

ALGEBRAIC ENDING LAMINATIONS AND QUASICONVEXITY

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ABSTRACT. We explicate a number of notions of algebraic laminations existing in the literature, particularly in the context of an exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

of hyperbolic groups. These laminations arise in different contexts: existence of Cannon-Thurston maps; closed geodesics exiting ends of manifolds; dual to actions on \mathbb{R} -trees.

We use the relationship between these laminations to prove quasiconvexity results for finitely generated infinite index subgroups of H , the normal subgroup in the exact sequence above.

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1. INTRODUCTION

A number of competing notions of laminations exist in literature. The weakest is that of an **algebraic lamination** [BFH97, CHL07, KL10, KL15, Mit97] for a hyperbolic group H : an H -invariant, flip invariant, closed subset $\mathcal{L} \subseteq \partial^2 H =$

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$(\partial H \times \partial H \setminus \Delta) / \sim$, where $(x, y) \sim (y, x)$ denotes the flip and Δ the diagonal in $\partial H \times \partial H$.

Several classes of algebraic laminations have come up in the study of automorphisms of hyperbolic groups, especially free and surface groups:

- (1) The dual lamination $\Lambda_{\mathbb{R}}$ arising from an action of H on an \mathbb{R} -tree [Thu80, BFH97, CHL07, CHL08a, KL10].
- (2) The ending lamination Λ_{EL} or Λ_{GEL} arising from closed geodesics an end of a 3-manifold [Thu80, Mit97].
- (3) The Cannon-Thurston lamination Λ_{CT} arising in the context of the existence of a Cannon-Thurston map [CT89, CT07, Mit97].

These different kinds of laminations play different roles.

- (1) The dual lamination $\Lambda_{\mathbb{R}}$ often has good mixing properties like arationality [Thu80] or minimality [CHR11] or the dual notion of indecomposability for the dual \mathbb{R} -tree [Gui08].
- (2) The Cannon-Thurston laminations Λ_{CT} play a role in determining quasiconvexity of subgroups [SS90, Mit99]. See Lemma 2.4 below.
- (3) The above two quite different contexts are mediated by ending laminations Λ_{EL} or Λ_{GEL} which may be intuitively described as Hausdorff limits of curves whose geodesic realizations exit an end.

We elaborate a bit the statement that Λ_{EL} mediates between Λ_{CT} and $\Lambda_{\mathbb{R}}$. It is easy to see that in various natural contexts the collection of ending laminations Λ_{EL} or Λ_{GEL} are contained in the collection of Cannon-Thurston laminations Λ_{CT} (Proposition 2.11 below) as well as in the dual laminations $\Lambda_{\mathbb{R}}$ (Proposition 4.3 below). Further, the (harder) reverse containment of Λ_{CT} in Λ_{EL} has been established in a number of cases (Theorem 2.8 below from [Mit97] for instance). What remains is to examine the reverse containment of $\Lambda_{\mathbb{R}}$ in Λ_{EL} in order to complete the picture. This is the subject of [KL15, DKT15] in the context of free groups and [Mj14] in the context of surface Kleinian groups.

What kicks in after this are the mixing properties of $\Lambda_{\mathbb{R}}$ established by various authors in particular arationality of ending laminations for surface groups [Thu80] or arationality in a strong form for free groups [Rey11, Rey12, BR12, CHR11, Gui08]. It follows that Λ_{CT} is arational in a strong sense—no leaf of Λ_{CT} is contained in a finitely generated infinite index subgroup K of H for various specific instances of H . Quasiconvexity of K in G (or more generally some hyperbolic metric bundle X) then follows from Lemma 2.4. Accordingly each of the Sections 3, 4 and 5 have two subsections each: one establishing arationality and the second combining arationality along with the general theory of Section 2 to prove quasiconvexity.

We are now in a position to state the main Theorems of this paper.

Theorem 1.1. *(See Theorems 3.7 and 4.12)*

Let

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence of hyperbolic groups, where H is either a free group or a (closed) surface group and Q is convex cocompact in Outer Space or Teichmüller space respectively (for the free group, we assume that Q is purely hyperbolic). Let K be a finitely generated infinite index subgroup of H . Then K is quasiconvex in G .

The case of H a closed surface group in Theorem 1.1 was obtained by Dowdall, Kent and Leininger recently in [DKL14] by different methods. Dowdall has communicated to the authors that in soon-forthcoming work [DT15], Dowdall and Taylor use the methods of their earlier work [DT14] on convex cocompact purely hyperbolic subgroups of $Out(F_n)$ to give a different proof of Theorem 1.1 when H is free.

For the statement of our next theorem, some terminology needs to be introduced. A Teichmuller geodesic ray $r(\subset Teich(S))$ is said to be **thick** [Min92, Min94, Min01, Raf14] if r lies in the thick part of Teichmuller space, i.e. there exists $\epsilon > 0$ such that $\forall x \in r$, the length of the shortest closed geodesic (or equivalently, injectivity radius for closed surfaces) on the hyperbolic surface S_x corresponding to $x \in Teich(S)$ is bounded below by ϵ . It follows (from [MM00, Min01, Raf14]) that the projection of r to the curve complex is a parametrized quasigeodesic and the universal curve U_r over r (associating S_x to x and equipping the resulting bundle with an infinitesimal product metric) has a hyperbolic universal cover \tilde{U}_r [Min94, Min01]. To emphasize this hyperbolicity we shall call these geodesic rays **thick hyperbolic rays**. We shall refer to \tilde{U}_r as the **universal metric bundle** (of hyperbolic planes) over r .

Analogously, we define a geodesic ray r in Culler-Vogtmann outer space [CV86] to be **thick hyperbolic** if

- (1) r projects to a parametrized quasigeodesic in the free factor complex \mathcal{F} .
- (2) the bundle of trees X over r (thought of as a metric bundle [MS12]) is hyperbolic.

In this case too, we shall refer to X as the universal metric bundle (of trees) over r .

Theorem 1.2. (See Theorems 3.6 and 4.11)

Let r be a thick hyperbolic quasigeodesic ray

- (1) either in $Teich(S)$ for S a closed surface of genus greater than one
- (2) or in the Outer space corresponding to F_n .

Let X be the universal metric bundle of hyperbolic planes or trees (respectively) over r . Let H denote respectively $\pi_1(S)$ or F_n and $i : \Gamma_H \rightarrow X$ denote the orbit map. Let K be a finitely generated infinite index subgroup of H . Then $i(K)$ is quasiconvex in X .

