DISCONTINUOUS MOTIONS OF LIMIT SETS

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ABSTRACT. We characterise completely when limit sets, as parametrised by Cannon-Thurston maps, move discontinuously for a sequence of algebraically convergent quasi-Fuchsian groups.

Contents

1. Introduction

In [Question 14] of [Thu82], Thurston raised a question about continuous motions of limit sets under algebraic deformations of Kleinian groups. This problem was formulated more precisely in [MS13, MS17] taking into account topologies of convergence of Kleinian groups and a parametrisation of limit sets using Cannon-Thurston maps [CT07, Mj14a]. The questions can be stated as follows.

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Question 1.1. [Thu82, MS13, MS17]

(1) *If a sequence of isomorphic Kleinian groups* (G_n) *converges to* G_∞ *algebraically then do the corresponding Cannon-Thurston maps converge pointwise?*
(2) *If* (G_n) *converges to* G_∞ *strongly then do the corresponding Cannon-Thurston maps converge uniformly?*

It was established, in [MS17] and [Mj17] (crucially using technology developed in [Mj14a], [Mj10]) that the second question has an affirmative answer. The answer to the first question has turned out to be considerably subtler. Indeed, interesting examples of both continuity and discontinuity occur naturally:

(1) In [MS13], it was shown that if the geometric limit Γ of (G_n) is geometrically finite, then the answer to Question 1.1 (1) is affirmative. In particular, this is true for the examples given by Kerckhoff-Thurston in [KT90].
(2) On the other hand, it was shown in [MS17], that for certain examples of quasi-Fuchsian groups converging to geometrically infinite groups constructed by Brock [Bro01], the answer to Question 1.1 (1) is negative.

In this paper, we shall complete the answer to Question 1.1 (1) for sequences of quasi-Fuchsian groups by characterising precisely when limit sets move discontinuously, i.e. we isolate the fairly delicate conditions that ensure the discontinuity phenomena illustrated in [MS17] for Brock's examples. In order to do this, we need to understand possible geometric limits of sequences of surface Kleinian groups. The necessary technology was developed in [OS10]. In particular, it was shown in [OS10] that there exists an embedding of any such geometric limit into $S \times (-1, 1)$.

Let $(\rho_n : \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{C})$ be a sequence of quasi-Fuchsian groups converging algebraically to $\rho_\infty : \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{C}$, where S is a hyperbolic surface of finite area. We set $G_n = \rho_n(\pi_1(S))$ and $G_\infty = \rho_\infty(\pi_1(S))$. Suppose that (G_n) converges geometrically to a Kleinian group Γ. In what follows, we need to consider ends of the non-cuspidal part $(\mathbb{H}^3/G_\infty)_0$ (resp. $(\mathbb{H}^3/\Gamma)_0$) of the algebraic (resp. geometric) limit. For an end e of either $(\mathbb{H}^3/G_\infty)_0$ or $(\mathbb{H}^3/\Gamma)_0$, if there is a $\mathbb{Z}$-cusp neighbourhood whose boundary A tends to e, we say that A (or the $\mathbb{Z}$-cusp) **abuts** on e. (Abusing terminology, we also say that for a neighbourhood E of e, the annulus A abuts on E in this situation.)

Theorem 1.2. Let $c_n : S^1(= \Lambda_{\pi_1(S)}) \to \Lambda_{\rho_n(\pi_1(S))}$ denote the Cannon-Thurston maps for the representations ρ_n $(n = 1, \cdots, \infty)$. Then (c_n) does not converge pointwise to c_∞ if and only if all of Conditions (1)-(3) below are satisfied.

(Condition 1:) The algebraic limit has a **coupled** end e in the following sense. There is a simply degenerate end e of $(\mathbb{H}^3/G_\infty)_0$ with a neighbourhood E homeomorphic to $\Sigma \times \mathbb{R}$ for an essential subsurface Σ of S. The end E projects homeomorphically to a neighbourhood $\bar{E}$ of an end $\bar{e}$ of $(\mathbb{H}^3/\Gamma)_0$, the non-cuspidal part of the geometric limit. The end $\bar{e}$ satisfies the following.
There is another end $\bar{e}'$ of $(\mathbb{H}^3/\Gamma)_0$, simply degenerate or wild, called a **partner** of $\bar{e}$. It is related to the latter as follows. The end $\bar{e}'$ has a neighbourhood $\bar{E}'$ such that if we pull back $\bar{E}$ and $\bar{E}'$ by approximate isometries to $\mathbb{H}^3/G_n$ for large n, then their images are both contained in a submanifold of the form $\Sigma \times (0, 1)$. Also $\mathrm{Fr}\,\Sigma \times (0, 1)$ lies on the boundaries of Margulis tubes. Finally, $\Sigma \times \{t\}$ is incompressible. (Note that $\bar{E}'$ might not be homeomorphic to $\Sigma \times \mathbb{R}$ but to $\Sigma' \times \mathbb{R}$ for some essential subsurface Σ' of Σ. We shall describe this condition more precisely in §2.6.)

(Condition 2:) There is a $\mathbb{Z}$-cusp corresponding to a parabolic curve σ, abutting on both $\bar{E}$ and $\bar{E}'$ (termed **conjoining**), such that the cusp is **untwisted**. More precisely, the Dehn twist parameters of the Margulis tubes corresponding to the $\mathbb{Z}$-cusp are bounded uniformly all along the sequence (ρ_n).

(Condition 3:) A crown domain for (μ, σ), where μ is the ending lamination of the end $\bar{e}$, is **well-approximated**, i.e. it is realised (with respect to the marking determined by approximate isometries) in $\bar{E}'$. (See §2.6 for a more precise definition.)

We also characterise the points where this discontinuity occurs (see Theorem 2.15). These turn out to be exactly the tips of crown domains as in [MS17].

Theorem 1.3. In the setting of Theorem 1.2, suppose that

1. $(\mathbb{H}^3/G_\infty)_0$ has a coupled simply degenerate end e with ending lamination λ,
2. an untwisted $\mathbb{Z}$-cusp (corresponding to a parabolic curve s) abutting on the image of e in $(\mathbb{H}^3/\Gamma)_0$,

(3) and a crown domain for (λ, s) which is well approximated.

Then for $\zeta \in S^1$, the images $c_n(\zeta)$ do not converge to $c_\infty(\zeta)$ if and only if ζ is a tip of a crown domain C for (μ, σ), where

(1) $\mu \cup \sigma$ is contained in either the union of the lower parabolic curves and the lower ending laminations (denoted by λ_-), or the union of the upper parabolic curves and the upper ending laminations, (denoted by λ_+);
(2) the crown domain C is well approximated;
(3) the simple closed curve σ corresponds to an untwisted conjoining cusp abutting on the projection of e in $(\mathbb{H}^3/\Gamma)_0$.

2. Preliminaries

2.1. **Relative hyperbolicity and electrocution.** We refer the reader to [Far98] and [Bow12] for generalities on relative hyperbolicity and to [Mj11] and [Mj14a] for the notions of electrocution and electro-ambient paths. We shall briefly recall the notion of electro-ambient quasi-geodesics, c.f. [Mj14a]. Let (X, d_X) be a δ-hyperbolic metric space. Bowditch showed in [Bow12] that if there are constants C, D and a family $\mathcal{K}$ of D-separated, C-quasi-convex sets in X, then X is (weakly) hyperbolic relative to $\mathcal{K}$. Now let $\mathcal{H}$ be a collection of C-quasi-convex sets in (X, d_X), without assuming the D-separated condition. Let $\mathcal{E}(X, \mathcal{H})$ denote the space obtained by electrocuting the elements of $\mathcal{H}$ in X: this space is a union of X and $\sqcup_{H\in\mathcal{H}} H \times [0, 1/2]$, where $H \times \{0\}$ is identified with H in X, each $\{h\} \times [0, 1/2]$ is isometric to $[0, 1/2]$, and $H \times \{1/2\}$ is equipped with the zero metric. Since $\{H \times \{1/2\}\}$ is 1-separated, we can apply Bowditch's result, and see that $\mathcal{E}(X, \mathcal{H})$ is Gromov hyperbolic.

Let $\alpha = [a, b]$ be a geodesic in (X, d_X), and β an electric quasi-geodesic without backtracking joining a, b in $\mathcal{E}(X, \mathcal{H})$, i.e. an electric quasi-geodesic which does not return to an element $H \in \mathcal{H}$ after leaving it. We further assume that the intersection of β and $H \times (0, 1/2)$ is either empty or a disjoint union of open arcs of the form $\{h\} \times (0, 1/2)$. We parametrise β, and consider the maximal subsegments of β contained entirely in some $H \times \{1/2\}$ (for some $H \in \mathcal{H}$). We extend each of such maximal subsegments by adjoining 'vertical' subsegments (of the form $h \times [0, 1/2]$) in β at both of its endpoints to obtain a subsegment of the form $\{p\} \times [0, 1/2] \cup [p, q] \times \{1/2\} \cup \{q\} \times [0, 1/2]$. We call these subpaths of β *extended maximal subsegments*. We replace each extended maximal subsegment in β by a geodesic path in (X, d_X) joining the same endpoints.

The resulting path β_q is called an **electro-ambient representative** of β in X. Also, if β is an electric P-quasi-geodesic without backtracking (in $\mathcal{E}(X, \mathcal{H})$), then β_q is called an **electro-ambient P-quasi-geodesic**. If β is an electric geodesic without backtracking, then β_q is simply called an **electro-ambient quasi-geodesic**. The following lemma says that hyperbolic geodesics do not go far from electro-ambient quasi-geodesic realisations.

Lemma 2.1. ([Kla99, Proposition 4.3], [Mj11, Lemma 3.10] and [Mj14a, Lemma 2.5]) *For given non-negative numbers δ, C and P, there exists R such that the following holds:*
Let (X, d_X) be a δ-hyperbolic metric space and $\mathcal{H}$ a family of C-quasi-convex subsets of X. Let $(\mathcal{E}(X, \mathcal{H}), d_e)$ denote the electric space obtained by electrocuting the elements of $\mathcal{H}$. Then, $(\mathcal{E}(X, \mathcal{H}), d_e)$ is Gromov hyperbolic, and if α, β_q denote respectively a geodesic arc with respect to d_X, and an electro-ambient P-quasi-geodesic with the same endpoints in X, then α lies in the R-neighbourhood of β_q with respect to d_X.

2.2. **Cannon-Thurston Maps.** We shall review known facts about Cannon-Thurston maps focusing on the case of interest in this paper. Let (Y, d_Y) be a Cayley graph of $\pi_1(S)$ for S a closed surface of genus at least 2 with respect to some finite generating system, and set $X = \mathbb{H}^3$. By adjoining the Gromov boundaries $\partial X (= S^2)$ and $\partial Y (= S^1)$ to X and Y respectively, we obtain their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively.

Suppose that $\pi_1(S)$ acts on $\mathbb{H}^3$ by isometries as a Kleinian group G via an isomorphism $\rho \colon \pi_1(S) \to G$, and let $i : Y \to X$ be a $\pi_1(S)$-equivariant injection.

Definition 2.2. *A* **Cannon-Thurston map** $\hat{i}$ *(for ρ) from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of i.*

The image of $\hat{i}$ coincides with the limit set of G. It is easy to see if a Cannon-Thurston map exists, it is unique.

The notion of a Cannon-Thurston map can be easily extended to the case where S is a hyperbolic surface of finite area. In this situation, it is a $\pi_1(S)$-equivariant continuous map from the relative (or Bowditch) boundary relative to the cusp subgroups, $\partial_\infty \widetilde{S} = \partial_\infty \mathbb{H}^2 = S^1$, onto the limit set in S^2. The first author [Mj14b] showed that for any Kleinian group isomorphic to a surface group (possibly with punctures), a Cannon-Thurston map always exists, and gave the following characterisation of non-injective points. Recall that an isomorphism from a Kleinian group to another Kleinian group is said to be weakly type-preserving when every parabolic element is sent to a parabolic element.

Theorem 2.3. [Mj14b] *Let $S = \mathbb{H}^2/F$ be a (possibly punctured) hyperbolic surface of finite area. Let $\rho : F \to \mathrm{PSL}_2\,\mathbb{C}$ be a weakly type-preserving discrete faithful representation with image G. Let λ_1 be the union of parabolic curves and ending laminations for upper ends and λ_2 be that of the lower ends, one (or both) of which might be empty. We regard λ_1 and λ_2 as geodesic laminations on S.*

For $k = 1, 2$, let $\mathcal{R}_k$ denote the relation on $\partial_\infty \mathbb{H}^2$ defined as follows: $\xi \mathcal{R}_k \eta$ if and only if ξ and η are either ideal endpoints of the same leaf of $\tilde{\lambda}_k$, or ideal boundary points of a complementary ideal polygon of $\tilde{\lambda}_k$, where $\tilde{\lambda}_k$ is the preimage of λ_k in $\mathbb{H}^2$. Denote the transitive closure of $\mathcal{R}_1 \cup \mathcal{R}_2$ by $\mathcal{R}$. Let $\hat{i}_\rho : \partial F \to \Lambda_G$ be the Cannon-Thurston-map for ρ. Then $\hat{i}(\xi) = \hat{i}(\eta)$ for $\xi, \eta \in \partial_\infty \mathbb{H}^2$ if and only if $\xi \mathcal{R} \eta$.

2.3. **Algebraic and Geometric Limits.** Let $(\rho_n : G \to \mathrm{PSL}_2\,\mathbb{C})$ be a sequence of weakly type-preserving, discrete, faithful representations of a fixed finitely generated torsion-free G converging to a discrete, faithful representation $\rho_\infty \colon G \to \mathrm{PSL}_2\,\mathbb{C}$. Also assume that $(\rho_n(G))$ converges to a Kleinian group Γ as a sequence of closed subsets of $\mathrm{PSL}_2\,\mathbb{C}$ in the Hausdorff topology. Then $\rho_\infty(G)$ is called the **algebraic limit** of the sequence and Γ the **geometric limit** of the sequence $(\rho_n(G))$. If $\rho_\infty(G) = \Gamma$, we say that the limit is **strong**.

There is a more geometric way of thinking of geometric limits (see [Thu80] and [CEG87]). A sequence of manifolds with basepoints $\{(N_i, x_i)\}$ converges geometrically to a manifold with basepoint (N, x_∞) if for any R, there are compact submanifolds $C_i \subset N_i$ and $C \subset N$ containing the r-balls centred at x_i and x_∞ respectively, and a K_i-bi-Lipschitz map $h_i \colon C_i \to C$ for any $i \geq i_0(R)$, with $K_i \to 1$ as $i \to \infty$. A sequence of Kleinian groups (G_n) converges geometrically to Γ if and only if for a fixed basepoint x and its projections x_n and x_∞ in $\mathbb{H}^3/G_n$ and $\mathbb{H}^3/\Gamma$, the sequence $\{(\mathbb{H}^3/G_n, x_n)\}$ converges geometrically to $(\mathbb{H}^3/\Gamma, x_\infty)$.

