

CUBULATING SURFACE-BY-FREE GROUPS

JASON F. MANNING, MAHAN MJ, AND MICHAH SAGEEV

ABSTRACT. Let

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence where $H = \pi_1(S)$ is the fundamental group of a closed surface S of genus greater than one, G is hyperbolic and Q is finitely generated free. The aim of this paper is to provide sufficient conditions to prove that G is cubulable and construct examples satisfying these conditions. The main result may be thought of as a combination theorem for virtually special hyperbolic groups when the amalgamating subgroup is not quasiconvex. Ingredients include the theory of tracks, the quasiconvex hierarchy theorem of Wise, the distance estimates in the mapping class group from subsurface projections due to Masur-Minsky and the model geometry for doubly degenerate Kleinian surface groups used in the proof of the ending lamination theorem.

CONTENTS

| | |
|--|----|
| 1. Introduction | 2 |
| 1.1. Motivation and context | 3 |
| 1.2. Statement of Results | 5 |
| 1.3. Scheme of the paper | 7 |
| 2. Graphs of spaces and tracks | 8 |
| 2.1. Graphs of Spaces | 8 |
| 2.2. Surface bundles over a graph | 9 |
| 2.3. Tracks | 9 |
| 2.4. Essentiality | 10 |
| 2.5. EIQ track implies cubulable | 11 |
| 2.6. Metric surface bundles | 12 |
| 3. Tight trees and models | 13 |
| 3.1. Tight trees of non-separating curves | 13 |
| 3.2. Topological building blocks from links: non-separating curves | 14 |
| 3.3. Model geometry | 15 |
| 3.4. A criterion for quasiconvexity | 20 |
| 4. The staircase construction | 22 |
| 4.1. The staircase construction in 3-manifolds | 22 |
| 4.2. The tree-stairstep | 23 |
| 4.3. Main Theorem | 23 |
| 5. Examples of EIQ tracks | 24 |
| 5.1. Stairsteps in 3-manifolds | 24 |
| 5.2. Tree stairsteps | 27 |
| 5.3. Stairsteps from regular tight trees | 28 |
| 6. Quasiconvexity of treads | 29 |

Date: August 6, 2019.

| | |
|---|----|
| 6.1. Laminations, Cannon-Thurston Maps and quasiconvexity | 29 |
| 6.2. Uniform quasiconvexity of subsurfaces in bounded geometry | 31 |
| 6.3. Uniform quasiconvexity of treads in split geometry manifolds | 33 |
| 6.4. Uniform quasiconvexity of treads in $(\widetilde{M}_T, d_{te})$: Proof of Theorem 6.1 | 37 |
| 7. Quasiconvexity of the track: Proof of Theorem 4.5 | 38 |
| 8. Generalization: separating curves | 40 |
| 8.1. Tight trees | 40 |
| 8.2. Balanced trees | 41 |
| 8.3. Virtual algebraic fibering | 43 |
| Acknowledgments | 43 |
| References | 43 |

1. INTRODUCTION

This paper lies at the interface of two themes in geometric group theory that have attracted a lot of attention of late: convex cocompact subgroups of mapping class groups, and cubulable hyperbolic groups. Let $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence with H a closed surface group and Q a convex cocompact subgroup of $MCG(S)$. It follows that G is hyperbolic. In fact convex cocompactness of Q is equivalent to hyperbolicity of G [FM02, Ham05, KL08]. The only known examples of convex cocompact subgroups Q of $MCG(S)$ are virtually free. Cubulable groups, by which we mean groups acting freely, properly discontinuously and cocompactly by isometries (cellular isomorphisms) on a CAT(0) cube complex, have been objects of much attention over the last few years particularly due to path-breaking work of Agol and Wise. In this paper, we shall address the following question that lies at the interface of these two themes:

Question 1.1. *Let*

$$(1) \quad 1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence of groups, where $H = \pi_1(S)$ is the fundamental group of a closed surface S of genus greater than one, G is hyperbolic and Q is a finitely generated free group of rank n .

- (i) *Does G have a quasiconvex hierarchy? Equivalently (by Wise's Theorem 1.2 below), is G virtually special cubulable?*
- (ii) *In particular, is G linear?*

We shall provide sufficient conditions on the exact sequence (1) guaranteeing an affirmative answer to Question 1.1. We shall also construct examples satisfying these conditions. A somewhat surprising consequence, using work of Kielak [Kie18] is the existence of groups G as in Question 1.1 that surject to \mathbb{Z} with finitely generated kernel (Section 8.3). Note that an affirmative answer to the first question in Question 1.1 implies an affirmative answer to the second. To the best of our knowledge, when the rank of Q is greater than one, there was no known example of a linear G as above, and the answer is not known in general. This is perhaps not too surprising as linearity even in the case $n = 1$ really goes back to Thurston's Double Limit Theorem [Thu86b] and the latter feeds into the cubulability of these 3-manifold groups.

The main theorem of this paper may also be looked upon as evidence for a combination theorem for cubulable groups along non-quasiconvex subgroups. Let us specialize to the case $n = 2$ in Question 1.1 for the time being. Let A (resp. B) be the fundamental group of a closed hyperbolic 3-manifold M_1 (resp. M_2) fibering over the circle with fiber a closed surface S of genus at least 2. Let $C = \pi_1(S)$ be the fundamental group of the fiber and we ask if $A *_C B$ is cubulable. We point out a preliminary caveat. Since the distortion of the fiber subgroup C in A is exponential, the double of A along C given by $G_0 = A *_C A$ has an exponential isoperimetric inequality. Since $CAT(0)$ groups satisfy a quadratic isoperimetric inequality, G_0 cannot be a $CAT(0)$ group; in particular G_0 is not cubulable. It therefore makes sense to demand that the group G resulting from the combination is hyperbolic. Unlike in the existing literature (see [HW12, HW15b, Wis11] for instance), the amalgamating subgroup C is not quasiconvex in A or B .

We briefly indicate the broader framework in which our results sit. The starting point of this work is Wise's quasiconvex hierarchy theorem [Wis11] for hyperbolic cubulable groups:

Theorem 1.2. [Wis11] *Let G be a finite graph of hyperbolic groups so that G is hyperbolic and the vertex groups are virtually special cubulable and quasiconvex in G . Then G is virtually special cubulable.*

Further, a celebrated Theorem of Agol [Ago13] proves a conjecture due to Wise [Wis11, Wis12] and establishes:

Theorem 1.3. [Ago13] *Let G be hyperbolic and cubulable. Then G is virtually special.*

The sufficient conditions we provide are a first attempt at relaxing the quasiconvexity hypothesis in Theorem 1.2: is there a combination theorem for cubulated groups along *non-quasiconvex* subgroups? We explicitly state the general question below:

Question 1.4. *Let G be a finite graph of hyperbolic groups (e.g. $G = A *_C B$ or $G = A *_C C$) so that the vertex groups are virtually special cubulable and G is hyperbolic. Is G virtually special cubulable?*

Related questions have been raised by Wise [Wis14, Problems 13.5, 13.15], for instance when each of A, B are hyperbolic free-by-cyclic groups of the form

$$1 \rightarrow F_k \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$$

and C is the normal subgroup F_k .

1.1. Motivation and context. The base case of Question 1.1 is when $Q = \mathbb{Z}$ and G is the fundamental group of a 3-manifold M fibering over the circle with fiber S . We briefly recall what goes into the proof [Ago13, Wis11] of the virtually special cubulability of such G . By Thurston's double limit theorem [Thu86b] M admits a hyperbolic structure. Then, by work of Kahn-Markovic [KM12] there are many immersed quasiconvex surfaces in M . These are enough to separate pairs of points on $\partial G = S^2$. Hence by work of Bergeron-Wise [BW12], G is cubulable. Finally, by Agol's theorem 1.3 [Ago13], G is virtually special. In the restricted case that the first Betti number $b_1(G) > 1$, a Thurston's norm argument guarantees the existence of an *embedded* quasiconvex surface in M and Wise's quasiconvex hierarchy theorem 1.2 guarantees that G is virtually special.

Yet another approach to the cubulation of G when $Q = \mathbb{Z}$ was given by Dufour [Duf12] where the cross-cut surfaces of Cooper-Long-Reid [CLR94] were used to manufacture enough codimension one quasiconvex subgroups. Dufour's approach essentially used the fact that the cross-cut surfaces of [CLR94] can be isotoped to be transverse to the suspension flow in M and are hence incompressible. Replacing H by a free group in Question 1.1, Hagen and Wise [HW16, HW15a] prove cubulability of hyperbolic G with $Q = \mathbb{Z}$. Their proof again uses a replacement of the suspension flow (a semi-flow).

Thus, in the general context of 3-manifolds fibering over the circle with pseudo-Anosov monodromy, there are two methods of proving the existence of codimension one quasiconvex subgroups:

- (1) Work of Cooper-Long-Reid [CLR94] that is special to fibered manifolds.
- (2) The general theorem of Kahn-Markovic [KM12] for hyperbolic 3-manifolds.

This uses real hyperbolicity of M in an essential way.

We do not know an answer to the following in this generality:

Question 1.5. *Let G be as in Question 1.1. Does G have a quasiconvex codimension one subgroup?*

Note that the spinning construction of Hsu-Wise [HW15b] requires quasiconvexity of the amalgamating subgroup. When Q has rank greater than one, we do not have an analog of Thurston's double limit theorem (or the geometrization theorem of Perelman) and hence we do not have an analog of the Kahn-Markovic theorem providing sufficiently many codimension one quasiconvex subgroups. We are thus forced to use softer techniques from the coarse geometry of hyperbolic groups, e.g. the Bestvina-Feighn combination theorem [BF92] giving necessary and sufficient conditions for the Gromov-hyperbolicity of G . We pose a general problem in this context that would help in addressing Question 1.4:

Question 1.6. *Let G be a finite graph of hyperbolic groups (e.g. $G = A*_C B$ or $G = A*_C$) so that the vertex groups are virtually special cubulable and G is hyperbolic. Is there a combination theorem for quasiconvex codimension one subgroups when the amalgamating subgroups are not necessarily quasiconvex? In particular, for G as in Question 1.1?*

The boundary of a G as in Question 1.1 is somewhat intractable. Abstractly, it may be regarded as a quotient of the circle (identified with the boundary ∂H of $H = \pi_1(S)$) under the Cannon-Thurston map [Mit97, MR18] that collapses a Cantor set's worth of ending laminations, where the Cantor set is identified with the boundary ∂Q of the quotient free group Q . It thus seems difficult to apply Bergeron-Wise's criterion for cubulability [BW12]. Further there is no natural replacement for the suspension flow: a flowline would have to be replaced by a tree and transversality breaks down, preempting any straightforward generalization of the techniques of [Duf12, HW16, HW15a].

This forces us to find sufficient conditions guaranteeing the existence of a quasiconvex hierarchy. The replacement of embedded incompressible surfaces in our context are tracks. Our main theorem 1.8 gives sufficient conditions to ensure the existence of embedded tracks. More surprising than the statement of Theorem 1.8 are the techniques that go into its proof. We draw from the model geometries that went into the proofs of the ending lamination theorem and the existence of Cannon-Thurston maps for Kleinian surface groups [Min94, Min03, Min10, BCM12, Mj14]

as also the hierarchy machinery of subsurface projections in the mapping class group [MM99, MM00]. Thus techniques that were developed essentially to address problems of infinite covolume surface group representations into $\mathrm{PSL}_2(\mathbb{C})$ (see [Thu82, Problems 6-14] for instance) are used to address a specific cubulability problem, whose underlying theory (especially Agol-Wise’s celebrated work [Ago13, Wis11]) was designed to address problems having to do with fundamental groups of closed (more generally finite volume) hyperbolic 3-manifolds (see [Thu82, Problems 15-18] for instance). In the interests of readability, the material that goes into proving geometric models is treated in the companion paper [Mj19].

1.2. Statement of Results. Even in the special case of a hyperbolic 3-manifold M fibering over the circle with monodromy Φ , our techniques yield mildly new results. In particular, we construct examples of Φ and embedded quasiconvex surfaces in the associated M that cannot be made transverse to the suspension flow corresponding to Φ (see Remark 4.3). Thus these surfaces need not realize the Thurston norm in their homology class (as in [CLR94]).

The curve graph $\mathcal{C}(S)$ of a closed surface of genus at least two [MM99] is a graph whose vertices are given by isotopy classes of simple closed curves, and whose edges are given by distinct isotopy classes of simple closed curves that can be realized disjointly on S . An element $\psi \in \mathrm{MCG}(S)$ is said to be a pseudo-Anosov homeomorphism in the complement of a simple closed curve α if it fixes α , restricts to a pseudo-Anosov on $(S \setminus \alpha)$, and further, the powers ψ^n are renormalized by Dehn twists $\mathrm{tw}_\alpha^{k_n}$ so that the renormalized powers $\overline{\psi}^n := \mathrm{tw}_\alpha^{k_n} \circ \psi^n$ do not twist about α (see Definitions 5.1 and 5.2 for details). The action of such a ψ on the curve complex $\mathcal{C}(S)$ fixes the vertex α . Thus renormalized large powers of ψ may be thought of as “large rotations” about α in $\mathcal{C}(S)$. A sequence of simple closed curves on S on a tight geodesic of length at least one in $\mathcal{C}(S)$ is called a **tight sequence**. Informally, Proposition 1.7 below says: The composition of large powers of pseudo-Anosovs in the respective complements of a finite tight sequence of homologous curves gives, via the mapping torus construction, a 3-manifold fibering over the circle with an embedded geometrically finite surface. Alternately, the composition of large rotations (cf. [DGO17]) about a finite tight sequence of homologous curves gives the monodromy of a 3-manifold fibering over the circle with an embedded geometrically finite surface.

More precisely, let $I = \{1, \dots, k\}$ denote a multi-index and p_I denote a k -tuple (p_1, \dots, p_k) of integers. We say that $p_I \geq p$ if for each i , $|p_i| \geq p$. Then (see Proposition 5.4 and Remark 8.5):

Proposition 1.7. *Let $v_I := v_1 \cdots, v_k \in \mathcal{C}(S)$, $k \geq 2$, be a sequence of vertices in $\mathcal{C}(S)$ forming a tight geodesic of homologous simple closed curves. Let $\psi_i : S \rightarrow S$ be a pseudo-Anosov homeomorphism in the complement of v_i . Let $\Phi(p_I) = \overline{\psi}_1^{p_1} \cdots \overline{\psi}_k^{p_k}$ and let $M(p_I)$ be the 3-manifold fibering over the circle with fiber S and monodromy $\Phi(p_I)$. Then there exists $p_0 > 1$, such that for all p_I satisfying $p_I \geq p_0$, $M(p_I)$ admits an embedded incompressible geometrically finite surface.*

For $k = 1$, a modified construction can be carried out; see Section 5.3 for details.

By Thurston’s theorem [Thu86a], it follows that $M(p_I)$ admits a quasiconvex hierarchy (in the terminology of Wise’s Theorem 1.2). Of course, Agol’s Theorem 1.3 shows that the manifolds $M(p_I)$ in Proposition 1.7 are virtually special cubulable and hence a finite cover of any such $M(p_I)$ does admit a quasiconvex

hierarchy. When the first Betti number $b_1(M(p_I))$ is at least 2, $M(p_I)$ itself admits an embedded geometrically finite surface by an argument involving the Thurston norm [Thu86c]. However, Proposition 1.7 furnishes a new sufficient condition on the monodromy $\Phi(p_I)$ to guarantee the existence of an embedded incompressible geometrically finite surface in the 3-manifold $M(p_I)$ even when $b_1(M(p_I)) = 1$. When $b_1(M(p_I)) = 1$, the surfaces we construct are necessarily separating. Sisto [Sis18] found sufficient conditions on Heegaard splittings of rational homology 3-spheres M to guarantee that they were Haken. Proposition 1.7 may be regarded as an analog of the main theorem of [Sis18] in the context of fibered manifolds M with $b_1(M) = 1$. We also mention work of Brock-Dunfield [BD17] in a similar spirit that uses model geometries of degenerate ends to extract information about closed manifolds.

Proposition 1.7 becomes an ingredient for the next theorem which provides some of the main new examples of this paper (see Theorems 4.5 and 5.8). We first provide a statement using the terminology of hierarchies [MM00] before giving an alternate description.

Theorem 1.8. *Let Q be a subgroup of $MCG(S)$ and σ a non-separating simple closed curve on S satisfying the following conditions:*

- (1) *Tight tree: The orbit map $q \rightarrow q.\sigma$, $q \in Q$ extends to a Q -equivariant isometric embedding i of a tree T_Q into $\mathcal{C}(S)$ such that T_Q/Q is a finite graph;*
- (2) *Large links: $d_{\mathcal{C}(S \setminus i(v))}(i(v_1), i(v_2)) \gg 1$, for any vertex v of T and distinct neighbors v_1, v_2 of v in T .*
- (3) *Homologous curves: All vertices of $i(T)$ are homologous to each other.*
- (4) *subordinate hierarchy paths small: Hierarchy geodesics supported in $\mathcal{C}(S \setminus w_1 \cup w_2)$ subordinate to the geodesics in $\mathcal{C}(S \setminus i(v))$ (Item (2) above) are uniformly bounded.*

Then Q is convex cocompact. For

$$1 \rightarrow \pi_1(S) \rightarrow G \rightarrow Q_n \rightarrow 1$$

the induced exact sequence of hyperbolic groups, G admits a quasiconvex hierarchy and hence is cubulable and virtually special.

We now describe fairly explicitly a way of constructing groups Q (and hence G) as in Theorem 5.8. We shall use the notion of subsurface projections from [MM00] (the relevant material is summarized in [Mj19, Section 2.2]). We also restrict ourselves here to the case $n = 2$ for ease of exposition. Let γ_1, γ_2 be two tight geodesics of homologous non-separating curves stabilized by pseudo-Anosov homeomorphisms Φ_1, Φ_2 constructed as in Proposition 1.7. Further assume without loss of generality that both γ_1, γ_2 pass through a common vertex v (this can be arranged after conjugating Φ_2 by a suitable element of $MCG(S)$ for instance). Note that $\gamma_1 = \gamma_2$ and $\Phi_1 = \Phi_2$ are allowed in the construction below. Thus, the data of a single 3-manifold constructed as in Proposition 1.7 above allows the construction below to go through. For $j = 1, 2$, we denote the vertex sequence of γ_j by v_{ij} , $i \in \mathbb{Z}$.

Let $MCG(S, v)$ denote the subgroup of $MCG(S)$ fixing the curve on S corresponding to v pointwise. We think of the elements $\Psi \in MCG(S, v)$ as rotations about v in the curve graph $\mathcal{C}(S)$ (cf. [DGO17]). Given $L, R > 0$, an element

$\Psi \in MCG(S, v)$ is said to be an (L, R) –**large rotation** about v for the pair γ_1, γ_2 (see Definition 5.6) if for any distinct $u, w \in \{v_{\pm 1, 1}, \Psi(v_{\pm 1, 2})\}$,

- (1) $d_{\mathcal{C}(S \setminus v)}(u, w) \geq L$,
- (2) subsurface projections (including annular projections) of any geodesic in $\mathcal{C}(S \setminus v)$ joining u, w are at most R .

We are now in a position to state a special case of one of the main theorems (see Theorem 5.8) of the paper. Informally, Theorem 1.9 below says that a pair of pseudo-Anosov homeomorphisms constructed as in Proposition 1.7 having axes passing through a common vertex $v \in \mathcal{C}(S)$ generate a convex cocompact free subgroup of $MCG(S)$ such that the resulting surface-by-free group is virtually special cubulable so long as the ‘angle’ between the axes at v is large (Theorem 1.9 below specializes Theorem 5.8 to the case $n = 2$). More precisely,

Theorem 1.9. *Let γ_1, γ_2 be two tight geodesics of homologous non-separating curves stabilized by pseudo-Anosov homeomorphisms Φ_1, Φ_2 constructed as in Proposition 1.7. Assume further that both γ_1, γ_2 pass through a common vertex v . There exist $L, R > 0$ such that if Ψ is an (L, R) –large rotation about v for the pair γ_1, γ_2 , then the group Q generated by $\Phi_1, \Psi\Phi_2\Psi^{-1}$ is a free convex cocompact subgroup of rank 2 in $MCG(S)$. For*

$$1 \rightarrow \pi_1(S) \rightarrow G \rightarrow Q_n \rightarrow 1$$

the induced exact sequence of hyperbolic groups, G admits a quasiconvex hierarchy and hence is cubulable and virtually special.

The hypothesis on the non-separating nature of the curves in γ_1, γ_2 can be relaxed; however the modified version of Theorem 1.9 becomes more technically involved to state. We refer the reader to Theorem 8.4 for the analog in this more general situation.

1.3. Scheme of the paper. The study of embedded incompressible surfaces has a long history in the study of 3-manifolds [Hem04]. Tracks (see [Sag95, Wis12] for instance) are the natural generalization of these to arbitrary cell complexes. Since our main motivation is to cubulate hyperbolic groups, we are interested primarily in quasiconvex tracks leading us naturally to the study of EIQ (essential incompressible quasiconvex) tracks in Section 2. An EIQ track in a 3-manifold is simply an embedded incompressible geometrically finite surface. The main content of Section 2 is a reduction theorem: Theorem 2.7. It says that if a hyperbolic bundle M over a finite graph \mathcal{G} with fiber a closed surface S admits an EIQ track, then it admits a quasiconvex hierarchy in the sense of Theorem 1.2. Theorem 2.7 thus reduces the problem of cubulating M to one of finding an EIQ track.

