Ending Laminations for Hyperbolic Group Extensions

Mahan Mitra
University of California at Berkeley, CA 94720
email: mitra@math.berkeley.edu

1 Introduction

Let H be a hyperbolic normal subgroup of infinite index in a hyperbolic group G. It follows from work of Rips and Sela [16] (see below), that H has to be a free product of free groups and surface groups if it is torsion-free. From [14], the quotient group Q is hyperbolic and contains a free cyclic subgroup. This gives rise to a hyperbolic automorphism [2] of H. By iterating this automorphism, and scaling the Cayley graph of H, we get a sequence of actions of H on δ_i-hyperbolic metric spaces, where $\delta_i \to 0$ as $i \to \infty$. From this, one can extract a subsequence converging to a small isometric action on a 0-hyperbolic metric space, i.e. an $\mathbb{R}$-tree. By the JSJ splitting of Rips and Sela [16], [17], the outer automorphism group of H is generated by internal automorphisms. One notes further, that a hyperbolic automorphism cannot preserve any splitting over cyclic subgroups and that the limiting action is in fact free. Hence, by a theorem of Rips [16], H has to be a free product of free groups and surface groups if it is torsion-free. Thus the collection of normal subgroups possible is limited. However, the class of groups G can still be fairly large. Examples can be found in [3], [5] and [13].

For the purposes of this paper we choose a finite generating set of G that contains a finite generating set of H. Let Γ_G and Γ_H be the Cayley graphs of G, H with respect to these generating sets. There is a continuous proper embedding i of Γ_H into Γ_G. Every hyperbolic group admits a compactification of its Cayley graph by adjoining the Gromov boundary consisting of asymptote-classes of geodesics [10]. Let $\widehat{\Gamma_H}$ and $\widehat{\Gamma_G}$ denote these compactifications. The main theorem of [12] states :

Theorem : $i : \Gamma_H \to \Gamma_G$ *extends to a continuous map* $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$.

This generalizes a theorem of Cannon and Thurston [5] for closed hyperbolic 3-manifolds fibering over the circle. Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber F. Let $\widetilde{F}$ and $\widetilde{M}$ denote the universal covers of F and M respectively. Then $\widetilde{F}$ and $\widetilde{M}$ are quasi-isometric to $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. Let $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ denote the standard compactifications. In [5] Cannon and Thurston show that the usual inclusion of $\widetilde{F}$ into $\widetilde{M}$ extends to a continuous map from $\mathbb{D}^2$ to $\mathbb{D}^3$. An explicit description of this map was also described in [5] in terms of 'ending laminations' [See [18] for definitions]. The explicit description depends on Thurston's theory of stable and unstable laminations for pseudo-anosov diffeomorphisms of surfaces [8]. In the case of normal hyperbolic subgroups of hyperbolic groups, though existence of a continuous extension $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$ was proven in [12], an explicit description was missing. To fill this gap in the theory, an analog of Thurston's theory of ending laminations is necessary. In this paper we generalize some parts of Thurston's theory of ending laminations to the context of normal hyperbolic subgroups of hyperbolic groups. Using this we give an explicit description of the map $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$ for H a normal hyperbolic subgroup of a hyperbolic group G. This paper is motivated largely by Cannon and Thurston's work and in the case of a closed hyperbolic 3-manifold fibering over the circle provides a different account of the analysis of $\hat{i}$ in [5]. The approach of [5] is thus reversed. We start off with the existence of a continuous extension (as shown in [12]) and using it we develop a theory of ending laminations in the context of normal hyperbolic subgroups of hyperbolic groups.

A brief outline of this paper follows. Some preliminary results about hyperbolic groups are reviewed in Section 2. In section 3, we define ending laminations for hyperbolic group extensions in a way that is similar to Thurston's definition of ending laminations for Kleinian groups arising as covers of 3 manifolds fibering over the circle (See [8] or [19] for Thurston's definitions). In the motivating example of a hyperbolic 3-manifold M fibering over the circle with fiber F there exist two ending laminations, corresponding to stable and unstable laminations of a pseudo-anosov diffeomorphism. In general, if

$$1 \to H \to G \to Q \to 1$$

is an exact sequence of finitely presented groups where H, G and hence Q (from [14]) are hyperbolic, one has ending laminations naturally parametrized by points in the boundary $\partial\Gamma_Q$ of the quotient group Q. It is also shown that $\hat{i}$ identifies end-points of leaves of the ending lamination. Section 4 is at the heart of this paper and contains a converse to the previous statement. It is shown that if $\hat{i}$ identifies a pair of points in $\partial\Gamma_H$, then a bi-infinite geodesic having these points as its end-points is a leaf of the ending lamination. Finally, in Section 5, we describe some properties of ending laminations.

2 Preliminaries

We recall some preliminaries about hyperbolic groups in the sense of Gromov [10]. For details, see [6], [9]. Let G be a hyperbolic group with Cayley graph Γ equipped with a word-metric d. The **Gromov boundary** of the Cayley graph Γ, denoted by $\partial\Gamma$, is the collection of equivalence classes of geodesic rays $r : [0, \infty) \to \Gamma$ with $r(0) = 1$, the identity element,where rays r_1 and r_2 are equivalent if $sup\{d(r_1(t), r_2(t))\} < \infty$. Let $\widehat{\Gamma}$=$\Gamma \cup \partial\Gamma$ denote the natural compactification of Γ topologized the usual way(cf.[9] pg. 124).

The **Gromov inner product** of elements a and b relative to c is defined by

$$(a, b)_c = \frac{1}{2}\ [d(a, c) + d(b, c) - d(a, b)].$$

Definitions: *A subset X of Γ is said to be k-***quasiconvex** *if any geodesic joining $a, b \in X$ lies in a k-neighborhood of X. A subset X is* **quasiconvex** *if it is k-quasiconvex for some k. A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a (K, ϵ)-***quasi-isometric embedding** *if*

$$\frac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \epsilon$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a **quasi-isometry**. *A (K, ϵ)-quasi-isometric embedding that is a quasi-isometry will be called a (K, ϵ)-quasi-isometry.*

A (K,ϵ)-**quasigeodesic** *is a (K,ϵ)-quasi-isometric embedding of a closed interval in $\mathbb{R}$. A $(K,0)$-quasigeodesic will also be called a K-quasigeodesic.*

Let G be a hyperbolic group and let H be a normal subgroup that is hyperbolic. We choose a finite symmetric generating set for H and extend it to a finite symmetric generating set for G. Assume also (for simplicity) that the generating set of G intersects H in the generating set of H. Let Γ_H and Γ_G denote the Cayley graphs of H, G respectively with respect to these generating sets. By adjoining the Gromov boundaries $\partial\Gamma_H$ and $\partial\Gamma_G$ to Γ_H and Γ_G, one obtains their compactifications $\widehat{\Gamma_H}$ and $\widehat{\Gamma_G}$ respectively [9].

Label the vertices of Cayley graphs by the corresponding group elements. G (resp. H) acts on Γ_G (resp. Γ_H) by left-translations. Denote the left action of g (resp. h) by t_g (resp. t_h).

Definition:*An* **edge-path** *in Γ representing a group element w is a sequence of vertices $v_0, v_1 \cdots v_n$ in Γ such that for all i, v_i, v_{i+1} are connected by edges in Γ and $v_n = v_0 w$.*

$|w|$ will denote the length of a shortest edge-path representing the group element w.

Unless otherwise stated $[x,y]$ will denote a geodesic segment joining x,y in Γ.

There is a natural embedding $i : \Gamma_H \rightarrow \Gamma_G$ sending a vertex of Γ_H labeled h to the vertex of Γ_G labeled h. The main theorem of [12] states that i extends to a continuous map $\hat{i}$ from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$. Let $\partial i : \partial\Gamma_H \rightarrow \partial\Gamma_G$ denote the restriction of $\hat{i}$ to $\partial\Gamma_H$. A large part of this paper is devoted to a study of the continuous map ∂i.

Some standard facts about hyperbolic groups will be needed in this paper and we mention them here (mostly without proof) for future reference.

Proposition 2.1 *Given $A \geq 0$ there exist $K \geq 1$ and $\epsilon \geq 0$ such that if a, b, c are vertices of Γ_H with $(a,c)_b \leq A$ then $[a,b] \cup [b,c]$ is a (K,ϵ)-quasigeodesic.*

Proof: It suffices to show that if $x \in [a,b]$ and $z \in [b,c]$ then $d_H(x,b) + d_H(b,z) \leq K d_H(x,z) + \epsilon$ for some K, ϵ depending only on δ.

Let u, v, w be points on edges $[x,b]$, $[b,z]$ and $[z,x]$ respectively such that $d_H(u,b) = d_H(v,b)$, $d_H(v,z) = d_H(w,z)$ and $d_H(w,x) = d_H(u,x)$. Then $d_H(u,b) = (x,z)_b \leq A$. Hence

$$d_H(x,b) + d_H(b,z) \leq d_H(x,z) + 2A$$

Choosing $K = 1$ and $\epsilon = 2A$ we are through. $\Box$

Proposition 2.2 *Given $A_0 \geq 0$ there exist $\beta>1$, $B > 0$, $K \geq 1$ and $\epsilon \geq 0$ such that if $[x,y],[y,z]$ and $[z,w]$ are geodesics in Γ_H with $(x,z)_y \leq A_0$, $(y,w)_z \leq A_0$ and $d_H(y,z) \geq B$ then*
1) Any path joining x to w and lying outside a D-neighborhood of $[y,z]$ has length greater than or equal to $\beta^D d_H(y,z)$,
where $D = min\{(d_H(x,[y,z]) - 1), (d_H(w,[y,z]) - 1)\}$.
2) $[x,y] \cup [y,z] \cup [z,w]$ is a (K,ϵ)-quasigeodesic.