The following Theorem generalizes the closed surface cases of Theorems 1.1 and 1.2 to surfaces with punctures.

Theorem 1.3. (See Theorems 5.4 and 5.1)

Let $H = \pi_1(S)$ for S a hyperbolic surface of finite volume. Let r be a thick hyperbolic ray in Teichmuller space $Teich(S)$ and let $r_\infty \in \partial Teich(S)$ be the limiting surface ending lamination. Let X denote the universal metric bundle over r minus a small neighborhood of the cusps and let \mathcal{H} denote the horosphere boundary components. Let K be a finitely generated infinite index subgroup of H . Then any orbit of K in X is relatively quasiconvex in (X, \mathcal{H}) .

Let $H = \pi_1(S^h)$ be the fundamental group of a surface with finitely many punctures and let H_1, \dots, H_n be its peripheral subgroups. Let Q be a convex cocompact subgroup of the pure mapping class group of S^h . Let

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

and

$$1 \rightarrow H_i \rightarrow N_G(H_i) \rightarrow Q \rightarrow 1$$

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}, i = 1, \dots, n$.

Let K be a finitely generated infinite index subgroup of H . Then K is relatively quasiconvex in G .

The first part of Theorem 1.3 is from [MS12]. The relative quasiconvexity statement (which requires relative hyperbolicity as its framework) is what is new.

1.1. Cannon-Thurston Maps. Let H be a hyperbolic subgroup of a hyperbolic group G (resp. acting properly on a hyperbolic metric space X). Let $\widehat{\Gamma}_H, \widehat{\Gamma}_G$ and \widehat{X} denote the Gromov compactifications. Further let $\partial\Gamma_H, \partial\Gamma_G$ and ∂X denote the boundaries [Gro85].

Definition 1.4. Let H be a hyperbolic subgroup of a hyperbolic group G (resp. acting properly on a hyperbolic metric space X). Let Γ_H, Γ_G denote the Cayley graphs of H, G . We assume that generating sets have been so chosen that the generating set of G contains the generating set of H .

Let $i : \Gamma_H \rightarrow \Gamma_G$ (resp. $i : \Gamma_H \rightarrow X$) denote the inclusion map (resp. the orbit map extended over edges).

A **Cannon-Thurston map** for the pair (H, G) (resp. (H, X)) is a map $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$ (resp. $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{X}$) which is a continuous extension of i .

The next two theorems establish the existence of Cannon-Thurston maps in closely related settings:

Theorem 1.5. [Mit98] Let G be a hyperbolic group and let H be a hyperbolic normal subgroup of G . Let $i : \Gamma_H \rightarrow \Gamma_G$ be inclusion map. Then i extends to a continuous map \hat{i} from $\widehat{\Gamma}_H$ to $\widehat{\Gamma}_G$.

Theorem 1.6. [MS12][Theorem 5.3] Let r be a thick hyperbolic quasigeodesic ray

- (1) either in $\text{Teich}(S)$ for S a closed surface of genus greater than one
- (2) or in the Outer space corresponding to F_n .

Let X be the universal metric bundle of hyperbolic planes or trees (respectively) over r . Let H denote respectively $\pi_1(S)$ or F_n . Then the pair (H, X) has a Cannon-Thurston map.

2. LAMINATIONS

An **algebraic lamination** [BFH97, CHL07, KL10, KL15, Mit97] for a hyperbolic group H is an H -invariant, flip invariant, closed subset $\mathcal{L} \subseteq \partial^2 H = (\partial H \times \partial H \setminus \Delta) / \sim$, where $(x, y) \sim (y, x)$ denotes the flip and Δ the diagonal in $\partial H \times \partial H$. Various classes of laminations exist in the literature and in this section, we describe three such classes that arise naturally.

2.1. Cannon-Thurston Laminations. In this section we shall define laminations in the context of a hyperbolic group H acting properly on a hyperbolic metric space X . For instance, X could be the Cayley graph of a hyperbolic group G containing H . The orbit map will be denoted by i . The laminations we describe in this section go back to [Mit97] and correspond intuitively to (limits) of geodesic segments in H whose geodesic realizations in X live outside large balls about a base-point.

We recall some basic facts and notions (cf. [Mit97, Mit99]). If λ is a geodesic segment in Γ_H a **geodesic realization** λ^r , of λ , is a geodesic in X joining the end-points of $i(\lambda)$.

Let $\{\lambda_n\}_n \subset \Gamma_H$ be a sequence of geodesic segments such that $1 \in \lambda_n$ and $\lambda_n^r \cap B(n) = \emptyset$, where $B(n)$ is the ball of radius n around $i(1 \in X)$. Take all bi-infinite subsequential limits (in the Hausdorff topology on closed subsets of Γ_H) of all such sequences $\{\lambda_i\}$ and denote this set by \mathcal{L}_0 . Let t_h denote left translation by $h \in H$.

Definition 2.1. *The set of Cannon-Thurston laminations $\Lambda_{CT} = \Lambda_{CT}(H, X)$ is given by*

$$\Lambda_{CT} = \{(p, q) \in \partial\Gamma_H \times \partial\Gamma_H \mid p \neq q \text{ and } p, q \text{ are the end-points of } t_h(\lambda) \text{ for some } \lambda \in \mathcal{L}_0\}$$

In the presence of a Cannon-Thurston map, we have an alternate description of Λ_{CT} .

Definition 2.2. *Suppose that a Cannon-Thurston map exists for the pair (H, X) . We define $\Lambda_{CT}^1 = \{(p, q) \in \partial\Gamma_H \times \partial\Gamma_H \mid p \neq q \text{ and } \hat{i}(p) = \hat{i}(q)\}$.*

Lemma 2.3. [Mit99] *If a Cannon-Thurston map exists, $\Lambda_{CT} = \Lambda_{CT}^1$.*

Note that for the definition of Λ_{CT} , one does not need the existence of a Cannon-Thurston map. The following Lemma characterises quasiconvexity in terms of Λ_{CT} .