2.4. **Criteria for Uniform/Pointwise convergence.** We recall some material from [MS13, MS17]. Let G be a fixed finitely generated Kleinian group and $(\rho_n(G) = G_n)$ be a weakly type-preserving sequence of Kleinian groups converging algebraically to $G_\infty = \rho_\infty(G)$. Also fix a basepoint $o_{\mathbb{H}^3} \in \mathbb{H}^3$. Let d_G denote the distance in a Cayley graph of G and d the distance in $\mathbb{H}^3$. Also $[g, h]$ denotes a geodesic in G joining g, h and $[\rho_n(g)(o_{\mathbb{H}^3}), \rho_n(h)(o_{\mathbb{H}^3})]$ denotes a geodesic in $\mathbb{H}^3$ joining $\rho_n(g)(o_{\mathbb{H}^3}), \rho_n(h)(o_{\mathbb{H}^3})$.

Definition 2.4. *The sequence (ρ_n) is said to have the* **Uniform Embedding of Points** *property (UEP for short) if there exists a non-negative function $f(N)$, with $f(N) \to \infty$ as $N \to \infty$, such that for all $g \in \Gamma$, $d_\Gamma(1, g) \geq N$ implies $d(\rho_n(g)(o_{\mathbb{H}^3}), o_{\mathbb{H}^3}) \geq f(N)$ for all $n = 1, \cdots, \infty$.*

The sequence (ρ_n) is said to have the **Uniform Embedding of Pairs of Points** *property (UEPP for short) if there exists a non-negative function $f(N)$, with $f(N) \to \infty$ as $N \to \infty$, such that for all $g, h \in \Gamma$, $d_\Gamma(1, [g, h]) \geq N$ implies $d([\rho_n(g)(o_{\mathbb{H}^3}), \rho_n(h)(o_{\mathbb{H}^3})], o_{\mathbb{H}^3}) \geq f(N)$ for all $n = 1, \ldots, \infty$.*

The property UEP is used in [MS13] to give a sufficient criterion to ensure that algebraic convergence is also geometric. The property UEPP is used to give the following criterion for proving uniform convergence of Cannon-Thurston maps.

Proposition 2.5 ([MS13])**.** *Let Γ be a geometrically finite Kleinian group and let $\rho_n : \Gamma \to G_n$ be weakly type-preserving isomorphisms to Kleinian groups. Suppose that (ρ_n) converges algebraically to a representation ρ_∞. If (ρ_n) satisfies UEPP, the corresponding Cannon-Thurston maps converge uniformly.*

Notation: We shall henceforth fix a complete hyperbolic structure of finite area on S and a Fuchsian group G corresponding to the hyperbolic structure. The limit set Λ_G is homeomorphic to S^1. Similarly, Λ_{G_n} will denote the limit set of G_n setting $G_n = \rho_n(G)$. For each G_n ($n \in \mathbb{N}$ or $n = \infty$), we shall denote the corresponding Cannon-Thurston map by $c_n : S^1 = \Lambda_G \to \Lambda_{G_n}$.

For pointwise convergence of Kleinian surface groups, a weaker condition called EPP is sufficient. This condition depends on points of Λ_G.

Proposition 2.6 ([MS13])**.** *Let $G \cong \pi_1(S)$ be a Fuchsian group corresponding to a hyperbolic surface S of finite area. Take $\xi \in \Lambda_G = S^1_\infty$ and let $[o_{\mathbb{H}^2}, \xi)$ be a geodesic ray in $\mathbb{H}^2$ from a fixed basepoint $o_{\mathbb{H}^2}$ to ξ. Let $o_{\mathbb{H}^3} \in \mathbb{H}^3$ be a fixed basepoint in $\mathbb{H}^3$. Suppose that*

1. *$(\rho_n : G \to \mathrm{PSL}_2\,\mathbb{C})$ is a sequence of weakly type-preserving discrete faithful representations converging algebraically to $\rho_\infty : G \to \mathrm{PSL}_2\,\mathbb{C}$. Set $G_n = \rho_n(G), n = 1, \cdots, \infty$ and $M_n = \mathbb{H}^3/G_n$.*
2. *Let $\phi_n : S \to M_n$ be an immersion inducing ρ_n at the level of fundamental groups. Further, assume that ϕ_n lifts to an embedding $\Phi_n : \mathbb{H}^2 \to \mathbb{H}^3$.*

Then the Cannon-Thurston maps for the ρ_n converge to the Cannon-Thurston map for ρ_∞ at ξ if

EPP: *There exists a proper function $g : \mathbb{N} \to \mathbb{N}$ such that for any geodesic subsegment $[a,b]$ of the ray $[o_{\mathbb{H}^2}, \xi) \subset \mathbb{H}^2$ lying outside $B_N(o_{\mathbb{H}^2})$ (the N-ball in $\mathbb{H}^2$ about $o_{\mathbb{H}^2}$), the geodesic in $\mathbb{H}^3$ joining $\Phi_n(a)$ with $\Phi_n(b)$ lies outside $B_{g(N)}(o_{\mathbb{H}^3})$, (the $g(N)$-ball about $o_{\mathbb{H}^3} \in \mathbb{H}^3$).*

2.5. **Models of Ends of Geometric Limits of Surface Groups.** We recall some material from [OS10], where a complete analysis of geometric limits of algebraically convergent quasi-Fuchsian groups was carried out by Ohshika and Soma.

Let (ρ_n) be a sequence of quasi-Fuchsian representations of $\pi_1(S)$ converging to ρ_∞ algebraically. Set $G_n = \rho_n(\pi_1(S))$ and $G_\infty = \rho_\infty(\pi_1(S))$. Further, (after passing to a subsequence if necessary) assume that (G_n) converges geometrically to Γ. Let $(\mathbb{H}^3/\Gamma)_0$ denote $\mathbb{H}^3/\Gamma$ minus a small cusp neighbourhoods. We call $(\mathbb{H}^3/\Gamma)_0$ the non-cuspidal part of $\mathbb{H}^3/\Gamma$.

In [OS10] it is shown that that there exists a bi-Lipschitz model manifold $\mathbf{M}_\Gamma$ of $\mathbb{H}^3/\Gamma$ admitting an embedding into $S \times (0,1)$. For simplicity, we regard $\mathbf{M}_\Gamma$ as a subset of $S \times (0,1)$. Denote by $f_\Gamma : M_\Gamma \to \mathbb{H}^3/\Gamma$ the model map. As was shown there, this model manifold and the model map can respectively be taken as the geometric limit of Minsky's model manifolds $\mathbf{M}_n$ of $\mathbb{H}^3/G_n$, and their model maps $f_n \colon \mathbf{M}_n \to \mathbb{H}^3/G_n$ as $n \to \infty$. We identify the model manifold $\mathbf{M}_\Gamma$ with its embedding into $S \times (0,1)$, and we regard f_Γ^{-1} as an embedding of $\mathbb{H}^3/\Gamma$ into $S \times (0,1)$. The embedding of the model manifold $\mathbf{M}_\Gamma$ and the model map f_Γ can be taken to have the following properties.

1. Each end e of $(\mathbb{H}^3/\Gamma)_0$ corresponds under f_Γ^{-1} to a level surface $\Sigma \times \{t\}$ for some incompressible subsurface Σ of S. More precisely, $\Sigma \times \{t\}$ lies in the frontier of the image of f_Γ^{-1}.
2. Every geometrically finite end is sent into $S \times \{0,1\}$ by f_Γ^{-1}.
3. There is an incompressible immersion ϕ of S into $\mathbb{H}^3/\Gamma$ such that the covering of $\mathbb{H}^3/\Gamma$ corresponding to $\phi_*\pi_1(S)$ coincides with $\mathbb{H}^3/G_\infty$.
4. The image under f_Γ^{-1} of the frontier of $(\mathbb{H}^3/\Gamma)_0$ consists of disjoint tori and open annuli built out of horizontal and vertical annuli. Each such torus component consists of two horizontal annuli and two vertical ones. Each open annulus component consists of either one horizontal annulus and two vertical annuli or two horizontal annuli and three vertical annuli.

An embedding f_Γ^{-1} and the corresponding model map f_Γ satisfying the above conditions is said to be **adapted to the product structure**.

A covering associated to the inclusion $G_\infty \subset \Gamma$ of $\mathbf{M}_\Gamma$ is homeomorphic to $\mathbb{H}^3/G_\infty$, and hence to $S \times (0,1)$. Its core surface projects to an immersion of S into $\mathbf{M}_\Gamma$, such that the immersion is homotopic to $f_\Gamma^{-1} \circ \phi$ with ϕ as in Condition (3) above. We call such an immersion of S into $\mathbf{M}_\Gamma$ an **algebraic locus**. We sometimes also refer to the immersion ϕ in Condition (3) as an algebraic locus.

If an end e of $(\mathbb{H}^3/\Gamma)_0$ corresponds to a level surface $\Sigma \times \{t\}$ for some proper subsurface Σ of S, then the boundary $\mathrm{Fr}\,\Sigma$ represents a parabolic element of Γ contained in a maximal parabolic group isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

2.5.1. *Brick Manifolds.* For later use, we shall give a more precise version of the discussion above in the form of Theorem 2.7 below. A **brick** B is a 3-manifold homeomorphic to $F \times J$, where F is an incompressible subsurface of S and J is either a closed or half-open interval. A **brick manifold** is a union of countably many bricks $F_n \times J_n$ glued to each other along essential connected subsurfaces of their horizontal boundaries $F_n \times \partial J_n$.

To any end of a half-open brick in a brick manifold we associate either a conformal structure (at infinity) or an ending lamination. In the first case, the brick is called geometrically finite and in the latter case, it is called simply degenerate. Accordingly, each half-open end of a brick constitutes a geometrically finite or simply degenerate end of M. The associated ending lamination or conformal structure is called the end invariant. The union of ideal boundaries corresponding to the geometrically finite ends thus carries a union of conformal structures and is called the **boundary at infinity** of M. Denoted the boundary at infinity by $\partial_\infty M$. A brick manifold equipped with end invariants is called **a labelled brick manifold**.

A labelled brick manifold is said to admit a block decomposition if the manifold can be decomposed into Minsky blocks [Min10] and solid tori such that

(1) Each block has horizontal and vertical directions coinciding with those of bricks.
(2) The block decomposition for a half-open brick agrees with the Minsky model corresponding to its end invariant.
(3) Blocks have standard metrics (as in [Min10]) and gluing maps are isometries.
(4) Solid tori are given the structure of Margulis tubes with coefficients determined by the block decomposition (as in [Min10]).

The resulting metric on the labelled brick manifold is called **a model metric**. The next theorem from [OS10] gives the existence of a model manifold corresponding to a geometric limit of Kleinian surface groups.

Theorem 2.7. [OS10] *Let S be a hyperbolic surface of finite area. Let $(\rho_n : \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{C})$ be a sequence of weakly type-preserving representations, converging geometrically to Γ. Set $N = \mathbb{H}^3/\Gamma$, and let N_0 denote the non-cuspidal part of N. Then there exists a labelled brick manifold $\mathbf{M}^0_\Gamma$ admitting a block decomposition and a K-bi-Lipschitz homeomorphism to N_0 such that the following hold:*

1. *The constant K depends only on $\chi(S)$.*
2. *Each component of $\partial\mathbf{M}^0_\Gamma$ is either a torus or an open annulus.*
3. *$\mathbf{M}^0$ has only countably many ends, and no two distinct ends lie on the same level surface $\Sigma \times \{t\}$.*
4. *There is no properly embedded incompressible annulus in $\mathbf{M}^0_\Gamma$ whose boundary components lie on distinct boundary components.*
5. *If there is an embedded, incompressible half-open annulus $S^1 \times [0, \infty)$ in $\mathbf{M}^0_\Gamma$ such that $S^1 \times \{t\}$ tends to a wild end e of $\mathbf{M}^0_\Gamma$ as $t \to \infty$ (see Remark 2.8 below), then its core curve is freely homotopic into an open annulus component of $\partial\mathbf{M}^0_\Gamma$ tending to e.*
6. *The manifold $\mathbf{M}^0_\Gamma$ is (not necessarily properly) embedded in $S \times (0, 1)$ in such a way that each brick has the form $F \times J$ where F is an incompressible subsurface of S and $J \subset (0, 1)$ is an interval. Also the product structure of $F \times J$ is compatible with that of $S \times (0, 1)$. Finally, the ends of geometrically finite bricks lie in $S \times \{0, 1\}$.*

The labelled brick manifold $\mathbf{M}^0_\Gamma$ of Theorem 2.7 is called a **model manifold** for N_0 – the non-cuspidal part of the geometric limit. As was explained in the previous section, the model manifold of $\mathbf{M}_\Gamma$ is obtained as a geometric limit of the model manifolds $\mathbf{M}_n$ of $\mathbb{H}^3/\rho_n(\pi_1(S))$. By removing cusp neighbourhoods from $\mathbf{M}_\Gamma$, we get $\mathbf{M}^0_\Gamma$.

Remark 2.8. In general, the non-cuspidal part of a geometric limit N with infinitely generated fundamental group as in Theorem 2.7 may contain an end all of whose neighbourhoods contain infinitely many distinct relative ends. We call such an end **wild**. In this case we have a sequence of relative ends accumulating (under the model map f_Γ) to some $\Sigma \times \{t\}$, where Σ is an essential subsurface of S.

Remark 2.9. In [OS10], the authors further show that given a family of end-invariants on a labelled brick manifold satisfying the conclusions of Theorem 2.7 above, there exists a model manifold with any given set of end-invariants (provided only that there are no two homotopic parabolic curves or two homotopic ending laminations). Further, such a manifold is unique up to isometry.

2.6. **Special Conditions on Ends.** Recall that we have fixed a Fuchsian group G with limit set Λ_G homeomorphic to S^1. Equivalently, if S is closed then $\partial_\infty G = S^1$ and if S is on-compact, then the relative hyperbolic boundary $\partial_h G = S^1$. Suppose that a sequence of quasi-Fuchsian groups $(G_n = \rho_n(G))$ converges geometrically to the geometric limit Γ. We denote by Λ_{G_n} the limit set of G_n, and by $c_n : S^1 = \Lambda_G \to \Lambda_{G_n}$ the corresponding Cannon-Thurston map. We assume that (ρ_n) converges to ρ_∞ algebraically, and set $G_\infty = \rho_\infty(G)$. Recall that we have a model manifold $\mathbf{M}_\Gamma$ with a model map $f_\Gamma \colon \mathbf{M}_\Gamma \to \mathbb{H}^3/\Gamma$ which are geometric limits of the model manifolds $\mathbf{M}_n$ of $\mathbb{H}^3/G_n$ and $f_n \colon \mathbf{M}_n \to \mathbb{H}^3/G_n$. We regard $\mathbf{M}_\Gamma$ as being embedded in $S \times (0,1)$. Since $(\mathbf{M}_n)$ converges to $\mathbf{M}_\Gamma$ geometrically, there exist K_n-bi-Lipschitz homeomorphisms $\mathbf{h}_n \colon B_{R_n}(\mathbf{M}_n, \mathbf{x}_n) \to B_{K_n R_n}(\mathbf{M}_\Gamma, \mathbf{x}_\infty)$, where $R_n \to \infty$ and $K_n \to 1$. We can also assume that $\mathbf{x}_\infty$ lies in the algebraic locus (i.e. an immersion of S into the geometric limit whose fundamental group corresponds to the algebraic limit, see §2.5). In the same way, corresponding to the geometric convergence of (G_n) to Γ, there exist K_n-bi-Lipschitz homeomorphisms $h_n \colon B_{R_n}(\mathbb{H}^3/G_n, x_n) \to B_{K_n R_n}(\mathbb{H}^3/\Gamma, x_\infty)$ with x_∞ lying on the algebraic locus.