The first part of the above Proposition states that it is exponentially more inefficient to travel outside large neighborhoods of geodesics (See [18] pg. 5.41 for instance). The second part follows from the first part and Proposition 2.1 above.

The following proposition is a 'quasification' of the fact that geodesics converging to the same point on $\partial\Gamma_H$ are asymptotic [See [9], pg. 117].

Proposition 2.3 *Given $K \geq 1, \epsilon \geq 0$ there exists α such that if $r_1, r_2 : [0,\infty) \to \Gamma_H$ are two (K,ϵ)-quasigeodesics converging to the same point in $\partial\Gamma_H$, then there exists $T \geq 0$ such that for all $t \geq T$*

$$max\ \{d_H(r_1(t), r_2), d_H(r_2(t), r_1)\} \leq \alpha$$

The following is again a simple 'quasification' of the standard fact that geodesics diverge exponentially in a δ-hyperbolic metric space [1].

Proposition 2.4 *Given $K \geq 1, \epsilon \geq 0, D \geq 0$ there exist $b > 1$, $A > 0$ and $C > 0$ such that the following holds:*
If r_1, r_2 are two (K,ϵ)-quasigeodesics with $d(r_1(0), r_2(0)) \leq D$ and there exists $T \geq 0$ with $d(r_1(T), r_2(T)) \geq C$ then any path joining $r_1(T+t)$ to $r_2(T+t)$ and lying outside the union of the $\frac{T+t-1}{K+\epsilon}$-balls around $r_1(0), r_2(0)$ has length greater than Ab^t for all $t \geq 0$.

The following Corollary can be obtained by combining Proposition 2.1 and Proposition 2.4.

Corollary 2.5 *Given $A_0 \geq 0$ there exist $B > 0$, $K \geq 1$ and $\epsilon \geq 0$ such that if $[x,y],[y,z]$ and $[z,w]$ are geodesics in Γ with $(x,z)_y \leq A_0$, $(y,w)_z \leq A_0$ and if there exist u,v in $[x,y]$ and $[z,w]$ respectively with $d(u,x) \geq B$, $d(v,y) = d(u,v) \geq B$ then $[x,y] \cup [y,z] \cup [z,w]$ is a (K,ϵ)-quasigeodesic.*

We need to establish a dictionary between certain objects in negatively curved manifolds and those in Cayley graphs.

Definitions: *An* **edge path representative** *of a word $w = h_1 h_2 \cdots h_n \in H$ (h_i's are in the generating set for H) starting at $v \in \Gamma_H$ is an edge-path $v_0 v_1 v_2 \cdots v_n$ in Γ_H such that $v_i^{-1} v_{i+1} = h_i$ where v_i's are vertices of Γ_H and $v_0 = v$.*

A **geodesic representative** *of $w \in H$ starting at v is a geodesic in Γ_H starting at v and ending at u, where $v^{-1}u = w$ as an element of H.*

A **cyclic conjugate** *of a word w is a word w_0 (not necessarily freely reduced) such that there exist words w_1 and w_2 with $w = w_1 w_2$ and $w_0 = w_2 w_1$ as freely unreduced words.*

A **cyclic conjugate** *of an edge-path μ is an edge-path representative of a cyclic conjugate of the word represented by μ.*

A **free homotopy representative** *of a word $w \in H$ is a geodesic $[a, aw_0]$ in Γ_H where w_0 is a shortest word in the conjugacy class of w in H.*

Edge path representatives, geodesic representatives and free homotopy representatives of words correspond to lifts (to the universal cover of a negatively curved manifold) of loops, based geodesic loops and shortest closed geodesics in free homotopy classes respectively.

In a closed negatively curved manifold a closed quasigeodesic lies close to the shortest closed geodesic in its free homotopy class. The following Proposition is the analog of this result for hyperbolic groups.

Proposition 2.6 *Let H be a δ-hyperbolic group. Given $K \geq 1, \epsilon \geq 0$ there exists $M \geq 0$ such that for any word $w \in H$ if all edge-path representatives of cyclic conjugates of w are (K,ϵ)-quasigeodesics then there exists a word c with length $|c| \leq M$ such that geodesic representatives of $c^{-1}wc$ are free homotopy representatives of w.*

Proof: Consider the set of pairs (h_1, h_2), where h_2 is a cyclic conjugate of w and $h_1 h_2 h_1^{-1} = u$ with u a shortest word in the conjugacy class of w. Choose (c, w_0) belonging to this set such that $|c|$ the minimum possible.

Consider a quadrilateral in Γ_H with vertices $1, c, uc, u$ where 1 is the identity. $1, c$ are joined by a geodesic representative $[1, c]$ of c starting at 1. u, uc are joined by $t_u([1, c])$. $1, u$ are joined by a geodesic representative $[1, u]$ of u and c, uc are joined by the edge-path representative ω corresponding to w_0 starting at c.

For $p \in \omega$ and $q \in [1, u]$, $d_H(p, q) \geq |c|$ as otherwise there would be an element $c_1 \in H$ with length less than $|c|$ conjugating a cyclic conjugate of w to a cyclic conjugate of u. In particular $d_H(c, q) \geq |c|$ for any $q \in [1, u]$ and $d_H(uc, q) \geq |c|$ for any $q \in [1, u]$. Let $[c, u]$ be a geodesic from c to u.

Let l, m, n be points on $[1, c]$, $[c, u]$ and $[1, u]$ respectively such that $d_H(1, n) = d_H(1, l)$, $d_H(c, l) = d_H(c, m)$ and $d_H(u, m) = d_H(u, n)$. Then $(u, c)_1 = d_H(l, 1)$. Also, since H is δ-hyperbolic, the diameter of the inscribed triangle with vertices l, m, n is less than or equal to 2δ (See [1] pg. 16).

$$\begin{aligned} d_H(c, l) + d_H(l, n) &\geq d_H(c, 1) = d_H(c, l) + d_H(l, 1) \\ \Rightarrow d_H(l, 1) &\leq d_H(l, n) \leq 2\delta \\ \Rightarrow (u, c)_1 &\leq 2\delta \end{aligned}$$

Similarly, $(uc, 1)_u \leq 2\delta$.

Hence, by Proposition 2.2 there exists $\beta > 1$ depending on δ alone such that the length of ω is greater than or equal to $\beta^{|c|}|u|$ for $|u|$ greater than some critical number B depending on δ. For $|u| \leq B$ the proposition is easy as there are only finitely many such elements. Else since ω is a (K, ϵ)-quasigeodesic, there exists $M_0 \geq 0$ depending on δ, K and ϵ alone such that $|c| \leq M_0$.

Also from Proposition 2.2 $[c, 1] \cup [1, u] \cup [u, uc]$ is a (k_0, ϵ_0)-quasigeodesic for some k_0, ϵ_0 depending only on δ. Hence ω lies in an M-neighborhood of $[c, 1] \cup [1, u] \cup [u, uc]$ for some M depending only on δ, K and ϵ. The Proposition follows easily. □

The proof above yields the following corollary.

Corollary 2.7 *There exists $M \geq 0$ such that the following holds:*
If v, w are words in the same conjugacy class such that all edge-path represen-

tatives of w or v are free homotopy representatives then there exists a cyclic conjugate v_1 of v and a word c with length $|c| \leq M$ such that $c^{-1}wc = v_1$.

Since quasi-isometries take geodesics to quasi-geodesics another easy corollary of Proposition 2.6 is the following.

Corollary 2.8 *Let H be a δ-hyperbolic group. Given $K \geq 1, \epsilon \geq 0$ there exists $M \geq 0$ such that for any element $h \in H$ and any automorphism ϕ of H inducing a (K, ϵ)-quasi-isometry of the vertex set of Γ_H there exists a word c with length $|c| \leq M$ such that a geodesic representative of $c^{-1}\phi(h)c$ is a free homotopy representative.*

3 Ending Laminations

In order to motivate the definition of ending laminations for hyperbolic group extensions, we recall (a slight modification of) Thurston's notion of a stable lamination for a pseudo-anosov diffeomorphism ϕ of a closed hyperbolic surface S. (See [8] pg 71 for a detailed discussion of Thurston's notion.) Let c be a simple closed geodesic on S. Let c_n be the shortest geodesic in the free homotopy class of $\phi^n(c)$ for $n \geq 0$. Consider all Hausdorff limits of convergent subsequences of $\{c_n\}$ as $n \to \infty$. Let Λ be the union of all such limits as c ranges over all simple closed geodesics on S. It is this construction of Λ that will be generalized to the case of a hyperbolic normal subgroup of a hyperbolic group.

Remark: Λ constructed above is closely related to laminations appearing in [3] and [8]. In [8] or [19] a limit is extracted in the space of projectivized measured laminations $\mathcal{PML}(S)$, which gives a (measured) geodesic lamination of S with complementary regions consisting of ideal polygons. However, limits in the Hausdorff topology may introduce diagonals of these ideal polygons and a union of these might contain intersecting diagonals. That this is the only possibility follows from [5] and Theorem 4.11. The laminations in [3] (H free) are exactly the same as those appearing here (see [3] for a proof).

A variant of the Hausdorff topology is defined below (see [4]).

Definition: *The* **Chabauty topology** *on the set $C(X)$ of closed subsets of a topological space X has a sub-basis given by sets of the following form:*
i) $O_1(K) = \{A | A \cap K = \emptyset\}$ *where K is compact.*

ii)$O_2(K) = \{A|A \cap U \neq \emptyset\}$ *where* U *is open.*

For a careful study of the relation between the Hausdorff topology and the Chabauty topology, see [11]. The two coincide for X compact Hausdorff. One advantage that the Chabauty topology has is that $C(X)$ is compact for any topological space X. The following lemma indicates the geometric nature of the topology.