In Section 3 we recall some of the essential features of the geometry of the hyperbolic bundle M over a finite graph \mathcal{G} from [Mj19]. An essential tool that is recalled in Section 3.1 is the notion of a tight tree T of non-separating curves in $\mathcal{C}(S)$ generalizing the notion of a tight geodesic. We also equip $M_T = S \times T$ with a metric d_{weld} using the model geometry of doubly degenerate hyperbolic 3-manifolds [Min10, BCM12, Mj19]. Further, the construction of an auxiliary ‘partially electrified’ pseudo-metric d_{te} on M_T is also recalled from [Mj19].

The tight tree T is then used in Section 4 to construct a track \mathcal{T}_T in M_T . The track we construct is of a special kind—a ‘stairstep’. This is fairly easy to describe in $S \times [0, n]$: it consists of essential horizontal subsurfaces called *treads*, denoted

Tread $_{vw}$, in $S \times \{i\}$ with boundary consisting of curves $v \times \{i\}, w \times \{i\}$ connected together by vertical annuli $v \times [i, i + 1], w \times [i - 1, i]$ called *risers* corresponding to the curves v and w . The sequence of simple closed curves thus obtained on S is required to be a tight geodesic in the curve graph $\mathcal{C}(S)$. Section 4 concludes with the statement of the main technical theorem 4.5 of the paper, whose proof is deferred to the later sections.

Section 5 then applies Theorem 4.5 to construct the main examples of the paper (Theorems 1.8 and 1.9) already described.

The next two sections prove that the track \mathcal{T}_T is incompressible in M_T and any elevation $\widetilde{\mathcal{T}}_T$ to the universal cover ($(\widetilde{M}_T, d_{weld})$ or $(\widetilde{M}_T, d_{te})$) is quasiconvex. Gromov-hyperbolicity (with constant of hyperbolicity depending only on genus of S , the maximal valence of T , and a parameter R as in Theorem 1.9) of $(\widetilde{M}_T, d_{te})$ was established in [Mj19]. Quasiconvexity of treads (with constant having the same dependence above) is established in Section 6 using the structure of Cannon-Thurston maps. In Section 7, the treads are pieced together via risers using a version of the local-to-global principle for quasigeodesics in δ -hyperbolic spaces to complete the proof of Theorem 4.5. Section 8 generalizes the main theorem by allowing tight trees of homologous separating curves.

2. GRAPHS OF SPACES AND TRACKS

After providing some background on graphs of spaces, surfaces bundles over graphs and tracks in this section, we prove a reduction step: we show that it suffices to construct an essential incompressible quasiconvex (EIQ) track.

2.1. Graphs of Spaces. For all of the discussion below, spaces are assumed to be connected and path connected. A **graph** \mathcal{G} we take to be a tuple (V, E, i, t) where V and E are sets (the **vertex set** and **edge set**, respectively), and $i : E \rightarrow V$ and $t : E \rightarrow V$ give the **initial** and **terminal** vertex of each edge. (Strictly speaking this is a *directed* graph.)

A graph of spaces is constructed from the following data [SW79]:

- A connected graph $\mathcal{G} = (V, E, i, t)$;
- a **vertex space** X_v for each $v \in V$ and an **edge space** X_e for each $e \in E$;
and
- continuous maps $f_e^- : X_e \rightarrow X_{i(e)}$ and $f_e^+ : X_e \rightarrow X_{t(e)}$ for each $e \in E$.

Given this data, we construct a space

$$X_{\mathcal{G}} = \bigcup_v X_v \cup \bigcup_e X_e \times I / \sim,$$

where $(x, 0) \sim f_e^-(x)$ and $(x, 1) \sim f_e^+(x)$ for each $x \in X_e, e \in E$. We say $X_{\mathcal{G}}$ is a **graph of spaces**. We say a homeomorphism $X \rightarrow X_{\mathcal{G}}$ is a **graph of spaces structure on X** .

Example 2.1. The trivial example in which every vertex and edge space is a single point yields a 1-complex, the geometric realization of \mathcal{G} . We abuse notation in the sequel and refer to this 1-complex also as \mathcal{G} .

When the maps f_e^{\pm} are all π_1 -injective then the fundamental group of the space $X_{\mathcal{G}}$ inherits a graph of groups structure. In particular, when the graph has a single edge, we obtain a decomposition of $\pi_1(X_{\mathcal{G}})$ as a free product with amalgamation

or as an HNN-extension, depending on whether or not the graph is an edge or a loop. (See [SW79] for more on graphs of spaces and graphs of groups.)

Conversely, if we are given a space X and a subspace $Y \subset X$ such that Y has a closed neighborhood N homeomorphic to $Y \times [0, 1]$, then we obtain a graph of spaces structure for X , namely the components of Y as the edge spaces and the components of $X \setminus \overset{\circ}{N}$ as the vertex spaces, where $\overset{\circ}{N} \cong Y \times (0, 1)$.

2.2. Surface bundles over a graph. In this paper the main objects of study will be surface bundles over graphs. Let \mathcal{G} be a graph, thought of as a 1-complex, and consider a bundle $E \rightarrow \mathcal{G}$ with fiber S a surface. Then it is easy to see that E has the structure of a graph of spaces (with graph \mathcal{G}) where every edge and vertex space is homeomorphic to S and every edge-to-vertex map is a homeomorphism.

In particular, we can describe the fundamental group of E as follows. Let $T \subset \mathcal{G}$ be a maximal tree. In the graph of spaces structure coming from the bundle, we may assume that for any edge $e \in T$, the gluing maps f_e^\pm are the identity map on S . Let e_1, \dots, e_n be the edges in $\mathcal{G} \setminus T$. For each $i \in \{1, \dots, n\}$, let $f_i = f_{e_i}^+$, and let $\phi_i = (f_i)_* : \pi_1 S \rightarrow \pi_1 S$. We can describe $\pi_1 E$ as a multiple HNN extension

$$\pi_1 E \cong \langle \pi_1 S, t_1, \dots, t_n \mid t_i s t_i^{-1} = \phi_i(s), \forall s \in \pi_1 S, i \in \{1, \dots, n\} \rangle.$$

We are particularly interested in the case that $\pi_1 E$ is Gromov hyperbolic. It is a theorem of Farb and Mosher [FM02] that such groups exist. As discussed in Section 2.5 below, our first step in cubulating such $\pi_1 E$ will be to find a quasiconvex subgroup of G over which G splits as an amalgam or HNN extension. The fiber group $\pi_1(S)$ is normal in $\pi_1 E$, in particular not quasiconvex, so we will need to look at other ways of expressing E as a graph of spaces.

2.3. Tracks. The following definition is not standard, but it is the most useful for our purposes.

Definition 2.2. Let X be a topological space. A closed connected subset $T \subset X$ is a **track** if there is a closed I -bundle neighborhood N of T in X . Thus, there is a deformation retraction $N \rightarrow T$ whose point-preimages are intervals.

A track is said to be **2-sided** if the I -bundle structure is trivial; otherwise it is said to be **1-sided**.

A 2-sided track T has a neighborhood homeomorphic to $T \times [0, 1]$. As we remarked already, such a subset of X gives a graph of spaces structure on X with underlying graph \mathcal{G} either an interval or a loop, so it gives a decomposition of $\pi_1 X$ as an amalgam or HNN extension over $\pi_1 T$.

Remark 2.3. (Historical remarks) Traditionally, the setting of tracks is simplicial complexes. In that setting a track is a connected subset of a simplicial complex whose intersection with each closed n -dimensional simplex is a disjoint union of intersections with finitely many n -dimensional hyperplanes missing the vertices. (We view each n simplex as a subspace of \mathbb{R}^{n+1} .) In particular, a track does not intersect the 0-skeleton, intersects each edge in a finite set of points and intersects each 2-simplex in a finite disjoint union of arcs, each of which has its endpoints in the 1-skeleton. We will refer to a track in this sense as a **combinatorial track**.

(Combinatorial) tracks were introduced by Dunwoody [Dun85] for 2-complexes and subsequently studied by him and others (see, for example, [Bow98, Del99, DS99, SS03, FP06]). In the case that X is a 3-dimensional manifold, combinatorial

tracks are known as normal surfaces, a subject which has been studied extensively over the past 30 years (see [Ken01] for a survey).

A track in our sense is more general, even in the presence of a simplicial structure. For example, the intersection with a simplex need not be composed of contractible pieces.

Tracks are not always π_1 -injective, but our aim will be to construct ones that are. In the main setting of this paper, X is a surface bundle over a graph, which decomposes as a union of copies of $S_g \times I$ glued together along their boundaries. The intersection of a track in X with a copy of $S_g \times I \subset X$ is necessarily a properly embedded surface. For the track to be two-sided it is necessary but not sufficient that these surfaces are two-sided. We will need to take further care to ensure the entire track is two-sided. If we can show a track is both two-sided and π_1 -injective, we obtain a decomposition of $\pi_1(X)$ as a free product with amalgamation or HNN-extension. The following lemma will be exploited throughout the work.

Lemma 2.4 (Dunwoody). *If τ is a track in a simply connected space X , then $X \setminus \tau$ has two path components.*

Proof. (See [DD89] for a proof in the case of 2-dimensional simplicial complexes. The proof in our setting is the same as Dicks–Dunwoody’s.) Since X and τ are path-connected and τ has a product neighborhood, it is clear that $X \setminus \tau$ has at most two path components. Suppose there is only one.

There is a closed neighborhood N of τ and a homeomorphism $\phi : \tau \times [0, 1] \rightarrow N$. By shrinking N if necessary we can assume that $X \setminus \text{Int}(N)$ is path-connected. Projection onto $[0, 1]$ and identifying the endpoints gives a continuous surjection $\pi : X \rightarrow S^1$.

Fix $x \in \tau$, and let σ_1 be the path $\phi(x \times [0, 1])$. Since $X \setminus \text{Int}(N)$ is path connected, there is a path σ_2 joining the endpoints of σ_1 in the complement of N . Putting σ_1 and σ_2 together we get a map $\psi : S^1 \rightarrow X$. The composition $\pi \circ \psi : S^1 \rightarrow S^1$ has degree one, so $\pi_1 X$ surjects \mathbb{Z} , in contradiction to the assumption that X was simply connected. \square

2.4. Essentiality. A track is said to be **inessential** if the associated graph of groups decomposition is a trivial splitting as a free product with amalgamation: namely $G = A *_C B$ where one of the maps $C \rightarrow A$ or $C \rightarrow B$ is an isomorphism. The reason such a splitting is considered trivial is that every group has one. Namely if G is any group and H is any subgroup then $G \cong G *_H H$. Consequently it does not provide any new information about the group (again, see Scott and Wall [SW79]).

In our setting, τ will be a track in a compact space, namely a surface bundle X over a graph. In this proper, cocompact setting, essentiality will correspond to the following: every elevation of τ to the universal cover \tilde{X} will separate \tilde{X} into two components each of which contains points arbitrarily far away from τ . (Here for the metric we may use any proper G -equivariant metric on X .) Such components are referred to as **deep components**.

Definition 2.5. Let X be a compact complex with hyperbolic fundamental group. A track $T \subset X$ is called an an **essential, incompressible quasiconvex track** or **EIQ track** if it is two-sided, essential and if the induced map on fundamental group is injective with quasiconvex image.

Since X is compact with hyperbolic fundamental group, its universal cover \tilde{X} endowed with any $\pi_1 X$ -equivariant geodesic metric is Gromov hyperbolic. Its Gromov boundary $\partial\tilde{X}$ is a compact metrizable space, and any subset $Y \subseteq \tilde{X}$ has a well-defined limit set $\Lambda(Y) \subseteq \partial\tilde{X}$. The next lemma explains how the limit set of an elevation of a track cuts the boundary in two.

Lemma 2.6. *Let X be a compact complex with hyperbolic fundamental group, and let $T \subseteq X$ be an EIQ track. Let \tilde{X} be the universal cover of X , and let \tilde{T} be an elevation of T to \tilde{X} .*

- (1) *The complement $\tilde{X} \setminus \tilde{T}$ consists of two components, H_1 and H_2 .*
- (2) *The limit set $\Lambda(\tilde{T}) = \Lambda(H_1) \cap \Lambda(H_2)$.*
- (3) *For $i \in \{1, 2\}$, $\text{Int}(\Lambda(H_1)) = \Lambda(H_1) \setminus \Lambda(\tilde{T})$ is nonempty.*

Proof. An elevation of a track is a track, so (1) follows directly from Lemma 2.4.

Since $\tilde{T} \subset \overline{H_i}$, $i = 1, 2$, it is clear that $\Lambda(\tilde{T}) \subset \Lambda(H_1) \cap \Lambda(H_2)$. We shall now prove the reverse inclusion. Let $p \in \Lambda(H_1) \cap \Lambda(H_2)$. Since \tilde{T} is quasiconvex, each H_i must also be quasiconvex. Fixing a base-point $o \in \tilde{T}$, there exists a geodesic ray γ based at o and tending to p which lies in a bounded neighborhood of both halfspaces. Any point at bounded distance from both halfspaces is also bounded distance from their intersection \tilde{T} , so there is a sequence of points on \tilde{T} converging to p . This proves $\Lambda(H_1) \cap \Lambda(H_2) \subset \Lambda(\tilde{T})$ and establishes (2).

Fix $i \in \{1, 2\}$. The track T is essential, which implies that there are points $\{x_j\}_{j \in \mathbb{N}}$ in H_i so $d(x_j, \tilde{T}) \rightarrow \infty$. The stabilizer of \tilde{T} acts cocompactly on T . Thus $\{x_j\}$ can be chosen so that the closest point projections to \tilde{T} lie in some compact set $K \subset \tilde{T}$. This implies that there exists an upper bound C_0 on the Gromov inner product of any x_j and any point in \tilde{T} with respect to some base-point in K . This means that, up to passing to a subsequence, x_j converges to a point p whose Gromov inner product with any point in \tilde{T} is at least C_0 . Therefore p is contained in an open subset of $\Lambda(H_i)$ missing $\Lambda(T)$. This establishes (3). \square

2.5. EIQ track implies cubulable. In this subsection we prove:

Theorem 2.7. *Let M be a closed surface bundle over a finite graph \mathcal{G} , so that $\pi_1 M$ is hyperbolic. Suppose that M contains an EIQ track \mathcal{T} . Then M is cubulable.*

Our proof relies heavily on the following consequence of Wise's Quasiconvex Hierarchy Theorem 1.2 and Agol's Theorem 1.3.

Proposition 2.8. *Suppose that G is a hyperbolic group which is either an HNN extension $A *_C$ or an amalgam $A *_C B$ so that:*

- (1) *The vertex groups (i.e. either A if G is an HNN-extension or A and B if G is an amalgam) are hyperbolic, cubulated; and*
- (2) *The edge group C is quasiconvex in G .*

Then G is virtually special cubulated.

Proof. Agol's Theorem 1.3 tells us that the vertex groups are virtually special. Then Wise's Quasiconvex Hierarchy Theorem 1.2 tells us that G must also be virtually special. In particular it is cubulated. \square

Now, let M be a surface bundle over a finite graph \mathcal{G} with fiber S . Also, let $\pi_1(M) = G$. The key point we want to show that is that the vertex groups coming from cutting along an EIQ track are themselves cubulated.

Lemma 2.9. *Let $0 \in \mathcal{G}$ be an arbitrary point and let S_0 be the fiber over $0 \in \mathcal{G}$. If \mathcal{T} is an EIQ in M , then $\mathcal{T} \cap S_0$ is nonempty.*

Proof. We use the observations in Lemma 2.6 to establish this lemma.

Suppose not. Then $S_0 \subset M \setminus \mathcal{T}$. Let M' be the component of $M \setminus \mathcal{T}$ containing S_0 . Since \mathcal{T} is an EIQ track, its quasiconvex universal cover $\tilde{\mathcal{T}}$ separates the universal cover \tilde{M} into more than one deep component. By quasiconvexity and essentiality of $\tilde{\mathcal{T}}$, the limit set $\partial\tilde{\mathcal{T}}$ separates $\partial\tilde{M} = \partial G$ into non-empty subsets $\partial_1 G, \partial_2 G$, both having non-empty interiors. We can further assume without loss of generality that

- (1) For $i = 1, 2$, $\partial_i G \cup \partial\tilde{\mathcal{T}}$ is the boundary of N_i , one of the (at least) two deep components of $\tilde{M} \setminus \tilde{\mathcal{T}}$.
- (2) an elevation \tilde{M}' of M' is contained in N_1 .

But, $\pi_1(S_0)$ is normal in G and hence its limit set is all of G . In particular (since $S_0 \subset M'$), the limit set of \tilde{M}' and hence N_1 is all of ∂G , contradicting the fact that $\partial_2 G$ has non-empty interior. \square

Since $0 \in \mathcal{G}$ was arbitrary, Lemma 2.9 shows that \mathcal{T} cuts every fiber S of M . To cut $M_c = M \setminus N_\epsilon(\mathcal{T})$ further along quasiconvex tracks, we shall need the following refinement of a theorem of Scott and Swarup [SS90] due to Dowdall, Kent and Leininger [DKL14] (see also [MR18]).

Theorem 2.10. *Let $1 \rightarrow \pi_1 S \rightarrow G \rightarrow F \rightarrow 1$ be an exact sequence of hyperbolic groups, where F is free. Let $H < \pi_1 S$ be a finitely generated infinite index subgroup of $\pi_1 S$. Then H is quasiconvex in G .*

Proposition 2.11. *Each component of $M \setminus N_\epsilon(\mathcal{T})$ admits a quasiconvex hierarchy.*

Proof. Let α be an edge of the graph \mathcal{G} and let 0 be an interior point of α . Let S_0 be the fiber above 0 . Then (after an isotopy of the fiber if necessary) we may assume that $S_0 \setminus N_\epsilon(\mathcal{T})$ is a finite disjoint union $\sqcup_i S_{0_i}$ of proper essential subsurfaces S_{0_i} of S_0 . Since S is an embedded incompressible track (not quasiconvex) in M , it follows that each S_{0_i} is an embedded incompressible track in $M \setminus N_\epsilon(\mathcal{T})$. Further, by Theorem 2.10, it follows that each $\pi_1(S_{0_i})$ is quasiconvex in G and hence in the subgroup $\pi_1(M \setminus N_\epsilon(\mathcal{T}))$ of G .

Cutting each component of $M \setminus N_\epsilon(\mathcal{T})$ along such subsurfaces of S_0 as S_0 ranges over fibers over mid-points of all edges of \mathcal{G} , the final pieces are homotopy equivalent to essential subsurfaces of S and hence have free fundamental group. Thus each component of $M \setminus N_\epsilon(\mathcal{T})$ admits a quasiconvex hierarchy, where the final pieces are finitely generated free. \square

Proposition 2.11 combined with Wise's Quasiconvex Hierarchy Theorem 1.2 immediately yields Theorem 2.7.

2.6. Metric surface bundles. If M is a surface bundle over a graph \mathcal{G} with fiber S , then the cover of M corresponding to $\pi_1(S)$ is again a surface bundle M_T over a tree, T , where $T = \tilde{\mathcal{G}}$ is the universal cover of \mathcal{G} .

We shall also have need to equip such surface bundles over graphs with a metric structure. Here, the underlying graph \mathcal{G} or tree T will be a metric tree.

Definition 2.12. Let (X, d) be a path-metric space equipped with the structure of a bundle $P : X \rightarrow \mathcal{G}$ over a graph \mathcal{G} with fiber a surface S (here we allow \mathcal{G} to be a tree T). Then $P : X \rightarrow \mathcal{G}$ will be called a **metric surface bundle** if

- (1) There exists a metric h on S and $\lambda \geq 1$ such that for all $x \in \mathcal{G}$, $P^{-1}(x) = S_x$ equipped with the induced path-metric induced from (X, d) is λ -bi-Lipschitz to (S, h) .
- (2) Further, for any isometrically embedded interval $I \subset \mathcal{G}$, with $I = [0, 1]$, $P^{-1}(I)$ is λ -bi-Lipschitz to $(S, h) \times [0, 1]$ by an λ -bi-Lipschitz fiber-preserving homeomorphism that is λ -bi-Lipschitz on the fibers.

Definition 2.12 above is a special case of the more general notion of metric bundles introduced in [MS12].

3. TIGHT TREES AND MODELS

Let $M \rightarrow \mathcal{G}$ be a surface bundle over a graph as in Section 2 where the edge and vertex spaces are all homeomorphic to a closed surface S . Then the cover of M corresponding to $\pi_1(S)$ is again a surface bundle over a graph with base graph the universal cover $\tilde{\mathcal{G}}$ of \mathcal{G} . In what follows in this section, we shall denote the tree $\tilde{\mathcal{G}}$ by T .

The **curve graph** $\mathcal{C}(S)$ of an orientable finite-type surface S is a graph whose vertices consist of free homotopy classes of simple closed curves and edges consist of pairs of distinct free homotopy classes of simple closed curves that can be realized by curves having minimal number of intersection points (2 for $S_{0,4}$, 1 for $S_{1,1}$ and 0 for all other surfaces of negative Euler characteristic). A fundamental theorem of Masur-Minsky [MM99] asserts that $\mathcal{C}(S)$ is Gromov-hyperbolic. In fact, Hensel, Przytycki and Webb [HPW15] establish that all curve graphs are uniformly hyperbolic. The Gromov boundary $\partial\mathcal{C}(S)$ may be identified with the space $\mathcal{EL}(S)$ of ending laminations [Kla99]. We shall be interested in surface bundles coming from trees T embedded in $\mathcal{C}(S)$. In this section, we will briefly recall from [Mj19] the construction of a geometric structure on such surface bundles. The ingredients of this construction are as follows:

- (1) A sufficient condition to ensure an isometric embedding of T into $\mathcal{C}(S)$.
- (2) The construction of an auxiliary metric tree $\text{BU}(T)$ from T where each vertex v is replaced by a finite *metric tree* T_v called the *tree-link* of v (see Definition 3.4). We refer to $\text{BU}(T)$ as the *blown-up tree*.
- (3) The construction of a surface bundle M_T over $\text{BU}(T)$. The metric tree $\text{BU}(T)$ captures the *geometry* of the base space of the bundle M_T , while the tree T only captures the topological features.
- (4) An effective construction of a metric on M_T such that the universal cover \tilde{M}_T is δ -hyperbolic, with δ depending only on some properties of T (see Theorems 3.10 and 3.13 below for precise statements).