We shall now describe the conditions that appear in the main theorem.

Coupled ends. Let e be a simply degenerate end of $(\mathbb{H}^3/\Gamma)_0$. Then there is a neighbourhood E of $f_\Gamma^{-1}(e)$ of the form $\Sigma \times (t_1, t_2)$ where Σ is an incompressible subsurface of S, and either $\Sigma \times \{t_1\}$ or $\Sigma \times \{t_2\}$ corresponds to e.

Alternately, let e be wild. Then there is a neighbourhood E homeomorphic to the complement of countably many pairwise disjoint neighbourhoods of (simply degenerate or wild) ends $\Sigma_k \times [s_k, s'_k]$ in $\Sigma \times (t_1, t_2)$. Here Σ_k is an incompressible subsurface of Σ and $t_1 < s_k < s'_k < t_2$. If e corresponds to $\Sigma \times t_1$, then both (s_k) and (s'_k) accumulates to t_1; similarly if $\Sigma \times \{t_2\}$ corresponds to e.

Definition 2.10. *Suppose that an end e of $(\mathbb{H}^3/\Gamma)_0$ corresponds (under f_Γ^{-1}) to $\Sigma \times \{t\}$ for some $t \in (0,1)$.*

(1) We say that an end of $(\mathbb{H}^3/\Gamma)_0$ is **algebraic** *if it has a neighbourhood which is a homeomorphic image of a neighbourhood of*

an end of $(\mathbb{H}^3/G_\infty)_0$ *under the covering projection* $q\colon \mathbb{H}^3/G_\infty \to \mathbb{H}^3/\Gamma$ *associated to the inclusion of* G_∞ *into* Γ.

(2) *We call the end* e **upward** *if* $f_\Gamma^{-1}(e)$ *corresponds to* $\Sigma \times \{t\}$ *and* $\Sigma \times (t-\epsilon, t)$ *intersects* $\mathbf{M}_\Gamma^0$ *for every* $\epsilon > 0$*, else it is called* **downward**.

(3) *When* e *is upward, we say that it is* **coupled** *if there are downward ends* $e'_1, \ldots, e'_d$ *of* $(\mathbb{H}^3/\Gamma)_0$ *which are either simply degenerate or wild, such that the following hold if we choose an embedding of* $\mathbf{M}_\Gamma$ *into* $S \times (0,1)$ *appropriately.*
   (a) $f_\Gamma^{-1}(e'_k)$ *corresponds to* $\Sigma'_k \times \{t'_k\}$ *for some* $t'_k > t$ *and an incompressible subsurface* Σ'_k *of* S *homeomorphic to a subsurface of* Σ;
   (b) *if we let* $A_1, \ldots, A_e$ *be the open annulus components of* $\mathrm{Fr}(\mathbb{H}^3/\Gamma)_0$ *that abut on* $e'_1, \ldots, e'_d$ *but not on* e*, then for any small* $\epsilon > 0$*, we can choose core annuli* $A'_1, \ldots, A'_e$*, one on each of them, so that* $\Sigma'_+ = \cup_{k=1}^d \Sigma'_k \times \{t'_k + \epsilon\} \cup \cup_{l=1}^e A'_l$ *is homeomorphic to an incompressible subsurface* Σ' *of* Σ;
   (c) *there is at least one open annulus boundary component of* $\mathbf{M}_\Gamma$ *which abuts on* e *as well as one of* $e'_1, \ldots, e'_d$;
   (d) *any open annulus boundary component of* $\mathbf{M}_\Gamma$ *that abuts on* $e'_1, \ldots, e'_d$ *only at one of its two ends and does not abut on* e *has the other end on* $S \times \{1\}$;
   (e) *the surface* $\mathbf{h}_n^{-1}(\Sigma'_+)$ *is parallel into* $\mathbf{h}_n^{-1}(\Sigma \times \{t-\epsilon\})$ *in* $\mathbf{M}_n$ *for* n *sufficiently large.*

   Similarly when e *is downward, we call it coupled if there are upward ends* $e'_1, \ldots, e'_d$ *satisfying analogous conditions to the upward case.*

(4) $e'_1, \ldots, e'_d$ *above are called the* **partners** *of* e.

(5) *We say that a simply degenerate end of the (non-cuspidal) algebraic limit* $(\mathbb{H}^3/G_\infty)_0$ *is coupled when the corresponding algebraic end of* $(\mathbb{H}^3/\Gamma)_0$ *is coupled.*

We note that we can change the embedding of $\mathbf{M}_\Gamma$ into $S \times (0,1)$ preserving the algebraic locus to another one adapted to the product structure without changing the combinatorial structure of the brick decomposition so that in (3), all t'_k are equal to t', the surface Σ'_+ is a subsurface of Σ, and $\Sigma \times [t, t']$ lies outside $\mathbf{M}_\Gamma$.

Now, suppose that e is algebraic, and let $\tilde{e}$ be a coupled end of $(\mathbb{H}^3/G_\infty)_0$ which is projected down to e by q. Let A be an open annulus boundary component of $(\mathbb{H}^3/\Gamma)_0$ such that one of its ends abuts on e whereas the other end abuts on one of its partners, say e'. The $\mathbb{Z}$-cusp corresponding to such an open annulus, as also the annulus itself, are called **conjoining**. A conjoining $\mathbb{Z}$-cusp lifts to a $\mathbb{Z}$-cusp U abutting on $\tilde{e}$. The cusp U corresponds to an annular neighbourhood A of a parabolic curve on S, and A has two sides on the surface S. Let e_0 be an end of $(\mathbb{H}^3/G_\infty)_0$ which lies on the other side of U (a priori e_0 may coincide with $\bar{e}$ itself if A is non-separating). If e_0 is geometrically infinite (i.e. simply degenerate), then, by the covering theorem [Thu80, Can96, Ohs92], this end also has a neighbourhood embedded homeomorphically into $(\mathbb{H}^3/\Gamma)_0$ under the covering projection q. In particular the projection $q(\operatorname{Fr} U)$ of the open annulus $\operatorname{Fr} U$ must abut on an algebraic end. This contradicts the assumption that $q(\operatorname{Fr} U)$ abuts on e', which cannot be algebraic. This forces e_0 to be geometrically finite, and in particular e_0 is distinct from $\tilde{e}$. We call a $\mathbb{Z}$-cusp of $(\mathbb{H}^3/G_\infty)_0$ separating a geometrically finite end from a geometrically infinite end **finite-separating**. What we have just shown implies that any conjoining cusp in $(\mathbb{H}^3/\Gamma)_0$ lifts to a finite-separating cusp in $(\mathbb{H}^3/G_\infty)_0$.

Twisted and Untwisted cusps. The $\mathbb{Z}$-cusps of $\mathbb{H}^3/\Gamma$ not corresponding to cusps of S are classified into two types: twisted and untwisted. We shall describe these now. Let P be a $\mathbb{Z}$-cusp neighbourhood of $\mathbb{H}^3/\Gamma$ corresponding to a maximal parabolic group generated by $\gamma \in \Gamma$, and not coming from a cusp of S.

Then, there is a sequence of loxodromic elements ($\gamma_n \in G_n$) converging to γ by the definition of geometric limits. Let U_n be a Margulis tube of $\mathbb{H}^3/G_n$ whose core curve is represented by γ_n. We can choose U_n so that they converge to P geometrically as $\mathbb{H}^3/G_n$ converges geometrically to $\mathbb{H}^3/\Gamma$. Take an annular core A of ∂P, and pull it back to an annulus A_n on ∂U_n by h_n^{-1}. Now, consider a meridian m_n on ∂U_n (i.e. an essential simple closed curve on ∂U_n bounding a disc in U_n) and a longitude l_n, i.e. a core curve of A_n generating $\pi_1(\partial U_n)$. Let s_n be a simple closed curve whose length with respect to the induced metric on ∂U_n is shortest among the simple closed curves intersecting l_n at one point. Then in $\pi_1(U_n)$, we can express $[s_n]$ as $[m_n] + \alpha_n[l_n]$ for some $\alpha_n \in \mathbb{Z}$. If α_n is bounded as $n \to \infty$, we say that the cusp P is **untwisted**; else it is said to be **twisted**.

A description of twisted and untwisted cusps may also be given using the hierarchy machinery of Masur-Minsky [MM00]. Let H_n be a hierarchy of tight geodesics in the curve complex of S corresponding to the quasi-Fuchsian group G_n having ρ_n as a marking. Then the cusp P is twisted if and only if H_n contains a geodesic g_n supported on an annulus whose core curve is freely homotopic to γ_n and whose length goes to ∞ as $n \to \infty$.

Crown domains and crown-tips. Let e be a simply degenerate end of $(\mathbb{H}^3/G_\infty)_0$ with ending lamination λ. Let Σ be the minimal supporting surface of λ, i.e. an incompressible subsurface of S containing λ and minimal with respect to inclusion (up to isotopy). Let σ be a component of $\mathrm{Fr}\,\Sigma$. Fixing a hyperbolic metric on S, we can assume that both σ and λ are geodesic on S. We consider their pre-images $\tilde{\sigma}$ and $\tilde{\lambda}$ in $\mathbb{H}^2$. A crown domain C for (λ, σ) is an ideal polygon in $\mathbb{H}^2 \setminus (\tilde{\sigma} \cup \tilde{\lambda})$ with countably many vertices bounded by a component σ_0 of $\tilde{\sigma}$ and countably many leaves of $\tilde{\lambda}$. A vertex of C which is not an endpoint of σ_0 is called **a tip** of the crown domain C or simply a **crown-tip** (for (λ, σ)).

Well-approximated crown domains in ending laminations. To state a sufficient condition for pointwise convergence, we need to introduce a subtle condition concerning geometric convergence to coupled geometrically infinite ends as follows. Let e be a coupled simply degenerate end of $(\mathbb{H}^3/G_\infty)_0$. We assume that e is an upward end. As mentioned earlier, e has a neighbourhood projecting homeomorphically to a neighbourhood of a simply degenerate end $\bar{e}$ of $(\mathbb{H}^3/\Gamma)_0$. We can define the same property when e is a lower end by turning everything upside down.

Recall that we have an embedding $f_\Gamma^{-1}\colon (\mathbb{H}^3/\Gamma)_0 \to S \times (0,1)$ adapted to the product structure as described above, and the end $f_\Gamma^{-1}(\bar{e})$ corresponds to a level surface $\Sigma \times \{t\}$ for some essential subsurface Σ of S. Since e is assumed to be coupled, there exists $t' > t$ such that $\Sigma'_1 \times \{t'\}, \ldots, \Sigma'_d \times \{t'\}$ correspond to downward ends $e'_1, \ldots, e'_d$ of $(\mathbb{H}^3/\Gamma)_0$ which are either simply degenerate or wild and $(\Sigma \times [t, t']) \cap \mathbf{M}_\Gamma = \emptyset$; further there is at least one conjoining annulus abutting on $\bar{e}$ as well as one of $e'_1, \ldots, e'_d$.

Now, as in the definition of a coupled end, pick a surface $\Sigma_- = \Sigma \times \{t_-\}$, with $t_- < t$, parallel to $\Sigma \times \{t - \epsilon\}$ in $\mathbf{M}^0_\Gamma$ for some small $\epsilon > 0$. Also assume that Σ_- is contained in the image of h_n for sufficiently large n. Define Σ'_+ to be $\cup_{k=1}^d \Sigma'_k \times \{t_+\} \cup \cup_{l=1}^e A'_l$ with $t_+ > t'$. The pre-image of Σ'_+ under $\mathbf{h}_n^{-1}$ is parallel into the pre-image of Σ_-. Let us denote by k_n an embedding of $\mathbf{h}_n^{-1}(\Sigma'_+)$ into $\mathbf{h}_n^{-1}(\Sigma_-)$ realising the parallelism. Then, $\mathbf{h}_n \circ k_n \circ \mathbf{h}_n^{-1}|\Sigma'_+$ gives an embedding of Σ'_+ into Σ_-. Denote this embedding by Ψ_n. Fix a complete hyperbolic structure on Σ_- making each component of $\mathrm{Fr}\,\Sigma_-$ a cusp, and isotope k_n so that each frontier component of $\Psi_n^{-1}(\Sigma'_+)$ is a closed geodesic in Σ_-.

Let λ be the ending lamination of e. Then Σ_- is regarded as the minimal supporting surface of λ. Let σ be a frontier component of Σ_- in S. Then $\rho_\infty(\sigma)$ represents a parabolic curve. Let P be a cusp in $\mathbb{H}^3/\Gamma$ corresponding to $\rho_\infty(\sigma)$, where G_∞ is regarded as a subgroup of Γ. Suppose that P is conjoining, and abuts on e and e'_k, (e'_k is one of the $e'_1, \ldots, e'_d$). Realise σ and λ as geodesics in S (with respect to a fixed hyperbolic metric) and consider their lifts $\tilde{\sigma}$ and $\tilde{\lambda}$ to $\mathbb{H}^2$. Let C be a crown domain of (λ, σ). Its projection $p(C)$ into S is an annulus having finitely many frontier components one of which is the closed geodesic σ and the others are bi-infinite geodesics. Let λ_C denote the union of the boundary leaves of $p(C)$ other than σ. We isotope k_n fixing the boundary of Σ'_+ so that $\Psi_n^{-1}(\lambda_C)$ is geodesic.

We say that the crown domain C is **well approximated** for (ρ_n) if $(\Psi_n^{-1}(\lambda_C))$ converges to geodesics which can be realised on a pleated subsurface. The pleated subsurface in question is thought of as a map from an incompressible subsurface Σ_+^0 of Σ'_+ containing Σ'_k, such that it is homotopic to the inclusion of Σ_+^0 into $(\mathbb{H}^3/\Gamma)_0$. If e'_k is simply degenerate, this condition is equivalent to saying that $(\Psi_n^{-1}(\lambda_C))$ does not converge to leaves of the ending lamination of e'_k. In particular, in the simply degenerate case, the choice of Σ'_+ is irrelevant since the parts of all such surfaces corresponding to Σ'_k are parallel in a neighbourhood of e'_k. The choice *is* however relevant if e'_k is wild.