Lemma 3.1 *[4] Suppose* X *is a locally compact metric space. A sequence* $\{A_n\}$ *of closed subsets of* X *converges in* $C(X)$ *(equipped with the Chabauty topology) to the closed subset* A *iff*
1) If $\{x_{n(k)} \in A_{n(k)}\}$ *converges to* $x \in X$ *then* $x \in A$.
2) If $x \in A$ *then there exists a sequence* $\{x_n\}$ *converging to* x *such that* $x_n \in A_n$.

Define $\widehat{\Gamma_H}^{(2)} = (\widehat{\Gamma_H} \times \widehat{\Gamma_H} \setminus \Delta)/(x,y)\sim(y,x)$ where $\Delta = \{(x,x)|x \in \widehat{\Gamma_H}\}$, and let $\partial^2\Gamma_H = (\partial\Gamma_H \times \partial\Gamma_H \setminus \Delta)/(x,y)\sim(y,x)$.

Two geodesics in Γ_H are defined to be equivalent if they have the same end-points in $\widehat{\Gamma_H}$. So points in $\widehat{\Gamma_H}^{(2)}$ correspond to equivalence classes of geodesics.

Both $\widehat{\Gamma_H}^{(2)}$ and $\partial^2\Gamma_H$ are locally compact and metrizable.

An essential ingredient of [12] will be recalled now. Let $P : G \to Q$ be a surjective homomorphism of finitely generated groups. Abusing notation slightly P will also denote the induced map from the vertex set of Γ_G to the vertex set of Γ_Q.

Definition:*A* (κ, ϵ_0)**-quasi-isometric section** *is a subset* $\Sigma \subset G$ *mapping onto* Q *such that for any* $g, g' \in \Sigma$,

$$\frac{1}{\kappa}d_Q(Pg, Pg') - \epsilon_0 \leq d_G(g, g') \leq \kappa d_Q(Pg, Pg') + \epsilon_0$$

where d_G *and* d_Q *are word metrics in* G, Q *respectively and* $\kappa \geq 1$, $\epsilon_0 \geq 0$ *are constants.*

A **single-valued quasi-isometric section** *is defined by choosing a single element of* Σ *representing each coset of the kernel of* P.

If a map σ from Γ_Q to Γ_G has a single-valued quasi-isometric section as its image, then σ will also be referred to as a quasi-isometric section. It will be clear from context whether we mean the map or its image.

The above notion is due to Lee Mosher [14]. A crucial lemma of [14] is the following.

Lemma 3.2 *(Quasi-isometric section Lemma [14])Given a non-elementary hyperbolic group H and a short exact sequence of finitely generated groups*

$$1 \to H \to G \to Q \to 1,$$

the map $P : G \to Q$ has a single-valued quasi-isometric section Σ. In fact, choosing a generating set B for G and letting $P(B)$ be the generating set for Q, we have for all $g, g' \in \Sigma$,

$$d_Q(Pg, Pg') \leq d_G(g, g') \leq \kappa d_Q(Pg, Pg') + \epsilon_0$$

for some constants $\kappa \geq 1$ and $\epsilon_0 \geq 0$.

Using a left translation t_h by an element $h \in H \subset G$, one can assume that Σ contains the identity element of G. $t_h(\Sigma)$ is still a single-valued quasi-isometric section as t_h preserves cosets.

Definition:*A* (κ, ϵ_0)**-quasi-isometric section** *of a subset U of Γ_Q is a map $\sigma : U \to \Gamma_G$ such that $P \cdot \sigma$ is the identity map on U and*

$$\frac{1}{\kappa} d_Q(k, k') - \epsilon_0 \leq d_G(\sigma(k), \sigma(k')) \leq \kappa d_Q(k, k') + \epsilon_0$$

where d_G and d_Q are word metrics in G, Q respectively and $\kappa \geq 1$, $\epsilon_0 \geq 0$ are constants.

Henceforth assume that H is a non-elementary hyperbolic group.

Corresponding to every element $g \in G$ there exists an automorphism of H taking h to $g^{-1}hg$ for $h \in H$. Such an automorphism induces a bijection ϕ_g of the vertices of Γ_H. This gives rise to a map from Γ_H to itself, sending an edge $[a, b]$ linearly to a shortest edge-path joining $\phi_g(a)$ to $\phi_g(b)$. Since ϕ_g is a quasi-isometry, this map extends to $\widehat{\Gamma_H}$ and acts by homeomorphisms on $\partial\Gamma_H$. Abusing notation slightly, we shall also call the extended map ϕ_g. Note that the action of ϕ_g on $\partial\Gamma_H$ and the extension of $t_{g^{-1}}$ to $\partial\Gamma_H$ coincide.

Fixing $z \in \partial\Gamma_Q$ for the time being (for notational convenience) we shall define the set of ending laminations corresponding to z.

Let $[1, z)$ be a semi-infinite geodesic ray in Γ_Q starting at the identity 1 and converging to $z \in \partial\Gamma_Q$. Let σ be a single-valued quasi-isometric section of Q into G. Let z_n be the vertex on $[1, z)$ such that $d_Q(1, z_n) = n$ and let $g_n = \sigma(z_n)$.

Given $h \in H$ let Σ_n^h be the (H-invariant) collection of all free homotopy representatives of $\phi_{g_n^{-1}}(h)$ in Γ_H. Identifying equivalent geodesics in Σ_n^h one obtains a subset S_n^h of $\widehat{\Gamma_H}^{(2)}$. The intersection with $\partial^2\Gamma_H$ of the union of all subsequential limits (in the Chabauty topology) of $\{S_n^h\}$ will be denoted by Λ_z^h.

Definition: *The set of* **ending laminations corresponding to** $z \in \partial\Gamma_Q$ *is given by*

$$\Lambda_z = \bigcup_{h\in H}\Lambda_z^h.$$

Definition: *The set* Λ *of all* **ending laminations** *is defined by*

$$\Lambda = \bigcup_{z\in\partial\Gamma_Q}\Lambda_z.$$

Note: Σ_n^h and therefore S_n^h are independent of the conjugacy class of h. Hence in the definition of Λ_z above it is enough to consider only smallest elements in conjugacy classes. Also from Corollary 2.7 it is enough, in the definition of Λ_z^h, to consider a single free homotopy representative of h and its translates by elements of H.

It needs to be checked that Λ_z is independent of the quasi-isometric section σ and the geodesic ray $[1, z)$.

For any two quasi-isometric sections σ and σ_1 of Q into G, and any $x \in Q$, $\sigma(x)$ and $\sigma_1(x)$ lie in the same coset of H in G. Hence for any $h \in H$, $\sigma(x)^{-1}h\sigma(x)$ and $\sigma_1(x)^{-1}h\sigma_1(x)$ are conjugate in H. Therefore Λ_z is independent of the quasi-isometric section σ.

Lemma 3.3 *Λ_z is independent of the geodesic ray $[1, z)$.*

Proof: Let μ_1 and μ_2 be two geodesic rays in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$. Suppose Q is δ_1-hyperbolic and σ is a (K, ϵ)-quasi-isometric section. Then $\sigma(\mu_1)$ and $\sigma(\mu_2)$ are quasigeodesics lying in a D-neighborhood of each other for some D dependent on K, ϵ, δ_1. Let Λ_z^1 and Λ_z^2 be the ending laminations given by these two quasi-geodesics. Let $h \in H$. Let $\{g_n\}$ be a sequence with $g_n \in \sigma(\mu_1)$ such that the set of all free homotopy representatives of $g_n h g_n^{-1}$ converge to a subset S of Λ_z^1. There exist $q_n \in G$ with $|q_n| \leq \delta_1(K+\epsilon)$ such that $g_n {q_n}^{-1} = r_n \in \sigma(\mu_2)$. Therefore $g_n h {g_n}^{-1} = r_n(q_n h {q_n}^{-1}) r_n^{-1}$ for $h \in H$. Since $|q_n|$ is bounded there exists a subsequence $\{n_i\}$ such that $q_{n_i}{=}q$ for some $q \in G$. Let $h_1 = qhq^{-1}$. Then $g_{n_i} h g_{n_i}^{-1} = r_{n_i} h_1 r_{n_i}^{-1}$ Since the set of all free homotopy representatives (in Γ_H) of $g_n h {g_n}^{-1}$ converges to S, so does every subsequence. Hence the set of free homotopy representatives of $\{r_{n_i} h_1 {r_{n_i}}^{-1}\}$ converges to S. Since Λ_z^1 is obtained by taking a union of all such subsets S over all $h \in H$, $\Lambda_z^1 \subset \Lambda_z^2$. Similarly $\Lambda_z^2 \subset \Lambda_z^1$. Hence Λ_z is independent of the geodesic ray $[1, z)$. $\Box$

We shall now provide some justification for the terminology. The main theorem of [12] is the following:

Theorem 3.4 *Let G be a hyperbolic group and H a hyperbolic normal subgroup. Then $i : \Gamma_H \to \Gamma_G$ extends uniquely to a continuous map $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$.*

Definition:*A* **leaf** *of an ending lamination Λ_z is a bi-infinite geodesic whose set of end-points is an element of Λ_z.*

Lemma 3.5 *If λ is a leaf of an ending lamination Λ_z with end-points u, v then $\hat{i}(u) = \hat{i}(v)$.*