3.1. Tight trees of non-separating curves. We refer the reader to [MM00] for details on subsurface projections (the necessary material is summarized in [Mj19, Section 2.1]).

Definition 3.1. [Mj19, Section 2.2] For any $L \geq 1$, an L -**tight tree of non-separating curves** in the curve graph $\mathcal{C}(S)$ consists of a simplicial tree T of bounded valence and a simplicial map $i : T \rightarrow \mathcal{C}(S)$ such that for every vertex v of T and for every pair of distinct vertices $u \neq w$ adjacent to v in T ,

$$d_{\mathcal{C}(S \setminus i(v))}(i(u), i(w)) \geq L.$$

An L -tight tree of non-separating curves for some $L \geq 3$ will simply be called a tight tree of non-separating curves. Such a tree is called a tight tree of homologous non-separating curves if, further, the curves $\{i(v) : v \in T\}$ are homologous.

Note that T need not be regular. We shall need the following condition guaranteeing that tight trees give isometric embeddings.

Proposition 3.2. [Mj19, Proposition 2.12] *There exists $L \geq 3$, such that the following holds. Let S be a closed surface of genus at least 2, and let $i : T \rightarrow \mathcal{C}(S)$ define an L -tight tree of non-separating curves. Then i is an isometric embedding.*

Chris Leininger told us a proof of the main technical Lemma that went into a proof of Proposition 3.2. A more general version (Proposition 8.2) due to Ken Bromberg will be given later.

3.2. Topological building blocks from links: non-separating curves. We recall the structure of building blocks from [Mj19, Section 2.3]. Let $i : T \rightarrow \mathcal{C}(S)$ be a tight tree of non-separating curves and let v be a vertex of T . The link of v in T is denoted as $lk(v)$. Let $S_v = S \setminus i(v)$. Then $i(lk(v))$ consists of a uniformly bounded number (depending only on the maximal valence of T) of vertices in $\mathcal{C}(S_v)$. Hence the weak convex hull $CH(i(lk(v)))$ of $i(lk(v))$ in $\mathcal{C}(S_v)$ admits a uniform approximating tree T_v . More precisely,

Lemma 3.3. *Let $i : T \rightarrow \mathcal{C}(S)$ be a tight tree of non-separating curves. There exists $k \geq 1$, depending only on the valence of vertices of T , such that for all $v \in T$ there exists a finite metric tree (T_v, d_{T_v}) and a surjective k -quasi-isometry¹*

$$\mathbb{P}_v : CH(i(lk(v))) \rightarrow T_v.$$

Further, \mathbb{P}_v maps the vertices of $i(lk(v))$ to the terminal vertices (leaves) of T_v such that for any pair of vertices $x, y \in i(lk(v))$,

$$d_{T_v}(\mathbb{P}_v(x), \mathbb{P}_v(y)) \leq d_{\mathcal{C}(S \setminus v)}(x, y) \leq d_{T_v}(\mathbb{P}_v(x), \mathbb{P}_v(y)) + k.$$

Definition 3.4. The finite tree T_v is called the **tree-link** of v .

Definition 3.5. For $i : T \rightarrow \mathcal{C}(S)$ a tight tree of non-separating curves and v any vertex of T , the **topological building block corresponding to v** is

$$M_v = S \times T_v.$$

The block M_v contains a distinguished subcomplex $i(v) \times T_v$ denoted as \mathcal{R}_v which we call the **Margulis riser** in M_v or the Margulis riser corresponding to v .

Note that in the definition of the Margulis riser, $i(v)$ is identified with a non-separating simple closed curve on S . The Margulis riser will take the place of Margulis tubes in doubly degenerate hyperbolic 3-manifolds. See also Definitions 4.1 and 4.4 below clarifying the use of the word ‘riser.’

Let $i : T \rightarrow \mathcal{C}(S)$ be a tight tree of non-separating curves. The **blow-up** $\text{BU}(T)$ of T is a *metric tree* obtained from T by replacing the $\frac{1}{2}$ -neighborhood of each $v \in T$ by the tree-link T_v (see [Mj19, Section 2.3] for a more detailed description).

Assembling the topological building blocks M_v according to the combinatorics of $\text{BU}(T)$, we get the following:

¹What we refer to as a k -quasi-isometry (resp. k -quasigeodesic) in this paper, is usually referred to as a (k, k) -quasi-isometry (resp. (k, k) -quasigeodesic) in the literature.

Definition 3.6. The **topological model corresponding to a tight tree** T of non-separating curves is

$$M_T = S \times \text{BU}(T).$$

$\Pi : M_T \rightarrow \text{BU}(T)$ will denote the natural projection giving M_T the structure of a surface bundle over the tree $\text{BU}(T)$.

For every v , the tree-link T_v occurs as a natural subtree of $\text{BU}(T)$ and M_v occurs naturally as the induced bundle $\Pi : M_v \rightarrow T_v$ after identifying $\Pi^{-1}(T_v)$ with M_v . The intersection of the tree-links $T_v, T_w \subset \text{BU}(T)$ of adjacent vertices v, w of T will be called a **mid-point vertex** vw . The pre-image $\Pi^{-1}(vw)$ of a mid-point vertex will be denoted as S_{vw} and referred to as a **mid-surface**.

3.3. Model geometry. We now recall from [Mj19, Section 3] the essential aspects of the geometry of M_T . To do this we need an extra hypothesis.

Definition 3.7. An L -tight tree $i : T \rightarrow \mathcal{C}(S)$ of non-separating curves is said to be R -**thick** if for any vertices u, v, w of T and any proper essential subsurface W of $S \setminus i(v)$ (including essential annuli),

$$d_W((i(u), i(w))) \leq R,$$

where $d_W(\cdot, \cdot)$ denotes distance between subsurface projections onto W .

Remark 3.8. The condition on R -thickness is really a local condition. By the Bounded Geodesic Image Theorem [MM00], it follows that it suffices to bound $d_W((i(u), i(w)))$ whenever u, w are adjacent to v .

Here, as elsewhere in this paper, a **Margulis tube** in a hyperbolic 3-manifold N will refer to a maximal solid torus $\mathbf{T} \subset N$, with $\text{inj}_x \leq \epsilon_M$ for all $x \in \mathbf{T}$, where ϵ_M is a Margulis constant for \mathbf{H}^3 fixed for the rest of the paper. In particular, all Margulis tubes are closed and embedded. For l a bi-infinite geodesic in T , let l_{\pm} denote the ending laminations given by the ideal end-points of $i(l) \subset \mathcal{C}(S)$ and let $\mathcal{V}(l)$ denote the vertices of T occurring along l . Let N_l denote the doubly degenerate hyperbolic 3-manifold with ending laminations l_{\pm} . Then l gives a bi-infinite geodesic in $\mathcal{C}(S)$ which is an L -tight R -thick tree with underlying space \mathbb{R} .

Definition 3.9. The manifold N_l will be called a doubly degenerate manifold of **special split geometry corresponding to the L -tight R -thick tree l** .

The reason for the terminology in Definition 3.9 will be explained in Proposition 3.15. For L large enough, if T is L -tight, each vertex $v \in \mathcal{V}(l)$ gives a Margulis tube in N_l [Mj19, Lemma 3.7]. Let \mathbf{T}_v denote the Margulis tube in N_l corresponding to $i(v)$ and $N_l^0 = N_l \setminus \bigcup_{v \in \mathcal{V}(l)} \mathbf{T}_v$. Let $\text{BU}(l)$ denote the bi-infinite geodesic in $\text{BU}(T)$ after blowing up l in T . Also let M_l denote the bundle over $\text{BU}(l)$ induced from $\Pi : M_T \rightarrow \text{BU}(T)$. Let $M_l^0 = M_l \setminus \bigcup_{v \in \mathcal{V}(l)} \mathcal{R}_v$ denote the complement of the risers in M_l .

Theorem 3.10. [Mj19] *Given $R \geq 0$ and $V_0 \in \mathbb{N}$, there exist $K \geq 1$ such that for an L -tight R -thick tree with $L \geq 3$ and valence bounded by V_0 , there exists a metric d_{weid} on M_T such that $\Pi : M_T \rightarrow \text{BU}(T)$ is a metric bundle of surfaces (cf. Definition 2.12) over the metric tree $\text{BU}(T)$ satisfying the following:*

- (1) *The induced metric for every Margulis riser \mathcal{R}_v is the metric product $S_e^1 \times T_v$, where S_e^1 is a unit circle.*

- (2) For any bi-infinite geodesic l in T , N_l^0 and M_l^0 are K -bi-Lipschitz homeomorphic by a homeomorphism that extends to their path-metric completions.
- (3) Further, if there exists a subgroup Q of $MCG(S)$ acting geometrically on $i(T)$, then this action can be lifted to an isometric fiber-preserving isometric action of Q on (M_T, d_{weld}) .

The bi-Lipschitz constant (in Definition 2.12) of the metric bundle $\Pi : M_T \rightarrow BU(T)$ is also at most K .

Henceforth we shall assume that a bi-Lipschitz homeomorphism as in Theorem 3.10 between N_l^0 and M_l^0 has been fixed. Theorem 3.10 establishes a bijective correspondence between the risers $R_v \cap M_l$ occurring along the geodesic $l \subset \mathcal{C}(S)$ and the Margulis tubes \mathbf{T}_v in N_l . We shall describe features of the hyperbolic geometry of the special split geometry manifold N_l in Proposition 3.15 below. For now, we dwell instead on the geometry of M_l . See Figure 1 below for a schematic representation of M_l in the special case that $T = \mathbb{R}$ with vertices at \mathbb{Z} . Also see Figure 2 for a description of the geometry of individual blocks. The edges of $BU(l)$ are at least $L - \kappa$ in length, where κ depends only on the maximal valence of a vertex of T (see Lemma 3.3). This is where the parameter L (for an L -tight tree) shows up in the model geometry.

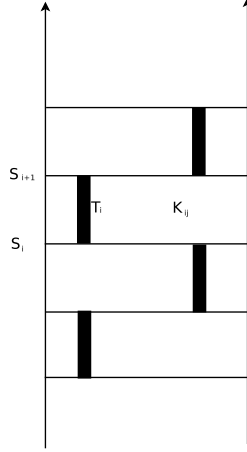


FIGURE 1. Model geometry for T a line

We draw the reader's attention to the fact that the topological building block M_i between S_i and S_{i+1} is a topological product (corresponding to the vertex i on the underlying tree $T = l = \mathbb{R}$ with vertices at \mathbb{Z}). Further, each such block contains a unique Margulis riser homeomorphic to $S^1 \times I$. Theorem 3.10 above shows that the complement M_l^0 of the Margulis risers in M_l and the complement N_l^0 of the Margulis tubes in the doubly degenerate hyperbolic manifold N_l are bi-Lipschitz homeomorphic. The place of building blocks M_i in M_l will be taken by split blocks in N_l (see Proposition 3.15 below).

Lemma 3.11. [Mj19, Corollary 3.25] *For a surface S , let ϕ be a pseudo-Anosov homeomorphism. Then there exists $R > 0$ such that any tight geodesic γ in $\mathcal{C}(S)$ preserved by ϕ is R -thick.*

More generally, let ϕ_1, \dots, ϕ_k freely generate a free convex cocompact subgroup $Q = F_k$. There exists R such that if Q preserves a quasi-isometrically embedded tree $T_Q \subset \mathcal{C}(S)$, then T_Q is also R -thick.

3.3.1. *Tube electrified metric.* It would be nice if $(\widetilde{M}_T, d_{weld})$ were δ -hyperbolic with a constant δ independent of L . This is simply not true as the Margulis risers \mathcal{R}_v lift to flat strips of the form $\mathbb{R} \times T_v$ and so the constant δ depends on the length of the largest isometrically embedded interval in T_v . There are a couple of ways to get around it. One way is to use relative hyperbolicity [Far98]. We shall use an alternate approach using pseudometrics and partial electrification [MP11] that preserves the T_v -direction in $\mathbb{R} \times T_v$.

An auxiliary pseudometric on M_T was defined in [Mj19] as follows. Equip each Margulis riser $\mathcal{R}_v = S^1 \times T_v$ with a product pseudometric that is zero on the first factor S^1 and agrees with the metric on T_v on the second. Let $(S^1 \times T_v, d_0)$ denote the resulting pseudometric.

Definition 3.12. [Mj14, p. 39] The **tube-electrified metric** d_{te} on M_v is the path-pseudometric defined as follows:

Paths that lie entirely within $M_v \setminus \mathcal{R}_v$ are assigned their d_{weld} -length. Paths that lie entirely within some \mathcal{R}_v are assigned their d_0 -length. The distance between any two points is now defined to be the infimum of lengths of paths given as a concatenation of subpaths lying either entirely \mathcal{R}_v or entirely outside \mathcal{R}_v except for end-points.

The **tube-electrified metric** d_{te} on M_T is defined to be the path-pseudometric that agrees with d_{te} on M_v for every v .

The lift of the metric d_{weld} (resp. d_{te}) on (M_T, d_{weld}) (resp. (M_T, d_{te})) to the universal cover \widetilde{M}_T is also denoted by d_{weld} (resp. d_{te}). Let $\widetilde{\mathcal{R}}_{\mathcal{M}}$ denote the collection of lifts of Margulis risers to \widetilde{M}_T . The main theorem of [Mj19] states:

Theorem 3.13. *Given $R > 0$, $V_0 \in \mathbb{N}$ there exists δ_0, L_0 such that the following holds. Let $i : T \rightarrow \mathcal{C}(S)$ be an L -tight R -thick tight tree of non-separating curves with $L \geq L_0$ such that the valence of any vertex of T is at most V_0 . Then*

- (1) $(\widetilde{M}_T, d_{te})$ is δ_0 -hyperbolic.
- (2) $(\widetilde{M}_T, d_{weld})$ is strongly hyperbolic relative to the collection $\widetilde{\mathcal{R}}_{\mathcal{M}}$.

Note that δ_0 in Theorem 3.13 depends on R but not on L provided L is large enough.

3.3.2. *Split geometry from a hyperbolic point of view.* For $l \subset T$ a bi-infinite geodesic and $v \in l$ a vertex, the tree link $T_{v,l}$ corresponding to the vertex v and tree l is an interval of length

$$h_v := d_{\mathcal{C}(S \setminus \{i(v)\})}(i(u), i(w)),$$

where u, w are the vertices on l adjacent to v . Note that since the underlying space of l regarded as a tree is the real line \mathbb{R} , we have an equality above. If on the other hand l is regarded as a bi-infinite geodesic in a tree T of bounded degree then the equality above is true only up to an error depending on the cardinality of v given by Lemma 3.3.

We fix a hyperbolic structure (S, h) on S for the rest of the discussion.

Definition 3.14. A D -bounded geometry surface in a hyperbolic manifold N is the image of a D -bi-Lipschitz embedding of (S, h) in N .

Proposition 3.15. [Mj19, Proposition 3.11] *Given $R > 0$, there exist $D \geq 1$ and $\epsilon > 0$ such that the following holds.*

Let N_l be a doubly degenerate manifold of special split geometry (see Definition 3.9) corresponding to an L -tight, R -thick tree $l \subset \mathcal{C}(S)$ with underlying space \mathbb{R} . Then

- (1) *there exists a sequence $\{S_i\}, i \in \mathbb{Z}$ of disjoint, embedded, incompressible, D -bounded geometry surfaces called **split surfaces** exiting the ends E_{\pm} as $i \rightarrow \pm\infty$ respectively. The surfaces are ordered so that $i < j$ implies that S_j is contained in the unbounded component of $E_+ \setminus S_i$.*
- (2) *Let B_i denote the topological product region between S_i and S_{i+1} . We call B_i a **split block**. Each split block B_i contains a distinguished Margulis tube \mathbf{T}_i^0 with core curve of length at most ϵ , such that for all $x \in N_l \setminus (\bigcup_i \mathbf{T}_i^0)$, the injectivity radius $\text{inj}_x(N_l) \geq \epsilon$.*

*There exists a solid torus neighborhood \mathbf{T}_i of \mathbf{T}_i^0 contained in a D -neighborhood of \mathbf{T}_i^0 , called a **splitting tube**, such that*

- (1) *$\mathbf{T}_i \cap S_i$ and $\mathbf{T}_i \cap S_{i+1}$ are annuli in S_i, S_{i+1} respectively. We can further assume that the annuli $\mathbf{T}_i \cap S_i$ and $\mathbf{T}_i \cap S_{i+1}$ contain the geodesic representatives of their core-curves in S_i, S_{i+1} respectively.*
- (2) *For $i \neq j$, $\mathbf{T}_i, \mathbf{T}_j$ are ϵ -separated from each other.*

Definition 3.16. The numbers $D \geq 1$ and $\epsilon > 0$ shall be called the parameters of special split geometry.

Thus the special split geometry manifold N_l can be decomposed as a union $N_l = \bigcup_i B_i$ of split blocks. See Figure 2 below, where a split block with a splitting tube is given. A section of the splitting tube T_i is drawn on the right side. Note the similarity between the block B_i and the region between S_i and S_{i+1} in Figure 1.

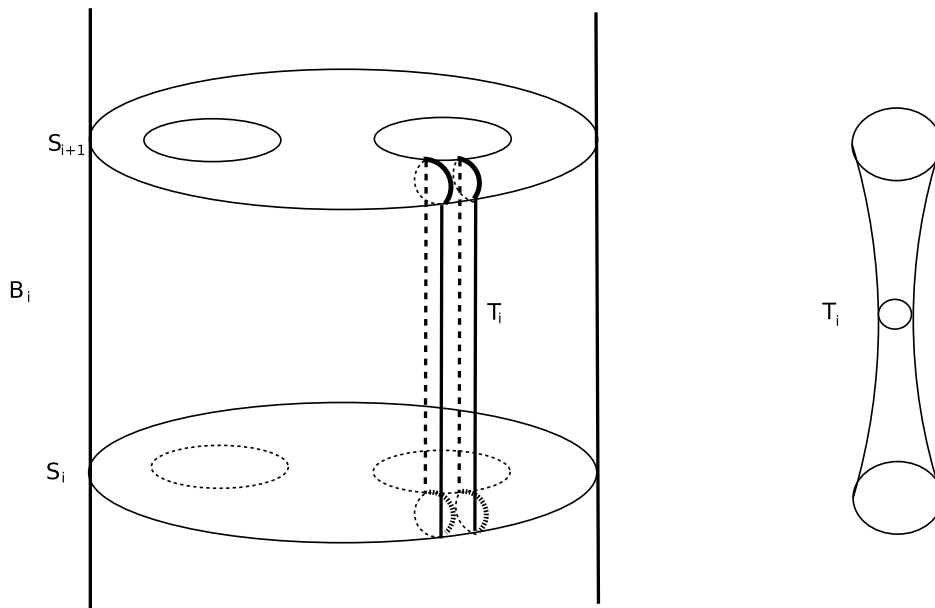


FIGURE 2. Split block B_i with splitting tube \mathbf{T}_i with core curve of length at most ϵ . Split surfaces S_i, S_{i+1} are of D -bounded geometry.

Ordering the vertices of l by $i \in \mathbb{Z}$, if v is the i -th vertex on l , we denote h_v by h_i and call it the **height** of the i -th split block.

3.3.3. Geometric limits. We shall need a few facts on geometric limits of doubly degenerate hyperbolic 3-manifolds N_l of special split geometry (see for instance [Thu80, Chapters 8, 9], [CEG87, Chapter I.3] and [Kap01, Chapters 8, 9] for details on geometric limits). In the proof of Proposition 6.13, we shall need to consider geometric limits of a sequence of special split geometry manifolds N_n with fixed parameters. For every n , fix a base split surface $S_{0,n} \subset N_n$ containing a base-point x_n . Assume without loss of generality (passing to a subsequence if necessary) that the triples $(N_n, S_{0,n}, x_n)$ converge geometrically. The base split surfaces $S_{0,n}$ are identified with each other by bi-Lipschitz homeomorphisms with bi-Lipschitz constant converging to 1. Let N_∞ denote the geometric limit of N_n with base surface $S_{0,n}$ (after passing to subsequences if necessary).

Denote the i -th split surface (resp. split block) of N_n as $S_{i,n}$ (resp. $B_{i,n}$). Denote the i -th splitting tube, i.e. the splitting tube in $B_{i,n}$, as $\mathbf{T}_{i,n}$. Let $h_{i,n}$ denote the height of $B_{i,n}$. Two cases arise for each i (passing to a further subsequence if necessary): Either $h_{i,n}$ remains bounded as n tends to infinity or $h_{i,n}$ tends to infinity as n tends to infinity.

We shall now describe geometric limits for the positive and negative ends N_n^+ and N_n^- of N_n as $n \rightarrow \infty$. If $h_{i,n}$ remains bounded for all $i \geq 0$ (resp. $i < 0$), then the positive (resp. negative) ends N_n^+ (resp. N_n^-) converge to a degenerate end of special split geometry. Else, let i be the least positive integer such that $h_{i,n}$ tends to infinity as n tends to infinity. Then the splitting tubes $\mathbf{T}_{i,n}$ converge to a rank one cusp. Similarly, for the negative ends.

Definition 3.17. If $h_{i,n}$ tends to infinity as n tends to infinity, the geometric limit $B_{i,\infty}$ of $B_{i,n}$ will be called a **limiting split block**.

Remark 3.18. Note that a limiting split block $B_{i,\infty}$ contains a rank one cusp arising as a limit of the splitting tubes $\mathbf{T}_{i,n}$.