Lemma 2.4. [Mit99] *H is quasiconvex in X if and only if $\Lambda_{CT} = \emptyset$*

We shall be requiring a generalization of Lemma 2.4 to relatively hyperbolic groups [Gro85, Far98, Bow12]. Let H be a relatively hyperbolic group, hyperbolic relative to a finite collection of parabolic subgroups \mathcal{P} . The relative hyperbolic (or Bowditch) boundary $\partial(H, \mathcal{P}) = \partial_r H$ of the relatively hyperbolic group (H, \mathcal{P}) was defined by Bowditch [Bow12]. The collection of bi-infinite geodesics $\partial^2 H$ is defined as $(\partial_r H \times \partial_r H \setminus \Delta) / \sim$ as usual. The existence of a Cannon-Thurston map in this setting of a relatively hyperbolic group H acting on a relatively hyperbolic space (X, \mathcal{H}) has been investigated in [Bow02, Mj09, MP11]. Such an H acts in a **strictly type preserving** manner on a relatively hyperbolic space (X, \mathcal{H}) if the stabilizer $Stab_H(Y)$ for any $Y \in \mathcal{H}$ is equal to a conjugate of an element of \mathcal{P} and if each conjugate of an element of \mathcal{P} stabilizes some $Y \in \mathcal{H}$. The notion of the Cannon-Thurston lamination $\Lambda_{CT} = \Lambda_{CT}(H, X)$ is defined as above to be the set of pairs $(x, y) \in \partial^2 H$ identified by the Cannon-Thurston map. The proof of Lemma 2.4 from [Mit99] directly translates to the following in the relative setup.

Lemma 2.5. *Suppose that the relatively hyperbolic group (H, \mathcal{P}) acts in a **strictly type preserving** manner on a relatively hyperbolic space (X, \mathcal{H}) such that the pair (H, X) has a Cannon-Thurston map. Let $\Lambda_{CT} = \Lambda_{CT}(H, X)$. Then any orbit of H is relatively quasiconvex in X if and only if $\Lambda_{CT} = \emptyset$.*

2.2. Algebraic Ending Laminations. In [Mit97], the first author gave a different, more group theoretic description of ending laminations motivated by Thurston's description in [Thu80]. Thurston's description uses a transverse measure which is eventually forgotten [Kla99, BR12], whereas the approach in [Mit97] uses Hausdorff limits and is purely topological in nature. We rename the ending laminations of [Mit97] **algebraic ending laminations** to emphasize the difference.

Thus some of the topological aspects of Thurston's theory of ending laminations were generalized to the context of normal hyperbolic subgroups of hyperbolic groups and used to give an explicit description of the continuous boundary extension $\hat{i} : \widehat{\Gamma}_H \rightarrow \widehat{\Gamma}_G$ occurring in Theorem 1.5.

Let

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

be an exact sequence of finitely presented groups where H , G and hence Q (from [Mos96]) are hyperbolic. In this setup one has algebraic ending laminations (defined below) naturally parametrized by points in the boundary $\partial\Gamma_Q$ of the quotient group Q .

Corresponding to every element $g \in G$ there exists an automorphism of H taking h to $g^{-1}hg$ for $h \in H$. Such an automorphism induces a bijection ϕ_g of the vertices of Γ_H . This gives rise to a map from Γ_H to itself, sending an edge $[a, b]$ linearly to a shortest edge-path joining $\phi_g(a)$ to $\phi_g(b)$.

Fix $z \in \partial\Gamma_Q$ and let $[1, z)$ be a geodesic ray in Γ_Q starting at the identity 1 and converging to $z \in \partial\Gamma_Q$. Let σ be a single-valued quasi-isometric section of Q into G . Let z_n be the vertex on $[1, z)$ such that $d_Q(1, z_n) = n$ and let $g_n = \sigma(z_n)$.

Definition 2.6. Given $h \in H$ let \mathcal{L}_n^h be the (H -invariant) collection of all free homotopy representatives (or shortest representatives in the same conjugacy class) of $\phi_{g_n^{-1}}(h)$ in Γ_H . Identifying equivalent geodesics in \mathcal{L}_n^h one obtains a subset L_n^h of (unordered) pairs of points in $\widehat{\Gamma}_H$. The intersection with $\partial^2\Gamma_H$ of the union of all subsequential limits (in the Chabauty topology) of $\{L_n^h\}$ is denoted by Λ_z^h .

The set of **algebraic ending laminations** corresponding to $z \in \partial\Gamma_Q$ is given by

$$\Lambda_{EL}^z = \bigcup_{h \in H} \Lambda_z^h.$$

Such algebraic ending laminations can be defined analogously for hyperbolic metric bundles [MS12] over $[0, \infty)$, where the vertex spaces correspond to the integers and edge spaces correspond to the intervals $[n-1, n]$ where $n \in \mathbb{N}$.

Definition 2.7. The set Λ_{EL} of all algebraic ending laminations for the triple (H, G, Q) is defined by

$$\Lambda_{EL} = \Lambda_{EL}(H, G, Q) = \bigcup_{z \in \partial\Gamma_Q} \Lambda_{EL}^z.$$

The main theorem of [Mit97] equates Λ_{EL} and Λ_{CT} .

Theorem 2.8. [Mit97] $\Lambda_{EL}(H, G, Q) = \Lambda_{EL} = \Lambda_{CT} = \Lambda_{CT}(H, G)$.

Remark 2.9. We shall be needing a slightly more general version of Theorem 2.8 later, where we consider hyperbolic metric bundles over rays $[0, \infty)$. We note here that the proof in [Mit97] goes through in this case, too, with slight semantic modifications.

2.2.1. Surface Ending Laminations. It is appropriate to explicate at this juncture the relation between the ending laminations introduced by Thurston in [Thu80][Chapter 9], which we call **surface ending laminations** henceforth, and the algebraic ending laminations we have been discussing. Work of several authors including [Min94, Kla99, Bow02, Mj14] explore related themes.

