\lambda}$ (resp. $\hat{\beta}$) denotes an electric geodesic joining the end-points of λ_h (resp. β_h) assuming that these end-points lie outside horospheres in the coned-off spaces $\hat{Y}$ and $\hat{X}$ respectively. We have thus shown the following.

Lemma 1.18. *A Cannon-Thurston map for $i\colon Y \to X$ exists if and only if there exists a non-negative function $M(N)$ with $M(N) \to \infty$ as $N \to \infty$ such that the following holds:*
Given $y_0 \in Y$, and an electric quasigeodesic segment $\hat{\lambda}$ in $\hat{Y}$ starting and ending outside horospheres, if $\lambda^b = \hat{\lambda} \setminus \mathcal{H}_{\mathcal{Y}}$ lies outside an N-ball around $y_0 \in Y$, then for any electric quasigeodesic $\hat{\beta}$ joining the end points of $\hat{i}(\hat{\lambda})$ in $\hat{X}$, $\beta^b = \hat{\beta} \setminus \mathcal{H}_{\mathcal{X}}$ lies outside an $M(N)$-ball around $i(y_0) \in X$.

Finally, we specialise Lemma 1.18 to the case we are interested in, viz. trees of relatively hyperbolic spaces. Recall that X is a tree (T) of relatively hyperbolic spaces, with vertex spaces X_v and edge spaces X_e. $X_{v\alpha}$ and $X_{e\alpha}$'s are the respective horosphere-like sets in X_v and X_e respectively. $\widehat{X}$ is the induced tree (T) of coned-off hyperbolic metric spaces, with vertex spaces $\widehat{X_v}$ and edge spaces $\widehat{X_e}$, $\mathcal{T}$ is the collection of maximal cone subtrees T_α in $\widehat{X}$ and $\mathcal{C}$ is the collection of maximal cone-subtrees of horosphere-like spaces C_α in X. Let $\widehat{\widehat{X}}$ denote $\widehat{X}$ with the T_α's coned off. Equivalently, $\widehat{\widehat{X}}$ may be regarded as X with the C_α's coned off. Let X_h denote X with hyperbolic cones $C_{\alpha h}$ attached. Let $\hat{\hat{\beta}}$ be an electric geodesic in $\widehat{\widehat{X}}$ and β be the relative geodesic with the same underlying subset. Also, let β_h denote the hyperbolic geodesic joining its end-points a, b in X_h and let β_{pel} be a (partially electrocuted) geodesic in $\widehat{X}$ joining a, b. (We assume that $a, b \in X \setminus \bigcup C_\alpha$.) Then, by Lemma 1.8 and Lemma 1.11 we find that $\hat{\hat{\beta}} \setminus \bigcup C_\alpha$, $\beta_{pel} \setminus \bigcup C_\alpha$, $\beta \setminus \bigcup C_\alpha$ and $\beta_h \setminus \bigcup C_{\alpha h}$ all track each other. Combining this with Lemma 1.18, we get the following.

Lemma 1.19. *A Cannon-Thurston map for $i \colon X_v \to X$ exists if and only if there exists a non-negative function $M(N)$ with $M(N) \to \infty$ as $N \to \infty$ such that the following holds:*
Given $y_0 \in X_v$, and an electric quasigeodesic segment $\hat{\lambda}$ in $\widehat{X_{vh}}$ with end-points a, b outside horospheres, if $\hat{\lambda} \setminus \bigcup X_{v\alpha}$ lies outside an N-ball around $y_0 \in X_{vh}$, then for any partially electrocuted quasigeodesic β_{pel} joining a, b in $\hat{X}$, $\beta^b_{pel} = \beta_{pel} \setminus \bigcup C_\alpha$ lies outside an $M(N)$-ball around $i(y_0) \in X$.

Thus, given a geodesic λ_h lying outside large balls in X_{vh}, we shall need to construct a partially electrocuted quasigeodesic β_{pel} in $\hat{X}$ satisfying the condition that it lies outside large balls in $\hat{X}$ off the sets C_α.