Thus, when $h_{i,n}$ tends to infinity as n tends to infinity, the split blocks $B_{j,n}$ converge in the geometric limit N_∞ , to split blocks $B_{j,\infty}$ for $j < i$, while the split blocks $B_{i,n}$ converge to a limiting split block $B_{i,\infty}$ containing a rank one cusp. In either case, the sequence of split surfaces $S_{i,n}$ converges to a D -bounded geometry surface $S_{i,\infty}$. We shall also refer to $S_{i,\infty}$ as a **split surface in N_∞** .

Geometric limits of (M_n, d_{weld}) : Next, let (M_n, d_{weld}) denote the metric surface bundle (cf. Definition 2.12) L -bi-Lipschitz to N_n away from risers and Margulis tubes (Theorem 3.10 Item (2)). Let (M_n, d_{te}) denote the corresponding tube-electrified metric. In what follows, by a geometric limit of (M_n, d_{weld}) we shall mean a pointed Gromov-Hausdorff limit (see [Jan17] for a careful discussion). There is a uniform (independent of n) lower bound ϵ_0 on systoles of (M_n, d_{weld}) .

Further, the local geometry of (M_n, d_{weld}) is controlled in the sense that $\frac{\epsilon_0}{2}$ -balls have the following geometry. Take an $\frac{\epsilon_0}{2}$ -ball B in \mathbf{H}^3 , remove from it part of a totally geodesic plane P bounded by a geodesic. Let \overline{B} denote the path-metric completion of $(B \setminus P)$. Then the boundary of \overline{B} looks like a part of a totally geodesic plane folded in half; equivalently it looks like two copies P_1, P_2 of $P \cap B$ glued along the boundary geodesic. Let E denote the inverse image of $P \cap B$ under the exponential map. Then gluing P_i to E under the inverse of the exponential map (for $i = 1, 2$) we have a new ball B' . It follows from Theorem 3.10 that all $\frac{\epsilon_0}{2}$ -balls have such a geometry. It is easy to see that the Gromov-Hausdorff limit of a sequence of such balls is again of the same type. This is enough to allow us to extract geometric (or equivalently pointed Gromov-Hausdorff) limits of (M_n, d_{weld}) .

Let (M_∞, d_{weld}) denote the metric corresponding to the metric surface bundle on the geometric limit of the sequence (M_n, d_{weld}) . Let (M_∞, d_{te}) denote the corresponding tube-electrified metric. Let N_∞^0 denote N_∞ minus the union of Margulis tubes and rank one cusps. Similarly, let M_∞^0 denote M_∞ minus the union of limits of Margulis risers.

From the way the metrics d_{weld} and d_{te} are constructed, we get the following Lemma as a consequence of Theorem 3.10 Item (2). It says that the metric d_{weld} on the geometric limit N_∞ is essentially the same as (M_∞, d_{weld}) :

Lemma 3.19. N_∞^0 and M_∞^0 are K -bi-Lipschitz homeomorphic.

Informally, Lemma 3.19 says that the welding procedure and geometric limits essentially commute.

3.4. A criterion for quasiconvexity. Finally, we recall from [Mj19, Section 4.5] a necessary and sufficient condition for promoting quasiconvexity in vertex spaces X_v to quasiconvexity in the total space $(\widetilde{M}_T, d_{te})$. We refer the reader to [BF92, MR08, Gau16] for the relevant background on trees of relatively hyperbolic spaces and the flaring condition. Let $P : (\widetilde{M}_T, d_{te}) \rightarrow \text{BU}(T)$ denote the usual projection map. Also (X_v, d_v) will denote $P^{-1}(v)$ for $v \in \text{BU}(T)$. We recall some of the necessary notions from [Mj19]. In [Mj19, Lemma 4.19], we showed that given a genus $g \geq 2$, there exists ρ_0 , depending only on g and the maximum valence of T ,

such that for all $x \in (\widetilde{M}_T, d_{weld})$ (resp. $x \in (\widetilde{M}_T, d_{te})$) there exists a ρ_0 -qi-section (i.e. a section that is a ρ_0 -qi-embedding) of the bundle $P : (\widetilde{M}_T, d_{weld}) \rightarrow \text{BU}(T)$ (resp. $P : (\widetilde{M}_T, d_{te}) \rightarrow \text{BU}(T)$). When we refer to qi-sections below in $(\widetilde{M}_T, d_{te})$ and $(\widetilde{M}_T, d_{weld})$, we assume that they are all ρ_0 -qi-sections.

Definition 3.20. A disk $f : [a, b] \times I \rightarrow (\widetilde{M}_T, d_{te})$ is a **qi-section bounded hallway** if:

- (1) for all $v \in \text{BU}(T)$, $f^{-1}(X_v) = \{t\} \times I$ for some $t \in [a, b]$. Further, f maps $t \times I$ to a geodesic in (X_v, d_v) . The length of the geodesic $f(t \times I)$ in X_v will be denoted by \mathcal{L}_t or, if there is no confusion, by \mathcal{L}_v .
- (2) For all $s \in [0, 1]$, $P \circ f$ is an isometry of $[a, b] \times \{s\}$ (with the Euclidean metric) onto a geodesic $\sigma \subset \text{BU}(T)$.
- (3) $f([a, b] \times \{0\})$ and $f([a, b] \times \{1\})$ are contained in ρ_0 -qi-sections; in particular, they are ρ_0 -qi-sections of $P \circ f : [a, b] \times \{0\} \rightarrow \text{BU}(T)$ and $P \circ f : [a, b] \times \{1\} \rightarrow \text{BU}(T)$.

The **girth** of such a hallway is $\min_t \mathcal{L}_t$.

Definition 3.21. The space $(\widetilde{M}_T, d_{te})$, is said to satisfy the **qi-section bounded hallways flare** condition with parameters $\lambda > 1$, $m \geq 1$ and $H \geq 0$ if for any qi-section bounded hallway $f : [a, b] \times I \rightarrow (\widetilde{M}_T, d_{te})$ of girth at least H and with $b - a \geq m$,

$$\lambda \mathcal{L}_{\frac{a+b}{2}} \leq \max \{ \mathcal{L}_a, \mathcal{L}_b \}.$$

Remark 3.22. In [Mj19] the equivalence of the flare condition above and hyperbolicity of $(\widetilde{M}_T, d_{te})$ was established. Thus, Theorem 3.13 implies the existence of constants $\lambda > 1$, $m \geq 1$, $H \geq 0$ as above such that $(\widetilde{M}_T, d_{te})$ satisfies qi-section bounded hallways flare condition with parameters λ, m, H .

Definition 3.23. Suppose that $(\widetilde{M}_T, d_{te})$ satisfies the qi-section bounded hallways flare condition with parameters $\lambda > 1$, $m \geq 1$ and $H \geq 0$. A subset $Y \subset (X_v, d_v)$ will be said to **flare in all directions with parameter K** if for any geodesic segment $[c, d] \subset (X_v, d_v)$ with $c, d \in Y$ and any qi-section bounded hallway $f : [0, k] \times I \rightarrow (\widetilde{M}_T, d_{te})$ of girth at least H satisfying

- (1) $f(\{0\} \times I) = [c, d]$,
- (2) $\mathcal{L}_0 \geq K$,
- (3) $k \geq K$,

the length \mathcal{L}_k of $f(\{k\} \times I)$ satisfies $\mathcal{L}_k \geq \lambda \mathcal{L}_0$.

Proposition 3.24. [Mj19, Proposition 4.27] *Given K, C, δ_0 , there exists C_0 such that the following holds.*

Suppose that $(\widetilde{M}_T, d_{te})$ is δ_0 -hyperbolic. Let $P : (\widetilde{M}_T, d_{te}) \rightarrow \text{BU}(T)$ and X_v be as above. If Y is a C -quasiconvex subset of (X_v, d_v) and flares in all directions with parameter K , then Y is C_0 -quasiconvex in $(\widetilde{M}_T, d_{te})$.

Conversely, given δ_0, C_0 , there exist K, C such that the following holds.

Suppose that $(\widetilde{M}_T, d_{te})$ is δ_0 -hyperbolic. If $Y \subset X_v$ is C_0 -quasiconvex in $(\widetilde{M}_T, d_{te})$, then it is a C -quasiconvex subset in (X_v, d_v) and flares in all directions with parameter K .

4. THE STAIRSTEP CONSTRUCTION

The rest of the paper is devoted to constructing π_1 -injective tracks \mathcal{T} in surface bundles M over graphs \mathcal{G} and proving that any elevation $\tilde{\mathcal{T}}$ is quasiconvex in the universal cover \tilde{M} .

4.1. The stairstep construction in 3-manifolds. In this section, we motivate the general stairstep construction by describing it in the simpler setting of a 3-manifold N fibering over the circle with fiber S .

Definition 4.1. A stairstep in $S \times [0, n]$ is constructed from the following:

- (1) A tight geodesic v_0, v_1, \dots, v_n in the curve graph $\mathcal{C}(S)$ of S such that the simple closed curves σ_i on S corresponding to v_i are homologous to each other.
- (2) An essential subsurface Tread_i of $S \times \{i\}$ with boundary components $(\sigma_{i-1} \cup \sigma_i) \times \{i\}$, $1 \leq i \leq n$. These horizontal subsurfaces shall be referred to as **treads**.
- (3) An annulus Riser_i given by $\sigma_i \times [i, i + 1]$, $0 \leq i \leq n - 1$, referred to as a **riser**.

The union of the treads and risers $\cup_i \text{Riser}_i \cup \cup \text{Tread}_i$ will be referred to as a **stairstep** in $S \times [0, n]$ and denoted as \mathcal{T} . See Figure 3 for a schematic, where treads are horizontal and risers are vertical.

Gluing $S \times \{0\}$ to $S \times \{n\}$ via a homeomorphism $\phi : S \rightarrow S$ taking σ_0 to σ_n we obtain a surface (we will also call this a stairstep) \mathcal{T}_N in the mapping torus $N = S \times [0, n] / \sim_\phi$.

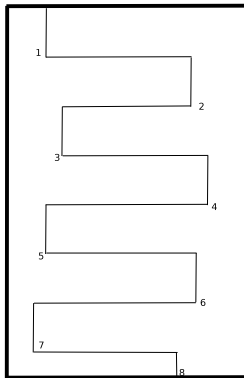


FIGURE 3. Stairstep in $S \times [0, n]$.

Example 4.2. An important motivating example is a geometrically finite surface constructed by Cooper-Long-Reid [CLR94, pp. 278-279]. In our language, what they build is a stairstep consisting of a single tread and riser. By taking care with orientations they ensure this stairstep \mathcal{T}_N is an orientable surface which can be isotoped to be transverse to the suspension flow. Such a surface must be incompressible, as can be seen by lifting it to the universal cover and observing that any flowline can intersect the lift at most once. (In fact, Cooper-Long-Reid prove

not just incompressibility, but the stronger assertion that \mathcal{T}_N is Thurston norm minimizing).

To ensure that the surface \mathcal{T}_N is geometrically finite (i.e. $\widetilde{\mathcal{T}}_N$ is quasiconvex in the universal cover \widetilde{N}), [CLR94] requires further that any elevation of \mathcal{T}_N to \widetilde{N} misses a flow-line. That this suffices to show geometric finiteness uses some machinery (either from the Thurston norm [CLR94, 3.14] or from Cannon-Thurston maps); we refer to [CLR94] for details.

Remark 4.3. Even if both the mapping torus N and stairstep \mathcal{T}_N are orientable, it may not be possible in general to isotope \mathcal{T}_N to be transverse to the suspension flow, when there are multiple treads.

Indeed, let $e(T\mathcal{F}) \in H^2(N)$ denote the Euler class of the tangent bundle to the foliation of N by the fibers $S \times \{t\}$. Fix an orientation on \mathcal{T}_N . Let \mathcal{T}_N^+ (resp. \mathcal{T}_N^-) denote the collection of treads that are positively (resp. negatively) co-oriented with respect to the suspension flow in N . Then $\langle e(T\mathcal{F}), \mathcal{T}_N \rangle = \sum_{\text{Tread}_i \in \mathcal{T}_N^+} \chi(\text{Tread}_i) - \sum_{\text{Tread}_i \in \mathcal{T}_N^-} \chi(\text{Tread}_i)$ [Thu86c]. If all the treads are not co-oriented in the same direction, it follows that $|\langle e(T\mathcal{F}), \mathcal{T}_N \rangle| < |\chi(\mathcal{T}_N)|$ and hence \mathcal{T}_N cannot be isotoped to be transverse to the suspension flow, cf. [FLP79, Exposé 14], especially the proof of [FLP79, Theorem 14.6].

We are sometimes able to use a different argument to prove incompressibility and quasi-convexity; see Sections 5.1 and 7.

4.2. The tree-stairstep.

Definition 4.4. A tree-stairstep in the topological model $M_T = S \times \text{BU}(T)$ corresponding to a tight tree T of homologous non-separating curves is built from the following data:

- (1) A tight tree $i : T \rightarrow \mathcal{C}(S)$ of non-separating curves such that the simple closed curves $\{i(v) : v \in V(T)\}$ on S corresponding to v are homologous to each other.
- (2) For every pair of adjacent vertices v, w of T , let Tread_{vw} be an essential subsurface of the mid-surface S_{vw} (cf. Definition 3.6) with boundary equal to $(i(v) \cup i(w))$. These subsurfaces Tread_{vw} shall be referred to as **treads**.
- (3) For each $v \in T$, a Margulis riser Riser_v in the topological building block M_v .

The union \mathcal{T}_T of the treads and risers $\cup_v \text{Riser}_v \cup \cup \text{Tread}_{vw}$ will be referred to as a **tree-stairstep** in $M_T = S \times \text{BU}(T)$ associated to $i : T \rightarrow \mathcal{C}(S)$.

4.3. Main Theorem. The following is the main theorem of this paper and will be proven in Section 7.

Theorem 4.5. *Given $R > 0$ and $V_0 \in \mathbb{N}$, there exists $\delta, L_0, C \geq 0$ such that the following holds. Let $i : T \rightarrow \mathcal{C}(S)$ be an L -tight R -thick tree of non-separating homologous curves with $L \geq L_0$ such that the valence of any vertex of T is at most V_0 . Let \mathcal{T}_T be a tree-stairstep associated to $i : T \rightarrow \mathcal{C}(S)$ and $\widetilde{\mathcal{T}}_T$ be an elevation to \widetilde{M}_T . Then*

- (1) $(\widetilde{M}_T, d_{te})$ is δ -hyperbolic.
- (2) $\widetilde{\mathcal{T}}_T$ is C -qi-embedded in $(\widetilde{M}_T, d_{te})$.
- (3) \mathcal{T}_T is incompressible in M_T , i.e. $\pi_1(\mathcal{T}_T)$ injects into $\pi_1(M_T)$.

- (4) If in addition there exists L_1 such that for every vertex v of T and for every pair of distinct vertices $u \neq w$ adjacent to v in T ,

$$d_{\mathcal{C}(S \setminus i(v))}(i(u), i(w)) \leq L_1,$$

then $(\widetilde{M}_T, d_{weld})$ is hyperbolic and $\widetilde{\mathcal{T}}_T$ is quasiconvex in $(\widetilde{M}_T, d_{weld})$.

5. EXAMPLES OF EIQ TRACKS

In this section, we shall give several families of examples to which Theorem 4.5 applies.

We start with some constructions of embedded geometrically finite surfaces in hyperbolic 3-manifolds. We shall then construct EIQ tracks in complexes fibering over finite graphs and hence by Theorem 2.7 construct cubulable complexes whose fundamental groups G are hyperbolic and fit into exact sequences of the form

$$1 \rightarrow \pi_1(S) \rightarrow G \rightarrow F_n \rightarrow 1.$$

5.1. Stairsteps in 3-manifolds. Our aim here is to construct a hyperbolic 3-manifold fibering over the circle and a stairstep in it. The first example of a stairstep we shall furnish has 2 treads and 2 risers. In a sense these are the simplest example of quasiconvex stairsteps. To the best of our knowledge, these are somewhat new even in the 3-manifold world inasmuch as they provide explicit examples of incompressible embedded surfaces in 3 manifolds fibering over the circle that cannot necessarily be isotoped to be transverse to the flow. We will indicate a generalization to k treads and k risers below. A somewhat different construction with 1 tread and 1 riser will be given later.

Let $MCG(S, \alpha)$ be the subgroup of $MCG(S)$ fixing a simple closed curve α , i.e. $MCG(S, \alpha) = Stab(\alpha) \subset MCG(S)$. Then $MCG(S, \alpha)$ is quasi-isometric to the product $MCG(S \setminus \alpha) \times \mathbb{Z}$, where \mathbb{Z} is the mapping class group of an annular neighborhood A of α (rel. boundary). To see this, note first that there is a central extension

$$1 \rightarrow \mathbb{Z} \xrightarrow{i} MCG(S, \alpha) \xrightarrow{q} MCG(S \setminus \alpha) \rightarrow 1.$$

It remains to show that there is a coarse surjection $MCG(S, \alpha) \xrightarrow{p} \mathbb{Z}$ such that $p \circ i$ is a bounded distance from the identity. To this end we shall use the the marking graph $\mathcal{M}(S)$ from [MM00], where it is shown that the mapping class group is quasi-isometric to $\mathcal{M}(S)$. Now, define $\mathcal{M}(S, \alpha)$ to be the subgraph of $\mathcal{M}(S)$ consisting of all markings that share the base-curve α . Let tw_α denote Dehn twist about α . We assume that tw_α is supported in a compact annulus contained in A . As in [MM00, Section 2.4] and [BM08, p. 1060], $\mathcal{M}(A)$ is identified with the curve complex $\mathcal{C}(A)$. In other words, we choose a transversal γ to α , fix $\gamma \cap A$ as a base-vertex in $\mathcal{C}(A)$, construct the remaining vertices $tw_\alpha^i(\gamma) \cap A$ by Dehn twists about α and join a pair of such vertices if the corresponding arcs can be isotoped to be disjoint (except for the end-points on the boundary of A). If $\mu \in \mathcal{M}(S, \alpha)$, observe that the collection $base(\mu)$ of base curves of μ has the form $base(\mu) = \alpha \sqcup \beta$, where β is a collection of curves arising as the vertex set of a maximal simplex in $\mathcal{C}(S \setminus \alpha)$. Also the transversal to any curve in β is disjoint from α . Hence, tw_α acts trivially on $\mathcal{M}(S \setminus \alpha)$. Clearly, tw_α acts by translation on $\mathcal{M}(A)$.

By [BM08, Lemma 1.7], a coarsely Lipschitz subsurface projection of $\mathcal{M}(S)$ onto $\mathcal{M}(A)$ exists, as also a coarsely Lipschitz subsurface projection of $\mathcal{M}(S)$ onto $\mathcal{M}(S \setminus \alpha)$. Further, a marking in $\mathcal{M}(S, \alpha)$ is coarsely determined by its projections

to $\mathcal{M}(S \setminus \alpha)$ and $\mathcal{M}(A)$ (see, in particular, [MM00, Lemma 2.4]). It follows that $\mathcal{M}(S, \alpha)$ is quasi-isometric to $\mathcal{M}(S \setminus \alpha) \times \mathcal{M}(A)$. Further, the quasi-isometry respects the action of tw_α . This proves the claim that $MCG(S, \alpha)$ is quasi-isometric to the product $MCG(S \setminus \alpha) \times \mathbb{Z}$. Let $d_A(\cdot, \cdot)$ denote distance between subsurface projections on A .

Definition 5.1. Let $R > 0$. For an element $\psi \in MCG(S, \alpha)$, the sequence ψ^n is said to be **renormalized by $\text{tw}_\alpha^{k_n}$** to have **R -bounded Dehn twist** along α if, for all $x \in \mathcal{M}(S)$,

$$d_A(x, \text{tw}_\alpha^{k_n} \circ \psi^n(x)) \leq R.$$

Since $\mathcal{M}(S, \alpha)$ is quasi-isometric to $\mathcal{M}(S \setminus \alpha) \times \mathcal{M}(A)$, it follows from the discussion above that there exists $R_0 > 0$ such that for all $R \geq R_0$ and $\psi \in MCG(S, \alpha)$, a sequence of renormalizing Dehn twists $\text{tw}_\alpha^{k_n}$ can always be found as in Definition 5.1 to ensure that $\text{tw}_\alpha^{k_n} \circ \psi^n(x)$ has R -bounded Dehn twist along α .

Definition 5.2. Let $R > 0$. Let $\sigma \subset S$ be a multicurve. Let $\psi : S \rightarrow S$ be a homeomorphism such that

- (1) ψ fixes σ point-wise.
- (2) $\psi : W \rightarrow W$ is a pseudo-Anosov homeomorphism for every component W of $(S \setminus \sigma_i)$.
- (3) For every component σ_i of σ , we choose k_{in} such that $\text{tw}_{\sigma_i}^{k_{in}} \circ \psi^n$ has R -bounded Dehn twist about σ_i .

Then $\text{tw}_{\sigma_i}^{k_{in}} \circ \psi^n$ is said to be a sequence of renormalized pseudo-Anosov homeomorphism of S **in the complement of σ** .

Since an $R > 0$ and a sequence k_{in} as above always exists, we abbreviate this by saying that ψ is a **pseudo-Anosov homeomorphism of S in the complement of σ** . Also, we shall denote the renormalized pseudo-Anosov homeomorphisms $\text{tw}_{\sigma_i}^{k_{in}} \circ \psi^n$ by $\overline{\psi}^n$.