Let r be a thick hyperbolic geodesic ray in Teichmüller space $Teich(S)$ where S is a surface possibly with punctures. The Thurston boundary $\partial Teich(S)$ consists of

projectivized measured laminations on S . Let $\Lambda_{EL}(r_\infty)$ be the geodesic lamination underlying the lamination $r_\infty \in \partial Teich(S)$. Let X_0 be the universal curve over r . Let X_1 denote X_0 with a small neighborhood of the cusps removed. Minsky proves [Min94] that X_1 is (uniformly) biLipschitz homeomorphic to the convex core minus (a small neighborhood of) cusps of the unique simply degenerate hyperbolic 3-manifold M with conformal structure on the geometrically finite end given by $r(0) \in Teich(S)$ and ending lamination of the simply degenerate end given by $\Lambda_{EL}(r_\infty)$. The convex core of M is denoted by Y_0 and let Y_1 denote Y_0 with a small neighborhood of the cusps removed. Thus X_1, Y_1 are (uniformly) biLipschitz homeomorphic. Let X denote the universal cover of X_1 and \mathcal{H} its collection of boundary horospheres. Then X is (strongly) hyperbolic relative to \mathcal{H} . Let $H = \pi_1(S)$ regarded as a relatively hyperbolic group, hyperbolic rel. cusp subgroups. The relative hyperbolic (or Bowditch) boundary $\partial_r H$ of the relatively hyperbolic group is still the circle (as when S is closed) and $\partial^2 H$ is defined as $(\partial_r H \times \partial_r H \setminus \Delta) / \sim$ as usual. The existence of a Cannon-Thurston map in this setting of a relatively hyperbolic group H acting on a relatively hyperbolic space (X, \mathcal{H}) has been investigated in [Bow02, Mj09].

The **diagonal closure** \mathcal{L}^d of a surface lamination is an algebraic lamination given by the transitive closure of the relation defined by \mathcal{L} on $\partial^2(H)$.

Theorem 2.10. [Min94, Bow02] *Let r be a thick hyperbolic geodesic in $Teich(S)$ and let $\Lambda_{EL}(r_\infty)$ denote its end-point in $\partial Teich(S)$ regarded as a surface lamination. Let X be the universal cover of X_1 . Then $\Lambda_{CT}(H, \widetilde{M}) = \Lambda_{CT}(H, (X, \mathcal{H})) = \Lambda_{EL}(r_\infty)^d$.*

Note that Theorem 2.10 holds both for closed surfaces as well as surfaces with finitely many punctures.

2.2.2. Generalised algebraic ending laminations. The setup of a normal hyperbolic subgroup of a hyperbolic subgroup is quite restrictive. Instead we could consider H acting geometrically on a hyperbolic metric space X . Let $Y = X/H$ denote the quotient. Let $\{\sigma_n\}$ denote a sequence of free homotopy classes of closed loops in Y (these necessarily correspond to conjugacy classes in H) such that the geodesic realisations of $\{\sigma_n\}$ in Y exit all compact sets. Then subsequential limits of all such sequences define again an algebraic lamination, which we call a **generalised algebraic ending lamination** and denote $\Lambda_{GEL}(= \Lambda_{GEL}(H, X))$.

Then Lemma 3.5 of [Mit97] (or Proposition 3.1 of [Mj14] or Section 4.1 of [Mj10]) gives

Proposition 2.11. *If the pair (H, X) has a Cannon-Thurston map, then*

$$\Lambda_{GEL}(H, X) \subset \Lambda_{CT}(H, X).$$

2.3. Laminations dual to an \mathbb{R} -tree. We refer the reader to [Bes02] for details on convergence of a sequence $\{(X_i, *_i, \rho_i)\}$ of based H -spaces for H a fixed group and recall from there some of the relevant notions.

Theorem 2.12. [Bes02]/[Theorem 3.3] *Let $(X_i, *_i, \rho_i)$ be a convergent sequence of based H -spaces such that*

- (1) *there exists $\delta \geq 0$ such that each X_i is δ hyperbolic,*

- (2) there exists $h \in H$ such that the sequence $d_i = d_{X_i}(*_i, \rho_i(h)(*))$ is unbounded.

Then there is a based H -tree $(T, *)$ and an isometric action $\rho : H \rightarrow \text{Isom}(T)$ such that $(X_i, *_i, \rho_i) \rightarrow (T, *, \rho)$.

In Theorem 2.12 above, the (pseudo)metric on T is obtained as the limit of pseudo-metrics $\frac{d_{(X_i, *_i, \rho_i)}}{d_i}$.

Definition 2.13. For a convergent sequence $(X_i, *_i, \rho_i)$ as in Theorem 2.12 above we define a dual algebraic lamination as follows:

Let h_i be any sequence such that $\frac{d_{(X_i, *_i, \rho_i)}(1, h_i)}{d_i} \rightarrow 0$. The collection of all limits of $(h_i^{-\infty}, h_i^{\infty})$ will be called the dual ending lamination corresponding to the sequence $(X_i, *_i, \rho_i)$ and will be denoted by $\Lambda_{\mathbb{R}}(X_i, *_i, \rho_i)$.

Next, let $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of hyperbolic groups. As in Section 2.2 let $z \in \partial\Gamma_Q$ and let $[1, z)$ be a geodesic ray in Γ_Q ; let σ be a single-valued quasi-isometric section of Q into G . Let z_i be the vertex on $[1, z)$ such that $d_Q(1, z_i) = i$ and let $g_i = \sigma(z_i)$. Now, let $X_i = \Gamma_H$, $*_i = 1 \in \Gamma_H$ and let $\rho_i(h)(*) = \phi_{g_i^{-1}}(h)(*)$. With this notation the following Proposition is immediate from Definition 2.6:

Proposition 2.14. $\Lambda_{EL}^z \subset \Lambda_{\mathbb{R}}(X_i, *_i, \rho_i)$.

An alternative description can be given directly in terms of the action on the limiting \mathbb{R} -tree in Theorem 2.12 as follows. The ray $[1, z) \subset Q$ defines a graph X_z of spaces where the underlying graph is a ray $[0, \infty)$ with vertices at the integer points and edges of the form $[n-1, n]$. All vertex and edge spaces are abstractly isometric to Γ_H . Let $e_n = g_{n-1}^{-1}g_n$. The edge-space to vertex space inclusions are given by the identity to the left vertex space and by ϕ_{e_n} to the right. We call X_z the **universal metric bundle** over $[1, z)$ (though it depends on the qi section σ of Q used as well). Hyperbolicity of X_z is equivalent to the *flaring* condition of Bestvina-Feighn [BF92] as shown for instance in [MS12] in the general context of metric bundles.

Suppose now that the sequence $(X_i, *_i, \rho_i)$ with $X_i = \Gamma_H$, $*_i = 1 \in \Gamma_H$, $\rho_i(h)(*) = \phi_{g_i^{-1}}(h)(*)$ converges as a sequence of H -spaces to an H -action on an \mathbb{R} -tree $T = T(X_i, *_i, \rho_i)$. Generalising the construction of Coulbois, Hilion and Lustig [CHL08a, CHL08b] to the hyperbolic group H we have the following notion of an algebraic lamination (contained in $\partial^2 H$) dual to T . The translation length in T will be denoted as l_T .