Remark 2.11. In the above definition, we assumed that the geodesics which are limits of $(\Psi_n^{-1}(\lambda_C))$ are realisable. In practice, it suffices to assume that for at least one leaf ℓ of λ_C, the limit geodesic of $(\Phi_n^{-1}(\ell))$ is realisable. Indeed, if a leaf ℓ_0 of $(\Psi_n^{-1}(\lambda_C))$ is not realisable, it is a leaf of an ending lamination of some simply degenerate end of $(\mathbb{H}^3/\Gamma)_0$. If ℓ_1 is the limit of $(\Psi_n^{-1}(\ell'))$ where ℓ' is a geodesic in λ_C adjacent to ℓ_0, then ℓ_1 is asymptotic to ℓ_0. Hence it is also a leaf of the ending lamination. Inductively, no geodesics in the limit of $(\Psi_n^{-1}(\lambda_C)$ are realisable.

Remark 2.12. In the study of Brock's examples [Bro01] carried out in [MS17], e has only one partner e'_1, and $\Sigma' = \Sigma'_1$ is homeomorphic to Σ. The partner e'_1 is simply degenerate. We can choose the embedding of $\mathbf{M}_\Gamma$ so that the ending lamination of e'_1 is distinct from the ending lamination λ of e. We can further take Ψ_n to be the identity. For a crown domain C of (λ, σ), any leaf of λ_C is a leaf of λ and is therefore dense in the latter. Since λ can be realised by a pleated surface homotopic to the inclusion map into a neighbourhood of e'_1, the crown domain C is well approximated. Thus Brock's examples satisfy the present condition.

Remark 2.13. In general, the condition that the limits of the geodesics in $(\Psi_n^{-1}(\lambda_C))$ are realisable is weaker than the condition that the Hausdorff limit of $\Psi_n^{-1}(\lambda)$ is realisable. For instance, it may be possible that the sequence $(\Psi_n^{-1}(\lambda_C))$ converges to geodesics both whose ends spiral around a frontier component c of Σ'_k and Σ'_k lies in the interior of Σ'. In such a case, the Hausdorff limit of the entire $\Psi_n^{-1}(\lambda)$ contains c as a leaf. But c is not realisable.

2.7. **Statements and scheme.** With the background and terminology above, we can restate Theorems 1.2 and 1.3 more precisely.

Theorem 2.14. *Let $S = \mathbb{H}^2/G$ be a hyperbolic surface of finite area, and let $(\rho_n : \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{C})$ be a sequence of quasi-Fuchsian groups (obtained as quasi-conformal deformations of G) converging algebraically to $\rho_\infty : \pi_n(S) \to \mathrm{PSL}_2\,\mathbb{C}$. We set $G_n = \rho_n(\pi_1(S))$ and $G_\infty = \rho_\infty(\pi_1(S))$. Suppose that (G_n) converges geometrically to a Kleinian group Γ. Then the Cannon-Thurston maps $c_n : S^1(= \Lambda_G) \to \Lambda_{\rho_n(\pi_1(S))}$ for ρ_n converge pointwise to the Cannon-Thurston map $c_\infty : S^1 \to \Lambda_{\rho_\infty(\pi_1(S))}$ for ρ_∞ if and only if there is no coupled simply degenerate end of $(\mathbb{H}^3/G_\infty)_0$ with ending lamination λ and an untwisted conjoining cusp U abutting on the projection of e to $(\mathbb{H}^3/\Gamma)_0$, such that U corresponds to a parabolic curve σ, and a crown domain for (λ, σ) is well approximated.*

Theorem 2.15. *In the setting of Theorem 2.14 above, suppose that $(\mathbb{H}^3/G_\infty)_0$ has a coupled simply degenerate end with an untwisted conjoining cusp abutting on it such that the corresponding crown domain is well approximated. Let λ_+ be the union of the upper parabolic curves and the upper ending laminations, and λ_- the union of the lower parabolic curves and the lower ending laminations for $\mathbb{H}^3/G_\infty$. Then for $\zeta \in \Lambda_G = S^1$, the sequence $(c_n(\zeta))$ does not converge to $c_\infty(\zeta)$ if and only if ζ is a tip of a crown domain C for (μ, σ) where*

1. *$\mu \cup \sigma$ is contained in either λ_- or λ_+;*
2. *σ corresponds to an untwisted conjoining cusp abutting on the end for which μ is the ending lamination; and*
3. *C is well-approximated.*

A word of clarification here. The algebraic limit may contain both upper and lower ending laminations corresponding to *different* subsurfaces. Each of these is a potential source of discontinuity *provided* they satisfy the second and third conditions. We now briefly describe the scheme we shall follow to prove the above two theorems:

1. In Section 3, we shall show that if there is a coupled geometrically infinite end $(\mathbb{H}^3/G_\infty)_0$ with an untwisted conjoining cusp

abutting on it, and the corresponding crown domain is well approximated, then at the corresponding crown-tips, the sequence of Cannon-Thurston maps *do not* converge.

(2) In Section 4, we shall show that at points other than crown-tips, the sequence of Cannon-Thurston maps *always* converge pointwise.

(3) Finally, in Section 5, we shall prove the remaining assertion: if a crown C
   - is either not well approximated
   - or does not come from the ending lamination of a simply degenerate end e of $(\mathbb{H}^3/G_\infty)_0$ and a parabolic curve corresponding to an untwisted conjoining cusp abutting on e,

   then at the tips of C the sequence of Cannon-Thurston maps *do* converge pointwise.

3. Necessity of conditions

In this section, we shall prove the 'only if' part of Theorem 2.14, and the 'if' part of Theorem 2.15. As in the definition of well-approximated crown domains in Section 2.6, we assume that $(\mathbb{H}^3/G_\infty)_0$ has a coupled simply degenerate end e with an untwisted conjoining cusp P abutting on its projection $\bar{e}$ in $(\mathbb{H}^3/\Gamma)_0$. Let σ be a parabolic curve representing P, and λ the ending lamination of e, both of which we realise as geodesics in S. We lift them to $\mathbb{H}^2$. Consider a crown domain C and let ζ be a tip of C. We assume that C is well approximated. We shall show that $(c_n(\zeta))$ does not converge to $c_\infty(\zeta)$. This will prove both the 'only if' part of Theorem 2.14, and the 'if' part of Theorem 2.15 at the same time.

The proof is similar to that of discontinuity for Brock's example dealt with in [MS17]. Let Σ denote the minimal supporting surface of λ, i.e. an incompressible subsurface of S containing λ and minimal up to isotopy with respect to the inclusion. Since σ is finite-separating, there exists an essential subsurface B of S such that

(1) $\Sigma \cap B = \sigma$
(2) B corresponds to an upper geometrically finite end in the algebraic limit $(\mathbb{H}^3/G_\infty)_0$

By assumption, the crown domain C is well approximated. Therefore, there are subsurfaces Σ_- and Σ_+ contained in neighbourhoods of e and e' respectively with the following properties. (Here instead of the model manifold $\mathbf{M}_\Gamma$ used in §2.6, we use $\mathbb{H}^3/\Gamma$ itself.) The surface $h_n^{-1}(\Sigma_+)$ is parallel into $h_n^{-1}(\Sigma_-)$ in $\mathbb{H}^3/G_n$. Let Ψ_n denote an embedding from Σ_+

to Σ_- induced by this parallelism in $\mathbb{H}^3/G_n$ via h_n as in Section 2.6. The assumption of well-approximated crown domain says that for the sides λ_C ($\neq \sigma$) of $p(C)$, the geodesics $(\Psi_n^{-1}(\lambda_C))$ converges to geodesics μ which may be realised by a pleated surface homotopic to the inclusion of Σ_+. Therefore, the realisation of λ_C by a pleated surface from S to $\mathbb{H}^3/G_n$ inducing ρ_n between fundamental groups can be further pushed forward by h_n to a quasi-geodesic realisation. The sequence of quasi-geodesic realisations thus obtained converge to geodesics realised in $\mathbb{H}^3/\Gamma$. We denote the latter by μ^*.

Now let (κ_n) be a sequence of bi-infinite geodesics in $\mathbb{H}^3/G_n$ asymptotic in one direction to the closed geodesic representing $\rho_n(\sigma)$ and in the other to (an end-point of a leaf of) the realisation of λ_C. Lift κ_n to a geodesic $\tilde{\kappa}_n$ in $\mathbb{H}^3$ asymptotic to a lift of a leaf of λ_C. Also assume that $\tilde{\kappa}_n$ contains a basepoint o_n^μ lying within a bounded distance of $o_{\mathbb{H}^3}$. Since $(\Psi_n^{-1}(\lambda_C))$ converges to μ^*, $(\tilde{\kappa}_n)$ converges to a geodesic with distinct endpoints. One of these, p_σ (say), corresponds to σ. The other, p_μ (say), is the endpoint of a lift of a leaf of μ^* and lies in the limit set of Γ.

We note that the basepoints which we need to consider for convergence of Cannon-Thurston maps should lie on the algebraic locus and its pre-image under h_n. If we try to connect this basepoint to a lift of the realisation of λ_C by an arc in the right homotopy class, it might land at a point whose distance from $o_{\mathbb{H}^3}$ goes to ∞. This is the point where the assumption that the cusp P corresponding to σ is *untwisted* is relevant. We shall explain this more precisely now.

Recall that the surface S has a subsurface B corresponding to an upper geometrically finite end. We can choose a basepoint $o_{\mathbb{H}^2}$ in a component $\tilde{B}$ of the pre-image of B in $\mathbb{H}^2$ so that on the other side of a component of $\mathrm{Fr}\,\tilde{B}$, there lies a crown domain C with tip ζ. We can assume, by perturbing Φ_n equivariantly that $\Phi_n(o_{\mathbb{H}^2}) = o_{\mathbb{H}^3}$ for all $n = 1, 2, \ldots, \infty$. Let ℓ be a side of C having ζ as its endpoint at infinity. Thus, ℓ is a lift of a component of λ_C. Now λ can be realised by a pleated surface, thought of as a map from S to $\mathbb{H}^3/G_n$ inducing ρ_n at the level of fundamental groups. Hence there is a realisation $\tilde{\ell}_n$ of ℓ in $\mathbb{H}^3$, given by a geodesic connecting the two endpoints of $\Phi_n(\ell)$.

Recall that we assumed that P is conjoining and untwisted. Let P_n be a Margulis tube in $\mathbb{H}^3/G_n$ converging to P geometrically as (G_n) converges geometrically to Γ. Then in the model manifold $\mathbf{M}_n$, the vertical annulus forming the part of ∂P_n on the $B-$side has bounded height as $n \to \infty$. This is because its limit is a conjoining annulus. Hence we can connect $o_{\mathbb{H}^3}$ with $\tilde{\ell}_n$ by a path a_ℓ^n 'bridging over' a lift $\tilde{P}_n$

of P_n (i.e. the path a_ℓ^n travels up the bounded height lift $\tilde{P}_n$ to move from $o_{\mathbb{H}^3}$ to $\widetilde{\ell}_n$).

The embedding $\Psi_n : \Sigma_+ \to \Sigma_-$ lifts to an embedding $\widetilde{\Psi_n} : \widetilde{\Sigma_+} \to \widetilde{\Sigma_-}$ of universal covers. Both $\widetilde{\Sigma_+}, \widetilde{\Sigma_-}$ may be regarded as embedded in $\mathbb{H}^2$. Also, ℓ lies in $\widetilde{\Sigma_-}$. Since C is well approximated (by assumption), $\widetilde{\Psi}_n^{-1}(\ell)$ is realised in $\mathbb{H}^3$ by a limit of the $\widetilde{\ell}_n$'s. This implies that the geodesic $\widetilde{\ell}_n$ passes at a bounded distance from $\widetilde{P}_n$. Furthermore, since P is untwisted $\widetilde{\ell}_n$ cannot move too far from $o_{\mathbb{H}^3}$ along $\widetilde{P}_n$. Therefore we can choose a_ℓ^n to have bounded length as $n \to \infty$. Thus we are in the situation of the previous paragraph whether or not ℓ is contained in $\widetilde{\Psi}_n(\widetilde{\Sigma}_+)$. Since a_ℓ^n has bounded length, by defining its end-point to be the basepoint o_μ^n of the last paragraph, we see that o_μ^n is at a uniformly bounded distance from $o_{\mathbb{H}^3}$. Hence the image of ζ by the Cannon-Thurston map c_n converges to an endpoint p_μ of a lift of μ^*. On the other hand, since ζ is a crown-tip, $c_\infty(\zeta)$ coincides with p_σ by Theorem 2.3. Since $p_\mu \neq p_\sigma$ as was shown above, we conclude that $\lim_{n\to\infty} c_n(\zeta) \neq c_\infty(\zeta)$, establishing the 'only if' part of Theorem 2.14, and the 'if' part of Theorem 2.15. □

4. Pointwise convergence for points other than tips of crowns

In this section, we shall prove that for any $\zeta \in S^1(= \Lambda_G)$ that is not a crown-tip, $c_n(\zeta) \to c_\infty(\zeta)$ where the c_n's denote Cannon-Thurston maps. Our argument follows the broad scheme of [MS17, Section 5.5] but technically is considerably more involved. In particular, we need to deal with several cases that did not arise in [MS17].

We consider the universal covering $p : \tilde{S} \to S$, identifying $\tilde{S}$ with $\mathbb{H}^2$ and the deck group with $\pi_1(S)$ as before. Fix basepoints $o_{\mathbb{H}^2} \in \mathbb{H}^2$ and $o_{\mathbb{H}^3} \in \mathbb{H}^3$ independent of n. Let ζ be a point in Λ_G ($= \partial G$ or $\partial_h G$ according as S is closed or finite volume non-compact) such that ζ is not a crown-tip. Let $r_\zeta : [0, \infty) \to \mathbb{H}^2$ be the geodesic ray from $o_{\mathbb{H}^2}$ to ζ. The representation ρ_n induces a map $\Phi_n : \mathbb{H}^2 \to \mathbb{H}^3$ such that $\Phi_n(\gamma x) = \rho_n(\gamma)\Phi_n(x)$. By Proposition 2.6, it suffices to show that the EPP condition holds for r_ζ.

Let us briefly recall the structure of ends of $(\mathbb{H}^3/G_\infty)_0$. Take a relative compact core C_∞ of $(\mathbb{H}^3/G_\infty)_0$. Identify C_∞ with $S \times [0, 1]$ preserving the orientations. Then $C_\infty \cap \mathrm{Fr}(\mathbb{H}^3/G_\infty)_0$ consists of annuli lying on $S \times \{0, 1\}$ whose core curves are parabolic curves. Let $\alpha_1, \ldots, \alpha_p$ be parabolic curves lying on $S \times \{0\}$. We call these lower parabolic curves. Let $\alpha_{p+1}, \ldots, \alpha_{p+q}$ be those lying on $S \times \{1\}$. We call these upper parabolic curves. Identifying S with $S \times \{0\}$ and $S \times \{1\}$, we may also regard these as curves on S. Recall that each component of $S \setminus \bigcup_{j=1\cdots p} \alpha_j$ corresponds to a lower end of $(\mathbb{H}^3/G_\infty)$ whereas each component of $S \setminus \bigcup_{j=p+1\cdots p+q} \alpha_j$ corresponds to an upper end. Each of these ends is either geometrically finite or simply degenerate. We let $\tilde{e}_1, \ldots, \tilde{e}_s$ be the lower simply degenerate ends and $\tilde{f}_1, \ldots, \tilde{f}_t$ be the upper ones.