Lemma 5.3. *Let $\sigma \subset S$ be a multicurve. Let $\psi : S \rightarrow S$ be a pseudo-Anosov homeomorphism of S in the complement of σ , and let A be an annular neighborhood of σ . Then there exists $R_0 > 0$ such that for W*

- (1) *either a connected component of A ,*
- (2) *or a proper essential subsurface of a component of $S \setminus A$,*

and any $n \in \mathbb{Z}$, there exist renormalizations $\overline{\psi}^n$ such that

$$d_W(\alpha, \overline{\psi}^n(\alpha)) \leq R_0.$$

Proof. For W a connected component of A , this follows from the hypothesis on ψ that for every component σ_i of σ , ψ has no Dehn twist about σ_i .

For W a proper essential subsurface of a component of $S \setminus A$, this follows from Lemma 3.11. \square

Stairsteps in 3-manifolds: 2 treads

Let σ_1, σ_2 be homologous disjoint non-separating simple closed curves on S . Let $\psi_i : S \rightarrow S$ be pseudo-Anosov homeomorphisms in the complement of σ_i , $i = 1, 2$. As in Definition 5.2, the renormalized pseudo-Anosov homeomorphisms will be denoted by $\overline{\psi}_i^p$. Define $\Phi_p = \overline{\psi}_1^p \cdot \overline{\psi}_2^p$ and let M_p be the 3-manifold fibering over the circle with fiber S and monodromy Φ_p . More generally, for $p_1, p_2 \in \mathbb{Z}$, define $\Phi(p_1, p_2) = \overline{\psi}_1^{p_1} \cdot \overline{\psi}_2^{p_2}$ and let $M(p_1, p_2)$ be the 3-manifold fibering over the

circle with fiber S and monodromy $\Phi(p_1, p_2)$. Since ψ_i acts by a pseudo-Anosov homeomorphism on $(S \setminus \sigma_i)$, the translation length of $\overline{\psi_i^p}$ on the curve graph $\mathcal{C}(S \setminus \sigma_i)$ is $O(p)$. In particular, for any $u \in \mathcal{C}(S \setminus \sigma_i)$, $d_{\mathcal{C}(S \setminus \sigma_i)}(u, \overline{\psi_i^p}(u))$ is $O(p)$. Further, by Lemma 5.3, there exists R such that any tight geodesic in $\mathcal{C}(S \setminus \sigma_i)$ preserved by ψ_i (and hence by any of its powers or renormalized powers) is R -thick. For $i = 1, 2$, let v_i be the vertex of $\mathcal{C}(S)$ corresponding to σ_i . Hence, given any $L > 1$ there exists $p_0 > 1$, such that for all $p \geq p_0$, the sequence

$$T = \cdots, \Phi_p^{-1}(v_1), \Phi_p^{-1}(v_2), v_1, v_2, \Phi_p(v_1), \Phi_p(v_2), \cdots$$

is an L -tight geodesic by Proposition 3.2. By Lemma 5.3, it is R -thick as well. More generally, for all p_1, p_2 with $|p_i| \geq p_0$, the sequence

$$T = \cdots, \Phi(p_1, p_2)^{-1}(v_1), \Phi(p_1, p_2)^{-1}(v_2), v_1, v_2, \Phi(p_1, p_2)(v_1), \Phi(p_1, p_2)(v_2), \cdots$$

is an L -tight R -thick geodesic. The cover of M_p (or more generally $M(p_1, p_2)$) corresponding to $\pi_1(S)$ is denoted as M_T . Let \mathcal{T}_T denote the associated stair-step in M_T constructed equivariantly with respect to the \mathbb{Z} -action generated by Φ_p (resp. $\Phi(p_1, p_2)$). Let \mathcal{T} denote the quotient stairstep in M_p (resp. more generally in $M(p_1, p_2)$). Then, by Theorem 4.5 \mathcal{T} is geometrically finite in M_p (resp. more generally in $M(p_1, p_2)$).

Stairsteps in 3-manifolds: k treads

There is a straightforward generalization of the above construction to a stairstep with k treads and k risers. Let $v_1 \cdots, v_k \in \mathcal{C}(S)$ be a sequence of vertices in $\mathcal{C}(S)$ forming a tight geodesic of simple closed non-separating curves σ_i that are homologous. As in the above construction, let $\psi_i : S \rightarrow S$ be pseudo-Anosov homeomorphisms in the complement of σ_i .

Let $I = \{1, \cdots, k\}$ denote a multi-index and p_I denote a k -tuple (p_1, \cdots, p_k) of integers. We say that $p_I \geq p$ if for each i , $|p_i| \geq p$. Define $\Phi(p_I) = \overline{\psi_1^{p_1} \cdots \psi_k^{p_k}}$ and let $M(p_I)$ be the 3-manifold fibering over the circle with fiber S and monodromy $\Phi(p_I)$. By Lemma 5.3, there exists R such that any tight geodesic in $\mathcal{C}(S \setminus \sigma_i)$ preserved by ψ_i (and hence by any of its renormalized powers) is R -thick (for $i = 1 \cdots, k$).

Again given any $L > 1$ there exists $p_0 > 1$, such that for all $p_I \geq p_0$, the sequence

$$\cdots, \Phi(p_I)^{-1}(v_1), \cdots, \Phi(p_I)^{-1}(v_k), v_1, \cdots, v_k, \Phi(p_I)(v_1), \cdots, \Phi(p_I)(v_k), \cdots$$

is an L -tight geodesic by Proposition 3.2. By Theorem 4.5 the associated stairstep surface \mathcal{T} in $M(p_I)$ is geometrically finite.

We restate the conclusion of the above construction as follows:

Proposition 5.4. *Let $v_1 \cdots, v_k \in \mathcal{C}(S)$ be a sequence of vertices in $\mathcal{C}(S)$ forming a tight geodesic of homologous simple closed non-separating curves σ_i . For $i = 1 \cdots, k$, let $\psi_i : S \rightarrow S$ be a pseudo-Anosov homeomorphism in the complement of σ_i . Let $\Phi(p_I) = \overline{\psi_1^{p_1} \cdots \psi_k^{p_k}}$ and let $M(p_I)$ be the 3-manifold fibering over the circle with fiber S and monodromy $\Phi(p_I)$. Then there exists $p_0 > 1$, such that for all $p_I \geq p_0$ the associated stairstep surface \mathcal{T} in $M(p_I)$ is geometrically finite.*

Remark 5.5. The existence of a quasiconvex hierarchy for such $M(p_I)$ now follows from Proposition 2.11. In fact, Thurston's theorem [Thu86a] shows that the Haken hierarchy is quasiconvex provided that the first surface is so. Cubulability of such manifolds now follows from Wise's Theorem 1.2). Theorem 8.4 will generalize

Theorem 4.5 to the case of a tree of homologous curves (possibly separating). Thus Proposition 5.4 can be extended to prove the existence of a geometrically finite surface in a class of fibered hyperbolic 3-manifolds with first Betti number one. In [Sis18], Sisto found sufficient conditions on Heegaard splittings of rational homology 3-spheres M to guarantee that they were Haken. The construction of Proposition 5.4 may be regarded as an analog of the main theorem of [Sis18] in the context of fibered manifolds M .

5.2. Tree stairsteps. We will now generalize the above construction to surface bundles over graphs. Given $R > 0$, choose $L \gg 1$ as in Theorem 4.5. For $j = 1, \dots, n$, let

$$\gamma_j := \dots, v_{-1,j}, v_0, v_{1,j}, \dots$$

be R -thick L -tight geodesics of homologous non-separating curves in $\mathcal{C}(S)$ invariant under pseudo-Anosov homeomorphisms Φ_j respectively (constructed as in Section 5.1). Note that v_0 belongs to γ_j for all $j = 1, \dots, n$. Since there is a unique non-separating curve on S up to the action of the mapping class group $MCG(S)$ any tight geodesic of homologous non-separating curves may be translated by an element of $MCG(S)$ so that the image contains v_0 (and thus this is a rather mild condition).

For $j = 2, \dots, n$, and $\Psi_j \in MCG(S, v_0)$, $\Psi_j \Phi_j \Psi_j^{-1}$ stabilize $\Psi_j(\gamma_j)$. We think of Ψ_j 's as *rotations* about v_0 (in the spirit of [DGO17]). For convenience of notation, let $\Psi_1 \in MCG(S, v_0)$ be the identity map. We shall choose the Ψ_j 's so that the 'angles' between the geodesics $\Psi_j(\gamma_j)$ at v_0 are large. See diagram below for $n = 2$, where the thick vertical and horizontal lines indicate tight geodesics passing through the origin v_0 . The thin broken lines indicate long tight geodesics in $\mathcal{C}(S \setminus v_0)$ joining $v_{\pm 1,1}$, $v_{\pm 1,2}$. Their lengths give a measure of the angles between the tight geodesics $\Psi_j(\gamma_j)$ at v_0 . The dotted lines indicate that all the vertices on the thin broken lines are at distance one from v_0 in $\mathcal{C}(S)$ (we have drawn these only in the positive quadrant to prevent cluttering up the figure).

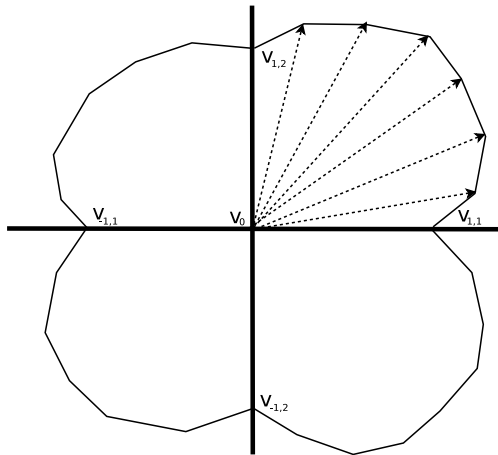


FIGURE 4. Two tight geodesics meeting at a large angle

We describe the construction more precisely.

Definition 5.6. Suppose that $\gamma_j = \{v_{ij}\}$, $i \in \mathbb{Z}$, $j = 1, \dots, n$ is a finite collection of L -tight R -thick geodesics, all passing through $v_0 = v_{0j}$. A collection $\Psi_j \in MCG(S, v_0)$, $j = 1, \dots, n$ is said to be a family of (L, R) -**large rotations** about v_0 for γ_j , $j = 1, \dots, n$ if for any distinct $v, w \in \bigcup_j \{\Psi_i v_{\pm 1, j}\}$,

- (1) $d_{\mathcal{C}(S \setminus v_0)}(v, w) \geq L$,
- (2) the geodesic in $\mathcal{C}(S \setminus v_0)$ joining v, w are R -thick.

Proposition 5.7. *Given $R > 0$, there exists $L_0 \geq 3$, such that the following holds: For $L \geq L_0$, and for $j = 1, \dots, n$, let γ_j be an R -thick L -tight-geodesic of homologous non-separating curves in $\mathcal{C}(S)$ invariant under pseudo-Anosov homeomorphisms Φ_j . Let Ψ_j 's be (L, R) -large rotations about v_0 for γ_j , $j = 1, \dots, n$.*

Let Q be the group generated by $\Psi_j \Phi_j \Psi_j^{-1}$, $j = 1, \dots, n$ (recall that Ψ_1 is the identity). Then the union of the Q -translates of $\Psi_j(\gamma_j)$ is an L -tight R -thick tree T . Associated to T is a Q -invariant track \mathcal{T}_T in M_T such that $\mathcal{T} := \mathcal{T}_T/Q$ is an EIQ track.

Proof. By choosing L sufficiently large (depending only the genus of S), Proposition 3.2 shows that Q is a convex cocompact free subgroup of $MCG(S)$ of rank n and T is an L -tight tree. R -thickness of T follows from R -thickness of the $\Psi_j(\gamma_j)$'s and the hypothesis that Ψ_j 's are (L, R) -large rotations about v_0 for γ_j , $j = 1, \dots, n$. The last statement now follows from Theorem 4.5. \square

We now have the following application of Theorem 4.5. It gives one of the main new examples of cubulable groups provided by this paper:

Theorem 5.8. *Let Q be a convex cocompact free subgroup of $MCG(S)$ of rank n as in Proposition 5.7 above. Let*

$$1 \rightarrow \pi_1(S) \rightarrow G \rightarrow Q_n \rightarrow 1$$

be the induced exact sequence of hyperbolic groups. Then G admits a quasiconvex hierarchy and is cubulable and virtually special.

Proof. Proposition 5.7 above furnishes the existence of an EIQ track \mathcal{T} . Theorem 2.7 now shows that G admits a quasiconvex hierarchy. Hence, by Wise's Theorem 1.2, G is cubulable and virtually special. \square

5.3. Stairsteps from regular tight trees. In Section 5.2 above, the tight tree T constructed in Proposition 5.7 is not regular. In this section, we sketch a construction of a regular tight tree and its associated tree-stairstep.

Stairstep in 3-manifolds with 1 riser: We first give a construction of an incompressible geometrically finite surface in a 3-manifold using one tread and one riser. Let $\sigma \subset S$ be a non-separating curve and $\psi : S \rightarrow S$ be a pseudo-Anosov homeomorphism in the complement of σ . Let $r : S \rightarrow S$ be an order 2-homeomorphism such that $r(\sigma)$ is homologous to σ , and $r(\sigma) \cap \sigma = \emptyset$. Let $\Phi_n = r \circ \bar{\psi}^n$ (recall from Definition 5.2 that $\bar{\psi}^n$ denotes a renormalized pseudo-Anosov). Then, $d_{\mathcal{C}(S \setminus \sigma)}(\psi^{-n}(r(\sigma)), (r(\sigma)))$ is $O(n)$. Hence there exists $R, N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, the sequence

$$\gamma(n) := \{\dots, \Phi_n^{-2}(\sigma), \Phi_n^{-1}(\sigma), \sigma, \Phi_n(\sigma), \Phi_n^2(\sigma), \dots\}$$

is an R -thick tight geodesic by Proposition 3.2. Given any $L \geq 3$, we can further choose N_0 large enough so that $\gamma(n)$ is L -tight for all $n \geq N_0$.

Let M_n denote the mapping torus of Φ_n . Let $\text{Tread} \subset S$ denote an essential subsurface with boundary equal to $\sigma \cup r(\sigma)$. Also, let $\text{Riser} = \sigma \times [0, 1] \subset S \times [0, 1]$. Then $\mathcal{T} = \text{Tread} \cup \text{Riser}$ gives a staircase surface in $S \times [0, 1]$. Further, in the mapping cylinder M_n , \mathcal{T} gives a staircase surface \mathcal{T}_n . By Theorem 4.5, \mathcal{T}_n is incompressible and geometrically finite for all large enough n .

Tree stairsteps with one riser: We now adapt the above example to construct tree-stairsteps. For ease of exposition, we will describe the construction for $n = 2$. Let σ be as above. For $j = 1, 2$, let $\psi_j : S \rightarrow S$ be pseudo-Anosov homeomorphisms in the complement of σ . If ψ_1, ψ_2 have distinct stable and unstable laminations in $(S \setminus \sigma)$, then by a ping-pong argument (see [FM02] for instance), ψ_1^m, ψ_2^m restricted to $(S \setminus \sigma)$ generate a convex cocompact rank 2 free subgroup of $MCG(S \setminus \sigma)$ for all large enough m . Assume without loss of generality therefore that $\langle \psi_1|_{(S \setminus \sigma)}, \psi_2|_{(S \setminus \sigma)} \rangle$ is a convex cocompact rank 2 free subgroup of $MCG(S \setminus \sigma)$.

Next, let r_1, r_2 be order two homeomorphisms of S satisfying the conditions on r above. We further require that $d_{\mathcal{C}(S \setminus \sigma)}(r_1(\sigma), r_2(\sigma))$ is large.

For m large enough, let $\Phi_i = \overline{\psi_i^m} \circ r_i$. Then the orbit of σ under $\langle \Phi_1, \Phi_2 \rangle$ gives an L -tight R -thick tree of homologous non-separating curves. We construct two treads Tread_i in the topological building block (Definition 3.5) M_v for $v = \sigma$ as follows: for $i = 1, 2$, Tread_i is defined to be an essential subsurface of S with $\sigma \cup r_i(\sigma)$ as its boundary. The tree-link T_v has, as its terminal vertices $\{\Phi_i^{\pm 1}\sigma\}$, $i = 1, 2$. The choices of r_i and m ensure that any two distinct $v, w \in \{\Phi_i^{\pm 1}\sigma\}$, $i = 1, 2$, satisfy the condition that $d_{\mathcal{C}(S \setminus \sigma)}(v, w)$ is large. A Margulis riser is constructed in M_v as in Definition 3.5. The tree-stairstep \mathcal{T}_T in M_T is obtained by equivariantly extending the track in M_v to M_T . Theorem 4.5 again guarantees that \mathcal{T}_T is incompressible and any elevation $\widetilde{\mathcal{T}}_T$ is quasiconvex. This gives a quotient track \mathcal{T} in $M_T / \langle \Phi_1, \Phi_2 \rangle$ as before and ensures the cubulability of the extension of $\pi_1(S)$ by $F_2 = \langle \Phi_1, \Phi_2 \rangle$.

6. QUASICONVEXITY OF TREADS

For the purposes of this section, we fix a closed surface S of genus at least 2. The main theorem of this section is the following:

Theorem 6.1. *Given $R > 0$, $V_0 \in \mathbb{N}$, there exists $C \geq 0$ such that the following holds.*

Let T be an L -tight R -thick tree of homologous non-separating curves in $\mathcal{C}(S)$ with valence at most V_0 and $L \geq 3$. Let \mathcal{T}_T be a tree-stairstep (cf. Definition 4.4) in M_T . Let v, w be a pair of adjacent vertices of T , S_{vw} be the corresponding mid-surface, and Tread_{vw} be a tread with boundary $i(v) \cup i(w)$. Then any elevation $\widetilde{\text{Tread}}_{vw}$ is C -quasiconvex in $(\widetilde{M}_T, d_{te})$.

The proof of Theorem 6.1 will occupy this section. We will need to recall some technology first.

6.1. Laminations, Cannon-Thurston Maps and quasiconvexity.

Definition 6.2. Let H be a hyperbolic subgroup of a hyperbolic group G . Let Γ_H, Γ_G denote Cayley graphs of H, G with respect to finite generating sets. Assuming that the generating set of G contains the generating set of H , let $i : \Gamma_H \rightarrow \Gamma_G$ denote the inclusion map. Let $\widehat{\Gamma}_H$ and $\widehat{\Gamma}_G$ denote the Gromov compactifications of Γ_H, Γ_G .

A **Cannon-Thurston map** for the pair (H, G) is a map $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$ which is a continuous extension of i .

We shall denote the Gromov boundaries of H, G by $\partial H, \partial G$ respectively. Note that these are independent of the choice of finite generating sets.

Theorem 6.3. [Mit98] *Let G be a hyperbolic group and let H be a hyperbolic normal subgroup of G . Let $i : \Gamma_H \rightarrow \Gamma_G$ be the inclusion map. Then a Cannon-Thurston map exists for the pair (H, G) , i.e. i extends to a continuous map \hat{i} from $\widehat{\Gamma}_H$ to $\widehat{\Gamma}_G$.*

Definition 6.4. An **algebraic lamination** [BFH97, CHL07, KL10, KL15, Mit97] for a hyperbolic group H is an H -invariant, flip invariant, closed subset $\mathcal{L} \subseteq \partial^2 H = (\partial H \times \partial H \setminus \Delta)$, where $(x, y) \rightarrow (y, x)$ is called the flip, and Δ is the diagonal in $\partial H \times \partial H$.

Definition 6.5. Suppose that a Cannon-Thurston map exists for the pair (H, G) . Let $\Lambda_{CT} = \{(p, q) \in \partial^2 H \mid \hat{i}(p) = \hat{i}(q)\}$. It is easy to see that Λ_{CT} is an algebraic lamination. We call it the **Cannon-Thurston lamination**.

Lemma 6.6. [Mit99] *Let G be a hyperbolic group and let H be a hyperbolic subgroup. Then H is quasiconvex in G if and only if a Cannon-Thurston map exists for the pair (H, G) and $\Lambda_{CT} = \emptyset$*

We shall use the following generalizations of Theorem 2.10.

Theorem 6.7. [Can96] *Let S be a closed surface and N be a doubly degenerate hyperbolic 3-manifold without parabolics homeomorphic to $S \times \mathbb{R}$. Let $H \subset \pi_1(S)$ be a finitely generated infinite index subgroup of $\pi_1(S)$. Then H is geometrically finite and any H -orbit is quasiconvex in \mathbf{H}^3 .*

Theorem 6.8. [DKL14] *Let*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence of hyperbolic groups, where H is a surface group. Let L be a finitely generated infinite index subgroup of H . Then L is quasiconvex in G .

We remark that the existence of an exact sequence of hyperbolic groups as in Theorem 6.8 forces H to be a free product of surface groups and a free group [RS94]. Thus the assumption that H is a surface group in Theorem 6.8 is not too restrictive.

For a convex cocompact subgroup Q of $MCG(S)$, it was shown in [FM02, KL08] that ∂Q embeds canonically in the projectivized measured lamination space $\mathcal{PML}(S) = \partial \text{Teich}(S)$ and also in the ending lamination space $\mathcal{EL}(S) = \partial \mathcal{C}(S)$. The associated map from ∂Q , thought of as a subset of $\mathcal{PML}(S)$, to ∂Q , thought of as a subset of $\mathcal{EL}(S)$, simply forgets the measure. Thus, $\partial Q \subset \mathcal{EL}(S)$ parametrizes a family of ending laminations. The ending lamination corresponding to $z \in \partial Q$ will be denoted as Λ_z .