Definition 2.15. Let $L_{\epsilon}(T) = \overline{\{(g^{-\infty}, g^{\infty}) \mid l_T(g) < \epsilon\}}$ (where \bar{A} denotes closure of A). Define $\Lambda_{\mathbb{R}}(T(X_i, *_i, \rho_i)) = \Lambda_{\mathbb{R}}(T_z) = \bigcap_{\epsilon > 0} L_{\epsilon}(T)$.

3. CLOSED SURFACES

3.1. Arationality. Establishing arationality of Λ_{CT} for surface laminations arising out of a thick hyperbolic ray or an exact sequence of hyperbolic groups really involves identifying the algebraic Cannon-Thurston lamination Λ_{CT} with (the original) geodesic laminations introduced by Thurston [Thu80]. To distinguish them from algebraic laminations, we shall refer to geodesic laminations on surfaces as surface laminations. A surface lamination $\mathcal{L} \subset S$ is **arational** if it has no closed leaves.

The results of this subsection hold equally for S compact or finite volume non-compact. We say that a bi-infinite geodesic l in \tilde{S} is **carried by a subgroup** $K \subset H (= \pi_1(S))$ if both end-points of l lie in the limit set $\Lambda_K \subset \partial\tilde{S}$. A surface lamination $\mathcal{L} \subset S$ is **strongly arational** if no leaf of \mathcal{L} or a diagonal in a complementary ideal polygon is carried by a finitely generated infinite index subgroup K of H .

Lemma 3.1. *Any arational lamination on a finite volume hyperbolic S is strongly arational.*

Proof. We assume that S is equipped with a finite volume hyperbolic metric. Consider a finitely generated infinite index subgroup K of H . By the LERF property of surface groups [Sco78], there exists a finite-sheeted cover S_1 of S such that K is a geometric subgroup of $\pi_1(S_1)$, i.e. it is the fundamental group of an embedded incompressible subsurface Σ of S_1 with geodesic boundary. Since \mathcal{L} has no closed leaves, nor does its lift \mathcal{L}_1 to S_1 . Hence no leaf of \mathcal{L}_1 , nor a diagonal in a complementary ideal polygon, is carried by an embedded subsurface of S_1 , in particular Σ . The result follows. \square

Theorem 3.2. [Kla99] *The boundary $\partial CC(S)$ of the curve complex $CC(S)$ consists of arational surface laminations.*

The following Theorem may be taken as a definition of convex cocompactness for subgroups of the mapping class group of a surface with (at most) finitely many punctures.

Theorem 3.3. [FM02, KL08, Ham08] *A subgroup Q of $MCG(S)$ is convex cocompact if and only if some (any) orbit of Q in the curve complex $CC(S)$ is qi-embedded.*

We identify the boundary $\partial Teich(S)$ of Teichmuller space with the space of projectivized measured laminations (the Thurston boundary). The following Theorem gives us the required strong arationality result.

Theorem 3.4. *Let r be a thick hyperbolic ray in Teichmuller space $Teich(S)$ and let $r_\infty \in \partial Teich(S)$ be the limiting surface lamination. Then r_∞ is strongly arational.*

In particular if Q is a convex cocompact subgroup of $MCG(S)$ and r is a quasi-geodesic ray in Q starting at $1 \in Q$, then its limit r_∞ in the boundary $\partial CC(S)$ of the curve complex is strongly arational.

Proof. By the definition of a thick hyperbolic ray r in $Teich(S)$, $r_\infty \in \partial CC(S)$. For the second statement of the Theorem, ∂Q embeds as a subset of $\partial CC(S)$ by Theorem 3.3 and hence the boundary point $r_\infty \in \partial CC(S)$ as well.

By Theorem 3.2, r_∞ is an arational lamination. Hence by Lemma 3.1, r_∞ is strongly arational. \square

3.2. Quasiconvexity. Let $1 \rightarrow H \rightarrow G \rightarrow Q$ be an exact sequence of hyperbolic groups with $H = \pi_1(S)$ for a closed hyperbolic surface S . Then Q is convex cocompact [KL08, Ham08] and its orbit in both $Teich(S)$ and $CC(S)$ are quasi-convex. By Theorem 2.8 $\Lambda_{EL}(H, G, Q) = \Lambda_{EL} = \Lambda_{CT} = \Lambda_{CT}(H, G)$. Further $\Lambda_{EL} = \cup_{z \in \partial Q} \Lambda_{EL}^z$. Recall that Λ_{EL}^z denotes the algebraic ending lamination corresponding to z and $\Lambda_{EL}(z)$ denotes the surface ending lamination corresponding to z . By Theorem 2.10, $\Lambda_{EL}^z = \Lambda_{EL}(z)^d$. We combine all this as follows.

Theorem 3.5. [Min94, Mit97] *If $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ is an exact sequence with Q convex cocompact and $H = \pi_1(S)$, and $z = r_\infty \in \partial Q \subset \partial CC(S)$ then any lift of $[1, z)$ to $\text{Teich}(S)$ is thick hyperbolic. Further, $\Lambda_{CT}(H, G) = \cup_{z \in \partial Q} \Lambda_{EL}(z)^d$.*

We are now in a position to prove the main Theorems of this Section.

Theorem 3.6. *Let $H = \pi_1(S)$ for S a closed surface of genus greater than one. Let r be a thick hyperbolic ray in Teichmuller space $\text{Teich}(S)$ and let $r_\infty \in \partial \text{Teich}(S)$ be the limiting surface ending lamination. Let X denote the universal metric bundle over r . Let K be a finitely generated infinite index subgroup of H . Then any orbit of K in X is quasiconvex.*

Proof. By Theorem 3.4, the lamination r_∞ is strongly arational. Hence no leaf or diagonal of r_∞ is carried by K . By Theorem 2.10, the Cannon-Thurston lamination $\Lambda_{CT}(H, X) = \Lambda_{EL}(r_\infty)^d$. Hence no leaf of $\Lambda_{CT}(H, X)$ is carried by K . By Lemma 2.4, any orbit of K in X is quasiconvex in X . \square

The next Theorem was proven by Dowdall, Kent and Leininger [DKL14][Theorem 1.3] by different methods.