Proof: Suppose $(u, v) \in \Lambda_z^h$. Since H is normal in G, $ghg^{-1} {\in} H$ for all $g \in G$. Let $g_n \in \sigma([1, z))$ such that some sequence of free homotopy representatives of $g_n h g_n^{-1}$ in Γ_H converges to λ. Since Λ_z is H-invariant assume without loss of generality that λ passes through the identity. Since Σ_n^h is independent of the conjugacy class of h in H, one can assume that h is a shortest element in its conjugacy class. Choose p_n in the same coset as g_n such that the geodesic representative (in Γ_H) of $p_n h p_n^{-1}$ (regarded as an element of H) is a free homotopy representative. Note also by Corollary 2.7, one can assume (after passing to a subsequence if necessary) that a sequence of geodesic representatives of $p_n h p_n^{-1}$ converges to λ. Since geodesic representatives of $p_n h p_n^{-1}$ and $(p_n h) h (p_n h)^{-1}$ coincide therefore $d_G(1, p_n h) \geq d_G(1, p_n)$. Hence $(1, p_n h)_{p_n} \leq \frac{1}{2}|h|$. Similarly, $(1, p_n h p_n^{-1})_{p_n h} \leq \frac{1}{2}|h|$. Also, the length of a geodesic representative (in Γ_H) of $p_n h p_n^{-1}$ tends to infinity as n tends to infinity. Hence by Corollary 2.5 $[1, p_n] \cup [p_n, p_n h] \cup [p_n h, p_n h p_n^{-1}]$ is a

(K_h, ϵ_h)-quasigeodesic in Γ_G where K_h, ϵ_h are independent of n. If $\{[u_n, v_n]\}$ is a sequence of geodesic representatives of $p_n h {p_n}^{-1}$ converging to λ then there exists $N \geq 0$ such that for all $n \geq N$, $[u_n, v_n]$ passes through the identity 1. Also $v_n = u_n(p_n h p_n^{-1})$. Then $t_{u_n}([1, p_n] \cup [p_n, p_n h] \cup [p_n h, p_n h p_n^{-1}])$ is a (K_h, ϵ_h)-quasigeodesic in Γ_G lying outside a D_n-ball around the identity, where $D_n \to \infty$ as $n \to \infty$. Hence any geodesic joining $i(u_n), i(v_n)$ in Γ_G lies outside a $(D_n - K_h')$-ball around the identity in Γ_G where K_h' is independent of n. As $(D_n - K_h') \to \infty$ as $n \to \infty$, Theorem 3.4 shows that $\hat{i}(u) = \hat{i}(v)$. $\Box$

4 Identification of Boundary Points

Let Λ' be the set of equivalence classes (x, y) of bi-infinite geodesics in Γ_H (regarded as elements of $\partial^2\Gamma_H$) such that $\hat{i}(u)=\hat{i}(v)$. A bi-infinite geodesic contained in Γ_H and belonging to such an equivalence class will be called a **leaf** of Λ'. The main aim of this section is to prove the following theorem.

Theorem 4.11 $\Lambda'=\bigcup_{z\in\partial\Gamma_Q}\Lambda_z$

Lemma 3.5 shows that $\bigcup_{z\in\partial\Gamma_Q}\Lambda_z \subset \Lambda'$. Hence it is enough to show that if $(u, v) \in \Lambda'$ then $(u, v) \in \Lambda_z$ for some $z \in \partial\Gamma_Q$.

There are two main ingredients of the proof. First, one observes that leaves of Λ' are aperiodic and consequences of this observation are obtained. The second ingredient is a coarse separation property (a notion introduced in [7]). These two properties are dealt with in the two subsections of this section. Theorem 4.11 will follow fairly easily.

4.1 Aperiodicity and its Consequences

For any $w \in H$ an edge-path representative of the bi-infinite word $w^* = \cdots www \cdots$ converges to some $x_w, y_w \in \partial\Gamma_H$ and includes into Γ_G as a quasigeodesic ([9], pg. 155).

Definition :*A bi-infinite geodesic $\mu \subset \Gamma_H$ is said to be* **periodic** *if there exists $w \in H$ such that μ lies in a bounded neighborhood of an edge-path representative of the bi-infinite word $w^* = \cdots www \cdots$. A path is* **aperiodic** *if it is not periodic.*

Since leaves of Λ' are not included as quasigeodesics in Γ_G, they are aperiodic.

This simple observation regarding aperiodicity of leaves of Λ' has several consequences which we now explore. First note that Λ' is H-invariant and (by Theorem 3.4) closed.

Lemma 4.1 *For all $w \in H$ there exists N_w such that if p, pw^n are vertices on a leaf of Λ', then $n \leq N_w$.*

Proof: Suppose not. Then for all n there exist leaves λ_n of Λ' containing geodesic subsegments $[p_n, p_n w^n]$. Since Λ' is H-invariant we can assume that for all n the identity 1 is a vertex in λ_n closest to the mid-point of $[p_n, p_n w^n]$.

An edge-path representative of the bi-infinite word $w^* = \cdots www \cdots$ converges to some $x_w, y_w \in \partial\Gamma_H$ and includes into Γ_G as a quasi-geodesic ([9] pg. 155). In particular, $\hat{i}(x_w) \neq \hat{i}(y_w)$. If x_n, y_n are end-points of λ_n in $\partial\Gamma_H$ then $(x_n, y_n) \to (x_w, y_w)$ in the Chabauty topology on $\partial^2\Gamma_H$. This implies that $(x_w, y_w) \in \Lambda'$ contradicting $\hat{i}(x_w) \neq \hat{i}(y_w)$. This proves the lemma. $\Box$

Definition: *A geodesic segment $[r, q]$ in Γ_H is said to be an* **extension** *of a geodesic segment $[a, b]$ in a bi-infinite geodesic λ if $[a, b] \subset [r, q] \subset \lambda$. (Here r, q, a, b are all vertices of Γ_H.)*

Recall that a cyclic conjugate of an edge-path μ is an edge-path representative of a cyclic conjugate of the word represented by μ.

Lemma 4.2 *There exist $K \geq 1$, $\epsilon \geq 0$ such that for any leaf λ of Λ' and any geodesic subsegment $[p, q]$ of λ there exists an extension $[r, q]$ of $[p, q]$ in λ with $d_H(p, r)$ equal to 0 or 1 such that edge-path representatives of all cyclic conjugates of $[p, r]$ are (K, ϵ)-quasigeodesics.*

Proof: Suppose not. Then for every pair $K \geq 1$, $\epsilon \geq 0$ there exists a leaf λ of Λ' and a geodesic subsegment $[p, q]$ of λ such that both extensions $[r, q]$ of $[p, q]$ in λ with $d_H(p, r) \leq 1$ have cyclic conjugates that are not (K, ϵ)-quasigeodesics. Since Λ' is H-invariant, we can assume that $q = 1$.

A cyclic conjugate of any geodesic $[r, 1]$ is of the form $[r_1, 1] \cup [1, r^{-1}r_1]$ where $r_1 \in H$ is the label of a vertex on $[r, 1]$.

Hence from Proposition 2.1 for all $j \geq 0$ there exist leaves λ_j of Λ', $[p_j, 1] \subset [q_j, 1] \subset \lambda_j$ and $s_j \in [p_j, 1]$, $t_j \in [q_j, 1]$ such that $d_H(p_j, q_j) = 1$, $(s_j, p_j^{-1}s_j)_1 \geq j$ and $(t_j, q_j^{-1}t_j)_1 \geq j$.

Also as in Proposition 2.1 there exist $w_j \in [s_j, 1] \cap [t_j, 1]$, $u_j \in [1, p_j^{-1}s_j]$, $v_j \in [1, q_j^{-1}t_j]$ such that $d_H(1, w_j) = j$, $d_H(w_j, u_j) \leq 2\delta$ and $d_H(w_j, v_j) \leq 2\delta$. Then $d_H(u_j, v_j) \leq 4\delta$ and hence $\Big||u_j| - |v_j|\Big| \leq 4\delta$.

Let $u_j{=}v_j g_j$ for some g_j with $|g_j| \leq 4\delta$. Then $x_j{=}p_j u_j \in [p_j, s_j] \subset \lambda_j$, $y_j{=}q_j v_j \in [q_j, t_j] \subset \lambda_j$ and $d_H(p_j, q_j) = 1$.

Since $\Big||u_j| - |v_j|\Big| \leq 4\delta$ $d_H(x_j, y_j) \leq 4\delta + 1$. Hence $[p_j, p_j u_j]$ and $[q_j, q_j v_j]$ are geodesic subsegments of λ_j starting at a distance one apart and ending at a distance less than or equal to $(4\delta + 1)$. Let $[q_j, q_j u_j]$ be a geodesic segment from q_j to $q_j u_j$. Then $[p_j, p_j u_j]$ and $[q_j, q_j u_j]$ are geodesic segments of length greater than or equal to $(j - 2\delta)$ starting at a distance one apart and ending at a distance less than or equal to $8\delta + 1$. Furthermore, (See [9] pg. 101) $[p_j, p_j u_j]$ and $[q_j, q_j u_j]$ lie in a $c(\delta)$-neighborhood of each other. Hence for all vertices h_j of $[1, u_j]$, $d_H(p_j h_j, q_j h_j) \leq c(\delta) + 1$.

This shows, (by the pigeon-hole principle) that for all $j \geq 0$, there exist leaves λ_j' of Λ', $[p_j', 1] \subset [{q_j}', 1] \subset \lambda_j'$ and $l_j, m_j \in H$ with $d_H(l_j, m_j) \geq j$ and $d_H(p_j', q_j') = 1$ such that

$$\begin{aligned} p_j' l_j, p_j' m_j &\in [p_j', 1] \qquad \text{and} \\ ({p_j}' l_j)^{-1}({q_j}' l_j) &= ({p_j}' m_j)^{-1}({q_j}' m_j) \qquad \text{and} \\ d_H({p_j}' l_j, {q_j}' m_j) &\leq c(\delta) + 1. \end{aligned}$$

Hence $l_j m_j^{-1}$ and $({p_j}')^{-1}{q_j}'$ commute.