Take disjoint annular neighbourhoods $A_1, \ldots, A_p$ of upper parabolic curves $\alpha_1, \ldots, \alpha_p$, and in the same way, $A_{p+1}, \ldots, A_{p+q}$ of lower parabolic curves $\alpha_{p+1}, \ldots, \alpha_{p+q}$. We number the components of $S \setminus \cup_{j=1}^{p} A_j$ and $S \setminus \cup_{j=p+1}^{p+q} A_j$ so that components $\Sigma_1, \ldots, \Sigma_s$ of $S \setminus \cup_{j=1}^{p} A_j$ correspond to simply degenerate ends $\tilde{e}_1, \ldots, \tilde{e}_s$ respectively, and in $S \setminus \cup_{j=p+1}^{p+q} A_j$, components $\Sigma'_1, \ldots, \Sigma'_t$ correspond to simply degenerate ends $\tilde{f}_1, \ldots, \tilde{f}_t$ respectively. Then, each of $\Sigma_1, \ldots, \Sigma_s; \Sigma'_1, \ldots, \Sigma'_t$ supports the ending lamination of the corresponding simply degenerate end. We denote the components of $S \setminus \cup_{j=1}^{p} A_j$ other than $\Sigma_1, \ldots, \Sigma_s$ by $T_1, \ldots, T_u$, and the components of $S \setminus \cup_{j=p+1}^{p+q} A_j$ other than $\Sigma'_1, \ldots, \Sigma'_t$ by $T'_1, \ldots, T'_v$. Each of $T_1, \ldots, T_u; T'_1, \ldots, T'_v$ corresponds to a component of $\Omega_{G_\infty}/G_\infty$– the surface at infinity of a geometrically finite end.

Let $q : \mathbb{H}^3/G_\infty \to \mathbb{H}^3/\Gamma$ be the covering map induced by the inclusion of G_∞ into Γ. By Thurston's covering theorem [Thu80, Can96, Ohs92], each simply degenerate end of $(\mathbb{H}^3/G_\infty)_0$ has a neighbourhood that projects homeomorphically down to the geometric limit $(\mathbb{H}^3/\Gamma)_0$ under q. We denote such neighbourhoods of $\tilde{e}_1, \ldots, \tilde{e}_s$ by $\tilde{E}_1, \ldots, \tilde{E}_s$, and those of $\tilde{f}_1, \ldots, \tilde{f}_t$ by $\tilde{F}_1, \ldots, \tilde{F}_t$. Recall also that $\mathbb{H}^3/\Gamma$ has a model manifold $\mathbf{M}_\Gamma$ embedded in $S \times (0,1)$, and we identify its non-cuspidal part $\mathbf{M}^0_\Gamma$ with $(\mathbb{H}^3/\Gamma)_0$ using the model map f_Γ. The corresponding ends of $(\mathbb{H}^3/\Gamma)_0$ in $S \times (0,1)$ and their neighbourhoods are denoted by the same symbols without tildes. By moving the model manifold as in Section 2.5 preserving the combinatorial structure of brick decomposition, we can assume that the model manifold $\mathbf{M}_\Gamma$, which is regarded as a subset in $S \times (0,1)$, and the model map f_Γ have the following properties:

- $\mathbf{M}^0_\Gamma$ is decomposed into 'bricks' each of which is defined to be the closure of a maximal family of parallel horizontal surfaces.
- There is a brick containing $S \times \{1/2\}$.
- The image of the inclusion of G_∞ into Γ corresponds to the fundamental group carried by an incompressible immersion of S into $\mathbf{M}_\Gamma$– the algebraic locus. The surface consists of horizontal subsurfaces lying on $S \times \{1/2\}$ and annuli going around torus boundary components.

- We can assume that the brick containing $S \times \{1/2\}$ has the form $S \times [1/3, 2/3]$. Corresponding to simply degenerate ends $\tilde{e}_1, \ldots, \tilde{e}_s$, there are bricks $\mathbf{E}_1 = \Sigma_1 \times [2/3, 5/6), \ldots, \mathbf{E}_s = \Sigma_s \times [2/3, 5/6)$ containing images of $\tilde{e}_1, \ldots, \tilde{e}_s$ under f_Γ.
- We also have bricks $T_1 \times [2/3, x_1], \ldots, T_u \times [2/3, x_u]$ with $x_1, \ldots, x_u \geq 5/6$.
- Similarly, corresponding to simply degenerate ends $f_1, \ldots, f_t$, there are bricks $\mathbf{F}_1 = \Sigma'_1 \times (1/6, 1/3], \ldots, \mathbf{F}_t = \Sigma'_t \times (1/6, 1/3]$, and $T'_1 \times [x'_1, 1/3], \ldots, T'_v \times [x'_v, 1/3]$ with $x'_1, \ldots, x'_v \leq 1/6$.
- Every end of $\mathbf{M}^0_\Gamma$ other than $e_1, \ldots, e_s; f_1, \ldots, f_t$ lies at a horizontal level in $(0, 1/6]$ or $(5/6, 1)$.
- There are cusp neighbourhoods in $\mathbf{M}_\Gamma$ containing $A_1 \times [2/3, 5/6), \ldots A_p \times [2/3, 5/6); A_{p+1} \times (1/6, 1/3], \ldots, A_{p+q} \times (1/6, 1/3]$, which we denote respectively by $U_1, \ldots, U_p; U_{p+1}, \ldots, U_{p+q}$.

The proof of the EPP condition that is required in order to establish that $c_n(\zeta) \to c_\infty(\zeta)$ splits into two cases.

(Case I:) The geodesic ray $r_\zeta \colon [0, \infty) \to \mathbb{H}^2$ that connects the basepoint $o_{\mathbb{H}^2}$ to the point at infinity ζ projected down to S enters and leaves each of the subsurfaces $\Sigma_i, \Sigma'_i, T_i, T'_i$ infinitely often.

(Case II:) The geodesic ray r_ζ projected down to S eventually lies inside one of the subsurfaces $\Sigma_i, \Sigma'_i, T_i, T'_i$.

4.1. **Case I: Infinite Electric Length.** We shall first consider Case I above and show that the EPP condition is satisfied. We can assume that at least one of s, t is positive since otherwise G_∞ is geometrically finite and this case has already been dealt with in [MS13].

Consider the pre-images $p^{-1}(\Sigma_1), \ldots, p^{-1}(\Sigma_s)$ in $\mathbb{H}^2$ of $\Sigma_1, \ldots, \Sigma_s$. Its union is denoted as $\tilde{\Sigma}$. In the same way, we denote by $\tilde{\Sigma}'$ the union of the pre-images of $\Sigma'_1, \ldots, \Sigma'_t$. Equip $\mathbb{H}^2$ with an electric metric d_Σ, by electrocuting components of $\tilde{\Sigma}$. Similarly, equip $\mathbb{H}^2$ with a different electric metric $d_{\Sigma'}$, by electrocuting components of $\tilde{\Sigma}'$. The hypothesis of Case I is equivalent to the following.

Assumption 4.1 (Infinite electric length)**.** *We assume (for the purposes of this subsection) that the lengths of r_ζ with respect to d_Σ and $d_{\Sigma'}$ are both infinite.*

In this case, we say that ζ satisfies the **IEL** *condition.*

Under Assumption 4.1, our argument is similar to that in §5.3.3 of [MS17].

Due to the IEL condition, r_ζ has infinite length for both d_Σ and $d_{\Sigma'}$. Hence the ray r_ζ goes in and out of components of $\tilde{\Sigma}$ (as also those of $\tilde{\Sigma}'$) infinitely many times. Therefore, we have the following. (See §5.4.4 in [MS17].)

Lemma 4.2. *Let d' equal either d_Σ or $d_{\Sigma'}$. Then there exists a function $f_\zeta : \mathbb{N} \to \mathbb{N}$ with $f_\zeta(t) \to \infty$ as $n \to \infty$ such that if $t \geq N$, then $d'(r_\zeta(0), r_\zeta(t)) \geq f_\zeta(N)$.*

To prove the EPP condition in Case I, we need to prove the following.

Proposition 4.3. *Suppose that r_ζ has infinite length in both d_Σ and $d_{\Sigma'}$ as in Assumption 4.1. Then, the EPP condition holds for ζ.*

The proof of Proposition 4.3 occupies the rest of this subsection.

Recall that we have a model manifold $\mathbf{M}_\Gamma$ for the geometric limit $\mathbb{H}^3/\Gamma$ with a bi-Lipschitz map $f_\Gamma^{-1} : \mathbb{H}^3/\Gamma \to \mathbf{M}_\Gamma$. Let $\tilde{f}_\Gamma^{-1} : \mathbb{H}^3 \to \tilde{\mathbf{M}}_\Gamma$ denote its lift to the universal cover. We denote the lift of the model metric on $\mathbf{M}_\Gamma$ to $\tilde{\mathbf{M}}_\Gamma$ by $d_{\mathbf{M}}$. To prove Proposition 4.3, we shall consider the behaviour of a ray in $\tilde{\mathbf{M}}_\Gamma$ with a new electric metric. Since (ρ_n) converges algebraically to ρ_∞, we have a ρ_∞-equivariant map $\Phi_\infty : \mathbb{H}^2 \to \mathbb{H}^3$. The ray $\Phi_\infty \circ r_\zeta$ is projected (under the covering projection) to a ray r_ζ^Γ in $\mathbb{H}^3/\Gamma$. Composing it further with $\tilde{f}_\Gamma^{-1}$, we get a ray $r_\zeta^{\mathbf{M}} : [0, \infty) \to \tilde{\mathbf{M}}_\Gamma$.

Recall that we have bricks $\mathbf{E}_1, \ldots, \mathbf{E}_s; \mathbf{F}_1, \ldots, \mathbf{F}_t$ in the model manifold $\mathbf{M}_\Gamma^0$. We say that $\mathbf{E}_j$ is **minimal** if none of the $\Sigma'_1, \ldots, \Sigma'_t$ can be isotoped into Σ_j. In the same way, we say that $\mathbf{F}_j$ among $\mathbf{F}_1, \ldots, \mathbf{F}_t$ is minimal if none of $\Sigma_1, \ldots, \Sigma_s$ can be isotoped into Σ'_j.

Let $\mathbf{E}$ be a neighbourhood of an end of $\mathbf{M}_\Gamma^0$, having a form $\Sigma \times J$ where J is a half-interval. A cusp neighbourhood in $\mathbf{M}_\Gamma$ is said to be **associated** to $\mathbf{E}$ if its core curve is freely homotopic into $\mathbf{E}$ in $\mathbf{M}_\Gamma$. Suppose that $\mathbf{E}_j$ is minimal. We first consider cusps among $U_1, \ldots, U_p$ whose corresponding annulus intersects Σ_j at both of its boundary components. (It may be possible that there are no such cusps.) We call these cusps **non-separating** for $\mathbf{E}_j$ and the others separating. We denote non-separating cusps by $U_{l_1}, \ldots, U_{l_v}$. We choose $\epsilon > 0$ small enough so that the ϵ-neighbourhoods of $U_{l_1}, \ldots, U_{l_v}$, which we denote by $U_{l_1}^\epsilon, \ldots, U_{l_v}^\epsilon$ respectively, are disjoint and $\mathbf{E}_j \setminus \cup_{i=1}^v U_{l_i}^\epsilon$ is a deformation retract of $\mathbf{E}_j$. Now, let $U_{j_1}, \ldots, U_{j_w}$ be the separating cusp neighbourhoods among $U_1, \ldots, U_p$ associated to $\mathbf{E}_j$ (i.e. the separating ones abutting on $\mathbf{E}_j$). Also let $U_{k_1}, \ldots, U_{k_\omega}$ be those among $U_{p+1}, \ldots, U_{p+q}$ which are associated to $\mathbf{E}_j$.

If the end e_j corresponding to $\mathbf{E}_j$ is *not coupled*, then we consider the union $(\mathbf{E}_j \setminus \cup_{i=1}^v U_{l_i}^\epsilon) \cup_{i=1}^w U_{j_i} \cup \cup_{i=1}^\omega (U_{k_i} \cup A_{k_i} \times [1/3, 2/3])$, and denote it by $\mathbf{E}_j^{\text{ext}}$, putting the superscript 'ext' to denote 'extension by adding cusps'. In the same way, we define $\mathbf{F}_j^{\text{ext}}$ for $F_1, \ldots, F_t$ exchanging the roles of upper cusp neighbourhoods and lower cusp neighbourhoods.

If e_j is *coupled*, we need to define $\mathbf{E}_j^{\text{ext}}$ to be something larger. Suppose that e_j is upward and coupled, and that $\mathbf{E}'_j$ is a neighbourhood of its partner. We take $\mathbf{E}'_j$ to have the form $\Sigma_j \times (11/12, 23/24]$. We can further isotope the embedding of $\mathbf{M}_\Gamma$ without changing the combinatorial structure of the brick decomposition so that every cusp neighbourhood associated to $\mathbf{E}'_j$ without abutting on it lies at the horizontal levels whose infimum is 47/48. We then add to $\mathbf{E}'_j$ all cusp neighbourhoods abutting on it, and for each cusp neighbourhood U associated to but not abutting on $\mathbf{E}'_j$, we add $U \cup A \times [23/24, 27/48]$, where A is an annulus which is the vertical projection of U into S, and denote by $\mathbf{E}''_j$ what we get. By adding the union of $\mathbf{E}''_j$ to $(\mathbf{E}_j \setminus \cup_{i=1}^{v} U_{l_i}^{\epsilon}) \cup \cup_{i=1}^{w} U_{j_i} \cup \cup_{i=1}^{\omega} (U_{k_i} \cup A_{k_i} \times [1/3, 2/3])$, we obtain $\mathbf{E}_j^{\text{ext}}$ in the case when e_j is coupled. We can define $\mathbf{F}_j^{\text{ext}}$ for the case when e'_j is coupled in the same way.

Let $\tilde{\mathbf{E}}_j^{\text{ext}}$ denote the pre-image of $\mathbf{E}_j^{\text{ext}}$ in the universal cover $\tilde{\mathbf{M}}_\Gamma$. Since $\mathbf{E}_j$ is parametrised as $\Sigma_j \times [2/3, 5/6)$ and $\Sigma_j \times \{2/3\}$ is homotopic to $\Sigma_j \times \{1/2\}$ in $\mathbf{M}_\Gamma$ lying on the algebraic locus, we see that there is a one-to-one correspondence between the components of $p^{-1}(\Sigma_j)$ and the components of $\tilde{\mathbf{E}}_j^{\text{ext}}$. In the same way, there is a one-to-one correspondence between the components of $p^{-1}(\Sigma'_j)$ and the components of the pre-image $\tilde{\mathbf{F}}_j^{\text{ext}}$ of $\mathbf{F}_j^{\text{ext}}$ in the universal cover $\tilde{\mathbf{M}}_\Gamma$. For a component R of $p^{-1}(\Sigma_j)$ or $p^{-1}(\Sigma'_j)$, we denote the corresponding component of $\tilde{\mathbf{E}}_j^{\text{ext}}$ or $\tilde{\mathbf{F}}_j^{\text{ext}}$ by $\tilde{E}_R$.