Theorem 3.7. *Let $1 \rightarrow H \rightarrow G \rightarrow Q$ be an exact sequence of hyperbolic groups with $H = \pi_1(S)$ and Q convex cocompact. Let K be a finitely generated infinite index subgroup of H . Then K is quasiconvex in G .*

Proof. As in the proof of Theorem 3.6 above the lamination $\Lambda_{EL}(z)$ is strongly arational for each $z \in \partial Q \subset \partial CC(S)$ (where we identify the boundary of Q with the boundary of its orbit in $CC(S)$). Hence for all $z \in \partial Q$, no leaf of $\Lambda_{EL}(z)^d$ is carried by K . By Theorem 3.5, $\Lambda_{CT}(H, G) = \Lambda_{EL}(H, G) = \cup_{z \in \partial Q} \Lambda_{EL}(z)^d$. Hence no leaf of $\Lambda_{CT}(H, G)$ is carried by K . By Lemma 2.4, K is quasiconvex in G . \square

4. FREE GROUPS

For the purposes of this section, $H = F_n$ is free.

4.1. Arationality. The (unprojectivized) Culler-Vogtmann Outer space corresponding F_n shall be denoted by cv_n [CV86] and its boundary by ∂cv_n . The points of ∂cv_n correspond to very small actions of F_n on \mathbb{R} -trees.

Definition 4.1. [Gui08] *An \mathbb{R} -tree $T \in \partial cv_n$ is said to be **indecomposable** if for any non-degenerate segments I and J contained in T , there exist finitely many elements $g_1, \dots, g_n \in F_n$ such that*

- (1) $I \subset \bigcup_{i=1 \dots n} g_i J$
- (2) $g_i J \cap g_{i+1} J$ is a non degenerate segment for any $i = 1, \dots, n-1$.

Dual to $T \in \partial cv_n$ is an algebraic lamination $\Lambda_{\mathbb{R}}(T)$ defined as follows:

Definition 4.2. [CHL08a, CHL08b] *Let $L_\epsilon(T) = \overline{\{(g^{-\infty}, g^\infty) | l_T(g) < \epsilon\}}$ (where \bar{A} deotes closure of A). Define $\Lambda_{\mathbb{R}}(T) := \cap_{\epsilon > 0} L_\epsilon(T)$.*

A geodesic or quasigeodesic ray $[1, z)$ in Outer Space cv_n defines a metric bundle X_z of spaces where the underlying graph is a ray $[0, \infty)$ with vertices at the integer points and edges of the form $[n-1, n]$. We refer the reader to [BF14, Section 2.4] for details on folding paths and geodesics in Outer Space. The material relevant to this paper is efficiently summarized in [DT14, Section 2.7]. We call X_z the **universal metric bundle** over $[1, z)$. We shall be interested in two cases:

- (Case 1:) $[1, z]$ is contained in a convex cocompact subgroup Q of $Out(F_n)$ and $\sigma([1, z])$ is identified with the corresponding quasigeodesic ray contained in an orbit of Q in cv_n . The universal metric bundle will (in this case) be considered over $\sigma([1, z])$.
- (Case 2:) $[1, z]$ is a thick geodesic ray in cv_n , i.e. a geodesic ray projecting to a parametrized quasigeodesic in the free factor complex \mathcal{F}_n .

This is the setup used in Proposition 4.3 below, which is extracted from Theorem 5.2 of [DKT15].

Proposition 4.3. *Let $[1, z]$ be as above and suppose that the universal metric bundle X_z is hyperbolic. Then $\Lambda_{EL}^z \subset \Lambda_{\mathbb{R}}(T_z)$.*

A lot of material feeds into Proposition 4.3 above and it is stated in a slightly more general form than the one used in [DKT15]. We therefore furnish detailed references for the convenience of the reader. Theorem 4.1 and Lemma 4.12 of [DT14] establish stability of \mathcal{F} -progressing quasigeodesics. This includes both cases of interest in Proposition 4.3 above. Lemma 5.5 of [DKT15] proves flaring for Case 1 above, while flaring is automatic for Case 2 by hyperbolicity of X_z [BF92, MS12]. Proposition 5.6 of [DKT15] then establishes thickness of $\sigma([1, z])$ in cv_n , thus furnishing what we need for Case 1. Thickness is by definition for Case 2. The crucial ingredient for Theorem 5.2 of [DKT15] is Proposition 5.8 of [DKT15] which, once Propositions 5.5 and 5.6 of [DKT15] are in place, makes no further use of the fact that $\sigma([1, z])$ comes from a ray in a convex cocompact Q (Case 1) but just that it is thick, stable and that the universal bundle over it satisfies flaring. Proposition 4.3 as we have stated it above, follows. Note that this part of the argument has nothing to do with identifying Λ_{EL} with Λ_{CT} (this latter is the content of Theorem 2.8 and Remark 2.9).

Remark 4.4. *A continuously parametrized version of the metric bundle described above occurs in the context of folding paths in Culler-Vogtmann Outer space cv_n converging to a point $z \in \partial cv_n$. Essentially the same proof furnishes the analogous result in this context.*

Definition 4.5. [BR12] *A leaf (p, q) of an algebraic lamination L is carried by a subgroup K of F_n if both p and q lie in the limit set of K . A lamination $\Lambda_{\mathbb{R}}$ is called arational (resp. strongly arational) if no leaf of L is carried by a proper free factor of F_n (resp. by a proper finitely generated infinite index subgroup of F_n).*

A tree $T \in \partial cv_n$ is called arational (resp. strongly arational) if $\Lambda_{\mathbb{R}}(T)$ is arational (resp. strongly arational).

Definition 4.6. [DT14, HH14] *A subgroup Q of $Out(F_n)$ is said to be convex cocompact in $Out(F_n)$ if some (and hence any) orbit of Q in \mathcal{F} is qi embedded.*

A subgroup Q of $Out(F_n)$ is said to be purely atoroidal if every element of Q is hyperbolic.

We collect together a number of Theorems establishing mixing properties for F_n -trees.

Theorem 4.7. [Rey11] *If T is a free indecomposable very small F_n -tree then no leaf of the dual lamination $L(T)$ is carried by a finitely generated subgroup of infinite index in F_n .*

Theorem 4.8. [Rey12] *Let $T \in \partial cv_n$. Then T is arational if and only if either*

- a) *T is free and indecomposable*
- b) *or T is dual to an arational measured foliation on a compact surface S with one boundary component and with $\pi_1(S) = F_n$.*

Let $\mathcal{AT} \subset \partial cv_n$ denote the set of arational trees, equipped with the subspace topology. Define a relation \sim on \mathcal{AT} by $S \sim T$ if and only if $L(S) = L(T)$, and give \mathcal{AT}/\sim the quotient topology.