Since $i_v \colon X_v \to X$ takes a horosphere-like set $X_{v\alpha}$ to a horosphere-like set C_α and image of no two horosphere-like sets in X_v lie in the same horosphere-like set C_α, i_v will induce an embedding $\hat{i}_v \colon \hat{X}_v \to \hat{X}$. Using the fact that all edge-to-vertex space inclusions are strictly type-preserving, it follows that the induced maps $\hat{i}_v \colon \hat{X}_v \to \hat{X}$ are uniformly proper embeddings, that is, for all $M > 0$, $v \in T$ and $x, y \in \hat{X}_v$, there exists $N > 0$ such that $d_{\hat{X}}(\hat{i}_v((x), \hat{i}_v(y)) \leq M$ implies $d_{\hat{X}_v}(x, y) \leq N$.

2. Existence of Cannon-Thurston Maps

Througout this section, we will assume that trees (T) of relatively hyperbolic spaces are as in Definition 1.14. We will also assume that horosphere-like sets are uniformly separated.

Recall that for a metric space X_v hyperbolic relative to a collection $\mathcal{H}$ of horosphere-like sets, there are two associated hyperbolic metric spaces:
1) The hyperbolic metric space X_{vh} obtained from X_v by attaching hyperbolic cones H_h to each horosphere-like sets H in X_v.

2) The coned-off or electrocuted space $\hat{X}_v$.

As in [12], the key step for proving the existence of a Cannon-Thurston map is to construct a hyperbolic ladder B_λ in $\hat{X}$, where $\hat{\lambda}$ is an electric geodesic segment in $\hat{X}_{v_0}$ for some $v_0 \in T$, and a large-scale Lipschitz retraction Π_λ from $\hat{X}$ onto B_λ. This proves the quasiconvexity of B_λ.

There is one real difference with the construction of B_λ in [12]. We consider here electric geodesics $\hat{\mu}$ in the coned-off vertex and edge-spaces $\widehat{X_v}$ and $\widehat{X_e}$. This is *not* a point of difference with [12] as the spaces $\widehat{X_v}$ and $\widehat{X_e}$ are hyperbolic and we have an induced tree (T) of (coned-off) hyperbolic metric spaces, exactly as in [12]. However, in [12], we need to find points in some C-neighborhood of λ to construct B_λ. Since there is only the usual (Gromov)-hyperbolic metric in [12], this creates no confusion. But, in the present situation, we have two metrics d_{X_v} and $d_{\hat{X}_v}$ on X_v. As electrically close (in the $d_{\hat{X}_v}$ metric) does not imply hyperbolically close (in the d_{X_v} metric), we cannot take a C-neighborhood in the $d_{\hat{X}_v}$ metric. Instead we will first construct an electroambient representative λ_{ea} of $\hat{\lambda}$ in the space X_{vh} and take a hyperbolic neighbourhood of λ_{ea} in X_{vh}. Also, by noting that X_{vh} with *horoballs* coned off is the same as $\widehat{X_v}$, we will be able to carry on the construction in [12] mutatis mutandis.

Finally we construct quasigeodesic rays in B_λ to show that if $(\lambda \setminus \bigcup X_{v\alpha})$ lies outside a large ball in X_v, then $(B_\lambda \setminus \bigcup C_\alpha)$ lies outside a large ball in X. The existence of a Cannon-Thurston map follows from Lemma 1.19.

2.1. Construction of Hyperbolic Ladder and Retraction map.

Given a geodesic segment $\hat{\lambda} \subset \hat{X}_{v_0}$ with end points lying outside horospheres-like sets, we now construct a quasiconvex set $B_{\hat{\lambda}} \subset \hat{X}$ containing $\hat{\lambda}$.

Lemma 2.1. *(Lemma 3.1 of [12])*
Given $\delta > 0$, there exist D, C such that if x, y are points of a $\delta-$hyperbolic metric space (X, d), λ is a hyperbolic geodesic in X, and π_λ is a nearest point projection of X onto λ with $d(\pi_\lambda(x), \pi_\lambda(y)) \geq D$, then $[x, (\pi_\lambda(x)] \cup [(\pi_\lambda(x), (\pi_\lambda(y)] \cup [(\pi_\lambda(y), y]$ lies in a $C-$neighbourhood of any geodesic joining x, y.