$({p_j}')^{-1}{q_j}'$ is a generator of H. Since there are only finitely many of these, after passing to a subsequence if necessary we can assume that $l_j {m_j}^{-1}$ commutes with a generator h_0 of H for all j. Since H is hyperbolic, any abelian subgroup is cyclic. Hence (after passing to a subsequence again if necessary) $l_j {m_j}^{-1} = h^{n_j}$ for some fixed $h \in H$ and $n_j \geq j$. Choosing $j \geq N_h{+}1$ (where N_h is as in Lemma 4.1) this contradicts Lemma 4.1, as some geodesic edge-path $[{p_j}' l_j, {p_j}' m_j]$ is contained in λ_j'. This contradiction proves the lemma. $\Box$

Let $\lambda \subset \Gamma_H$ be a geodesic (finite, semi-infinite or bi-infinite). Let a, b denote the end-points of λ and let $\lambda_g \subset \Gamma_H$ denote a geodesic joining $\phi_g(a)$ to $\phi_g(b)$ in Γ_H. Note that $a, b \in \widehat{\Gamma_H}$ and may lie in Γ_H or $\partial\Gamma_H$.

Lemma 4.3 *Given $K \geq 1$, $\epsilon \geq 0$ there exists $C \geq 0$ such that for any* *geodesic ray* $[a, z) \subset \Gamma_Q$ *starting at* a*, converging to* $z \in \partial\Gamma_Q$ *and passing through* $1 \in \Gamma_Q$ *the following holds:*

If λ *is a geodesic segment in* Γ_H *with end-points* $1, h \in \Gamma_H$ *all whose cyclic conjugates are* (K, ϵ)*-quasigeodesics then there exists a* $(C, 0)$*-quasi-isometric section* σ_0 *of* $[a, z)$ *into* Γ_G *containing* $1 \in \Gamma_G$ *such that for all* $g \neq 1$ *in* $\sigma[a, z)$*,* λ_g *is a free homotopy representative (i.e. all cyclic conjugates of* λ_g *are geodesics for* $g \neq 1$*).*

Proof: Firstly, by Corollary 2.7 there exist $M \geq 0$ (depending on K, ϵ) and $c \in H$ with $|c| \leq M$ such that any geodesic representative of $c^{-1}hc$ is a free homotopy representative. Let λ' be a geodesic segment from c to hc. Then λ' is a free homotopy representative. If we can construct a $(C_0, 0)$-quasi-isometric section σ_0 of $[a, z)$ through c such that for all $g \in \sigma_0([a, z))$, λ'_g is a free homotopy representative, then by taking $C = C_0 + M$ and changing $\sigma_0(1)$ from c to 1 we would have a quasi-isometric section satisfying the conclusions of the Lemma.

Hence assume for the sake of simplicity that λ is a free homotopy representative. Let z_n be a point in $[1, z) \subset [a, z)$ such that $d_Q(1, z_n) = n$ and let z_{-n} be a point on $[a, 1] \subset [a, z)$ such that $d_Q(1, z_{-n}) = n$.

Let σ be a (κ, ϵ_0)-quasi-isometric section of Γ_Q obtained from Lemma 3.2. Let

$$\begin{array}{lrcll} & (\sigma(z_n))^{-1}(\sigma(z_n)) & = & s_n & \text{for} \quad n \geq 0, \\ \text{and} & (\sigma(z_n))^{-1}(\sigma(z_n)) & = & s_{n-1} & \text{for} \quad n \leq 0. \end{array}$$

Then for all j, $|s_j| \leq \kappa + \epsilon_0$ and ϕ_{s_j} is a (K_1, ϵ_1)-quasi-isometry for some K_1, ϵ_1 depending only on κ, ϵ_0. The quasi-isometric section σ_0 of $[a, z)$ will now be constructed inductively. Put $\sigma(z_0) = 1$.

Assume that $\sigma(z_j) = g_j$ has been constructed for $-m \leq j \leq n$ satisfying the conclusions of the lemma.

Since ϕ_{s_n} is a (K_1, ϵ_1)-quasi-isometry, and λ has been assumed to be a free homotopy representative, $\lambda_{g_n s_n}$ is a geodesic all whose cyclic conjugates are (K_1, ϵ_1)-quasigeodesics. Hence from Corollary 2.8 there exist M_1 (depending on K_1, ϵ_1) and $h_n \in H$ such that $|h_n| \leq M_1$ and $\lambda_{g_n s_n h_n}$ is a free homotopy representative. Put $g_{n+1} = \sigma(z_{n+1}) = g_n s_n h_n$. Similarly, one can construct $g_{-m-1} = \sigma(z_{-m-1})$.

Note that $d_G(z_n, z_{n+1}) \leq \kappa + \epsilon_0 + M_1$. Choose $C_0 = \kappa + \epsilon_0 + M_1$. Then $\sigma(z_j) = g_j$ gives a $(C_0, 0)$-quasi-isometric section and the Lemma is proven. $\square$

Remark: Note that C in Lemma 4.3 above is greater than or equal to $(\kappa + \epsilon_0)$.

Translating the quasi-isometric section obtained in Lemma 4.3 above by an element of G, one obtains the following Corollary.

Corollary 4.4 *Given $K \geq 1$ and $\epsilon \geq 0$ there exists $C \geq (\kappa + \epsilon_0)$ such that for any geodesic ray $[1, z)$ in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$, and any $g \in P^{-1}([1, z))$ the following holds :*

If λ is a geodesic segment in Γ_H with end-points $1, h \in \Gamma_H$ all whose cyclic conjugates are (K, ϵ)-quasigeodesics then there exists a $(C, 0)$-quasi-isometric section σ_0 of $[1, z)$ into Γ_G containing $g \in \Gamma_G$ such that for all $g' \neq g$ in $\sigma_0([1, z))$ $\lambda_{g^{-1}g'}$ is a free homotopy representative (i.e. all cyclic conjugates of $\lambda_{g^{-1}g'}$ are geodesics for $g' \neq g$).

From Lemma 4.2 and Corollary 4.4 above we obtain the following Lemma needed to prove Theorem 4.11.

Lemma 4.5 *There exists $C' \geq \kappa + \epsilon_0$ such that for any leaf λ of Λ', any geodesic ray $[1, z)$ in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$ and any geodesic subsegment $[p, q]$ of λ_g for some $g \in P^{-1}([1, z))$ the following holds: There exists an extension $[r, q] = \mu$ of $[p, q]$ in λ_g with $d_H(p, r)$ equal to 0 or 1 and a $(C', 0)$-quasi-isometric section σ:$[1, z) \to \Gamma_G$ such that $gr \in \sigma([1, z))$ and $\mu_{r^{-1}g^{-1}g'}$ is a free homotopy representative for all $g' \neq gr$ in $\sigma([1, z))$ (i.e. all cyclic conjugates of $\mu_{r^{-1}g^{-1}g'}$ are geodesics for $g' \neq g$).*

4.2 Coarse Separation and its Consequences

Recall that if $\lambda \subset \Gamma_H$ is a geodesic (finite, semi-infinite or bi-infinite) with end-points a, b, then $\lambda_g \subset \Gamma_H$ denotes a geodesic joining $\phi_g(a)$ to $\phi_g(b)$ in Γ_H. Note that $a, b \in \widehat{\Gamma_H}$ and may lie in Γ_H or $\partial\Gamma_H$. Let $\sigma : \Gamma_Q \to \Gamma_G$ be a quasi-isometric section through the identity. Define

$$B(\lambda, \sigma) = \bigcup_{g \in \sigma(Q)} t_g \cdot i(\lambda_g).$$

Mark that $B(\lambda, \sigma)$ contains $i(\lambda)$ as $1 \in \sigma(Q)$. It is important to note also that λ_g is contained in Γ_H and not in Γ_G. Its only after acting on λ_g by $t_g \cdot i$ that we obtain a subset of Γ_G. Note that if λ is bi-infinite, $B(\lambda, \sigma)$ can be assumed to be independent of the quasi-isometric section σ, as for any $h \in H$ and $g \in \sigma$, $t_g \cdot t_h \cdot i \cdot t_{h^{-1}} \cdot \phi_g(\lambda) = t_g \cdot i \cdot \phi_g(\lambda)$, and $t_{gh} \cdot i \cdot \phi_{gh}(\lambda)$ and $t_g \cdot t_h \cdot i \cdot t_{h^{-1}} \cdot \phi_g(\lambda)$ lie in a δ-neighborhood of each other.

There is a natural identification of Γ_H with $t_g(i(\Gamma_H))$ taking h to $t_g(i(h))$. On Γ_H define a map $\pi_{g,\lambda} : \Gamma_H \to \lambda_g$ taking h to one of the points on λ_g closest to h in the metric d_H. Strictly speaking, $\pi_{g,\lambda}$ is defined only on the vertex set, but this is enough for our purposes. Now define

$$\Pi_\lambda^\sigma \cdot t_g \cdot i(h) = t_g \cdot i \cdot \pi_{g,\lambda}(h) \text{ for } g \in \sigma(Q).$$

For every $g' \in \Gamma_G$, there exists a unique $g \in \sigma(Q)$ such that $g' \in t_g(i(h))$ as σ is a single-valued section. Hence, Π_λ^σ is well-defined on the entire vertex set of Γ_G.

A crucial theorem of [12] states the following:

Theorem 4.6 *There exists a constant C independent of λ such that for $x, y \in \Gamma_G$,*

$$d_G(\Pi_\lambda^\sigma(x), \Pi_\lambda^\sigma(y)) \leq C d_G(x, y).$$

In [12] the above theorem was stated and proven only for finite geodesic segments, but the proof goes through for semi-infinite or bi-infinite geodesics as well.