Theorem 4.9. [BR12] *The space $\partial\mathcal{F}$ is homeomorphic to \mathcal{AT}/\sim . In particular, all boundary points are arational trees.*

Combining the above Theorems we obtain the crucial mixing property we need.

Theorem 4.10. *Let r be a thick hyperbolic ray in Outer space and let $r_\infty \in \partial cv_n$ be the limiting \mathbb{R} -tree. Then $\Lambda_{\mathbb{R}}(r_\infty)$ is strongly arational.*

In particular if Q is a convex cocompact purely hyperbolic subgroup of $Out(F_n)$ and r is a quasigeodesic ray in Q starting at $1 \in Q$, then its limit r_∞ in the boundary $\partial\mathcal{F}$ of the free factor complex is strongly arational.

Proof. By Theorem 4.9 every point in $\partial\mathcal{F}$ comes from an arational \mathbb{R} -tree. Hence r_∞ is arational.

Since r is hyperbolic, it follows from Theorem 4.8 that r_∞ is indecomposable free. It finally follows from Theorem 4.7 that r_∞ is strongly arational.

Next, suppose that Q is a convex cocompact purely atoroidal subgroup of $Out(F_n)$ and r a quasigeodesic ray in Q starting at $1 \in Q$. By Theorem 4.1 of [DT14], an orbit of Q is quasiconvex (in a strong symmetric sense). Then the limit point r_∞ of r lies in $\partial\mathcal{F}$ since the orbit map from Q to \mathcal{F} is a qi-embedding and is therefore arational. Since Q is purely atoroidal quasiconvex, r_∞ is indecomposable free by Theorem 4.8. Again, r_∞ is strongly arational. \square

4.2. Quasiconvexity.

Theorem 4.11. *Let $H = F_n$ and let r be a thick hyperbolic ray in Outer space cv_n and let $r_\infty \in \partial cv_n$ be the limiting \mathbb{R} -tree. Let X denote the universal metric bundle over r . Let K be a finitely generated infinite index subgroup of H . Then any orbit of K in X is quasiconvex.*

Proof. By Theorem 4.10, the tree $T = r_\infty$ is strongly arational. Hence no leaf of $\Lambda_{\mathbb{R}}(T)$ is carried by K . By Proposition 4.3 and Remark 4.4, the algebraic ending lamination $\Lambda_{EL}(H, X) = \Lambda_{EL} \subset \Lambda_{\mathbb{R}}(T)$. Further by Theorem 2.8 and Remark 2.9, $\Lambda_{CT}(H, X) = \Lambda_{EL}(H, X)$. Hence no leaf of $\Lambda_{CT}(H, X)$ is carried by K . By Lemma 2.4, any orbit of K in X is quasiconvex in X . \square

Theorem 4.12. *Let $1 \rightarrow H \rightarrow G \rightarrow Q$ be an exact sequence of hyperbolic groups with $H = F_n$ and Q convex cocompact. Let K be a finitely generated infinite index subgroup of H . Then K is quasiconvex in G .*

Proof. As in the proof of Theorem 4.11 above the tree T_z is strongly arational for each $z \in \partial Q \subset \partial\mathcal{F}$ (where we identify the boundary of Q with the boundary of its orbit in \mathcal{F}). Hence for all $z \in \partial Q$, no leaf of $\Lambda_{\mathbb{R}}(T_z)$ is carried by K . By Proposition 4.3, the algebraic ending lamination $\Lambda_{EL}^z \subset \Lambda_{\mathbb{R}}(T_z)$. Further by Theorem 2.8, $\Lambda_{CT}(H, G) = \Lambda_{EL}(H, G) = \cup_z \Lambda_{EL}^z$. Hence no leaf of $\Lambda_{CT}(H, G)$ is carried by K . By Lemma 2.4, K is quasiconvex in G . \square

5. PUNCTURED SURFACES

For the purposes of this section, $H = \pi_1(S)$ and S is finite volume hyperbolic, noncompact.

5.1. Quasiconvexity for rays.

Theorem 5.1. *Let $H = \pi_1(S)$ for S a hyperbolic surface of finite volume. Let r be a thick hyperbolic ray in Teichmüller space $\text{Teich}(S)$ and let $r_\infty \in \partial\text{Teich}(S)$ be the limiting surface ending lamination. Let X denote the universal metric bundle over r minus a small neighborhood of the cusps and let \mathcal{H} denote the horosphere boundary components. Let K be a finitely generated infinite index subgroup of H . Then any orbit of K in X is relatively quasiconvex in (X, \mathcal{H}) .*

Proof. The proof is essentially the same as that of Theorem 3.6. By Theorem 3.4 (which, recall, holds for punctured surfaces), the lamination r_∞ is strongly arational. Hence no leaf or diagonal of r_∞ is carried by K . By Theorem 2.10, (which, recall, holds for punctured surfaces as well), the Cannon-Thurston lamination $\Lambda_{CT}(H, X) = \Lambda_{EL}(r_\infty)^d$. Hence no leaf of $\Lambda_{CT}(H, X)$ is carried by K . By Lemma 2.5, any orbit of K in X is relatively quasiconvex in $(X, n\mathcal{H})$. \square

5.2. Quasiconvexity for Exact sequences. We shall require a generalization of Theorem 3.5 to punctured surfaces.

Let $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of hyperbolic groups with $H = \pi_1(S^h)$ for a finite volume hyperbolic surface S^h with finitely many peripheral subgroups H_1, \dots, H_n and Q a convex cocompact subgroup of $MCG(S^h)$, where MCG is taken to be the pure mapping class group, fixing peripheral subgroups (this is a technical point and is used only for expository convenience). Note that the normalizer $N_G(H_i)$ is then isomorphic to $H_i \times Q (\subset G)$. The following characterizes convex cocompactness

Proposition 5.2. [MS12][Proposition 5.17] *Let $H = \pi_1(S^h)$ be the fundamental group of a surface with finitely many punctures and let H_1, \dots, H_n be its peripheral subgroups. Let Q be a convex cocompact subgroup of the pure mapping class group of S^h . Let*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

and

$$1 \rightarrow H_i \rightarrow N_G(H_i) \rightarrow Q \rightarrow 1$$

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}, i = 1, \dots, n$.

Conversely, if G is (strongly) hyperbolic relative to the collection $\{N_G(H_i)\}, i = 1, \dots, n$ then Q is convex-cocompact.