The quasi-isometric embeddings $f_v \colon X_e \to X_v$ induce quasi-isometric embeddings $f_{vh} \colon X_{eh} \to X_{vh}$. Thus for every vertex v and edge e in T, $f_{vh}(X_{eh})$ will be C_2 quasiconvex for some $C_2 > 0$. Assuming that every X_{vh}, X_{eh} is δ-hyperbolic let C_1 be as in Lemma 2.1 above. Let $C = C_1 + C_2$

For $Z \subset X_{vh}$, let $N_C(Z)$ denote the C-neighbourhood of Z in X_{vh}, where C is as in above lemma.

Hyperbolic Ladder B_λ

Identify $\widehat{X_v}$ with the space obtained from X_{vh} by coning off the 'horoballs' H_{vh}. Similarly for edge spaces $\widehat{X_e}$. The tree (T) of coned-off spaces is thus looked upon as a tree of these coned-off spaces. Recall that $P : \widehat{X} \to T$ is the usual projection to the base tree.

For convenience of exposition, T shall be assumed to be rooted, i.e. equipped with a base vertex v_0. Let $v \neq v_0$ be a vertex of T. Let v_- be the penultimate vertex on the geodesic edge path from v_0 to v. Let e denote the directed edge from v_- to v.

Define $\phi_v \colon f_e(X_e \times \{0\}) \to f_e(X_e \times \{1\})$ as follows:
If $p \in f_e(X_e \times \{0\}) \subset X_{v_-}$, choose $x \in X_e$ such that $p = f_e(x \times \{0\})$ and define $\phi_v(p) = f_e(x \times \{1\})$.

Note that in the above definition, x is chosen from a set of bounded diameter.

Since $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ are quasi-isometric embeddings into their respective vertex spaces ϕ_v's are uniform quasi-isometries for all vertices.

Step 1

Let $\hat{\mu} \subset \widehat{X_v}$ be a geodesic segment in $(\widehat{X_v}, d_{\widehat{X_v}})$ with starting and ending points lying outside horoballs and μ be the corresponding electroambient quasi-geodesic in X_{vh} (cf Lemma 1.13). Then $P(\hat{\mu}) = v$. For each edge e incident on v , but not lying on the geodesic (in T) from v_0 to v, choose $p_e, q_e \in N_C(\mu) \cap f_v(X_e)$ such that $d_{X_{vh}}(p_e, q_e)$ is maximal. Let $\{v_i\}$ be the terminal vertices of edges e_i for which $d_{\widehat{X_v}}(p_{e_i}, q_{e_i}) > D$, where D is as in Lemma 2.1 above.

Now $\hat{f}_v(\widehat{X_{e_i}})$ is quasiconvex in $\widehat{X_v}$, so we can take an k-electric quasi-geodesic $\hat{\mu}_{v_i}$ in $\widehat{X_v}$ joining p_{e_i} and q_{e_i} without backtracking and lying totally in $\hat{f}_v(\widehat{X_{e_i}})$. Note that since $\hat{f}_v(\widehat{X_e})$ are uniformly quasi-convex k is independent of vertex and edge spaces.
Define $B^1(\hat{\mu}) = i_v(\hat{\mu}) \cup \bigcup_i \hat{\Phi}_{v_i}(\hat{\mu}_{v_i})$,
where $\hat{\Phi}_{v_i}(\hat{\mu}_{v_i})$ is an electric geodesic joining $\phi_{v_i}(p_{e_i})$ and $\phi_{v_i}(q_{e_i})$.

Step 2

Step 1 above constructs $B^1(\hat{\lambda})$ in particular. We proceed inductively. Suppose that $B^m(\hat{\lambda})$ has been constructed such that the convex hull of $P(B^m(\hat{\lambda})) \subset T$ is a tree. Let $\{w_i\}_i = P(B^m(\hat{\lambda})) \setminus P(B^{m-1}(\hat{\lambda}))$.