If σ is a quasi-isometric section, then from Theorem 4.6, $\Pi_\lambda^\sigma \cdot \sigma$ is also a quasi-isometric section. Moreover, $\Pi_\lambda^\sigma \cdot \sigma(Q) \subset B(\lambda, \sigma)$.

The following Lemma is an easy consequence of Lemma 4.1 of [12].

Lemma 4.7 *For all $K \geq 1$ and $\epsilon \geq 0$ there exists $A \geq 1$ such that if σ is a (K, ϵ)-quasi-isometric section then for all $p, q \in \sigma(Q)$ and $x \in t_p \cdot i(\lambda_p)$ there exists $y \in t_q \cdot i(\lambda_q)$ such that $d_G(x, y) \leq A d_Q(Px, Py) = A d_Q(Pp, Pq)$.*

Suppose now that λ is a bi-infinite geodesic in Γ_H joining $u, v \in \partial\Gamma_H$. Let $a \in i(\lambda)$ and let $\sigma(Q) \subset B(\lambda, \sigma)$ be a (K, ϵ)-quasi-isometric section through

$i(a)$. Let λ^- and λ^+ denote the semi-infinite geodesics contained in λ starting at a and converging to u, v respectively. Then

$$\begin{aligned} B(\lambda,\sigma) &= B(\lambda^-,\sigma) \cup B(\lambda^+,\sigma) \\ \text{and} \qquad \sigma(Q) &= B(\lambda^-,\sigma) \cap B(\lambda^+,\sigma) \end{aligned}$$

Definition : *Let X, Y, Z be geodesically complete metric spaces such that $X \subset Y \subset Z$. X is said to coarsely separate Y into Y_1 and Y_2 if*
(1) $Y_1 \cup Y_2 = Y$
(2) $Y_1 \cap Y_2 = X$
(3) For all $M \geq 0$, there exist $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $d(y_1, Y_2) \geq M$ and $d(y_2, Y_1) \geq M$
(4) There exists $C \geq 0$ such that for all $y_1 \in Y_1$ and $y_2 \in Y_2$ any geodesic in Z joining y_1, y_2 passes through a C-neighborhood of X.

We will show that σ coarsely separates $B(\lambda,\sigma)$ into $B(\lambda^-,\sigma)$ and $B(\lambda^+,\sigma)$. The first three conditions follows easily from the construction. The following Lemma verifies condition (4) above.

Lemma 4.8 *For all $K \geq 1$ and $\epsilon \geq 0$ there exists $M \geq 0$ such that the following holds:*

If λ is a bi-infinite geodesic in Γ_H, a is a vertex on λ splitting λ into semi-infinite geodesics λ^- and λ^+ and $\sigma(Q) \subset B(\lambda,\sigma)$ is a (K,ϵ)-quasi-isometric section through $i(a)$ then any geodesic joining a point in $B(\lambda^-,\sigma)$ to a point in $B(\lambda^+,\sigma)$ passes through an M-neighborhood of $\sigma(Q)$.

Proof: Let $[p,q]$ be a geodesic in Γ_G joining $p \in B(\lambda^-,\sigma)$ to $q \in B(\lambda^+,\sigma)$. Let $\mu = \Pi^\sigma_\lambda[p,q]$. Then μ is a C-quasigeodesic contained in $B(\lambda,\sigma)$. Let $p = p_0, p_1, \cdots, p_n = q$ be the sequence of vertices (in order) on μ.

There exists j such that $p_j \in B(\lambda^-,\sigma)$ and $p_{j+1} \in B(\lambda^+,\sigma)$. Then $d_G(p_j, p_{j+1}) \leq C$. By Lemma 4.7 there exists $q_j \in B(\lambda^+,\sigma)$ in the same coset as p_j such that $d_G(q_j, p_{j+1}) \leq AC$ where A depends only on K, ϵ. Hence, $d_G(q_j, p_j) \leq (A+1)C$.

Let $x_j \in \sigma(Q)$ be in the same coset as p_j. Let $[p_j, q_j]$ be the subsegment of $t_{x_j} \cdot i(\lambda_{x_j})$ joining p_j and q_j. Since Γ_H is properly embedded in Γ_G there exists a constant M depending only on A, C (and hence on K, ϵ alone) such that the length of the edge-path $[p_j, q_j]$ is less than or equal to M. Since x_j lies on this edge-path, $d_G(p_j, x_j) \leq M$ and the Lemma is proven. $\Box$

Let λ be a leaf of Λ' joining $u, v \in \partial\Gamma_H$. Let p_n, q_n be vertices on λ such that $p_n \to u$ and $q_n \to v$. Recall that $B(\lambda, \sigma)$ can be assumed to be independent of the quasi-isometric section. Let σ be a (K, ϵ)-quasi-isometric section contained in $B(\lambda, \sigma)$. Then, by Lemma 4.8 there exist $k_n \in \Gamma_Q$ and $M \geq 0$ such that any geodesic in Γ_G joining $i(p_n)$, $i(q_n)$ passes through an M-neighborhood of $\sigma(k_n)$. Since $\hat{i}(u) = \hat{i}(v)$, and $\hat{i}$ is continuous (Theorem 3.4), the sequences $\{i(p_n)\}, \{i(q_n)\}$ and $\{\sigma(k_n)\}$ converge to the same point in $\partial\Gamma_G$. Since σ is a quasi-isometric section, it extends to an embedding of $\partial\Gamma_Q$ into $\partial\Gamma_G$. Hence there exists $z \in \partial\Gamma_Q$ such that $\{k_n\} \to z$. Also $\sigma([1, z))$ is a quasi-geodesic contained in $B(\lambda, \sigma)$.

If σ' is another quasi-isometric section contained in $B(\lambda, \sigma)$ then the above argument shows that there exists a sequence ${k_n}'$ such that $\{i(p_n)\}, \{i(q_n)\}$ and $\{\sigma'({k_n}')\}$ converge to the same point in $\partial\Gamma_G$ and ${k_n}' \to z'$ for some $z' \in \partial\Gamma_Q$. So $\sigma([1, z))$ and $\sigma'([1, z'))$ are quasi-geodesics converging to the same point in $\partial\Gamma_G$. Moreover, by Proposition 2.3, they are asymptotic.

Since σ and σ' are quasi-isometric sections $z = z'$.

The above discussion proves the following.

Lemma 4.9 *Given $K \geq 1, \epsilon \geq 0$ there exists α such that if λ is a leaf of Λ' and σ a quasi-isometric section then there exists $z \in \partial\Gamma_Q$ satisfying the following:*
If σ_1 and σ_2 are (K, ϵ)-quasi-isometric sections contained in $B(\lambda, \sigma)$ then there exists $z_n \in [1, z)$ such that

$$max \, \{d_G(\sigma_1([z_n, z)), \sigma_2([1, z))), d_G(\sigma_2([z_n, z)), \sigma_1([1, z)))\} \leq \alpha$$

For any ray $[1, z)$ in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$, let z_n be the point on $[1, z)$ such that $d_Q(1, z_n) = n$.

Combining Lemma 4.9 above with Lemma 4.7 we obtain the following Corollary.

Corollary 4.10 *Given $K \geq 1, \epsilon \geq 0$ there exists α such that if λ is a leaf of Λ' then there exists $z \in \partial\Gamma_Q$ satisfying the following:*
If σ and σ' are (K, ϵ)-quasi-isometric sections such that $B(\lambda, \sigma) = B(\lambda, \sigma')$ and σ, σ' are contained in $B(\lambda, \sigma)$ then there exists $N \geq 0$ such that for all $n \geq N$

$$d_G(\sigma(z_n), \sigma'(z_n)) \leq \alpha$$

We are now in a position to prove the main Theorem of this section.

Theorem 4.11 $\Lambda' = \bigcup_{z \in \partial\Gamma_Q} \Lambda_z$

Proof: Lemma 3.5 shows that $\bigcup_{z\in\partial\Gamma_Q}\Lambda_z \subset \Lambda'$. Hence it is enough to show that if λ is a leaf of Λ' then λ is a leaf of Λ_z for some $z \in \partial\Gamma_Q$. Since Λ' is invariant under H, λ can be assumed to be a leaf of Λ' passing through $1 \in \Gamma_H$.

Let σ_0 be a (κ, ϵ_0)-quasi-isometric section obtained from Lemma 3.2. Also suppose $\sigma_0(1) = 1$. Let $z \in \partial\Gamma_Q$ be as in Corollary 4.10 above and let $A_\lambda = B(\lambda, \sigma) \cap P^{-1}([1, z))$. ($A_\lambda$ can be thought of as a 'quasi-horodisk'.)

From Lemma 4.5, there exists $C' \geq \kappa + \epsilon_0$ such that for any geodesic subsegment $[p, q]$ of λ_g (for some $g \in \sigma_0([a, z))$) there exists an extension $[r, q] = \mu$ of $[p, q]$ in λ_g with $d_H(p, r)$ equal to 0 or 1 and a $(C', 0)$-quasi-isometric section $\sigma{:}[1, z) \to \Gamma_G$ such that $gr \in \sigma([1, z))$ and $\mu_{r^{-1}g^{-1}g'}$ is a free homotopy representative for all $g' \neq gr$ in $\sigma([1, z))$.

If σ is a $(C', 0)$-quasi-isometric section of $[1, z)$ into Γ_G then from Theorem 4.6, $\Pi^\sigma_\lambda{\cdot}\sigma$ is a $(CC', 0)$-quasi-isometric section of $[1, z)$ into A_λ. Let $C_2 = CC'$.