Theorem 6.9. [MR18] *Let $1 \rightarrow H \rightarrow G \rightarrow F_n \rightarrow 1$ be an exact sequence of hyperbolic groups, with H a closed surface group. Then a Cannon-Thurston map exists for the pair (H, G) and the Cannon-Thurston lamination is given as a union of ending laminations parametrized by ∂F_n :*

$$\Lambda_{CT} = \bigcup_{z \in \partial F_n} \Lambda_z.$$

6.2. Uniform quasiconvexity of subsurfaces in bounded geometry. We motivate the proof of Theorem 6.1 by proving the corresponding statement for bounded geometry doubly degenerate hyperbolic 3-manifolds and bounded geometry bundles over trees. Lemma 6.10 and Proposition 6.11 below are not logically necessary for the rest of the paper. However, especially for those familiar with the model of bounded geometry manifolds [Min94], it gives a simplified version of one of the main steps of Theorem 4.5. To state it correctly we must use some tools and terminology not otherwise used in the paper. For a geodesic l in Teichmüller space there is a metric bundle $M_l \rightarrow l$ (called the **universal curve over l**) whose fiber over x is a hyperbolic surface representing the point x . See [Mj19, Section 3.1.2] for more details.

It is shown in [Min94] that given $\epsilon > 0$ there exists $e > 0$ and $L > 1$ such that the following holds:

Let N be a doubly degenerate hyperbolic 3-manifold with injectivity radius bounded below by $\epsilon > 0$ and ending laminations N_\pm . Then there is a bi-infinite Teichmüller geodesic l in $\text{Teich}_e(S)$ with endpoints N_\pm , and the universal curve M_l over l is L -bi-Lipschitz homeomorphic to N .

We will call M_l equipped with this metric the **Minsky model for N** . We shall say that an incompressible surface (possibly with boundary) in M_l is geometrically finite if the corresponding surface in N is geometrically finite.

Lemma 6.10. *Given $\epsilon, c > 0$, there exists $e, C > 0$ such that the following holds: Let N be a doubly degenerate hyperbolic 3-manifold with injectivity radius bounded below by $\epsilon > 0$ and ending laminations N_\pm . For e as above, let l be the bi-infinite Teichmüller geodesic in $\text{Teich}_e(S)$ with end-points N_\pm and let M_l be the Minsky model. For any $x \in l$, let S_x be the fiber over x and let v be an essential null-homologous multicurve on S_x such that v has length at most c . Let Σ be a subsurface of S_x with boundary $i(v)$. Then any elevation $\tilde{\Sigma}$ is C -quasiconvex in \tilde{M}_l .*

Proof. The existence of $e > 0$ follows from the discussion preceding the Lemma. We argue by contradiction. Suppose that the statement is not true. Then there exists a sequence of worse and worse counterexamples with data given by bi-infinite geodesics $l_i \subset \text{Teich}_e(S)$, M_i the universal curve over l_i , $x_i \in l_i$, S_i the fiber over x_i , v_i null-homologous multicurves in S_i of length at most c and an essential subsurface Σ_i of S_i with boundary v_i . Since $MCG(S)$ acts cocompactly on $\text{Teich}_e(S)$, we can assume further that all the x_i 's lie in a compact subset of $\text{Teich}_e(S)$. Choosing base-points $y_i \in S_i$, we pass to a geometric limit of a based sequence (M_i, y_i) . Such a limit exists because of the following. Firstly, a geometric limit N_∞ of the corresponding N_i 's exist and has the same lower bound on injectivity radius. Note also that, in fact, the geometric limit is a strong limit, i.e. the corresponding Kleinian groups converge algebraically. Hence there exists a Minsky model M_l such that N_∞ is bi-Lipschitz to M_l where l is a bi-infinite geodesic in Teich_δ for some $\delta > 0$ depending on e .

Secondly, the homeomorphisms between M_i and N_i are uniformly bi-Lipschitz (with bi-Lipschitz constant depending only on ϵ). The argument below shows that if the limit of a sequence Σ_i of essential subsurfaces is quasiconvex in M_∞ , then Σ_i 's are uniformly quasiconvex (with quasiconvexity constant depending only on ϵ). After passing to a subsequence if necessary, we have the following:

- (1) $(M_i, y_i) \rightarrow (M_\infty, y_\infty)$, where M_∞ is of bounded geometry. Further, M_∞ is bi-Lipschitz homeomorphic to M_l via their bi-Lipschitz homeomorphisms to N_∞ .
- (2) $(S_i, \Sigma_i, v_i) \rightarrow (S_\infty, \Sigma_\infty, v_\infty)$ for some S_∞ in M_∞ . Hence via the bi-Lipschitz homeomorphism to M_l we have an $x \in l$ such that the surface $S_x \subset M_l$ is at a uniformly bounded distance from the image of S_∞ (the bound depends only the bi-Lipschitz constant above and hence only on ϵ). We can, in fact assume, after passing to a further subsequence, that as vertices of a simplex in the curve-complex, the multicurves v_i form a constant sequence.
- (3) A null-homologous essential multicurve v on S_x freely homotopic to v_∞ via a uniformly bounded homotopy (the bound on the lengths of tracks of the homotopy depends only on ϵ). Hence there exists c' (depending on c and the bi-Lipschitz constant in Step 1) such that v has length at most c' on S .
- (4) An essential subsurface Σ of S_x freely homotopic via a uniformly (depending only on ϵ) bounded homotopy to Σ_∞ with v as boundary.

Let $\Sigma_i^h \subset N_i$ be the image of Σ_i in N_i under the bi-Lipschitz homeomorphism between M_i and N_i . Let $CC(\Sigma_i^h)$ denote the convex cores of the corresponding Kleinian groups. Abusing notation slightly, we denote the elevation of Σ_i^h in $CC(\Sigma_i^h)$ also as Σ_i^h . Since $\{\Sigma_i\}$ is a sequence of worse and worse counterexamples to the statement of the Lemma, it follows that there exist points in $CC(\Sigma_i^h)$ at distance d_i from Σ_i^h , where $d_i \rightarrow \infty$ as $i \rightarrow \infty$. Passing to the limit as $i \rightarrow \infty$, we note that $\Sigma_\infty \subset M_\infty$, and hence $\Sigma \subset M_l$ is *not* geometrically finite. But this contradicts Theorem 6.7 and proves the Lemma. \square

Recall [FM02] that a subgroup Q of $MCG(S)$ is K_0 -**convex cocompact** if the weak hull in $\text{Teich}(S)$ of its limit set in $\mathcal{PML}(S)$ lies in the K_0 -neighborhood of a point $o \in \text{Teich}(S)$. Let $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ be the associated exact sequence of hyperbolic groups. Let $Hull(\partial Q) \subset \text{Teich}_\epsilon(S)$ denote the weak hull of ∂Q in $\text{Teich}_\epsilon(S)$. Then there exists a tree T equipped with a geometric action of Q and a Q -equivariant K -qi-embedding of T into $Hull(\partial Q)$, where K depends on K_0 alone. We denote the image of T under the qi-embedding also as T .

Proposition 6.11. *Given $\epsilon, c, K_0 > 0$, there exists $C > 0$ such that the following holds:*

Let Q be a K_0 -convex cocompact free subgroup of $MCG(S)$, let M_Q denote the universal bundle over T , and let \widetilde{M}_Q denote its universal cover. Let $x \in Hull(\partial Q)$, S_x the fiber over x in M_Q and Tread_{vw} a subsurface of S_x with boundary $\widetilde{(i(v) \cup i(w))}$ such that each of v, w has length at most c . Then any elevation $\widetilde{\text{Tread}_{vw}}$ of Tread_{vw} is C -quasiconvex in \widetilde{M}_Q . Hence the subgroup of H carried by Tread_{vw} is C -quasiconvex in G .

Proof. Lemma 6.10 shows that there exists C_0 such that $\widetilde{\text{Tread}_{vw}}$ is C_0 -quasiconvex in \widetilde{M}_l for any bi-infinite $l \subset \text{Teich}_\epsilon(S)$ passing through (or more generally, within a uniformly bounded distance, depending only on ϵ , of) x .

We refer the reader to [Mj19, Proposition 4.27] for a general version of Proposition 3.24 that holds in the context of metric bundles like \widetilde{M}_Q with Q free (in addition to the bundle $(\widetilde{M}_T, d_{te})$ mentioned in Proposition 3.24). The converse direction of [Mj19, Proposition 4.27] now shows that there exists K such that $\widetilde{\text{Tread}_{vw}}$ flares in all directions with parameter K . The forward direction of [Mj19, Proposition 4.27]

finally shows that there exists C such that $\widetilde{\text{Tread}}_{vw}$ is C -quasiconvex in \widetilde{M}_Q . The last statement of the proposition follows from this. \square

6.3. Uniform quasiconvexity of treads in split geometry manifolds. Let l be an L -tight R -thick tree whose underlying topological space is homeomorphic to \mathbb{R} . Recall from Section 3.3.3 that the associated doubly degenerate 3-manifold N_l is of special split geometry. As in Section 3.3.3, let $\{B_i\}$ denote split blocks, $S_i = B_{i-1} \cap B_i$ denote split surfaces, and $\tau_i(+), \tau_i(-)$ be the geodesics on S_i corresponding to the core curves of the splitting tubes $\mathbf{T}_i, \mathbf{T}_{i-1}$. For convenience of notation, we shall also refer to the splitting tubes $\mathbf{T}_i, \mathbf{T}_{i-1}$ as $\mathbf{T}_i(+), \mathbf{T}_i(-)$ respectively. Also recall that there exists $D \geq 1$ such that any split surface is of D -bounded geometry.

Let $S_{c,i} \subset S_i \setminus (\tau_i(+)\cup\tau_i(-))$ denote a component of the surface S_i cut open along the curves $\tau_i(+), \tau_i(-)$. Let $\widetilde{S}_{c,i}$ be an elevation of $S_{c,i}$ to the universal cover \widetilde{N}_l . Let $\widetilde{S}_i \subset \widetilde{N}_l$ denote the set obtained by adjoining to $\widetilde{S}_{c,i}$ all the elevations of $\mathbf{T}_i(+)$ and $\mathbf{T}_i(-)$ that abut $\widetilde{S}_{c,i}$. Recall that l_i is the ‘height’ of the standard annulus in the welded split block B_i . By the construction of tree-links, l_i is approximately equal to $d_{\mathcal{C}(S \setminus v_i)}(v_{i-1}, v_{i+1})$ (up to an additive constant depending on R alone).

Remark 6.12 (Dependence of constants). Before we state the next Lemma, we briefly recount for the convenience of the reader, the implicit dependence of constants involved. Uniformity of the quasiconvexity constant C in Lemma 6.13 below is crucial in our argument. The statement of Lemma 6.13 shows that it depends on R (the parameter determining R -thickness). It also depends on the genus of the surface S ; but this has been fixed at the outset. In the final proof of Theorem 6.1, the quasiconvexity constant C will depend also on the valence of the tree T . The constant C certainly depends on the parameters D, ϵ of special split geometry in Definition 3.16; but as shown in Proposition 3.15, the parameters D, ϵ depend in turn only on R (and implicitly on the genus of S).

We shall prove that

Lemma 6.13. *Given $R > 0$, there exists $C \geq 0$ such that the following holds.*

Let l be an L -tight R -thick tree of homologous non-separating curves in $\mathcal{C}(S)$ whose underlying topological space is homeomorphic to \mathbb{R} and let N_l be the corresponding doubly degenerate manifold of special split geometry. Let \widetilde{S}_i be as above. Then, for all i , \widetilde{S}_i is C -quasiconvex in \widetilde{N}_l .

Proof. Note that quasiconvexity of \widetilde{S}_i was already known by Theorem 6.8. The effective dependence of C on R is what we establish now.

We argue by contradiction. Suppose that the statement is not true. We carry out a geometric limit argument and use the description of geometric limits from Section 3.3.3. Let $\{(N_m, x_m)\}$ be a sequence of worse and worse counterexamples, where we assume that x_m lies on a split surface $S(m) \subset N_m$ (we use the notation $S(m)$ to distinguish from a sequence S_i of split surfaces exiting the end of a fixed N_m). Then, (after passing to a subsequence if necessary), a geometric limit N_∞ exists, $S(m)$ converges to a split surface $S(\infty) \subset N_\infty$ (see paragraph following Definition 3.17 for terminology). Let $\tau(m+), \tau(m-)$ denote the distinguished curves on $S(m)$ homotopic to core curves of splitting tubes $\mathbf{T}(m+), \mathbf{T}(m-)$ abutting $S(m)$ (recall that splitting tubes are neighborhoods of Margulis tubes). Let $B(m+), B(m-)$

denote the split blocks of N_m containing $\mathbf{T}(m+), \mathbf{T}(m-)$ respectively. Also, let $l(m+), l(m-)$ denote the heights of $B(m+), B(m-)$ respectively.

A connected component $S_c(m)$ of $\widetilde{S}(m) \setminus \tau(m+) \cup \tau(m-)$ converges to a connected subsurface S_c of $S(\infty)$. Let \widetilde{S}_c denote an elevation of S_c to $\widetilde{N}(\infty)$. Let $\mathbf{T}(\infty, +), \mathbf{T}(\infty, -)$ denote the geometric limits of $\mathbf{T}(m+), \mathbf{T}(m-)$ respectively in N_∞ . Then $\mathbf{T}(\infty, -), \mathbf{T}(\infty, +)$ are either splitting tubes in split blocks or rank one cusps in limiting split blocks (Definition 3.17). Construct \widetilde{S} from \widetilde{S}_c by adjoining abutting elevations of $\mathbf{T}(\infty, -), \mathbf{T}(\infty, +)$ as in the discussion preceding the Lemma. Note also that the pair of simple closed curves $\tau(m+), \tau(m-)$ converge to a pair of simple closed curves $\tau(+), \tau(-)$ on $S(\infty)$ corresponding to the core curves of $\mathbf{T}(\infty, -), \mathbf{T}(\infty, +)$. Also, $\tau(+), \tau(-)$ are the boundary curves of S_c . As elements of the curve graph $\mathcal{C}(S)$, $\tau(m+) = \tau(+), \tau(m-) = \tau(-)$ for m large enough. We shall now prove the following claim before returning to the proof of the Lemma.

Claim 6.14. *There exists an ending lamination of $N(\infty)$ supported strictly in the interior of S_c .*

Proof of Claim 6.14. We now pass to the cover of N_∞ corresponding to $\pi_1(S_c)$ and apply the failure of quasiconvexity of \widetilde{S} . Since $\{(N_m, x_m)\}$ is a sequence of worse and worse counterexamples to the statement of the Lemma, the limiting connected subsurface S_c cannot be geometrically finite with possible cusps only along $\tau(+), \tau(-)$. Hence S_c is either geometrically infinite or there exists a non-peripheral curve α on S_c that is parabolic in N_∞ . In the latter case, we say that S_c has an **accidental parabolic**. Since $\pi_1(S_c)$ is a subgroup of $\pi_1(N(\infty))$, it is geometrically tame as a Kleinian group [Bon86, Can96, Can93]. Hence, either there exists an accidental parabolic on S_c or there exist simple closed curves α_n on S_c such that

- (Condition 1:) α_n is not homotopic to either of the peripheral curves $\tau(+), \tau(-)$, and
- (Condition 2:) The geodesic realization of α_n in the geometric limit $N(\infty)$ lies at distance at least n from S_c .

In either case, there exists an ending lamination of $N(\infty)$ supported strictly in the interior of S_c . \square

We return to the proof of Lemma 6.13: $N(\infty)$ can have one of the following types according to the cases enumerated below (see Section 3.3.3):

- (1) The heights $l(m+), l(m-)$ of the split blocks $B(m+), B(m-)$ remain bounded along the sequence. Two subcases arise. If $N(\infty)$ is doubly degenerate, quasiconvexity of \widetilde{S} follows from Theorem 6.7. This is a contradiction.

In case $N(\infty)$ has rank one cusps, then these must necessarily correspond to curves v in the curve graph $\mathcal{C}(S)$ at distance at least 2 from either $\tau(+)$ or $\tau(-)$ or both. In particular v cannot have distance 1 from both $\tau(+)$ and $\tau(-)$ and cannot be supported by S_c . Without loss of generality, assume that v lies in the $+$ end of $N(\infty)$ so that it has distance at least 2 from $\tau(-)$. Further, there is an ending lamination \mathcal{L}_v supported on $S \setminus v$. Neither v nor \mathcal{L}_v can be supported strictly in the interior of S_c .

The $-$ end is either degenerate or has a rank one cusp. A similar argument shows that no ending lamination or rank one cusp for the $-$ end can be supported strictly in the interior of S_c . This violates Claim 6.14.

- (2) Exactly one of the heights $l(m+), l(m-)$ of the standard annuli corresponding to the curves $\tau(m+), \tau(m-)$ remains bounded. Without loss of generality assume that $l(m-)$ is unbounded. This gives us a rank one cusp in the $-$ end of $N(\infty)$ corresponding to $\tau(-)$ and a (geometrically infinite) degenerate $-\infty$ end corresponding to the subsurface $S \setminus \tau(-)$. In particular, the ending lamination for the $-\infty$ end cannot be supported strictly in the interior of S_c .

We proceed as in Item (1) for the $+$ end. It is either degenerate (no rank one cusps) in which case its ending lamination cannot be supported strictly in the interior of S_c . Else, if the $+$ end has rank one cusps, then these must necessarily lie along curves v in $\mathcal{C}(S)$ at distance at least 2 from $\tau(-)$. In particular v cannot be supported by S_c . As before, there cannot exist an ending lamination of the $+$ end of $N(\infty)$ supported strictly in the interior of S_c .

- (3) This leaves us to deal with the case where both $l(m+), l(m-)$ tend to ∞ . In this case $N(\infty)$ has two rank one cusps corresponding to the boundary curves $\tau(+), \tau(-)$ of $S_c \subset S_g$. We shall deal with this case below.

Let $N(\infty)^0$ denote $N(\infty)$ minus a neighborhood of the two rank one cusps $T_{g,-}, T_{g,+}$. The manifold $N(\infty)^0$ has two ends:

- (1) $N(\infty)^+$ corresponding to $(S \setminus \tau(+)) \times [0, \infty)$
- (2) $N(\infty)^-$ corresponding to $(S \setminus \tau(-)) \times [0, \infty)$

The ends $N(\infty)^+$ and $N(\infty)^-$ are degenerate ends and have ending laminations $\mathcal{L}_+, \mathcal{L}_-$ supported on $(S \setminus \tau(+))$ and $(S \setminus \tau(-))$ respectively. Further, each leaf of \mathcal{L}_+ (resp. \mathcal{L}_-) is dense in \mathcal{L}_+ (resp. \mathcal{L}_-). Hence no leaf of \mathcal{L}_+ (or \mathcal{L}_-) is carried by S_c , again contradicting Claim 6.14. This completes the proof of the Lemma. \square

Lemma 6.13 establishes uniform quasiconvexity (see Remark 6.12 for dependence of constants) of the \tilde{S}_i 's in \tilde{N}_l . We would like now to transfer this quasiconvexity to the universal covers (\tilde{M}_l, d_{te}) . Recall that we have fixed a bi-Lipschitz homeomorphism between N_l^0 and M_l^0 after Theorem 3.10. Recall also that in (M_l, d_{weld}) , a splitting tube T of N_l gets replaced by a standard annulus A . Further, (see Definition 3.12) let (M_l, d_{te}) denote the tube-electrified metric obtained by electrifying the \mathbb{R} -direction of the universal cover $\tilde{A} = \mathbb{R} \times I$ of A . Lemma 3.19 allows us to pass between N_l and the corresponding metric surface bundle (M_l, d_{weld}) . Let $S_{c,i}$ be as in Lemma 6.13 and the discussion preceding it. Let $\tilde{S}_{c,i}$ be an elevation of $S_{c,i}$ to \tilde{M}_l . Let $\{A_{ij}\}$ denote the collection of elevations of standard annuli abutting $\tilde{S}_{c,i}$. Abusing notation slightly, let

$$(\tilde{S}_i, d_{weld}) = (\tilde{S}_{c,i} \bigcup \bigcup_j \tilde{A}_{ij}, d_{weld}),$$

and let

$$(\tilde{S}_i, d_{te}) = (\tilde{S}_{c,i} \bigcup \bigcup_j \tilde{A}_{ij}, d_{te}),$$

denote the corresponding tube-electrified metric.

We shall need a slightly modified version of [Mj14, Lemma 8.3] that says that partial electrification preserves quasiconvexity. Since this argument will appear again in the proof of Theorem 4.5 in Section 7, we briefly discuss the transfer of information from relatively hyperbolicity to tube-electrified spaces in our context

by specializing the general discussion in [MP11]. The spaces we shall be dealing with are as follows:

- (1) $(\widetilde{M}_T, d_{weld})$, which is δ -hyperbolic relative to the collection $\widetilde{\mathcal{R}}_{\mathcal{M}}$ of elevations of risers to \widetilde{M}_T by Theorem 3.13.
- (2) $\mathcal{E}((\widetilde{M}_T, d_{weld}), \widetilde{\mathcal{R}}_{\mathcal{M}})$, which is the electric space obtained from $(\widetilde{M}_T, d_{weld})$ by electrifying the elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$.
- (3) $\mathcal{E}((\widetilde{M}_T, d_{te}), \widetilde{\mathcal{R}}_{\mathcal{M}})$, which is the electric space obtained from $(\widetilde{M}_T, d_{te})$ by electrifying the elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$. Note that $\mathcal{E}((\widetilde{M}_T, d_{weld}), \widetilde{\mathcal{R}}_{\mathcal{M}})$ is quasi-isometric to $\mathcal{E}((\widetilde{M}_T, d_{te}), \widetilde{\mathcal{R}}_{\mathcal{M}})$.