Assume further that $P^{-1}(w_k) \cap B^m(\hat{\lambda})$ is a path of the form $i_{w_k}(\hat{\lambda}_{w_k})$, where $\hat{\lambda}_{w_k}$ is a geodesic in $(\hat{X}_{w_k}, d_{\hat{X}_{w_k}})$.
Define
$B^{m+1}(\hat{\lambda}) = B^m(\hat{\lambda}) \cup \bigcup_k (B^m(\hat{\lambda}_{w_k}))$,
where $B^1(\hat{\lambda}_{w_k})$ is defined in step 1 above.
Define
$B_{\hat{\lambda}} = \cup_{m\geq 0} B^m(\hat{\lambda})$.
Observe that the convex hull of $P(B_{\hat{\lambda}})$ in T is a subtree, say, T_1.

Definition 2.2 (Retraction Map)**.** *Let* $\hat{\pi}_{\hat{\lambda}_v} : \hat{X}_v \to \hat{\lambda}_v$ *be a nearest point projection of* $\hat{X}_v$ *onto* $\hat{\lambda}_v$. $\hat{\Pi}_{\hat{\lambda}}$ *is defined on* $\bigcup_{v\in T_1} \hat{X}_v$ *by* $\hat{\Pi}_{\hat{\lambda}}(x) = \hat{i}_v(\hat{\pi}_{\hat{\lambda}_v}(x))$ *for* $x \in \hat{X}_v$. *If* $x \in P^{-1}(T \setminus T_1)$, *choose* $x_1 \in P^{-1}(T_1)$ *such that* $d(x, x_1) = d(x, P^{-1}(T_1))$ *and define* $\hat{\Pi}_{\hat{\lambda}}(x) = x_1$. *Next define* $\hat{\Pi}_{\hat{\lambda}}(x) = \hat{\Pi}_{\hat{\lambda}}(\hat{\Pi}'_{\hat{\lambda}}(x))$.

The following is the main technical theorem of [12].

Theorem 2.3. (Theorem 3.8 of [12]**)**
There exists $C_0 \geq 0$ *such that*

$$d_{\hat{X}}(\hat{\Pi}_{\hat{\lambda}}(x), \hat{\Pi}_{\hat{\lambda}}(y)) \leq C_0 d_{\hat{X}}(x, y) + C_0 \text{ for } x, y \in \hat{X}.$$

In particular, if $\widehat{X}$ *is hyperbolic, then* $B_{\hat{\lambda}}$ *is uniformly (independent of* λ*) quasiconvex.*

2.2. **Quasigeodesic Rays.** Let $\hat{\lambda}_{v_0}$ be an electric geodesic segment from a to b in $\hat{X}_{v_0}$ with a and b lying outside horosphere-like sets and λ_{v_0} denotes its electroambient quasigeodesic representative in X_{v_0h}.

Note that $B_{\hat{\lambda}_{v_0}} = \bigcup_{v\in T_1} \hat{i}_v(\hat{\lambda}_v)$, is the quasiconvex set constructed above, where $P(B_{\hat{\lambda}_{v_0}}) = T_1$.

Let
- λ_v^c be the union of geodesic subsegments of the electroambient quasigeodesic λ_v lying inside the horoball-like sets penetrated by λ_v.
- $\lambda_v^b = \lambda_v \setminus \lambda_v^c$.

(Note that $\lambda_v^b \subset \hat{\lambda}_v$).
- $B_{\lambda_{v_0}^b} = \bigcup_{v\in T_1} i_v(\lambda_v^b)$.

(Then $B_{\lambda_{v_0}^b} \subset B_{\hat{\lambda}_{v_0}}$).

If $x \in B_{\lambda_{v_0}^b}$, then there exists $v \in T_1$ such that $x \in \lambda_v^b$. Let $[v = v_n, v_{n-1}] \cup ... \cup [v_1, v_0]$ be the geodesic edge path in T_1 joining v and v_0. Let $S = [v_n, v_{n-1}] \cup ... \cup [v_1, v_0]$.