For σ_0 the above-mentioned (κ, ϵ_0)-quasi-isometric section let σ_e be the restriction of $\Pi^{\sigma_0}_\lambda \cdot \sigma_0$ to $[1, z)$. Let $g_n = \sigma_e(z_n)$ (where z_n is the point on $[1, z)$ such that $d_Q(1, z_n) = n$). Then σ_e is a $(C_2, 0)$-quasi-isometric section of $[1, z)$ into A_λ.

From Corollary 4.10 above there exists α depending on C_2 such that for σ' any $(C_2, 0)$-quasi-isometric section of $[1, z)$ contained in A_λ there exists $N \geq 0$ such that for all $n \geq N$

$$d_G(g_n, \sigma'(z_n)) \leq \alpha$$

From Proposition 2.4 there exists $b > 1$, $A > 0$ and $\eta > 0$ depending on α such that if $\sigma'([1, z))$ is (the image of) a $(C_2, 0)$-quasi-isometric section of $[1, z)$ contained in A_λ with $d_G(\sigma'(z_n), g_n) \geq \eta$ then any path in $i(\Gamma_H)$ joining $\sigma'(1)$ and $\sigma(1) = 1$ has length greater than or equal to Ab^n.

Let λ^+ and λ^- denote the closures of the two components of $\lambda \backslash \{1\}$. Recall that the generating set of G intersects H in the generating set for H. Hence for all $n > 0$,there exist $p_n \in t_{g_n} \cdot i(\lambda^-_{g_n})$ and $q_n \in t_{g_n} \cdot i(\lambda^+_{g_n})$ such that

$$\begin{aligned} d_G(p_n, g_n) &= \eta + 1 \\ d_G(q_n, g_n) &= \eta \end{aligned}$$

From Lemma 4.5, there exists $r_n \in t_{g_n} \cdot i(\lambda^-_{g_n})$ with $d_G(r_n, p_n)$ equal to 0 or 1, such that there exists a $(C', 0)$-quasi-isometric section σ_n of $[1, z)$ into Γ_G satisfying the following two conditions:

(1) $r_n = \sigma_n(z_n)$

(2) If $\mu^{(n)}$ is the subsegment of λ_{g_n} joining $(t_{g_n} \cdot i)^{-1}(r_n)$ and $(t_{g_n} \cdot i)^{-1}(q_n)$ then $\mu^{(n)}_{r_n^{-1} g_n}$ is a free homotopy representative.

${\sigma_n}' = \Pi^{\sigma_e}_\lambda \cdot \sigma_n$ is a $(C_2, 0)$-quasi-isometric section. And $d_G({\sigma_n}'(z_n), \sigma_e(z_n)) = d_G(r_n, g_n) \geq \eta + 1 - 1 = \eta$.

Hence from, Proposition 2.4 the length of the subsegment of $i(\lambda)$ joining $\sigma'_n(1)$ and $\sigma_e(1) = 1$ is greater than or equal to Ab^n.

Let $\tau_n = t_{q_n r_n^{-1}} \cdot \sigma_n$ - a $(C', 0)$-quasi-isometric section with $\tau_n(z_n) = q_n$. As in the case of ${\sigma_n}'$, ${\tau_n}' = \Pi^{\sigma_e}_\lambda \cdot \tau_n$ is a $(C_2, 0)$-quasi-isometric section such that $d_G({\tau_n}'(z_n), \sigma_e(z_n)) = d_G(q_n, g_n) \geq \eta$.

Hence from, Proposition 2.4 (as before) the length of the subsegment of $i(\lambda)$ joining $\tau'_n(1)$ and $\sigma_e(1) = 1$ is greater than or equal to Ab^n.

Note that $\sigma_n(1)$, ${\sigma_n}'(1)$, $\tau_n(1)$, ${\tau_n}'(1)$ all lie in $i(\Gamma_H)$ for all n. Let $[{\sigma_n}'(1), {\tau_n}'(1)]$ denote the subsegment of λ joining $i^{-1} \cdot {\sigma_n}'(1)$ and $i^{-1} \cdot {\tau_n}'(1)$. Then the sequence $\{[{\sigma_n}'(1), {\tau_n}'(1)]\}$ converges to λ in the Chabauty topology on Γ_H.

Since $d_G(r_n, q_n) \leq 2\eta + 1$, there exists $\rho > 0$ such that $r_n^{-1} q_n$ is an element of H with $|r_n^{-1} q_n| \leq \rho$. Since there are only finitely many of these, pass to a subsequence n_j such that $r_{n_j}^{-1} q_{n_j} = h$ where h is some element of H.

Since the sequence $\{[{\sigma_n}'(1), {\tau_n}'(1)]\}$ converges to λ in the Chabauty topology on Γ_H, so does the subsequence $\{[{\sigma_{n_j}}'(1), {\tau_{n_j}}'(1)]\}$.

Let $[\sigma_n(1), {\sigma_n}'(1)]$ denote a geodesic segment in Γ_H joining $i^{-1} \cdot \sigma_n(1)$ and $i^{-1} \cdot {\sigma_n}'(1)$ and let $[{\tau_n}'(1), \tau_n(1)]$ denote a geodesic segment in Γ_H joining $i^{-1} \cdot {\tau_n}'(1)$ and $i^{-1} \cdot \tau_n(1)$.

Let $\pi_\lambda = i^{-1} \cdot \Pi^{\sigma_e}_\lambda \cdot i$. Since $\pi_\lambda \cdot i^{-1} \sigma_n(1) = i^{-1} {\sigma_n}'(1)$ and $\pi_\lambda \cdot i^{-1} \tau_n(1) = i^{-1} {\tau_n}'(1)$ therefore by Proposition 2.2 and using the fact that quasi-geodesics lie in a bounded neighborhood of geodesics there exists $B \geq 0$ such that $[\sigma_n(1), {\sigma_n}'(1)] \cup [{\sigma_n}'(1), {\tau_n}'(1)] \cup [{\tau_n}'(1), \tau_n(1)]$ is a quasi-geodesic lying in a B-

neighborhood of the geodesic $[\sigma_n(1), \tau_n(1)]$ in Γ_H joining $i^{-1}\cdot\sigma_n(1)$ and $i^{-1}\cdot\tau_n(1)$.

The sequence $\{[\sigma_{n_j}'(1), \tau_{n_j}'(1)]\}$ converges to λ in the Chabauty topology. Therefore the sequence $\{[\sigma_{n_j}(1), \sigma_{n_j}'(1)]\cup[\sigma_{n_j}'(1), \tau_{n_j}'(1)]\cup[\tau_{n_j}'(1), \tau_{n_j}(1)]\}$ also converges to λ in the Chabauty topology. In particular the sequences $\{i^{-1}\cdot\sigma_{n_j}(1)\}$ and $\{i^{-1}\cdot\tau_{n_j}(1)\}$ converge to the end-points of λ in $\partial\Gamma_H$. Since $[\sigma_n(1), \tau_n(1)] = \mu^{(n)}_{r_n^{-1}g_n}$ is a free homotopy representative for all $n > 0$, λ is a leaf of Λ_z^h and hence of Λ_z. This proves the theorem. $\Box$

Remark: Note that the proof of Theorem 4.11 above shows that in the definition of Λ_z it is enough to consider only finitely many elements h of H. That is $\Lambda_z = \bigcup_{h\in H}\Lambda_z^h$ is actually equal to $\bigcup_{h\in B_{(2\eta+1)}}\Lambda_z^h$ where $B_{(2\eta+1)}$ is the ball of radius $(2\eta + 1)$ around the identity in Γ_H. Since η can be seen to be dependent on δ alone (by unravelling definitions of the constants involved) η works equally well for all $z \in \partial\Gamma_Q$. Since Λ_z^h is closed for all h, Λ_z is also closed, being a finite union of closed sets.

5 Properties of Ending Laminations

The purpose of this section is to investigate some basic properties of the ending laminations Λ_z. If Λ_s and Λ_u are stable and unstable laminations of a pseudo-anosov diffeomorphism acting on a closed hyperbolic surface, then leaves of Λ_s and Λ_u intersect (if they do) at an angle bounded away from 0 or π (See [8] pg. 71). The following proposition generalizes this fact to the case of normal hyperbolic subgroups of hyperbolic groups.

Proposition 5.1 *Given $D \geq 0$ and a pair of distinct points p, q in $\partial\Gamma_Q$ there exists $L > 0$ such that if l_p and l_q are leaves of Λ_p and Λ_q then $l_q\cap N_D(l_p)$ has diameter less than L, where $N_D(l_p)$ is the D-neighborhood of l_p in Γ_H.*

Proof: Suppose not. Then for some $D \geq 0$ and $p \neq q$ in $\partial\Gamma_Q$ and all positive integers n, there exist leaves l_p^n and l_q^n such that the set $l_q^n\cap N_D(l_p^n)$ has diameter greater than n.

Let u_n, v_n be points in $l_q^n\cap N_D(l_p^n)$ such that $d_H(u_n, v_n) \geq n$. Since both Λ_p and Λ_q are H-invariant, assume without loss of generality that the geodesic segment $[u_n, v_n]\subset l_q^n$ has the identity 1 as its mid-point (or a vertex closest to its mid-point). After passing to a subsequence if necessary, l_p^n (resp. l_q^n)

converges to a leaf l_p^0 (resp. l_q^0) of Λ_p (resp. Λ_q), since Λ_p and Λ_q are closed, by the Remark following Theorem 4.11.

After passing to a further subsequence if necessary, the segments $[u_n, v_n]$ converge to l_q^0 in the Chabauty topology.

Since $[u_n, v_n] \subset (N_D l_p^n)$, $l_q^0 \subset N_D(l_p^0)$. Hence l_p^0 and l_q^0 are geodesics with the same end-points in $\partial\Gamma_Q$. Let u, v be these end-points.