We shall need the following lemma, which is a slight generalisation of [DM16, Lemma 2.25].

Lemma 4.4. *Let* $\mathbf{E}$ *be one of* $\mathbf{E}_1, \ldots, \mathbf{E}_s; \mathbf{F}_1, \ldots, \mathbf{F}_t$, *and assume that it is minimal. Let* $U_{l_1}, \ldots, U_{l_v}$ *be the non-separating cusps for* $\mathbf{E}$. *We electrocute* $\tilde{\mathbf{M}}_\Gamma$ *at the pre-images of* $\sqcup_{i=1}^{v} U_{l_i}$, *and get a Gromov hyperbolic metric* $d_{\mathbf{M}}^0$. *Then, there is a constant* K *depending only on* Γ *such that each component of the pre-image* $\tilde{\mathbf{E}}^{\text{ext}}$ *of* $\mathbf{E}^{\text{ext}}$ *in* $(\tilde{\mathbf{M}}_\Gamma, d_{\mathbf{M}}^0)$ *is* K*-quasi-convex.*

Proof. By definition, $\mathbf{E}$ has a form $\Sigma \times J$, where Σ is one of $\Sigma_1, \ldots, \Sigma_s; \Sigma'_1, \ldots, \Sigma'_t$ and J is either $[2/3, 5/6)$ or $(1/6, 1/3]$. For simplicity, we assume that J is $[2/3, 5/6)$. The case where J is $(1/6, 1/3]$ can be dealt with just by turning everything upside down.

Consider a covering of $\mathbb{H}^3/\Gamma$ corresponding to $(f_\Gamma)_*(\pi_1(\Sigma))$, where $\pi_1(\Sigma)$ is regarded as a subgroup of $\pi_1(\mathbf{M}_\Gamma)$ by identifying $S \times \{1/2\}$

with S. (Note that the identification of S with $S \times \{1/2\}$ may be different from that of S with the algebraic locus. Still, if we restrict them to Σ, we get the same identification, for the algebraic locus lies on $S \times \{1/2\}$ by the covering theorem.) We denote the covering by $\mathbb{H}^3/\Gamma^\Sigma$. The simply degenerate end of $\mathbb{H}^3/\Gamma$ corresponding to $\Sigma \times \{5/6\}$ lifts to $\mathbb{H}^3/\Gamma^\Sigma$. On the other hand, if there is a lower simply degenerate end of $(\mathbb{H}^3/\Gamma^\Sigma)_0$, it must correspond to $\Sigma' \times \{1/6\}$ for a Σ' that can be isotoped into Σ. Since we assumed that Σ is minimal, this cannot happen. Therefore every lower end of $(\mathbb{H}^3/\Gamma^\Sigma)_0$ is geometrically finite.

Since $\mathbf{E}$ is a deformation retract of $\mathbf{E}^{\text{ext}}$, we see that f_Γ induces a Lipschitz embedding of $\mathbf{E}^{\text{ext}}$ into $\mathbb{H}^3/\Gamma^\Sigma$. We denote this Lipschitz embedding by f_Γ^Σ. The map f_Γ^Σ extends to a bi-Lipschitz homeomorphism to the convex core of $\mathbb{H}^3/\Gamma^\Sigma$ just by adding geometrically finite bricks corresponding to the lower geometrically finite ends and isometric lifts of non-separating cusp neighbourhoods for $\mathbf{E}$. Therefore, each component of the pre-image of $f_\Gamma^\Sigma(\mathbf{E}^{\text{ext}})$ in $\mathbb{H}^3$ is quasi-convex modulo pre-images of the cusps corresponding to $U_{l_1}, \ldots, U_{l_v}$ with a constant depending only on Γ. Since the bi-Lipschitz constant of f_Γ depends only on S (see [OS10]), this implies that each component of $\tilde{\mathbf{E}}^{\text{ext}}$ is quasi-convex in $(\tilde{\mathbf{M}}_\Gamma, d^0_{\mathbf{M}})$. □

It follows, as was explained in §2.1, that we can electrocute the components of the pre-image of $\mathbf{E}^{\text{ext}}$ in $(\tilde{\mathbf{M}}_\Gamma, d^0_{\mathbf{M}})$ and get a Gromov hyperbolic (pseudo-)metric on $\tilde{\mathbf{M}}_\Gamma$. We denote the new electric metric by $d^1_{\mathbf{M}}$, and use $\tilde{\mathbf{M}}^1_\Gamma$ as a shorthand for $(\tilde{\mathbf{M}}_\Gamma, d^1_{\mathbf{M}})$. In this new metric, geodesic arcs homotopic to subarcs of $r^{\mathbf{M}}_\zeta$ going deep into $\tilde{\mathbf{E}}$ may be conveniently ignored, which is what electrocution allows us to do. However, we need to handle arcs that go deep into the pre-images of the other ends. For this, we need the second electrocution process as follows.

Let $\mathbf{E}'$ be one of $\mathbf{E}_1, \ldots, \mathbf{E}_s; \mathbf{F}_1, \ldots, \mathbf{F}_t$ other than the $\mathbf{E}$ we took in Lemma 4.4. We say that $\mathbf{E}' = \Sigma \times [2/3, 5/6)$ among $\mathbf{E}_1, \ldots, \mathbf{E}_s$ is second minimal if there is no Σ'_j $(j = 1, \ldots, t)$ that can be isotoped into Σ except for the case when $\mathbf{F}_j$ is the $\mathbf{E}$ chosen above. In the same way, we define $\mathbf{E}' = \Sigma \times (1/6, 1/3]$ among $\mathbf{F}_1, \ldots, \mathbf{F}_t$ to be second minimal.

Suppose that $\mathbf{E}'$ is second minimal. For simplicity of description, we assume that $\mathbf{E}'$ is one of $\mathbf{E}_1, \ldots, \mathbf{E}_s$. We define $\mathbf{E}'^{\text{ext}}$ as follows. We first fix a positive number ϵ such that the ϵ-neighbourhoods of $\mathbb{Z}$-cusps abutting on $\mathbf{E}'$ are disjoint, and if we delete the union of such neighbourhoods from $\mathbf{E}'$, we get a deformation retract of $\mathbf{E}'$. If ϵ is small enough, it has these properties, and we can choose ϵ depending only on Γ. Suppose that a $\mathbb{Z}$-cusp U among $U_1, \ldots, U_p$ is associated to

$\mathbf{E}'$, i.e., abuts on $\mathbf{E}'$. Then, if U is separating for $\mathbf{E}'$ and not contained in $\mathbf{E}^{\text{ext}}$, we add it to $\mathbf{E}'$. If U is either non-separating for $\mathbf{E}'$ or contained in $\mathbf{E}^{\text{ext}}$, we delete the ϵ-neighbourhood of U from $\mathbf{E}'$. Next suppose that U is a $\mathbb{Z}$-cusp neighbourhood among $U_{p+1}, \ldots, U_{p+q}$ associated to $\mathbf{E}'$, i.e. it does not abut on $\mathbf{E}'$. If U is contained in $\mathbf{E}^{\text{ext}}$, we do nothing. Otherwise, we add $U \cup A \times [1/3, 2/3]$ to $\mathbf{E}'$, where A denotes A_j defined before if U is U_j.

Let $\tilde{\mathbf{E}}'^{\text{ext}}$ be the pre-image of $\mathbf{E}'^{\text{ext}}$ in $\tilde{\mathbf{M}}_\Gamma$. Then by the same argument as Lemma 4.4, we can show that each component of $\tilde{\mathbf{E}}'^{\text{ext}}$ is K'-quasi-convex in $\tilde{\mathbf{M}}^1_\Gamma$ after electrocuting the pre-images of non-separating cusps for $\mathbf{E}'$. Here K' is a constant depending only on Γ, and ϵ is also a constant chosen as above depending only on Γ. Therefore (c.f. the discussion in §2.1), we can again electrocute $\tilde{\mathbf{E}}'^{\text{ext}}$ in $(\tilde{\mathbf{M}}_\Gamma, d^1_{\mathbf{M}})$ to get a new Gromov hyperbolic metric denoted by $d^2_{\mathbf{M}}$.

We repeat the same procedure inductively. Assume that we have defined the electric metric $d^{m-1}_{\mathbf{M}}$. Then as the next step of induction, we consider an m-th minimal $\mathbf{E}$ among $E_1, \ldots, E_s; F_1, \ldots, F_t$. We construct $\mathbf{E}^{\text{ext}}$ in the same way as we defined $\mathbf{E}'^{\text{ext}}$, paying attention to already electrocuted cusp neighbourhoods. We define a new electric metric $d^m_{\mathbf{M}}$, electrocuting the pre-image of $\mathbf{E}^{\text{ext}}$. The new metric is again Gromov hyperbolic. Finally, we get a hyperbolic electric metric $d^{s+t}_{\mathbf{M}}$ and denote it by $\bar{d}_{\mathbf{M}}$.

Recall that we have a ray $r^{\mathbf{M}}_\zeta : [0, \infty) \to \tilde{\mathbf{M}}_\Gamma$. For $s_1, s_2 \in [0, \infty)$ with $s_1 < s_2$, we denote by $r^{\mathbf{M}}_\zeta(s_1, s_2)^*$ the geodesic arc, parametrised by length, path-homotopic to $r^{\mathbf{M}}_\zeta|[s_1, s_2]$. If we connect $r_\zeta(s_1)$ and $r_\zeta(s_2)$ by a geodesic arc with respect to $\bar{d}_{\mathbf{M}}$, it fellow-travels $r_\zeta(s_1, s_2)$ in $(\tilde{\mathbf{M}}_\Gamma, \bar{d}_{\mathbf{M}})$ since all geometrically infinite ends into which the geodesic may escape are electrocuted. Therefore, by Lemma 2.1, we have the following.

Lemma 4.5. *There is a constant L depending only on Γ with the following property. Let $\{R_\alpha\}_\alpha$ be the collection of all components of $p^{-1}(\Sigma_j)\,(j = 1, \ldots, s)$ and $p^{-1}(\Sigma'_j)\,(j = 1, \ldots t)$ that $r_\zeta([s_1, s_2])$ intersects essentially (relative to the endpoints). We consider $\mathbf{E}^{\text{ext}}_j$ or $\mathbf{F}^{\text{ext}}_j$ for each Σ_j or Σ'_j, and let $\tilde{E}_{R_\alpha}$ be a component of the pre-image of one of them for some j corresponding to R_α. Then $r^{\mathbf{M}}_\zeta(s_1, s_2)^*$ is contained in the L-neighbourhood of $\cup_\alpha \tilde{E}_{R_\alpha}$ with respect to the metric $d_{\mathbf{M}}$.*

To show the EPP condition, we need to consider the behaviour of geodesics in the universal cover of the model manifold $\mathbf{M}_n$ of $\mathbb{H}^3/G_n$

with the model metric. Recall that corresponding to the geometric convergence of $\mathbf{M}_n$ to $\mathbf{M}_\Gamma$,, there is an approximate isometry $\mathbf{h}_n\colon B_{R_n}(\mathbf{M}_n, \mathbf{x}_n) \to B_{K_n R_n}(\mathbf{M}_\Gamma, \mathbf{x}_\infty)$. Also $\mathbf{h}_n$ is a K_n-bi-Lipschitz homeomorphism, where $R_n \to \infty$, $K_n \to 1$, and $\mathbf{x}_\infty$ lies on the algebraic locus. The cusp neighbourhoods $U_1, \ldots, U_{p+q}$ correspond to Margulis tubes in $\mathbf{M}_n$ for large n, and we denote them by $U_1^n, \ldots, U_{p+q}^n$. Their complement $\mathbf{M}_n^0 = \mathbf{M}_n \setminus (\cup_{k=1}^{p+q} U_k^n)$ can also be regarded as a brick manifold by defining a brick to be the closure of a maximal collection of parallel horizontal surfaces. Each $\mathbf{E}_j$ or the union of $\mathbf{E}_j$ and its partner when it is coupled is a geometric limit of a brick $\mathbf{E}_j^n$ in $\mathbf{M}_n^0$ as $n \to \infty$. We also define a brick $\mathbf{F}_j^n$ in the same way from $\mathbf{F}_j$.

Recall that $\mathbf{E}_j^{\mathrm{ext}}$ is composed of possibly 'shrunken' $\mathbf{E}_j$ along with some associated cusp neighbourhoods and thickened annuli connecting $\mathbf{E}_j$ with associated cusp neighbourhoods. We define $\mathbf{E}_j^{n\mathrm{ext}}$ by shrinking $\mathbf{E}_j^n$ correspondingly, and adding Margulis tubes and thickened annuli corresponding to the associated cusp neighbourhoods and thickened annuli in $\mathbf{E}_j^{\mathrm{ext}}$. In the same way, we also define $\mathbf{F}_j^{n\mathrm{ext}}$. In the process of shrinking, we arrange so that there is a positive constant $\epsilon > 0$ independent of n such that any two of $\mathbf{E}_1^{n\mathrm{ext}}, \ldots, \mathbf{E}_s^{n\mathrm{ext}}; \mathbf{F}_1^{n\mathrm{ext}}, \ldots, \mathbf{F}_t^{n\mathrm{ext}}$ are at least ϵ-apart from each other.

Then by the same argument as in Lemma 4.4, we see that we can electrocute the pre-images of $\mathbf{E}_1^{n\mathrm{ext}}, \ldots, \mathbf{E}_s^{n\mathrm{ext}}; \mathbf{F}_1^{n\mathrm{ext}}, \ldots, \mathbf{F}_t^{n\mathrm{ext}}$ in the universal cover $\tilde{\mathbf{M}}_n$ of $\mathbf{M}_n$ with the lifted model metric (denoted as $d_{\mathbf{M}}^n$). After electrocution we get a Gromov hyperbolic metric denoted as $\bar{d}^n$. In the above construction of $\mathbf{E}_1^{n\mathrm{ext}}, \ldots, \mathbf{E}_s^{n\mathrm{ext}}; \mathbf{F}_1^{n\mathrm{ext}}, \ldots, \mathbf{F}_t^{n\mathrm{ext}}$, we can arrange so that $(\tilde{\mathbf{M}}_n, \bar{d}^n)$ converges geometrically to $(\tilde{\mathbf{M}}_\Gamma, \bar{d}_{\mathbf{M}})$.