We will construct a map $r_x : S \to B_{\lambda_{v_0}^b}$ satisfying

$d_S(w,w') \le d(r_x(w), r_x(w') \le C d_S(w,w')$ for all $w, w' \in S$. r_x will be called a *quasigeodesic ray*.

- For $v_n \in S$, define $r_x(v_n) = x$

Let $w = v_{n-1}$ and $\psi_{v_i} = \phi_{v_i}^{-1} \colon X_{v_i} \to X_{v_{i-1}}$ for all $i = 1, ..., n$.

Then $\psi_v \colon X_v \to X_w$ is a quasi-isometry. Since x lies outside horosphere-like sets and ψ_v preserves horosphere-like sets, $\psi_v(x)$ will lie outside horosphere-like sets.

Let $[a,b]$ be the maximal connected component of λ_v^b on which x lies. Then there exists two horosphere-like sets H_1 and H_2 such that $a \in H_1$ (or is a initial point of λ_v) and $b \in H_2$ (or is a terminal point of λ_v). Since ψ_v preserves horosphere-like sets, $\psi_v([a,b])$ will lie outside horosphere-like sets.

As ψ_v is a quasi-isometry, $\psi_v([a,b])$ will be a quasi-geodesic in X_{wh}. Let $\Psi_v^h([a,b])$ be the hyperbolic geodesic in X_{wh} joining $\psi_v(a)$ and $\psi_v(b)$. Then $\psi_v([a,b])$ will lie in a bounded neighbourhood of $\Psi_v^h([a,b])$. Therefore there exists $C_1 > 0$ such that $d(\psi_v(x_1), \Psi_v^h([a,b])) \le C_1$. By Lemma 1.8, there exists an upper bound on how much $\Psi_v^h([a,b])$ can penetrate horoball-like sets, that is, for all $z \in \Psi_v^h([a,b])$ there exists $z' \in \Psi_v^h([a,b])$ lying outside horoball-like sets such that $d(z,z') \le D$. Hence there exists $y_1 \in \Psi_v^h([a,b])$ such that $d(\psi_v(x_1), y_1) \le D + C_1$ and y_1 lies outside horosphere-like sets.

Again, $\Psi_v^h([a,b])$ lies at a uniformly bounded distance $\le C_2$ from μ_v (the electroambient quasigeodesic representative of $\hat{\mu}_v$ in the construction of $B_{\hat{\lambda}_{v_0}}$)). Let $c, d \in \mu_v$ such that $d(a,c) \le C_2$ and $d(b,d) \le C_2$. Then $\Psi_v^h([a,b])$ and the quasigeodesic segment $[c,d] \subset \mu_v$ have similar intersection patterns (Lemma 1.8) with horoball-like sets. Therefore $[c,d]$ can penetrate only a bounded distance $\le D$ into any horoball-like set. Hence there exists $y_2 \in \mu_v$ and y_2 lies outside horosphere-like sets such that $d(y_1, y_2) \le C_2 + D$.

Since end points of μ_v lie at a bounded neighbourhood of λ_w, there exists $C_3 > 0$ such that μ_v will lie at a C_3 neighbourhood of λ_w. Therefore there exists $y_3 \in \lambda_w$ such that $d(y_2, y_3) \le C_3$. Now y_3 may lie in a horoball-like set. Since μ_v and $\pi_{\lambda_w}(\mu_v)$ lies in a bounded neighbourhood of each other, by Lemma 1.8 they have similar intersection patterns with horoball-like sets. Therefore there exists $D > 0$ and $y \in \lambda_w$ such that y lies outside horosphere-like sets and $d(y_3, y) \le D$.

Hence $d(x,y) \le 1 + C_1 + C_2 + C_3 + 3D = C$(say).

- If $w = v_{n-1}$, we define $r_x(v_{n-1}) = y$.

Thus we have $1 \le d(r_x(v_n), r_x(v_{n-1})) \le C$.

Using the above argument repeatedly (inductively replacing x with $r_x(v_i)$ in each step we get the following.