Since Q is convex cocompact, its orbits in both $\text{Teich}(S^h)$ and $CC(S^h)$ are quasiconvex and qi-embedded [KL08, Ham08]. Identify Γ_Q with a subset of $\text{Teich}(S^h)$ by identifying the vertices of Γ_Q with an orbit $Q.o$ of Q and edges with geodesic segments joining the corresponding vertices.

Let X_0 be the universal curve over Γ_Q . Let X_1 denote X_0 with a small neighborhood of the cusps removed. Then X_1 is a union $\cup_{q \in \partial\Gamma_Q} X_q$, where X_q is a bundle over the quasigeodesic $[1, q) (\subset \Gamma_Q \subset \text{Teich}(S^h))$ with fibers hyperbolic surfaces diffeomorphic to S^h with a small neighborhood of the cusps removed. Minsky proves

[Min94] that X_q is (uniformly) biLipschitz homeomorphic to the convex core minus (a small neighborhood of) cusps of the unique simply degenerate hyperbolic 3-manifold M with conformal structure on the geometrically finite end given by $o = 1.o \in \text{Teich}(S)$ and ending lamination of the simply degenerate end given by $\Lambda_{EL}(q)$. The convex core of M is denoted by Y_{q0} and let Y_{q1} denote Y_{q0} with a small neighborhood of the cusps removed. Thus X_q, Y_{q1} are (uniformly) biLipschitz homeomorphic. Let \tilde{X}_q denote the universal cover of X_q and \mathcal{H}_q its collection of boundary horospheres. Then \tilde{X}_q is (strongly) hyperbolic relative to \mathcal{H}_q . Let $H = \pi_1(S^h)$ regarded as a relatively hyperbolic group, hyperbolic relative to the cusp subgroups $\{H_i\}, i = 1, \dots, n$. The relative hyperbolic (or Bowditch) boundary $\partial_r H$ of the relatively hyperbolic group is still the circle (as when S is closed) and $\partial^2 H$ is defined as $(\partial_r H \times \partial_r H \setminus \Delta) / \sim$ as usual. The existence of a Cannon-Thurston map in this setting from the relative hyperbolic boundary of H to the relative hyperbolic boundary of $(\tilde{X}_q, \mathcal{H}_q)$ has been proved in [Bow02, Mj09]. Also, it is established in [Bow02, Mj14] (see Theorem 2.10) that the Cannon-Thurston lamination for the pairs H, \tilde{X}_q is given by

$$\Lambda_{CT}(H, \tilde{X}_q) = \Lambda_{EL}(q)^d,$$

where $\Lambda_{EL}(q)^d$ denotes the diagonal closure of the ending lamination $\Lambda_{EL}(q)$.

Next, by Proposition 5.2 G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}, i = 1, \dots, n$. Note that the inclusion of H into G is strictly type-preserving as an inclusion of relatively hyperbolic groups. The existence of a Cannon-Thurston map for the pair (H, G) is established in [Pal10]. The description of the Cannon-Thurston lamination $\Lambda_{CT}(H, G)$ for the pair (H, G) can now be culled from [Mit97] and [Pal10]. The latter [Pal10] guarantees the existence of a qi-section and allows the ladder construction in [Mit97] (which does not require hyperbolicity of G but only that of H , which is free in this case) to go through.

The proof of the description of the ending lamination in [Mit97] (using the ladder constructed there) now shows that the Cannon-Thurston lamination $\Lambda_{CT}(H, G)$ for the pair (H, G) is the union $\cup_{q \in \partial Q} \Lambda_{CT}(H, \tilde{X}_q)$ of Cannon-Thurston laminations for the pairs (H, \tilde{X}_q) . We elaborate on this a bit. Recall that X_1 is a union $\cup_{q \in \partial \Gamma_Q} X_q$, and that the universal cover of X_1 is naturally quasi-isometric to G . Thus Γ_G can be thought of as a union (non-disjoint) of the metric bundles over $[1, q)$, as q ranges over ∂Q . In fact if $P : G \rightarrow Q$ denotes projection, then \tilde{X}_q is quasi-isometric to $P^{-1}([1, q))$. The construction of the ladder and a coarse Lipschitz retract of Γ_G onto it then shows that a leaf of the Cannon-Thurston lamination $\Lambda_{CT}(H, G)$ arises as a concatenation of at most two infinite rays, each of which lies in a leaf of the Cannon-Thurston lamination $\Lambda_{CT}(H, P^{-1}([1, q)))$ for some q . Thus $\Lambda_{CT}(H, G)$ is the (transitive closure of) union $\cup_{q \in \partial Q} \Lambda_{CT}(H, \tilde{X}_q)$.

We combine all this in the following.

Theorem 5.3. [Min94, Mit97, MS12] *Let $H = \pi_1(S^h)$ be the fundamental group of a surface with finitely many punctures and let H_1, \dots, H_n be its peripheral subgroups. Let Q be a convex cocompact subgroup of the pure mapping class group of S^h . Let*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

and

$$1 \rightarrow H_i \rightarrow N_G(H_i) \rightarrow Q \rightarrow 1$$

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}, i = 1, \dots, n$.

Further, $\Lambda_{CT}(H, G) = \cup_{z \in \partial Q} \Lambda_{EL}(z)^d$.

We can now prove our last quasiconvexity Theorem:

Theorem 5.4. *Let $H = \pi_1(S^h)$ be the fundamental group of a surface with finitely many punctures and let H_1, \dots, H_n be its peripheral subgroups. Let Q be a convex cocompact subgroup of the pure mapping class group of S^h . Let*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

and

$$1 \rightarrow H_i \rightarrow N_G(H_i) \rightarrow Q \rightarrow 1$$

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}, i = 1, \dots, n$.

Let K be a finitely generated infinite index subgroup of H . Then K is relatively quasiconvex in G .

Proof. As in the proof of Theorem 5.1 above the lamination $\Lambda_{EL}(q)$ is strongly arational for each $q \in \partial Q$. Hence for all $q \in \partial Q$, no leaf of $\Lambda_{EL}(q)^d$ is carried by K . By Theorem 5.3, $\Lambda_{CT}(H, G)$ is the transitive closure of $\cup_{z \in \partial Q} \Lambda_{EL}(z)^d$. Hence no leaf of $\Lambda_{CT}(H, G)$ is carried by K . By Lemma 2.5, K is relatively quasiconvex in G . \square

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