Recall that we have a ρ_n-equivariant map $\Phi_n\colon \mathbb{H}^2 \to \mathbb{H}^3$, and a ray $\Phi_n \circ r_\zeta$ in $\mathbb{H}^3$. By pulling back this ray by the lift of the model map f_n, we get a ray $r_\zeta^{\mathbf{M}_n}\colon [0,\infty) \to \tilde{\mathbf{M}}_n$, where $\tilde{\mathbf{M}}_n$ is the universal cover of $\mathbf{M}_n$. For $s < t \in [0,\infty)$, we let $r_n(s,t)^*$ be the geodesic arc in $\mathbb{H}^3$ connecting $\Phi_n \circ r_\zeta(s)$ and $\Phi_n \circ r_\zeta(t)$. Recall also that each component R of $p^{-1}(\Sigma_j)$ or $p^{-1}(\Sigma_j')$ corresponds to a component of the pre-image of $\mathbf{E}_j^{\mathrm{ext}}$ or $\mathbf{F}_j^{\mathrm{ext}}$. Each component of the pre-image of $\mathbf{E}_j^{\mathrm{ext}}$ or $\mathbf{F}_j^{\mathrm{ext}}$ in turn corresponds to one of $\mathbf{E}_j^{n\mathrm{ext}}$ or $\mathbf{F}_j^{n\mathrm{ext}}$ in $\tilde{\mathbf{M}}_n$. We denote the component of $\mathbf{E}_j^{n\mathrm{ext}}$ or $\mathbf{F}_j^{n\mathrm{ext}}$ corresponding to R by $\tilde{E}_R^n$. Then by pulling back the situation of Lemma 4.5 above by $\mathbf{h}_n$, we get the following.

Lemma 4.6. *There is a constant L' independent of n with the following property. Let $\{R_\alpha\}_\alpha$ be the collection of all components of $p^{-1}(\Sigma_j)\,(j = 1, \ldots, s)$ and $p^{-1}(\Sigma_j')\,(j = 1, \ldots t)$ that $r_\zeta([s_1, s_2])$ intersects essentially (relative to the endpoints) as in Lemma 4.5. Then*

$r_n(s_1, s_2)^*$ is contained in the L'-neighbourhood of $\cup_\alpha \tilde{E}^n_{R_\alpha}$ with respect to the metric $d^n_{\mathbf{M}}$.

Proof of Proposition 4.3:
Since the action of $\pi_1(S)$ on $\mathbb{H}^2$ corresponding to G is properly discontinuous, Assumption 4.1 implies that there is a function $g\colon [0,\infty) \to [0,\infty)$ with $g(x) \to \infty$ as $x \to \infty$ such that $r_\zeta|[s,t]$ intersects only components of $p^{-1}(\Sigma_j)$ that are at distance greater than $g(s)$ from the origin $o_{\mathbb{H}^2}$. We now fix lifts of the basepoints $\mathbf{o}_\Gamma$ in $\tilde{\mathbf{M}}_\Gamma$ and $\mathbf{o}_n$ in $\tilde{\mathbf{M}}_n$. These are lifts of $\mathbf{x}_\infty$ and $\mathbf{x}_n$ respectively. By proper discontinuity of the action of Γ on $(\tilde{\mathbf{M}}_\Gamma, \bar{d}_\Gamma)$, we also see that there is a function $k_\Gamma\colon [0,\infty) \to [0,\infty)$ with $k_\Gamma(x) \to \infty$ as $x \to \infty$ such that if R is a component of $p^{-1}(\Sigma_j)$ that $r_\zeta|[s,t]$ intersects, then $\tilde{E}_R$ is at distance greater than $k_\Gamma(s)$ from $\mathbf{o}_\Gamma$. By pulling this back by $\mathbf{h}_n$ for large n, we see that there is a function $k : [0,\infty) \to [0,\infty)$ with $k \to \infty$ as $x \to \infty$ such that if R is a component of $p^{-1}(\Sigma_j)$ that $r_\zeta|[s,t]$ intersects, then $\tilde{E}^n_R$ is at $\bar{d}_n$-distance greater than $k(s)$ from $\mathbf{o}_n$. By Lemma 4.6, this implies that any point of the geodesic arc $r_n(s_1, s_2)^*$ is at distance greater than $k(s) - L'$ from $\mathbf{o}_n$. Since $f_n\colon \mathbf{M}_n \to \mathbb{H}^3/G_n$ is a bi-Lipschitz map whose Lipschitz constant can be chosen independently of n, this concludes the proof of Proposition 4.3. □

4.2. **Case II: $\bar{r}_\zeta$ is eventually contained in one subsurface.** Now, we turn to Case II, i.e. we suppose that Assumption 4.1 does not hold. Then eventually $r_\zeta(t)$ stays in one component R of $p^{-1}(\Sigma_j)$ or $p^{-1}(\Sigma'_j)$ or $p^{-1}(T_j)$ or $p^{-1}(T'_j)$. We now assume that R is a component of $p^{-1}(\Sigma_j)$. We can argue in the same way also for the case when R is a component of $p^{-1}(\Sigma'_j)$ just turning $\mathbf{M}_\Gamma$ upside down, whereas for the case when R is a component of $p^{-1}(T_j)$ or $p^{-1}(T'_j)$ we need a little modification of the argument, which we shall mention at the end of this subsection. We can also assume that if $r_\zeta(t)$ is also eventually contained in a component of $p^{-1}(\Sigma'_k)$, then Σ'_k is *not* contained in Σ_j up to isotopy, by choosing a minimal element among components containing $r_\zeta(t)$ eventually.

4.2.1. *Case II A: when ζ is an endpoint of a lift of a boundary parabolic curve.* We first consider the special case when ζ is an endpoint of a lift $\tilde{c}$ of a component c of $\operatorname{Fr}\Sigma_j$. Then $c_\infty(\zeta)$ is a parabolic fixed point of $\rho_\infty(\gamma_c)$, where γ_c is an element of G corresponding to c. We consider the axis geodesic a_c of $\rho_n(\gamma_c)$, and let $\mathcal{N}_n$ be its ϵ-neighbourhood in $\mathbb{H}^3$. Correspondingly, there is a horoball $\mathcal{N}_\infty$ stabilised by $\rho_\infty(\gamma_c)$ to which $(\mathcal{N}_n)$ converges geometrically. Since there is an upper bound for the distance from any point on $\Phi_\infty \circ \tilde{c}$ to $\mathcal{N}_\infty$, there is an upper bound independent of n for the distance from any point on the image of $\Phi_n \circ r_\zeta$ to $\mathcal{N}_n$.

We define a broken geodesic arc $r_n^+(s,t)$ consisting of three geodesic arcs as follows. Consider the shortest geodesic arcs δ_s^n and δ_t^n that connect respectively $\Phi_n \circ r_\zeta(s)$ and $\Phi_n \circ r_\zeta(t)$ with $\mathcal{N}_n$, and then the geodesic arc $\delta_{s,t}^n$ in $\mathcal{N}_n$ connecting the endpoint of δ_s on $\operatorname{Fr}\mathcal{N}_n$ with that of δ_t. We define $r_n^+(s,t)$ to be the concatenation $\delta_s^n * \delta_{s,t}^n * \delta_t^n$. Then the observation in the previous paragraph implies that $r_n^+(s,t)$ is a uniform quasi-geodesic, i.e., there are constants A, B independent of s, t and n such that $r_n^+(s,t)$ is an (A,B)-quasi-geodesic.

Now, by the convexity of Margulis tubes and the properness of $\Phi_n \circ r_\zeta$, it is easy to check that there is a function $g^+ \colon [0,\infty) \to [0,\infty)$ with $g^+(u) \to \infty$ as $u \to \infty$ such that any point in $r_n^+(s,t)$ lies outside the $g^+(s)$-ball centred at $o_{\mathbb{H}^3}$. Since $r_n^+(s,t)$ is uniformly quasi-geodesic, the geodesic arc connecting $\Phi_n \circ r_\zeta(s)$ with $\Phi_n \circ r_\zeta(t)$ is contained in a uniform neighbourhood of $r_n^+(s,t)$. This shows the EPP condition for r_ζ.

4.2.2. *Case II B: when ζ is neither an endpoint of the lift of a boundary parabolic curve nor a crown-tip.* Now, we assume that ζ is neither an endpoint of a lift of a component of $\operatorname{Fr}\Sigma_j$ nor a crown-tip. Note that the latter is the standing assumption of this section.

Let G^j be a subgroup of $G = \pi_1(S)$ corresponding to $\pi_1(\Sigma_j)$, and define G^j_∞ to be $\rho_\infty(G^j)$. The non-cuspidal part of the hyperbolic 3-manifold $\mathbb{H}^3/G^j_\infty$ has a geometrically infinite end e with a neighbourhood homeomorphic to $\Sigma_j \times (0,\infty)$.

The proof splits into further subcases.

Subcase II B (i):
We first prove the EPP condition for the following special (sub)case. We say that the *geodesic realisation* of a geodesic (finite or infinite) in (the intrinsic metric on) $\widetilde{S}(\subset \widetilde{\mathbf{M}_\Gamma})$ (resp. $S(\subset \mathbf{M}_\Gamma)$) is the geodesic in $\widetilde{\mathbf{M}_\Gamma}$ (resp. $\mathbf{M}_\Gamma$) joining its end points and path-homotopic to it.

Lemma 4.7. *Suppose that there exists a Σ'_k contained in Σ_j (up to isotopy) and let $\mathbf{F_k}$ be the end corresponding to it. If the geodesic realisation $r_\zeta^{\mathbf{M}}$ in $\mathbf{M}_\Gamma$ is not eventually disjoint from $\mathbf{F_k}$, then the EPP condition holds.*

Proof. By our choice of Σ_j, it is impossible that the geodesic realisation of $r_\zeta^{\mathbf{M}}$ is eventually contained in one component of the pre-image of $\mathbf{F_k}$ in $\widetilde{\mathbf{M}}_\Gamma$, as this would imply that $p \circ r_\zeta$ is eventually contained in Σ'_k contradicting our choice of Σ_j. Therefore the only possibility under the hypothesis is that $r_\zeta^{\mathbf{M}}$ intersects infinitely many components of the pre-image of $\mathbf{F_k}$. Then the argument in the previous subsection goes through to show the EPP condition. □

Subcase II B (ii):
Next we consider the subcase when there exists a T'_k contained in Σ_j and $p \circ r_\zeta$ is *not* eventually disjoint from T'_k.

Lemma 4.8. *Suppose that there is a T'_k contained in Σ_j up to isotopy. Let $U_1, \dots, U_q$ be the cusps abutting on the geometrically finite end of $(\mathbb{H}^3/\Gamma)_0$ corresponding to T'_k. If the geodesic realisation of the ray r_ζ^Γ is not eventually disjoint from the pre-images of $U_1 \cup \dots \cup U_q$, then the EPP condition holds.*

Proof. Under this assumption, the geodesic realisation of r_ζ^Γ intersects infinitely many horoballs that are lifts of $U_1, \dots, U_q$. We can assume that the same holds for the geodesic realisation of $r_\zeta^{\mathbf{M}}$ in the universal cover of the model manifold. (We can properly homotope the ray through a bounded distance if necessary.) By the approximate isometry $\mathbf{h}_n$, these cusps correspond to Margulis tubes $U_1^n, \dots, U_q^n$. In this situation, we can electrocute $\tilde{\mathbf{M}}_\Gamma$ at the pre-images of $U_1, \dots, U_q$ and $\tilde{\mathbf{E}}_j$. In $\tilde{\mathbf{M}}_n$, we also electrocute

(1) pre-images of the corresponding Margulis tubes,
(2) pre-images of the product region geometrically converging to $\mathbf{E}_j$ or the union of $\mathbf{E}_j$ and a neighbourhood of its partner.

Then by repeating the argument in the previous section, the EPP condition follows. □

Subcase II B (iii):
Next we consider the (sub)case when there exists a Σ'_k contained in Σ_j up to isotopy, but $p \circ r_\zeta$ is eventually disjoint from Σ'_k, and also there is no T'_k as in Lemma 4.8.

Lemma 4.9. *Suppose that there exists at least one Σ'_k contained in Σ_j up to isotopy. Suppose that $p \circ r_\zeta$ is eventually disjoint from any Σ'_k that is contained in Σ_j, and that moreover it is not in the situation of Lemma 4.8. Then, there are constants R and t_0 independent of n such that for any $s, t \in [0, \infty)$ both greater than t_0, the geodesic arc $r_n(s,t)^*$ is contained in the R-neighbourhood of $\Phi_n(\mathbb{H}^2)$.*

Proof. Since $p \circ r_\zeta$ is eventually disjoint from any Σ'_k, there is t_0 such that $r_\zeta(t)$ is contained in a component F of $\Sigma_j \setminus (\cup_k \Sigma'_k)$ if $t \geq t_0$. Let G^F be a subgroup of G corresponding to $\pi_1(F)$, and set G_n^F, G_∞^F to be the corresponding subgroup of G_n and G_∞ respectively, and we denote by Γ_F the geometric limit of (G_n^F). Then by the covering theorem, we see that G_∞^F is geometrically finite and only parabolic classes are those represented by the components of $\operatorname{Fr}\Sigma_k'$ and those of $\operatorname{Fr}T_l'$. Since $\Phi_n \circ r_\zeta$ converges to r_ζ^Γ under the geometric convergence of G_n to Γ, we can assume by taking larger t_0 if necessary, that $\Phi_n \circ r_\zeta(t)$ is contained in the ϵ-neighbourhood of the Nielsen convex hull of G_n^F for $t \geq t_0$. Furthermore, since we are not in the situation of Lemma 4.8, the geodesic realisation of r_ζ^Γ is eventually disjoint from the horoballs corresponding to parabolic curves lying on $\operatorname{Fr}T_l$. This implies that there is a constant δ such that the geodesic realisation of r_ζ^Γ is within a bounded distance of the image $q \circ \Phi_\infty(\mathbb{H}^2)$. Since $\Phi_n(\mathbb{H}^2)$ converges geometrically to $q \circ \Phi_\infty(\mathbb{H}^2)$, the geodesic arc connecting two points of $\Phi_n \circ r_\zeta(s)$ and $\Phi_n \circ r_\zeta(t)$ with $s, t \geq t_0$ is also within uniformly bounded distance from $\Phi_n(\mathbb{H}^2)$. □

The EPP condition in the situation of Lemma 4.9 is now a replica of the proof of Theorems A, B in [MS13] as we are essentially reduced to the geometrically finite case.

Subcase II B (iv):
We have now come to the remaining subcase. Here, the geodesic realisation of $p_{\mathbf{M}} \circ r_\zeta^{\mathbf{M}}$ in $\mathbf{M}_\Gamma$ is eventually contained **inside** a fixed degenerate end $\mathbf{E}$ of $\mathbf{M}_\Gamma$, where $p_{\mathbf{M}} \colon \tilde{\mathbf{M}}_\Gamma \to \mathbf{M}_\Gamma$ is the universal covering.

We assume that $\mathbf{E}$ corresponds to Σ_j. The case when it corresponds to Σ' can also be dealt with in the same way. We can then, by moving basepoints, assume that r_ζ is entirely contained in Σ_j. Let G^j be the subgroup of $\pi_1(S)$ associated to $\pi_1(\Sigma_j)$. We shall use G_n^j, G_∞^j respectively for $\rho_n(G^j), \rho_\infty(G^j)$. Let Γ^j be a geometric limit of (a subsequence of) $(G_n^j)_n$. Then Γ^j can be regarded as a subgroup of Γ.