The following Lemma is now a special case of [MP11, Lemmas 1.20, 1.21]:

Lemma 6.15. *Given $\delta_1, D_1 > 0$, there exists $\alpha_1 > 0$ such that the following holds. Suppose that $\mathcal{E}((\widetilde{M}_T, d_{weld}), \widetilde{\mathcal{R}}_{\mathcal{M}})$ is δ_1 -hyperbolic, and that any two distinct elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$ are at distance at least D_1 from each other in $(\widetilde{M}_T, d_{weld})$. Let γ be a geodesic in $\mathcal{E}((\widetilde{M}_T, d_{te}), \widetilde{\mathcal{R}}_{\mathcal{M}})$ joining a, b . Let γ_e^1 be an electric geodesic in $\mathcal{E}((\widetilde{M}_T, d_{weld}), \widetilde{\mathcal{R}}_{\mathcal{M}})$ joining a, b . Let γ_e^2 be an electric geodesic in $\mathcal{E}((\widetilde{M}_T, d_{te}), \widetilde{\mathcal{R}}_{\mathcal{M}})$ joining a, b . Let γ_h be a geodesic in $(\widetilde{M}_T, d_{weld})$ joining a, b . Then $\gamma, \gamma_e^1, \gamma_e^2$ and γ_h track each other away from $\widetilde{\mathcal{R}}_{\mathcal{M}}$ with tracking constant α_1 .*

Let γ and γ_e^1 be as in Lemma 6.15. We shall now modify γ_e^1 to a more canonical representative. Let x_v, y_v be the entry and exit point of the electric geodesic γ_e^1 for an elevation $v \times T_v$ of a Margulis riser. Interpolating a geodesic segment η_v in $v \times T_v$ between x_v, y_v , for every $v \times T_v$ that γ_e^1 meets, we obtain a path

$$\bar{\gamma} = (\gamma_e^1 \setminus \bigcup_{\widetilde{\mathcal{R}}_{\mathcal{M}}} v \times T_v) \bigcup_v \eta_v.$$

where the first union ranges over all elevations of risers in $\widetilde{\mathcal{R}}_{\mathcal{M}}$ and the second union ranges over all v such that γ_e^1 meets the Margulis riser $v \times T_v$.

Since γ_e^1 and γ_h track each other away from $\widetilde{\mathcal{R}}_{\mathcal{M}}$, we have the following consequence of Lemma 6.15:

Corollary 6.16. *Given $\delta_1, D_1 > 0$, there exists $\alpha_1 > 0$ such that the following holds. Suppose that $\mathcal{E}((\widetilde{M}_T, d_{weld}), \widetilde{\mathcal{R}}_{\mathcal{M}})$ is δ_1 -hyperbolic, and that any two distinct elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$ are at distance at least D_1 from each other in $(\widetilde{M}_T, d_{weld})$. Let γ_h be as in Lemma 6.15 and $\bar{\gamma}$ be as in the discussion preceding the Corollary. Then $\bar{\gamma}$ and γ_h track each other away from $\widetilde{\mathcal{R}}_{\mathcal{M}}$ with tracking constant α_1 .*

Lemma 6.15 and Corollary 6.16 above recast the ‘‘Bounded Coset Penetration’’ property of relative hyperbolicity [Far98] in our context.

Lemma 6.17. *Given $R > 0$ there exists $\delta \geq 0, C > 0$ such that the following holds. Let N be a doubly degenerate manifold of special split geometry corresponding to an L -tight R -thick tree with underlying space \mathbb{R} . Let (M, d_{weld}) (resp. (M, d_{te})) denote the corresponding metric surface bundle with the welded (resp. tube-electrified) metric. Let $\widetilde{\mathcal{R}}_{\mathcal{M}}$ denote the collection of elevations of Margulis risers. Then*

- (1) (\widetilde{M}, d_{te}) is δ -hyperbolic and $(\widetilde{M}, d_{weld})$ is strongly δ -hyperbolic relative to the collection $\widetilde{\mathcal{R}}_{\mathcal{M}}$.
- (2) $(\widetilde{S}_i, d_{te})$ is C -qi-embedded in (\widetilde{M}, d_{te}) .

Proof. The first conclusion follows from Theorem 3.13. The proof of the second conclusion follows that of [Mj14, Lemma 8.3] and we indicate the slight modification to the setup we need. Let \mathcal{T} denote the collection of splitting tubes in N . Uniform separation of splitting tubes (see Proposition 3.15, where the separation constant ϵ is shown to depend only on R) shows that \tilde{N} is strongly hyperbolic relative to the collection $\tilde{\mathcal{T}}$ of elevations of $\mathbf{T} \in \mathcal{T}$, with constant of relative hyperbolicity depending only on d_0 and the constant D appearing in Proposition 3.15 defining the relationship between Margulis tubes and splitting tubes (note that D depends only on R).

Let N_0 denote $N \setminus \bigcup_{\mathbf{T} \in \mathcal{T}} \mathbf{T}^0$, where \mathbf{T}^0 denotes the interior of \mathbf{T} . Let \tilde{N}_0 denote the elevation of N_0 to \tilde{N} and let $\partial\tilde{\mathcal{T}}$ denote the collection of elevations of $\partial\mathbf{T}$ for $\mathbf{T} \in \mathcal{T}$. Equipping N_0 with the induced path metric d_p , it follows that \tilde{N}_0 is strongly hyperbolic relative to the collection $\partial\tilde{\mathcal{T}}$ (with the same constant as the strong relative hyperbolicity constant of \tilde{N} relative to $\tilde{\mathcal{T}}$). Since (\tilde{M}, d_{te}) can be obtained by tube electrification of (\tilde{N}_0, d_p) , Lemma 6.15 and Corollary 6.16 show that geodesics in (\tilde{M}_T, d_{weld}) and (\tilde{M}_T, d_{te}) track each other away from the elements of $\tilde{\mathcal{R}}_{\mathcal{M}}$.

Next observe that $(\tilde{\mathcal{S}}_i, d_{te})$ in (\tilde{M}, d_{te}) is obtained by the tube-electrification procedure applied to $\tilde{\mathcal{S}}_i$ in \tilde{N} . By Lemma 6.13, there exists C_0 depending on R alone such that $\tilde{\mathcal{S}}_i$ is C_0 -qi-embedded in \tilde{N} . Identifying \tilde{N}_0 and \tilde{M}_0 via the bi-Lipschitz homeomorphism of Theorem 3.10, the previous paragraph now shows that geodesics in \tilde{N} and those in (\tilde{M}, d_{te}) track each other away from the elements of $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{R}}_{\mathcal{M}}$. Hence, under tube-electrification, quasigeodesics in $\tilde{\mathcal{S}}_i \subset \tilde{N}$ go to quasigeodesics in $(\tilde{\mathcal{S}}_i, d_{te})$ (as in the proof of [Mj14, Lemma 8.3]) proving the second assertion. \square

Proposition 6.18. *Given $R > 0$, there exists $C \geq 0$ such that the following holds. Let l be an L -tight R -thick tree in $\mathcal{C}(S)$ whose underlying topological space is homeomorphic to \mathbb{R} and let (M_l, d_{weld}) be the corresponding metric surface bundle. Let (\tilde{M}_l, d_{te}) denote the corresponding tube-electrified metric. Let $S_{c,i}$ be as above. Then any elevation $\tilde{S}_{c,i}$ is C -quasiconvex in (\tilde{M}_l, d_{te}) .*

Proof. During the course of the proof of this proposition, ‘uniform’ will mean ‘depending only on R ’. Observe that in the (pseudo)metric space (\tilde{M}_l, d_{te}) , each elevation (A_{ij}, d_{te}) of a standard annulus is uniformly quasi-isometric to the interval $[0, l_i]$, where l_i is its height. Since each $\{A_{ij}\}$ is uniformly quasi-isometric to an interval, $(\tilde{S}_{c,i}, d_{te})$ is uniformly quasiconvex in $(\tilde{\mathcal{S}}_i, d_{te})$ (in fact, it is a uniform Lipschitz retract obtained by projecting each $\tilde{A}_{ij} = \mathbb{R} \times [0, l_i]$ onto $\mathbb{R} \times \{0\}$). Hence by the second conclusion of Lemma 6.17, $(\tilde{S}_{c,i}, d_{te})$ is uniformly quasiconvex in (\tilde{M}_l, d_{te}) . \square

6.4. Uniform quasiconvexity of trends in (\tilde{M}_T, d_{te}) : Proof of Theorem 6.1.
We now turn our attention to $P : (\tilde{M}_T, d_{te}) \rightarrow \text{BU}(T)$ with T an L -tight R -thick tree.

We restate the main theorem of this section for convenience.

Theorem 6.1. *Given $R > 0$, $V_0 \in \mathbb{N}$, there exists $C \geq 0$ such that the following holds.*

Let T be an L -tight R -thick tree of homologous non-separating curves in $\mathcal{C}(S)$ with valence at most V_0 and $L \geq 3$. Let \mathcal{T}_T be a tree-stairstep (cf. Definition 4.4) in M_T . Let v, w be a pair of adjacent vertices of T , S_{vw} be the corresponding mid-surface, and Tread_{vw} be a tread with boundary $i(v) \cup i(w)$. Then any elevation $\widetilde{\text{Tread}}_{vw}$ is C -quasiconvex in $(\widetilde{M}_T, d_{te})$.

Proof. With Proposition 6.18 in place, the proof is a replica of that of Proposition 6.11. Proposition 6.18 shows that there exists C_0 such that $\widetilde{\text{Tread}}_{vw}$ is C_0 -quasiconvex in $(\widetilde{M}_l, d_{te})$ for any bi-infinite geodesic l in $\text{BU}(T)$ passing through the midpoint vertex vw .

The converse direction of Proposition 3.24 now shows that there exists K such that $\widetilde{\text{Tread}}_{vw}$ flares in all directions with parameter K . The forward direction of Proposition 3.24 finally shows that there exists C such that $\widetilde{\text{Tread}}_{vw}$ is C -quasiconvex in $(\widetilde{M}_T, d_{te})$. \square

7. QUASICONVEXITY OF THE TRACK: PROOF OF THEOREM 4.5

We are now in a position to prove the main technical Theorem of this paper, Theorem 4.5, which we restate for convenience:

Theorem 4.5. *Given $R > 0$ and $V_0 \in \mathbb{N}$, there exists $\delta, L_0, C \geq 0$ such that the following holds. Let $i : T \rightarrow \mathcal{C}(S)$ be an L -tight R -thick tree of non-separating homologous curves with $L \geq L_0$ such that the valence of any vertex of T is at most V_0 . Let \mathcal{T}_T be a tree-stairstep associated to $i : T \rightarrow \mathcal{C}(S)$ and $\widetilde{\mathcal{T}}_T$ be an elevation to \widetilde{M}_T . Then*

- (1) $(\widetilde{M}_T, d_{te})$ is δ -hyperbolic.
- (2) $\widetilde{\mathcal{T}}_T$ is C -qi-embedded in $(\widetilde{M}_T, d_{te})$.
- (3) \mathcal{T}_T is incompressible in M_T , i.e. $\pi_1(\mathcal{T}_T)$ injects into $\pi_1(M_T)$.
- (4) If in addition there exists L_1 such that for every vertex v of T and for every pair of distinct vertices $u \neq w$ adjacent to v in T ,

$$d_{\mathcal{C}(S \setminus i(v))}(i(u), i(w)) \leq L_1,$$

then $(\widetilde{M}_T, d_{weld})$ is hyperbolic and $\widetilde{\mathcal{T}}_T$ is quasiconvex in $(\widetilde{M}_T, d_{weld})$.

Proof. The first conclusion of the theorem follows from Theorem 3.13.

We turn now to the second assertion. Note that any elevation $\widetilde{\mathcal{T}}_T$ of the track \mathcal{T}_T to $(\widetilde{M}_T, d_{te})$ consists of

- elevations $\widetilde{\text{Tread}}_{vw}$ of treads that are (uniformly) C_0 -quasiconvex in $(\widetilde{M}_T, d_{te})$ by Theorem 6.1 (C_0 depends only on R and V_0),
- attached to one another via elevations of Margulis risers $(v \times T_v, d_{te})$. Note that these are uniformly (depending only on V_0) quasi-isometric to the tree-link T_v since the direction corresponding to v (and hence its universal cover \mathbb{R} is electrified in the d_{te} metric).

Note also that by the hypothesis of L -tightness, the distance between any two terminal vertices of T_v (for any v) is at least L . Thus, we have a collection of C_0 -quasiconvex treads meeting elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$ at large distances (at least L , up to an additive constant depending on V_0) from each other. Thus any geodesic γ in $\widetilde{\mathcal{T}}_T$ is built up of alternating segments of the following types:

- geodesics in (the intrinsic metric on) elevations $\widetilde{\text{Tread}}_{vw}$ of treads Tread_{vw} ,
- geodesics in Margulis risers $(v \times \widetilde{T}_v, d_{te})$ of length at least L (up to an additive constant depending on V_0).

Recall the local-global principle for quasigeodesics in hyperbolic spaces (see [BH99, p. 407] for instance): Given $\delta > 0$ and $C' \geq 1$, there exists $\Upsilon > 0$ and $C \geq 1$ such that if a parametrized path β in a δ -hyperbolic metric space satisfies the property that each subpath of β of length Υ is a C' -quasigeodesic, then β is a C -quasigeodesic. By Theorem 3.13 or the first statement of the theorem, $(\widetilde{M}_T, d_{te})$ is δ -hyperbolic. By Lemma 6.17 any tread along with abutting risers is also C -qi-embedded as is any riser. Hence there exists C' depending on L such that any subpath of γ of length Υ is contained in

- (1) a tread along with abutting risers, or
- (2) a riser.

Hence, by the local-global principle for quasigeodesics γ is a C -quasigeodesic provided L is sufficiently large. This proves the second conclusion of the theorem.

We now prove the third assertion of Theorem 4.5. Fix an elevation $\widetilde{\mathcal{T}}_T$ of \mathcal{T}_T . We denote the elevations $\widetilde{\text{Tread}}_{vw}$ contained in $\widetilde{\mathcal{T}}_T$ by \widetilde{Tr}_i where i ranges over some countable set. Since any two elevations $\widetilde{Tr}_1, \widetilde{Tr}_2$ lying over distinct mid-point vertices of $\text{BU}(T)$ are separated by at least L , and since each \widetilde{Tr}_i is simply connected, any closed essential loop σ in $\widetilde{\mathcal{T}}_T$ must have at least one geodesic segment in the elevation $(v \times \widetilde{T}_v, d_{te})$ of a riser. We can thus put σ in standard form so that it is a union of geodesic segments in \widetilde{Tr}_i and geodesic segments in elevations of risers. But then the local-global principle for quasigeodesics in hyperbolic spaces again shows that σ is a C -quasigeodesic for L sufficiently large; in particular, if $L \gg C$, it cannot begin and end at the same point. This contradiction proves the third conclusion of the theorem.

To prove the last assertion, assume now that there is a uniform upper bound L_1 on the diameters of tree-links T_v . Observe first that $(\widetilde{M}_T, d_{weld})$ is δ -hyperbolic relative to the elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$ by Theorem 3.13. The elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$ are now uniformly hyperbolic, since the upper bound L_1 furnishes uniform quasi-isometries of $v \times \widetilde{T}_v$ with \mathbb{R} . Hence, $(\widetilde{M}_T, d_{weld})$ is hyperbolic by [Bow12] (see also [Mj08, Proposition 2.9] or [DM17, Proposition 6.1]).

For $a, b \in \widetilde{\mathcal{T}}_T$, let γ_h be a geodesic in $(\widetilde{M}_T, d_{weld})$ joining a, b . Also, let $\bar{\gamma}$ be constructed from the geodesic in $(\widetilde{M}_T, d_{te})$ joining a, b as in the discussion preceding Corollary 6.16. From the second assertion of this theorem, it follows that $\widetilde{\mathcal{T}}_T$ is C -qi-embedded in $(\widetilde{M}_T, d_{te})$. Hence $\bar{\gamma}$ lies in a bounded neighborhood of $\widetilde{\mathcal{T}}_T$ in $(\widetilde{M}_T, d_{te})$. By Lemma 6.15 and Corollary 6.16, geodesics in $(\widetilde{M}_T, d_{weld})$ and geodesics in $(\widetilde{M}_T, d_{te})$ track each other away from elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$; also, $\bar{\gamma}$ and γ_h track each other away from elements of $\widetilde{\mathcal{R}}_{\mathcal{M}}$. Using strong relative hyperbolicity of $(\widetilde{M}_T, d_{weld})$ relative to the collection $\widetilde{\mathcal{R}}_{\mathcal{M}}$, it follows that γ_h lies in a bounded neighborhood of $\widetilde{\mathcal{T}}_T$ in $(\widetilde{M}_T, d_{weld})$. This proves the fourth assertion of the theorem and completes the proof of Theorem 4.5. \square

Remark 7.1. A part of the fourth assertion of Theorem 4.5, viz. quasiconvexity of $\widetilde{\mathcal{T}}_T$ in $(\widetilde{M}_T, d_{weld})$ actually holds (as the proof above shows) without the extra hypothesis of the existence of L_1 . However, in the absence of such an upper bound L_1 , $(\widetilde{M}_T, d_{weld})$ is no longer hyperbolic, but only strongly hyperbolic relative to the collection $\widetilde{\mathcal{R}}_{\mathcal{M}}$.

8. GENERALIZATION: SEPARATING CURVES

The purpose of this section is to summarize some notions from [Mj19] in order to extend the discussion in Section 3 to allow the possibility of multicurves, as well as separating curves.

8.1. Tight trees. The collection of complete graphs in the curve graph $\mathcal{C}(S)$ will be denoted as $\mathcal{C}_\Delta(S)$. (Equivalently, $\mathcal{C}_\Delta(S)$ is the collection of 1-skeleta of simplices in the curve complex. However, since we have only used the curve graph and not the curve complex in this paper we opt for the earlier point of view.) Let $\gamma = \{\cdots, v_{-1}, v_0, v_1, \cdots\}$ be a geodesic (finite, semi-infinite, or bi-infinite) in a tree T and $i : V(T) \rightarrow \mathcal{C}_\Delta(S)$ a map. A path in $\mathcal{C}(S)$ **induced by** γ is a choice of simple closed curves $\sigma_i \in i(v_i)$. The map i will be called an **isometric embedding** if any path induced in $\mathcal{C}(S)$ by a geodesic γ in T is a geodesic in $\mathcal{C}(S)$. We now generalize the notion of tight trees of non-separating curves (Definition 3.1) to multicurves.

Definition 8.1. [Mj19] An L -**tight tree** in the curve graph $\mathcal{C}(S)$ consists of a (not necessarily regular) simplicial tree T of bounded valence and a map $i : V(T) \rightarrow \mathcal{C}_\Delta(S)$ such that

- (1) for every vertex v of T , $S \setminus i(v)$ consists of exactly one or two components. Further, if $S \setminus i(v)$ consists of two components and $i(v)$ contains more than one simple closed curve, then each component of $i(v)$ is individually non-separating. If $S \setminus i(v)$ consists of two components, v is called a **separating vertex** of T .
- (2) for every pair of adjacent vertices $u \neq v$ in T , and any vertices u_0, v_0 of the simplices $i(u), i(v)$ respectively,

$$d_{\mathcal{C}(S)}(u_0, v_0) = 1.$$

- (3) There is a **distinguished component** Y_v of $S \setminus i(v)$ such that for any vertex u adjacent to v in T , $i(u) \subset Y_v$ (automatic if $i(v)$ is non-separating). For $i(v)$ separating, we shall refer to $Y'_v := S \setminus Y_v$ as the **secondary component** for v .
- (4) for every pair of distinct vertices $u \neq w$ adjacent to v in T , and any vertices u_0, w_0 of the simplices $i(u), i(w)$ respectively,

$$d_{\mathcal{C}(Y_v)}(u_0, w_0) \geq L.$$

An L -tight tree for some $L \geq 3$ will simply be called a tight tree.

We recall the following from [Mj19] due to Bromberg.

Proposition 8.2. *There exists $L > 1$ such that if S is a closed surface of genus at least 2, and $i : V(T) \rightarrow \mathcal{C}_\Delta(S)$ defines an L -tight tree, then i is an isometric embedding.*

In fact, as shown in [Mj19], $L \geq \max(2M, 4D)$ suffices, where M is a constant given by the Bounded Geodesic Image Theorem [MM00] and D is the Behrstock constant. We shall not need this.

8.2. Balanced trees. A special class of tight trees will be required to generalize Theorem 4.5. For $i : V(T) \rightarrow \mathcal{C}_\Delta(S)$ a tight-tree, tree-links T_v are defined as in Definition 3.4 with the qualifier that for v a separating vertex, the weak convex hull $CH(i(lk(v)))$ is constructed in the curve graph $\mathcal{C}(Y_v)$ of the distinguished component Y_v of $S \setminus i(v)$. The “balanced” condition we shall introduce now essentially guarantees that for v a separating vertex, T_v serves as the tree-link of the secondary component Y'_v as well. For v a separating vertex, the tree link T_v furnishes, a priori, a way of constructing a model geometry for $Y_v \times T_v$. To extend this to a model geometry for $S \times T_v$, we need the balanced condition below.

For w adjacent to v let T'_w denote the connected component of $T \setminus \{v\}$ containing w . Let $\Pi'_v(T'_w)$ denote the subsurface projection of $i(V(T'_w))$ onto $\mathcal{C}(Y'_v)$.

Definition 8.3. [Mj19] A tight tree $i : T \rightarrow \mathcal{C}(S)$ is said to be a **balanced** tree with parameters D, k if

- (1) For every separating vertex v of T , and every adjacent vertex w ,

$$\text{diam}(\Pi'_v(T'_w)) \leq D.$$

- (2) For the secondary component Y'_v , let $\text{sec}(v) \subset \mathcal{C}(Y'_v)$ denote the collection of curves in $\Pi'_v(T'_w)$ as w ranges over all vertices adjacent to v in T . Let $CH(\text{sec}(v))$ denote the weak hull of $\text{sec}(v)$ in $\mathcal{C}(Y'_v)$. We demand that there exists a surjective k -quasi-isometry

$$\mathbb{P}'_v : CH(\text{sec}(v)) \rightarrow T_v$$

to the tree-link T_v , such that for any vertex w of T adjacent to v ,

$$\mathbb{P}'_v(\Pi'_v(T'_w)) = \mathbb{P}_v(w),$$

(where \mathbb{P}_v is the projection defined in Definition 3.4).