Lemma 2.4. *There exists $C \geq 0$ such that for all $x \in \lambda_v^b \subset B^b_{\lambda_{v_0}}$ there exists a C- quasigeodesic ray $r_x \colon S \to B^b_{\lambda_{v_0}}$ such that $r_x(v) = x$ and $d_S(v,w) \leq d(r_x(v), r_x(w) \leq C d_S(v,w)$, where S is the geodesic edge path in T_1 joining v and v_0 and $w \in S$.*

The following is the concluding Lemma of this subsection.

Lemma 2.5. *There exists $C > 0$ such that for all $x \in \lambda_v^b \subset B^b_{\lambda_{v_0}} \subset B_{\hat{\lambda}_{v_0}}$ if $\lambda^b_{v_0}$ lies outside $B_n(p)$ for a fixed reference point $p \in X_{v_0}$ (so entry and exit points of $\hat{\lambda}$ to a horosphere lie outside $B_n(p)$) and assuming that in fact the reference point p lies outside the horospheres, then x lies outside an $n/(C+1)$ ball about p in X.*

Proof: Since $\lambda^b_{v_0}$ is a part of $\hat{\lambda}_{v_0}$, $r_x(v_0)$ lies outside $B_n(p)$.

Let m be the first non-negative integer such that $v \in P(B^m(\hat{\lambda}_{v_0})) \setminus P(B^{m-1}(\hat{\lambda}_{v_0}))$. Then $d_{T_1}(v_0, v) = m$, and $d(x,p) \geq m$ (since $r_x(v) = x \in \lambda_v^b$).
From Lemma 2.4, $m \leq d(r_x(v), r_x(v_0)) \leq Cm$.
Since $r_x(v_0)$ lies outside $B_n(p)$, $d(r_x(v_0), p) \geq n$.
$n \leq d(r_x(v_0), p) \leq d(r_x(v_0), r_x(v)) + d(r_x(v), p) \leq mC + d(r_x(v), p)$.
Therefore, $d(r_x(v), p) \geq n - mC$ and $d(r_x(v), p) \geq m$.
Hence $d(x,p) = d(r_x(v), p) \geq \frac{n}{C+1}$. $\square$

2.3. **Proof of Main theorem.** First we recall the following notation:
- $\hat{\lambda}_{v_0}$ = electric geodesic in $\widehat{X}_{v_0}$ joining a, b with $\lambda^b_{v_0}$ lying outside $B_n(p)$ for a fixed reference point $p \in X_{v_0}$ lying outside horosphere-like sets.
- λ_{v_0} = electroambient quasi-geodesic in $X_{v_0 h}$ constructed from $\hat{\lambda}_{v_0}$.
- β_{pel} = quasi-geodesic in the partial electrocuted space $\widehat{X}$ joining a, b.
- β = electroambient quasi-geodesic in X_h corresponding to β_{pel}.
- $\beta'_{pel} = \hat{\Pi}_{\hat{\lambda}_{v_0}}(\beta_{pel})$, where $\hat{\Pi}_{\hat{\lambda}_{v_0}}$ is the retraction map from the vertex space of $\widehat{X}$ to the quasi-convex set $B_{\hat{\lambda}_{v_0}}$.

β'_{pel} is a dotted quasi-geodesic in the partial electrocuted space $\widehat{X}$ lying on $B_{\hat{\lambda}_{v_0}}$.
So β'_{pel} lies in a K-neighbourhood of β_{pel} in $\widehat{X}$. But β'_{pel} might backtrack. β'_{pel} can be modified to form a quasigeodesic in $\widehat{X}$ of the same type (i.e. lying in a K-neighbourhood of β_{pel} in $\widehat{X}$) without backtracking with end points remaining the same. Let u and v be the first and last points on a horosphere-like set C_α penetrated by β'_{pel}. Replace β'_{pel}

on $[u,v]_{\widehat{X}}(\subset \beta'_{pel})$ by a partially electrocuted geodesic from u to v in $\widehat{X}$. Since β'_{pel} has finite length in $\widehat{X}$, it jumps from one horosphere to another a finite number of times, so after a finite number of modification we will have produced a path that does not backtrack. In this process, we have if anything, reduced the length of β'_{pel}, so the new path, γ_{pel} (say), is again a partially electrocuted quasigeodesic of the same type as β'_{pel}.