The geometric limit Γ^j may be larger than G_∞^j, but by the covering theorem (see [Can96] and [Ohs92]), there is a neighbourhood $\bar{E}_j$ of $\bar{e}_j$, that projects homeomorphically to a neighbourhood $\hat{E}_j$ of a geometrically infinite end of $(\mathbb{H}^3/\Gamma^j)_0$. Renumbering the parabolic curves on S, we assume that $c_1, \ldots, c_p$ are the parabolic curves on Σ_j including those corresponding to components of $\operatorname{Fr}\Sigma_j$. Each c_k corresponds to either a $\mathbb{Z}$-cusp or a $\mathbb{Z} \times \mathbb{Z}$-cusp in $\mathbb{H}^3/\Gamma^j$, whose neighbourhood we denote by U_k. Recall that by the geometric convergence of $\mathbb{H}^3/G_n^j$ to $\mathbb{H}^3/\Gamma^j$, these cusp neighbourhoods correspond to Margulis tubes in $\mathbb{H}^3/G_n$ for large n. We denote the corresponding Margulis tubes in $\mathbb{H}^3/G_n^j$ by $U_1^n, \ldots, U_p^n$.

We now electrocute $U_1, \ldots, U_p$ in $\mathbb{H}^3/\Gamma^j$ to get a new metric $\bar{d}_{\Gamma^j}$, and accordingly we electrocute $U_1^n, \ldots, U_p^n$ in $\mathbb{H}^3/G_n^j$ to get a new metric $\bar{d}_n$ in such a way that $(\mathbb{H}^3/G_n^j, \bar{d}_n)$ converges geometrically to $(\mathbb{H}^3/\Gamma^j, \bar{d}_{\Gamma^j})$. Let $r_\zeta^\Gamma(s,t)^*$ be the geodesic arc in $\mathbb{H}^3$ connecting $r_\zeta^\Gamma(s)$ with $r_\zeta^\Gamma(t)$, and $r_\zeta^\Gamma(s,t)^\times$ be the geodesic arc with respect to the electric metric $\bar{d}_\Gamma^j$. As per our previous terminology, $r_\zeta^\Gamma(s,t)^*$ is the geodesic realisation of $r_\zeta^\Gamma([s,t])$. The hypothesis of Subcase II B (iv) can then be restated as follows:

Assumption 4.10. *There is a constant R' such that for any $s, t \in [0, \infty)$, the geodesic realisation $r_\zeta^\Gamma(s,t)^*$ is contained in the R'-neighbourhood of $\hat{E}_j$ with respect to the (electric) metric $\bar{d}_\Gamma$.*

Under Assumption 4.10, we are reduced to the case where the geodesic realisations $r_\zeta^\Gamma(s,t)^*$ can go deep into only the end $\bar{e}^j$. This is exactly the situation dealt with in [Mj14a, Corollary 6.13] or in the proof of [MS17, Theorem A] in Section 4.2.3 of that paper:
We approximate the geodesics $r_\zeta^\Gamma(s,t)^*$ or $r_\zeta^\Gamma(s,t)^\times$ by quasi-geodesics using the construction of model manifolds. Geometric convergence to the end $\mathbf{E}$ of the approximants $\mathbf{E}^n$ ensures the geometric convergence of their model manifolds. Corollary 6.13 [Mj14a] now translates to the EPP condition as in the proof of [MS17, Theorem A].

4.2.3. *The case when $p \circ r_\zeta$ is eventually contained in T_j.* Recall that T_j corresponds to a geometrically finite end. This case is simpler than previous one and we can repeat most of the arguments of the previous cases. If ζ is an endpoint of a lift of a parabolic curve, the argument in Case II A goes through without modification as we did not use the assumption that Σ_j corresponds to a simply degenerate end there. For the analogues of Cases II B-(i, ii, iii) too we can argue in the same way as there to show the EPP condition.

In the remaining case, since the end e corresponding to T_j is geometrically finite, the geodesic realisation of r_ζ^Γ cannot escape deep into e. Therefore the only possibility is that it lies in a neighbourhood of $q \circ \Phi_\infty(\mathbb{H}^2)$. Thus, this is the same situation as Case II B (iii).

5. Pointwise convergence for tips of crown domains

We shall now prove what remains in order to complete the proofs of Theorems 2.14 and 2.15:
If ζ is a tip of a crown domain C for (λ, σ), where σ is a parabolic curve and λ is an ending lamination, and either the cusp corresponding to σ

is not conjoining, or e is not coupled, or σ is twisted, or C is not well approximated, then the Cannon-Thurston maps *do* converge at ζ.

The proof splits into three cases:

(1) Either the cusp corresponding to σ is not conjoining or e is not coupled. This case will be dealt with in Proposition 5.1 below, and for some special case in Proposition 5.2.
(2) e is coupled but the crown domain C is *not* well approximated. This will be dealt with in Proposition 5.2 below.
(3) The cusp corresponding to σ is twisted. This will be dealt with in Proposition 5.3 below.

Now we consider the case when either the cusp corresponding to σ is not conjoining or e is a simply degenerate end that is not coupled. We shall consider the point-wise convergence of Cannon-Thurston maps at crown-tips (λ, σ). Without loss of generality, we assume that e is upward as usual. Let U be a $\mathbb{Z}$-cusp neighbourhood of $\mathbb{H}^3/\Gamma$ corresponding to σ. Let A be an open annulus bounding U. Let e' be another end on which U abuts. By assumption, e' is not coupled with e. Then

(a) Either e' is also an upward algebraic end (if U is not conjoining this is the only case), or
(b) e' is downward, but by taking the approximate isometry h_n^{-1}, a neighbourhood of e' cannot be parallel into a neighbourhood of e.

The proof of the case (b) is deferred and will be dealt with in Proposition 5.2 together with the case when the crown domain C is not well approximated.

We now state the point-wise continuity in the case (a) above.

Proposition 5.1. *Let e be a simply degenerate end of $(\mathbb{H}^3/G_\infty)_0$ corresponding to a subsurface Σ of S, and let σ be a component of $\mathrm{Fr}\,\Sigma$. Suppose that either e is not coupled or the cusp U of $\mathbb{H}^3/\Gamma$ corresponding to σ is not conjoining. In the former case, we assume moreover that condition (a) above holds. Let λ be the ending lamination for e, whose minimal supporting surface is Σ. Let ζ be a crown-tip of (σ, λ). Also, let c_n $(n = 1, \cdots, \infty)$ denote the Cannon-Thurston maps for the representations ρ_n $(n = 1, \cdots, \infty)$. Then we have $c_n(\zeta) \to c_\infty(\zeta)$.*

Proof. The cusp U corresponds to a Margulis tube U^n in $\mathbb{H}^3/G_n$ under the approximate isometry h_n^{-1} for large n. The core curve of U^n is freely homotopic to $\rho_n(\sigma)$. We denote by $\mathbf{U}$ and $\mathbf{U}^n$ their counterparts in the model manifolds $\mathbf{M}_\Gamma$ and $\mathbf{M}_n$. In the model manifold $\mathbf{M}_n$, the Margulis tube $\mathbf{U}^n$ is bounded by a torus $\mathbf{T}_n$ consisting of two horizontal annuli and two vertical annuli. These in turn correspond to the boundary T_n

of U^n. Let A^n_l, A^n_r be the two vertical annuli in $\mathbf{T}_n$, thought of as the left and right vertical annulus respectively. Under condition (a), $\mathbf{T}_n$ converges to neither a torus nor a conjoining annulus. Therefore, we see that the moduli of both A^n_l and A^n_r go to ∞ as $n \to \infty$. Let $\mathbf{E}$ be a brick of $\mathbf{M}^0_\Gamma$ containing the end $f_\Gamma^{-1}(e)$. We denote by $\mathcal{U}$ the union of cusp neighbourhoods of $\mathbf{M}_\Gamma$ abutting on e, and by $\mathcal{U}^n$ the union of Margulis tubes corresponding to $\mathcal{U}$ under the approximate isometry $\mathbf{h}_n$. Then, if we regard $\mathbf{M}_n \setminus \mathcal{U}^n$ as a brick manifold with bricks consisting of maximal sets of parallel horizontal surfaces, there is a brick $\mathbf{E}_n$ of $\mathbf{M}_n \setminus \mathcal{U}^n$ which corresponds to $\mathbf{E}$ under $\mathbf{h}_n$. Since T_n converges to A geometrically, we see that $\mathbf{E}_n$ converges geometrically to $\mathbf{E}$ wherever we choose a basepoint within a bounded distance from the algebraic locus. We denote the counterpart of $\mathbf{E}_n$ and $\mathbf{E}$ in $\mathbb{H}^3/G_n$ and $\mathbb{H}^3/\Gamma$ by E_n and E respectively.

Now consider a pleated surface $f_n \colon S \to \mathbb{H}^3/G_n$ realising λ and inducing ρ_n between the fundamental groups. Let $\tilde{f}_n \colon \mathbb{H}^2 \to \mathbb{H}^3$ be its lift. The endpoint at infinity $c_n(\zeta)$ is also an endpoint at infinity of a leaf ℓ_n of the pleating locus of $\tilde{f}_n$. It is in fact a lift of a leaf of λ_C. Therefore, the geodesic realisation r^n_ζ of $\Phi_n \circ r_\zeta$ (setting its starting point to be $o_{\mathbb{H}^3}$) is asymptotic to this leaf ℓ_n. Since (E_n) converges geometrically to E and λ_C, which consists of leaves of λ, is not realisable in E, we see that for any compact set K, there exists n_0 such that for $n \geq n_0$, the surface $f_n(S) \cap E_n$ is disjoint from $h_n^{-1}(K)$. Let $\tilde{U}$ be a lift of U corresponding to a lift of σ lying on the boundary of the crown domain C having ζ as a vertex. Then $\tilde{U}$ is a horoball touching S^2_∞ at $c_\infty(\zeta)$. The geodesic ray r^n_ζ can then be properly homotoped to a quasi-geodesic ray $\bar{r}^n_\zeta$ consisting of two geodesics; one connecting $o_{\mathbb{H}^3}$ to the point z_n on ℓ_n nearest to $\tilde{U}$, and a geodesic ray connecting z_n to $c_n(\zeta)$ lying in the image of $\tilde{f}_n$. Since the moduli of both A^n_l and A^n_r go to ∞, the exterior angle of the two constituents of $\bar{r}^n_\zeta$ is bounded away from π. Hence $(\bar{r}^n_\zeta)$ is uniformly quasi-geodesic. Since the time $\bar{r}^n_\zeta$ spends in U^n goes to ∞, the same holds for r^n_ζ. Finally note that U^n converges to U and the latter lifts to a horoball $\tilde{U}$ touching S^2_∞ at only one point $c_\infty(\zeta)$. Hence $(c_n(\zeta))$ converges to $c_\infty(\zeta)$. □

Now we consider the second case when P is conjoining but e is not coupled or C is not well approximated.

Proposition 5.2. *Let e be a coupled simply degenerate end of $(\mathbb{H}^3/G_\infty)_0$ corresponding to the subsurface Σ of S. Let λ be the ending lamination for e supported on Σ and σ a parabolic curve whose corresponding cusp abuts on e. Suppose that*

(1) Either e is not coupled and the condition (b) before Proposition 5.1 holds,

(2) Or, a crown domain C for (λ, σ) is not *well approximated.*

Let ζ be a tip of C. Also, let $c_n\,(n = 1, \cdots, \infty)$ denote the Cannon-Thurston maps for the representations $\rho_n\,(n = 1, \cdots, \infty)$. Then $(c_n(\zeta))$ converges to $c_\infty(\zeta)$.

Proof. We retain the notation in the proof of Proposition 5.1 and assume that e is upward. We take a pleated surface $f_n\colon S \to \mathbb{H}^3/G_n$ realising $\lambda \cup \operatorname{Fr}\Sigma$. Since the pleated surface f_n realises λ, it follows that if $(f_n|\Sigma)$ converges geometrically, then after passing to a subsequence, we obtain a pleated surface map from Σ to $\mathbb{H}^3/\Gamma$. But this implies that e is coupled and C is well approximated, contradicting the assumption. Therefore, (f_n) cannot converge geometrically.

Let m_n be the hyperbolic structure on Σ induced from $\mathbb{H}^3/G_n$ by f_n. If (m_n) is bounded in the moduli space then by the compactness of unmarked pleated surfaces, (f_n) must converge geometrically (after passing to a subsequence) as above. Therefore (m_n) is unbounded. Hence there is an essential simple closed curve s_n in Σ such that $\mathrm{length}_{m_n}(s_n) \to 0$. We can choose s_n so that for a component W_n of $\Sigma \setminus s_n$ containing σ in its boundary, (W_n, m_n) converges geometrically to a complete hyperbolic surface. If there is a limiting pleated surface realising λ_C in the partner(s) of e, then e is coupled and C is well approximated, contradicting our assumption.

It follows that the only possibility is that the distance from the basepoint and $f_n(\Sigma)$ goes to ∞. By repeating the argument in the proof of Proposition 5.1, we see that $(c_n(\zeta))$ converges to $c_\infty(\zeta)$. □

Now we turn to the last case when the cusp P is twisted although the end is coupled, P is conjoining, and C is well approximated.

Proposition 5.3. *Let e be a coupled simply degenerate end of $(\mathbb{H}^3/G_\infty)_0$ corresponding to the subsurface Σ of S. Let λ be the ending lamination for E supported on Σ and let σ be a parabolic curve corresponding to a twisted cusp P abutting on e. Let ζ be a tip of a crown domain C of (λ, c). We assume that C is well approximated. Let $c_n\,(n = 1, \cdots, \infty)$ denote the Cannon-Thurston maps for the representations ρ_n, $n = 1, \cdots, \infty$. Then $(c_n(\zeta))$ converges to $c_\infty(\zeta)$.*

Proof. We shall use notation from §3. In particular, we consider a leaf ℓ of the pre-image of λ in $\mathbb{H}^2$, whose endpoint at infinity is ζ. The only difference from the situation in §3 is that the cusp P is twisted now. As in §3, we consider the realisation $\tilde{\ell}_n$ of ℓ and an arc a^n_ℓ in $\mathbb{H}^3$ connecting $o_{\mathbb{H}^3}$ to $\tilde{\ell}_n$ that bridges over $\tilde{P}^n$. Then, since P is twisted, the landing point y_n of $a^n_{\tilde{\ell}}$ has distance from $o_{\mathbb{H}^3}$ going to ∞, and the time it spends in a lift $\tilde{P}^n$ of the Margulis tube P^n also goes to ∞. Recall that $\tilde{P}^n$ converges to a horoball $\tilde{P}$ touching S^2_∞ at $c_\infty(\zeta)$. This shows that $c_n(\zeta)$, which is the endpoint at infinity of ℓ, converges to $c_\infty(\zeta)$. □

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