The notions of topological building block M_v (Definition 3.5), blow-up $\text{BU}(T)$, topological model (Definition 3.6) and tree-stairstep (Definition 4.4) go through exactly as in Section 3.2. Definition 8.3 guarantees that the weak hulls $CH(i(lk(v))) \subset \mathcal{C}(Y_v)$ and $CH(\text{sec}(v)) \subset \mathcal{C}(Y'_v)$ are coarsely quasi-isometric to each other and to the tree-link T_v . With these constructions, the proof of Theorem 4.5 goes through and we have:

Theorem 8.4. *Given $R, D, k > 0$, $V_0 \in \mathbb{N}$ there exists $\delta_0, L_0, C \geq 0$ such that the following holds. Let $i : T \rightarrow \mathcal{C}_\Delta(S)$ be an L -tight R -thick balanced tree of homologous curves with parameters D, k such that the valence of any vertex of T is at most V_0 . Then, for $L \geq L_0$,*

- (1) $(\widetilde{M}_T, d_{te})$ is δ_0 -hyperbolic.
- (2) $\widetilde{\mathcal{T}}_T$ is C -qi-embedded in $(\widetilde{M}_T, d_{te})$.
- (3) \mathcal{T}_T is incompressible in M_T , i.e. $\pi_1(\mathcal{T}_T)$ injects into $\pi_1(M_T)$.
- (4) If in addition there exists L_1 such that for every vertex v of T and for every pair of distinct vertices $u \neq w$ adjacent to v in T ,

$$d_{\mathcal{C}(S \setminus i(v))}(i(u), i(w)) \leq L_1,$$

then $\widetilde{\mathcal{T}}_T$ is quasiconvex in $(\widetilde{M}_T, d_{weld})$.

Item (1) was proven in [Mj19]. The proof of Theorem 4.5 in Sections 6 and 7 goes through mutatis mutandis to establish the remaining conclusions. We briefly refer the reader to the relevant sections in the text:

- 1) For the construction of a topological building block M_v associated to a tree-link T_v , see Definition 3.5. For the blown-up tree $\text{BU}(T)$, the whole topological model $S \times \text{BU}(T)$ (Definition 3.6) see Section 3.2.
- 2) The construction of a tree-stairstep (Definition 4.4) goes through as the metric on the riser \mathcal{R}_v is coarsely well-defined thanks to the tree-link T_v being coarsely well-defined.
- 3) We now turn to the proof of quasiconvexity of $\widetilde{\mathcal{T}}_T$ in Theorem 8.4. Quasiconvexity of treads (Theorem 6.1) follows as before. There are two points to note. First, since the tree-links T_v are now only coarsely well-defined, the parameters D, k of Definition 8.3 will be involved in Remark 6.12. Second, the geometric limit argument in Lemma 6.13 really only used the fact that no component of the ending laminations of the geometric limit can be contained in the subsurface S_c obtained by cutting S along $\tau(\pm)$. This allows Theorem 6.1 to go through.
- 4) Finally, with Theorem 6.1 in place, Section 7 goes through without change.

Remark 8.5. With Theorem 8.4 in place, Propositions 5.4 and 5.7 now generalize in a straightforward way simply by dropping the hypothesis that the curves are non-separating. We illustrate this first by the construction of a stairstep in a 3-manifold with two treads. Note first that Lemma 5.3 allows for σ to be separating. We now let σ_1, σ_2 be a pair of disjoint null-homologous simple closed curves. Let $S \setminus \sigma_i = W_{i1} \cup W_{i2}$. Define $\psi_i : S \rightarrow S$ such that ψ_i is a pseudo-Anosov homeomorphism in the complement of σ_i (see Definition 5.2 and the conventions therein). Then for large enough p_1, p_2 the 3-manifold with monodromy $\overline{\psi_1^{p_1}} \cdot \overline{\psi_2^{p_2}}$ is hyperbolic. Further it admits a stairstep surface \mathcal{T} with two risers corresponding to σ_1, σ_2 and two treads. Theorem 8.4 now shows that \mathcal{T} is incompressible and geometrically finite. There is now a straightforward generalization to the case with k treads and k risers, allowing us to drop the hypothesis that the curves are non-separating in Proposition 5.4.

To extend this to stairsteps as in Section 5.2, it remains only to check the balanced condition. (The large rotations argument is as before.) This needs to be checked only locally at a vertex v_0 . Let σ be a separating simple closed curve corresponding to a vertex $v_0 \in \mathcal{C}(S)$. Let $S \setminus \sigma = W_1 \sqcup W_2$. It suffices to construct (for $i = 1, 2$) $\Phi_i = \phi_{i1} \sqcup \phi_{i2}$ —two pseudo-Anosov homeomorphisms in the complement of σ ensuring the balanced condition. Thus $\phi_{ij} : W_j \rightarrow W_j$ are pseudo-Anosov homeomorphisms and we renormalize by Dehn twists about σ if necessary to ensure that Φ_i has no Dehn twist about σ . Let $\gamma_{ij} \subset \mathcal{C}(W_j)$ be the axes of ϕ_{ij} . Translating γ_{ij} suitably by a power of an auxiliary pseudo-Anosov homeomorphism of W_j , we can ensure that $d_{\mathcal{C}(W_1)}(\gamma_{11}, \gamma_{21})$ and $d_{\mathcal{C}(W_2)}(\gamma_{12}, \gamma_{22})$ are comparable:

$$\frac{1}{2} d_{\mathcal{C}(W_2)}(\gamma_{12}, \gamma_{22}) \leq d_{\mathcal{C}(W_1)}(\gamma_{11}, \gamma_{21}) \leq d_{\mathcal{C}(W_2)}(\gamma_{12}, \gamma_{22}).$$

We can also ensure that the shortest geodesic in $\mathcal{C}(W_j)$ between γ_{1j}, γ_{2j} realizing $d_{\mathcal{C}(W_j)}(\gamma_{1j}, \gamma_{2j})$ is thick (since we have used a power of an auxiliary pseudo-Anosov homeomorphism of W_j to translate one away from the other). Finally, suppose that the translation lengths of ϕ_{ij} are within a multiplicative factor $k \geq 2$ of each other. Let T_v denote the (primary) tree-link constructed for W_1 from ϕ_{11}^p, ϕ_{21}^p . Assuming that $N \gg p$, T_v looks like the English letter H , where the horizontal bar has length approximately $d_{\mathcal{C}(W_1)}(\gamma_{11}, \gamma_{21})$ and the two verticals have length of the order of p (up to a multiplicative factor k). For the secondary subsurface W_2 , the secondary

weak hull $CH(\sec(v))$ constructed for W_2 from ϕ_{12}^p, ϕ_{22}^p is then $2k$ -bi-Lipschitz homeomorphic to T_v ensuring the balanced condition.

8.3. Virtual algebraic fibering. We say that a group G **virtually algebraically fibers** if it admits a surjective homomorphism to \mathbb{Z} with finitely generated kernel. We recall the following Theorem of Kielak [Kie18]:

Theorem 8.6. [Kie18] *Let G be cubulable. Then G virtually algebraically fibers if and only if the first ℓ^2 -betti number $\beta_1^{(2)}(G)$ vanishes.*

As a consequence we have:

Proposition 8.7. *Given $R > 0$, there exists $L \geq 3$ such that the following holds. Let G be a hyperbolic group admitting an exact sequence*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1,$$

where $H = \pi_1(S)$ is the fundamental group of a closed surface and Q acts freely and cocompactly by isometries on an R -thick, L -tight tree of homologous curves (separating or non-separating). Then G virtually algebraically fibers.

Proof. G is cubulable by Proposition 5.7 and Remark 8.5. Since G contains an infinite index finitely generated normal subgroup H , $\beta_1^{(2)}(G) = 0$ [Gab02, PT11]. Hence G virtually algebraically fibers by Theorem 8.6. \square

Remark 8.8. Recent work of Kropholler and Walsh [KW19] establishes that G as in Proposition 8.7 is incoherent provided $b_1(G)$ is strictly greater than $b_1(Q)$. We expect that the finitely generated normal subgroup of G that Proposition 8.7 furnishes is not finitely presented and hence the hypothesis $b_1(G) > b_1(Q)$ in [KW19] may not be necessary.

ACKNOWLEDGMENTS

We thank Chris Leininger for telling us the proof of Proposition 3.2 and Ken Bromberg for helpful correspondence and the proof of Proposition 8.2. MM gratefully acknowledges several extremely helpful conversations with Dani Wise who generously shared many of his ideas on cubulations with him and for the hospitality the McGill Mathematics Department extended during May 2015. All authors were supported in part by the National Science Foundation under Grant No. DMS-1440140 at the Mathematical Sciences Research Institute in Berkeley during Fall 2016 program in Geometric Group Theory where this collaboration began. This work was also partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund. The second and third authors acknowledge the hospitality of the International Centre for Theoretical Sciences during a workshop on Groups, Geometry and Dynamics in November 2017. Research of JFM supported by Simons Collaboration Grant #524176. Research of MM partly supported by a DST JC Bose Fellowship, Matrics research project grant MTR/2017/000005 and CEFIPRA project No. 5801-1. Research of MS partly supported by the Israel Science Foundation (grant 1026/15).

REFERENCES

- [Ago13] I. Agol. The virtual Haken conjecture (With an appendix by Ian Agol, Daniel Groves, and Jason Manning). *Doc. Math.* 18, pages 1045–1087, 2013.

- [BCM12] J. F. Brock, R. D. Canary, and Y. N. Minsky. The Classification of Kleinian surface groups II: The Ending Lamination Conjecture. *Ann. of Math.* 176 (1), *arXiv:math/0412006*, pages 1–149, 2012.
- [BD17] Jeffrey F. Brock and Nathan M. Dunfield. Norms on the cohomology of hyperbolic 3-manifolds. *Invent. Math.*, 210(2):531–558, 2017.
- [BF92] M. Bestvina and M. Feighn. A Combination theorem for Negatively Curved Groups. *J. Diff. Geom.*, vol 35, pages 85–101, 1992.
- [BFH97] M. Bestvina, M. Feighn, and M. Handel. Laminations, trees, and irreducible automorphisms of free groups. *Geom. Funct. Anal.*, 7(2):215–244, 1997.
- [BH99] M. Bridson and A. Haefliger. Metric spaces of non-positive curvature. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 319. Springer-Verlag, Berlin, pages xxii+643 pp., 1999.
- [BM08] Jason A. Behrstock and Yair N. Minsky. Dimension and rank for mapping class groups. *Ann. of Math.* (2), 167(3):1055–1077, 2008.
- [Bon86] F. Bonahon. Bouts de varietes hyperboliques de dimension 3. *Ann. Math. vol.124*, pages 71–158, 1986.
- [Bow98] Brian H. Bowditch. Cut points and canonical splittings of hyperbolic groups. *Acta Math.*, 180(2):145–186, 1998.
- [Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra and Computation*. 22, 1250016, 66pp, 2012.
- [BW12] N. Bergeron and D. T. Wise. A boundary criterion for cubulation. *Amer. J. Math.* 134, no. 3, pages 843–859, 2012.
- [Can93] R. D. Canary. Ends of hyperbolic 3 manifolds. *J. Amer. Math. Soc.*, pages 1–35, 1993.
- [Can96] R. D. Canary. A covering theorem for hyperbolic 3 manifolds. *Topology* 35, pages 751–778, 1996.
- [CEG87] R. D. Canary, D. B. A. Epstein, and P. Green. Notes on Notes of Thurston. in *Analytical and Geometric Aspects of Hyperbolic Spaces*, pages 3–92, 1987.
- [CHL07] T. Coulbois, A. Hilion, and M. Lustig. Non-unique ergodicity, observers’ topology and the dual algebraic lamination for R-trees. *Illinois J. Math.*, 51(3), pages 897–911, 2007.
- [CLR94] D. Cooper, D. D. Long, and A. Reid. Bundles and finite foliations. *Invent. Math.* 118, pages 255–283, 1994.
- [DD89] Warren Dicks and M. J. Dunwoody. *Groups acting on graphs*, volume 17 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1989.
- [Del99] Thomas Delzant. Sur l’accessibilité acylindrique des groupes de présentation finie. *Ann. Inst. Fourier (Grenoble)*, 49(4):1215–1224, 1999.
- [DGO17] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.*, 245(1156):v+152, 2017.
- [DKL14] S. Dowdall, R. P. Kent, and C. Leininger. Pseudo-anosov subgroups of fibered 3-manifold groups. *Groups Geom. Dyn.* 8 no. 4, pages 1247–1282, 2014.
- [DM17] F. Dahmani and M. Mj. Height, graded relative hyperbolicity and quasiconvexity. *J. Éc. polytech. Math.*, 4:515–556, 2017.
- [DS99] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.*, 135(1):25–44, 1999.
- [Duf12] G. Dufour. Cubulations de varietes hyperboliques compactes. *PhD thesis, Universite Paris-Sud XI*, <http://tel.archives-ouvertes.fr/tel-00690334/>, 2012.
- [Dun85] M. J. Dunwoody. The accessibility of finitely presented groups. *Invent. Math.*, 81(3):449–457, 1985.
- [Far98] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.* 8, pages 810–840, 1998.
- [FLP79] A. Fathi, M. Laudenbach, and V. Poenaru. Travaux de Thurston sur les surfaces. *Asterisque* 66-67, pages 1–284, 1979.
- [FM02] B. Farb and L. Mosher. Convex cocompact subgroups of mapping class groups. *Geom. Topol.* 6, pages 91–152, 2002.
- [FP06] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. *Geom. Funct. Anal.*, 16(1):70–125, 2006.
- [Gab02] Damien Gaboriau. Invariants l^2 de relations d’équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, (95):93–150, 2002.

- [Gau16] F. Gautero. Geodesics in Trees of Hyperbolic and Relatively Hyperbolic Spaces. *Proc. Edinburgh. Math. Soc.* 59(3), pages 701–740, 2016.
- [Ham05] U. Hamenstaedt. Word hyperbolic extensions of surface groups . *preprint, arXiv:math/0505244*, 2005.
- [Hem04] John Hempel. *3-manifolds*. AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original.
- [HPW15] Sebastian Hensel, Piotr Przytycki, and Richard C. H. Webb. 1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs. *J. Eur. Math. Soc. (JEMS)*, 17(4):755–762, 2015.
- [HW12] F. Haglund and D. T. Wise. A combination theorem for special cube complexes. *Ann. of Math. (2)* 176, no. 3, pages 1427–1482, 2012.
- [HW15a] Mark F. Hagen and Daniel T. Wise. Cubulating hyperbolic free-by-cyclic groups: the general case. *Geom. Funct. Anal.*, 25(1):134–179, 2015.
- [HW15b] T. Hsu and D. T. Wise. Cubulating malnormal amalgams. *Invent. Math.* 199, no. 2, pages 293–331, 2015.
- [HW16] Mark F. Hagen and Daniel T. Wise. Cubulating hyperbolic free-by-cyclic groups: the irreducible case. *Duke Math. J.*, 165(9):1753–1813, 2016.
- [Jan17] D. Jansen. Notes on pointed Gromov-Hausdorff convergence. *preprint, arXiv:1703.09595*, 2017.
- [Kap01] M. Kapovich. Hyperbolic Manifolds and Discrete Groups. *Progress in Mathematics, vol. 183, Birkhauser*, 2001.
- [Ken01] A. Kent. The theory of normal surfaces. *based on Lecture Notes by Cameron McA. Gordon, https://www.math.wisc.edu/~kent/normal.pdf*, 2001.
- [Kie18] Dawid Kielak. Residually finite rationally-solvable groups and virtual fibring. *preprint, arXiv:1809.09386*, 2018.
- [KL08] R. P. Kent and C. Leininger. Shadows of mapping class groups: capturing convex cocompactness. *Geom. Funct. Anal.* 18 no. 4, pages 1270–1325, 2008.
- [KL10] I. Kapovich and M. Lustig. Intersection form, laminations and currents on free groups. *Geom. Funct. Anal.* 19, no. 5, pages 1426–1467, 2010.
- [KL15] I. Kapovich and M. Lustig. Cannon-Thurston fibers for iwip automorphisms of F_N . *J. Lond. Math. Soc. (2)* 91, no. 1, pages 203–224, 2015.
- [Kla99] E. Klarreich. The boundary at infinity of the curve complex and the relative Teichmüller space. *preprint, http://nasw.org/users/klarreich/research.htm*, 1999.
- [KM12] J. Kahn and V. Markovic. Immersing almost geodesic surfaces in a closed hyperbolic three manifold. *Ann. of Math. (2)* 175, no. 3, pages 1127–1190, 2012.
- [KW19] P. Kropholler and G. Walsh. in preparation. 2019.
- [Min94] Y. N. Minsky. On Rigidity, Limit Sets, and End Invariants of Hyperbolic 3-Manifolds. *J. Amer. Math. Soc.*, vol.7, pages 539–588, 1994.
- [Min03] Y. N. Minsky. End invariants and the classification of hyperbolic 3-manifolds. *Current Developments in Mathematics, 2002, Int. Press Somerville, MA*, pages 181–217, 2003.
- [Min10] Y. N. Minsky. The Classification of Kleinian surface groups I: Models and Bounds. *Ann. of Math.* 171(1), *math.GT/0302208*, pages 1–107, 2010.
- [Mit97] M. Mitra. Ending Laminations for Hyperbolic Group Extensions. *Geom. Funct. Anal. vol.7 No. 2*, pages 379–402, 1997.
- [Mit98] M. Mitra. Cannon-Thurston Maps for Hyperbolic Group Extensions. *Topology* 37, pages 527–538, 1998.
- [Mit99] M. Mitra. On a theorem of Scott and Swarup. *Proc. Amer. Math. Soc. v. 127 no. 6*, pages 1625–1631, 1999.
- [Mj08] M. Mj. Relative Rigidity, Quasiconvexity and C-Complexes. *Algebraic and Geometric Topology* 8, pages 1691–1716, 2008.
- [Mj14] M. Mj. Cannon-Thurston Maps for Surface Groups. *Ann. of Math.*, 179(1), pages 1–80, 2014.
- [Mj19] M. Mj. Tight trees and effective hyperbolization of surface bundles. *preprint, arXiv:1901.02170*, 2019.
- [MM99] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Invent. Math.* 138, pages 103–139, 1999.

- [MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves II: Hierarchical structure. *Geom. Funct. Anal.* 10, pages 902–974, 2000.
- [MP11] M. Mj and A. Pal. Relative Hyperbolicity, Trees of Spaces and Cannon-Thurston Maps. *Geom. Dedicata* 151, pages 59–78, 2011.
- [MR08] M. Mj and L. Reeves. A Combination Theorem for Strong Relative Hyperbolicity. *Geom. Topol.* 12, pages 1777 – 1798, 2008.
- [MR18] M. Mj and K. Rafi. Algebraic ending laminations and quasiconvexity. *Algebr. Geom. Topol.*, 18(4):1883–1916, 2018.
- [MS12] M. Mj and P. Sardar. A combination theorem for metric bundles. *Geom. Funct. Anal.* 22, no. 6, pages 1636–1707, 2012.
- [PT11] Jesse Peterson and Andreas Thom. Group cocycles and the ring of affiliated operators. *Invent. Math.*, 185(3):561–592, 2011.
- [RS94] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. *Geom. Funct. Anal.*, 4(3):337–371, 1994.
- [Sag95] M. Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc.* 71, pages 585–617, 1995.
- [Sis18] A. Sisto. Many Haken Heegaard splittings. *J. Topol. Anal. (to appear)*, 2018.
- [SS90] G. P. Scott and G. A. Swarup. Geometric finiteness of certain Kleinian groups. *Proc. Amer. Math. Soc.*, 109(3):765–768, 1990.
- [SS03] Peter Scott and Gadde A. Swarup. Regular neighbourhoods and canonical decompositions for groups. *Astérisque*, pages vi+233, 2003.
- [SW79] P. Scott and C. T. C. Wall. Topological Methods in Group Theory. *Homological group theory (C. T. C. Wall, ed.), London Math. Soc. Lecture Notes Series, vol. 36, Cambridge Univ. Press*, 1979.
- [Thu80] W. P. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes*, 1980.
- [Thu82] W. P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.*, pages 357–382, 1982.
- [Thu86a] W. P. Thurston. Hyperbolic Structures on 3-Manifolds,I: Deformations of acylindrical manifolds. *Ann. of Math.* 124, pages 203–246, 1986.
- [Thu86b] W. P. Thurston. Hyperbolic Structures on 3-Manifolds,II: Surface Groups and 3-Manifolds which Fiber over the Circle. *preprint, arXiv:math.GT/9801045*, 1986.
- [Thu86c] William P. Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 59(339):i–vi and 99–130, 1986.
- [Wis11] D. T. Wise. The structure of groups with a quasiconvex hierarchy. *preprint*, 2011.
- [Wis12] Daniel T. Wise. *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012.
- [Wis14] Daniel T. Wise. The cubical route to understanding groups. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1075–1099. Kyung Moon Sa, Seoul, 2014.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, 573 MALOTT HALL, ITHACA, NY 14853, U.S.A.

E-mail address: `jfmanning@math.cornell.edu`

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, MUMBAI-40005, INDIA

E-mail address: `mahan@math.tifr.res.in`

MATHEMATICS DEPARTMENT, TECHNION - ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, 32000, ISRAEL

E-mail address: `sageevmtx.technion.ac.il`