Now a, b are end points of $\hat{\lambda}_{v_0}$, therefore $a, b \in B_{\hat{\lambda}_{v_0}}$ and end points of γ_{pel} are a, b. Hence, β_{pel} and γ_{pel} are two quasigeodesics in $\widehat{X}$ without backtracking joining the same pair of points. By Lemma 1.11, β_{pel} and γ_{pel} have similar intersection patterns with the maximal horospheres-like sets C_α. Let γ be the corresponding electroambient quasigeodesics in X_h, where X_h is a hyperbolic metric space obtained from X by attaching the hyperbolic cone $C_{\alpha h}$ to C_α, then β lies in a bounded neighbourhood of γ. Due to similar intersection patterns of β_{pel} and γ_{pel}, if γ_{pel} penetrates a horosphere C_α and β_{pel} does not, then the length of geodesic traversed by γ inside C_α is uniformly bounded.

Thus there exists $C_1 \geq 0$ such that if $x \in \beta^b_{pel} = \beta_{pel} \setminus \mathcal{C}$, then there exists $y \in \gamma^b_{pel} = \gamma_{pel} \setminus \mathcal{C}$ such that $d(x,y) \leq C_1$.

Since $\gamma^b_{pel} \subset B_{\lambda^b_{v_0}}$, it follows from lemma 2.5 that
$d(y,p) \geq \frac{n}{C+1}$.

So, $\frac{n}{C+1} \leq d(x,y) + d(x,p) \leq C_1 + d(x,p)$,

i.e. $d(x,p) \geq \frac{n}{C+1} - C_1$ (=$M(n)$,say).

Thus we have the following proposition :

Proposition 2.6. *For every point x on β^b, x lies outside an $M(n)$-ball around p in X, such that $M(n) \to \infty$ as $n \to \infty$.*

It is now easy to assemble the pieces to deduce the existence of Cannon-Thurston maps.

Theorem 2.7. *Let (X,d) be a tree (T) of relatively hyperbolic spaces satisfying the quasi-isometrically (qi) embedded condition. Further suppose that the inclusion of edge-spaces into vertex spaces is strictly type-preserving, and the induced tree of coned-off spaces continue to satisfy the qi-embedded condition. If X is strongly hyperbolic relative to the family $\mathcal{C}$ of maximal cone-subtrees of horosphere-like sets, then a Cannon-Thurston map exists for the proper embedding $i_{v_0}\colon X_{v_0} \to X$, where v_0 is a vertex of T and (X_{v_0}, d_{v_0}) is the relatively hyperbolic metric space corresponding to v_0.*

Proof: A Cannon-Thurston map exists if it satisfies the condition of Lemma 1.19.

So for a fixed reference point $p \in X_{v_0}$ with p lying outside horosphere-like sets, we assume that $\hat{\lambda}_{v_0}$ is an electric geodesic in $\widehat{X}_{v_0}$ such that $\lambda^b_{v_0} \subset X_{v_0}$ lies outside an n-ball $B_n(p)$ around p.

Since i_{v_0} is a proper embedding, $\lambda^b_{v_0} = \hat{\lambda}_{v_0} \setminus \mathcal{H}_{\sqsubseteq}$, lies outside an $f(n)$-ball around p in X such that $f(n) \to \infty$ as $n \to \infty$

From the Proposition 2.6, if β_{pel} is an electrocuted geodesic joining the end points of λ_{v_0}, then β^b lies outside an $M(f(n))$-ball around p in X such that $M(f(n)) \to \infty$ as $n \to \infty$.

From Lemma 1.19, a Cannon-Thurston map for $i\colon X_{v_0} \to X$ exists. □

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RKM Vivekananda University, Belur Math, WB-711 202, India

Stat-Math Unit, Indian Statistical Institute, 203 B.T.Road, Kolkata 700108