Let σ be a quasi-isometric section through the identity contained in $B(l_p^0, \sigma)$. σ extends to an embedding $\partial\sigma$ of Γ_Q into Γ_G. Then from the proof of Lemma 4.9, $\hat{i}(u) = \hat{i}(v) = \partial\sigma(p)$.

Since l_q^0 too has end-points u, v, $\hat{i}(u) = \hat{i}(v) = \partial\sigma(q)$. Hence $\partial\sigma(p) = \partial\sigma(q)$. Since $\partial\sigma$ is an embedding, this implies that $p = q$, contradicting the hypothesis and proving the Proposition. $\Box$

Since Λ_z depends only on $z \in \partial\Gamma_Q$, there is a well-defined map from $\partial\Gamma_Q$ to $C(\partial^2\Gamma_H)$ - the set of closed subsets of $\partial^2\Gamma_H$ equipped with the Chabauty topology. Let F denote this map. Then $F(z) = \Lambda_z$. Proposition 5.1 above shows in particular that F is injective. It would be interesting to know if F is continuous. Proposition 5.3 below provides some weak positive evidence.

The following Lemma will be required for the proof:

Lemma 5.2 *Let l_i be a sequence of bi-infinite geodesics converging (in the Chabauty topology) to a bi-infinite geodesic l. If $\sigma(Q)$ is a (K, ϵ)-quasi-isometric section contained in $B(l, \sigma)$ and $\sigma_i = \Pi_{l_i}^{\sigma} \cdot \sigma$, then for any $x \in \partial\Gamma_Q$, $\partial\sigma_i(x) \to \partial\sigma(x)$ in $\partial\Gamma_G$, where $\partial\sigma_i$ (resp. $\partial\sigma$) denotes the continuous extension of σ_i (resp. σ) to $\partial\Gamma_Q$.*

Proof: Assume first that for any two points in $\partial\Gamma_H$ there exists a unique bi-infinite geodesic joining them in Γ_H. Then $l_i \to l$ implies that $\phi_g(l_i) \to \phi_g(l)$ for any $g \in G$. Recall that $t_g \cdot i \cdot \phi_g(l_i)$ and $t_{gh} \cdot i \cdot \phi_{gh}(l_i)$ are asymptotic in both directions for $h \in H$. Using the assumption that there exists a unique geodesic joining a pair of points in $\partial\Gamma_H$, $t_g \cdot i \cdot \phi_g(l_i) = t_{gh} \cdot i \cdot \phi_{gh}(l_i)$. Hence $B(l_i, \sigma_i) = B(l_i, \sigma)$ converges to $B(l, \sigma)$ in the Chabauty topology on Γ_G.

Let $[1, x)$ denote a geodesic ray in Γ_Q starting at 1 and converging to $x \in \partial\Gamma_Q$. Then for all $u \in [1, x)$, there exists $N \geq 0$ such that $\sigma[1, u] \subset B(l_i, \sigma_i)$ for all $i \geq N$. Hence $\sigma[1, u] = \sigma_i[1, u]$ for all $i \geq N$.

Since σ_i's are $(CK, C\epsilon)$-quasi-isometric sections (from Theorem 4.6), $\sigma_i([1, x))$ converges to $\sigma([1, x))$ in the Chabauty topology on Γ_G. Hence $\partial\sigma_i(x) \to \partial\sigma(x)$ in $\partial\Gamma_G$.

In general, if Γ_H is δ-hyperbolic, any geodesic bigon is δ-thin. Then the above argument shows that there exists $c = c(\delta)$, such that for all $u \in [1, x)$ the following holds :
There exists $N \geq 0$ such that $\sigma[1, u]$ lies in a $c(\delta)$-neighborhood of $B(l_i, \sigma_i)$ for all $i \geq N$. Therefore $\partial\sigma_i(x) \to \partial\sigma(x)$. $\Box$

Proposition 5.3 *If $z_i \to z$ in $\partial\Gamma_Q$ and l_i's are leaves of Λ_{z_i} converging to a bi-infinite geodesic leaf l then l is a leaf of Λ_z.*

Proof: Let u_i, v_i be the end-points of l_i in $\partial\Gamma_H$ and let u, v be the end-points of l in $\partial\Gamma_H$. Then $\hat{i}(u_i) = \hat{i}(v_i)$ for all i. Hence by Theorem 3.4 $\hat{i}(u) = \hat{i}(v)$. Moreover, by Theorem 4.11 l is a leaf of Λ_{z_0} for some $z_0 \in \partial\Gamma_Q$. It is enough to show that $z = z_0$.

Since Λ_{z_i}'s and Λ_z are H-invariant, assume (after passing to a subsequence if necessary) that l_i's and l pass through the identity. Let $\sigma(Q)$ be a (K, ϵ)-quasi-isometric section through 1 contained in $B(l, \sigma)$. Also, let $\sigma_i = \Pi^{\sigma}_{l_i} \cdot \sigma$. Then σ and σ_i's are $(CK, C\epsilon)$-quasi-isometric sections (Theorem 4.6) and extend continuously to embeddings - $\partial\sigma$ and $\partial\sigma_i$ respectively - of $\partial\Gamma_Q$ to Γ_G. As in the proof of Lemma 4.9, $\hat{i}(u_i) = \hat{i}(v_i) = \partial\sigma_i(z_i)$ and $\hat{i}(u) = \hat{i}(v) = \partial\sigma(z_0)$. By Theorem 3.4, $\partial\sigma_i(z_i) \to \partial\sigma(z_0)$. From Lemma 5.2, $\partial\sigma_i(x) \to \partial\sigma(x)$ for any $x \in \partial\Gamma_Q$. Since in addition σ_i's are all $(CK, C\epsilon)$-quasi-isometric embeddings, $\partial\sigma_i(z_i) \to \partial\sigma(z)$. Hence $\partial\sigma(z) = \partial\sigma(z_0)$. Since $\partial\sigma$ is an embedding, $z = z_0$ and l is a leaf of Λ_z. $\Box$

Concluding Remarks : Laminations Λ_z constructed in this paper are parametrized by $z \in \partial\Gamma_Q$. $\partial\Gamma_Q$ can also be used to parametrize limiting actions of H on $\mathbb{R}$-trees [15]. Every direction in Q gives rise to a sequence of automorphisms of H. As in [15] this gives a sequence of actions of H on δ_i-hyperbolic metric spaces, where $\delta_i \to 0$ as $i \to \infty$. From this, one can extract a subsequence [15] converging (in the sense of Gromov) to a small isometric action on a 0-hyperbolic metric space, i.e. an $\mathbb{R}$-tree. The laminations of this paper and small actions on $\mathbb{R}$-trees can be regarded as dual objects. In fact, the results of Section 4 provide convergent sequences of H-actions. Such convergent sequences follow from work of Bestvina, Feighn and Handel [3] when H is free, from work of Thurston [19] when H is a surface group and from the JSJ splitting of Rips and Sela [16], [17] when H is freely indecomposable.

Acknowledgments: The author would like to thank his advisor Andrew Casson, Sergio Fenley, Curt McMullen and the referee for helpful comments.

References

[1] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short. Notes on word hyperbolic groups. *Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky eds.)*, pages 3–63, 1991.

[2] M. Bestvina and M. Feighn. A Combination theorem for Negatively Curved Groups. *J. Diff. Geom., vol 35*, pages 85–101, 1992.

[3] M. Bestvina, M. Feighn, and M. Handel. Laminations, trees and irreducible automorphisms of free groups. *preprint*.

[4] R. D. Canary, D. B. A. Epstein, and P. Green. Notes on Notes of Thurston. *in Analytical and Geometric Aspects of Hyperbolic Spaces, Cambridge University Press*, pages 3–92, 1987.

[5] J. Cannon and W. P. Thurston. Group Invariant Peano Curves. *preprint*.

[6] M. Coornaert, T. Delzant, and A.Papadopoulos. Geometrie et theorie des groupes. *Lecture Notes in Math.,vol.1441,Springer Verlag*, 1990.

[7] B. Farb and R. E. Schwartz. Quasi-isometric Rigidity of Hilbert Modular Groups. *preprint*.

[8] A. Fathi, M. Laudenbach, and V. Poenaru. Travaux de Thurston sur les surfaces. *Asterisque 66-67*, pages 1–284, 1979.

[9] E. Ghys and P. de la Harpe(eds.). Sur les groupes hyperboliques d'apres Mikhael Gromov. *Progress in Math. vol 83, Birkhauser, Boston Ma.*, 1990.

[10] M. Gromov. Hyperbolic Groups. *in Essays in Group Theory, ed. Gersten, MSRI Publ.,vol.8, Springer Verlag,1985*, pages 75–263.

[11] C. Mcmullen. Renormalization and 3-Manifolds which Fiber over the Circle. *preprint*.

[12] M. Mitra. Cannon-Thurston Maps for Hyperbolic Group Extensions. *preprint.*

[13] L. Mosher. A hyperbolic-by-hyperbolic hyperbolic group. *preprint.*

[14] L. Mosher. Hyperbolic Extensions of Groups. *preprint.*

[15] F. Paulin. Outer Automorphisms of Hyperbolic Groups and Small Actions on R-trees. *in Arboreal Group Theory (ed.R. C. Alperin),MSRI Publ., vol. 19, Springer Verlag,1991*, pages 331–344.

[16] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. *GAFA v.4 no.3*, pages 337–371, 1994.

[17] Z. Sela. Structure and rigidity in (gromov) hyperbolic groups and discrete subgroups in rank 1 lie groups. *preprint.*

[18] W. P. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes.*

[19] W. P. Thurston. On the Geometry and Dynamics of Diffeomorphisms of Surfaces. *Bull. A.M.S. vol. 19*, pages 417–431, 1987.