THE FUJIKI CLASS AND POSITIVE DEGREE MAPS

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Abstract. We show that the collection of Fujiki-class groups coincides with the class of Kähler groups. We further show that a map between complex-analytic manifolds, at least one of which is in the Fujiki class, is a biholomorphism under a natural condition on the second cohomologies. We use this to establish that, with mild restrictions, a certain relation of "domination" introduced by Gromov is in fact a partial order.

1. Introduction

This article is motivated by the following loosely-worded question: Let X and Y be two compact complex manifolds and let $f : Y \longrightarrow X$ be a surjective holomorphic map. If one of these manifolds is Kähler, then how close is the other to being a Kähler manifold ? (In this article, it is implicitly assumed that a manifold is connected.)

It turns out that if Y is a Kähler manifold, then quite a lot is already known about this question. It follows from [F, Lemma 4.6] by Fujiki that X belongs to the Fujiki class $\mathcal{C}$. Any non-projective Moishezon manifold is in the class $\mathcal{C}$ but is non-Kähler. Thus, without any additional conditions, X is *not* necessarily Kähler. Varouchas showed [Va1] that, with Y, X and f as in the question above and Y Kähler, if every fiber of f has the same dimension, then X is Kähler. In a related work, Varouchas [Va2] showed that that a compact complex manifold belongs to $\mathcal{C}$ if and only if it is *bimeromorphic* to a compact Kähler manifold. Our first theorem asserts that the fundamental group of a manifold in $\mathcal{C}$ is a Kähler group. This might be thought of as a Kähler analogue of the assertion that the fundamental group of a Moishezon manifold is projective.

Theorem 1.1. *A finitely-presented group Γ is a Fujiki group if and only if it is a Kähler group.*

We recall that a finitely presented group Γ is called a *Kähler group* if Γ is the fundamental group of a compact connected Kähler manifold. Likewise, we will call a finitely presented group Γ a *Fujiki group* it is the fundamental group of a compact connected complex manifold in the Fujiki class $\mathcal{C}$.

If, in the motivating question above, X is Kähler, then the question of whether Y is Kähler is much trickier. In contrast to the result of Varouchas, Y need not be Kähler even if $f : Y \longrightarrow X$ is a submersion. The simplest example of this is to take X to be Kähler, $Y = X \times M$, where M is a non-Kähler complex manifold, and let f be the projection onto X. A subtler example— where Y this time is not a product manifold—is the Hopf surface $Y = (\mathbb{C}^2 \setminus \{0\})/2\mathbb{Z}$, which is non-Kähler, but admits a submersion onto $X = \mathbb{P}^1$. However, we have the following positive result.

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Theorem 1.2. *Let X and Y be compact connected complex manifolds satisfying*

$$\dim X = \dim Y \quad and \quad \dim H^2(X, \mathbb{Q}) = \dim H^2(Y, \mathbb{Q}).$$

Let $\varphi : Y \longrightarrow X$ be a surjective holomorphic map of degree one. If at least one of X and Y is in the Fujiki class $\mathcal{C}$, then φ is a biholomorphism.

Theorem 1.2 is a variation of a result of [BB] which is recalled in Section 3.

We shall consider an application of Theorem 1.2 in Section 4 by establishing that a relation—defined on the set of all closed smooth manifolds of a fixed dimension—introduced by Gromov is in fact a partial order when restricted to certain subcollections. We denote this relation by $\geq$, and refer the reader to Section 4 for its definition. In the notation of Section 4, we have the following consequence of Theorem 1.2, and a key step to the main result of Section 4:

Proposition 1.3. *Let X and Y be compact connected complex manifolds with $\dim X = \dim Y$ such that at least one of X, Y belongs to the Fujiki class $\mathcal{C}$. Further, suppose that $X \geq Y$ and $Y \geq X$. If X, Y are not biholomorphic, then X admits a self-endomorphism of degree greater than one.*

2. Fundamental groups of Fujiki-class manifolds

We need the following lemma before we can provide a proof of Theorem 1.1.

Lemma 2.1. *Let X be a compact connected complex manifold in the Fujiki class $\mathcal{C}$. Then there exists a pair (Y, f), where Y is a compact connected Kähler manifold with $\dim Y = \dim X$, and*

$$f : Y \longrightarrow X$$

is a surjective holomorphic map of degree one.

Proof. Let d be the complex dimension of X. A theorem of Varouchas in [Va2] says that X is bimeromorphic to a compact connected Kähler manifold of complex dimension d — see [Ba, p. 31, Theorem 10] for a short proof). Let

$$\phi : Z \dashrightarrow X$$

denote such a bimeromorphic map from a compact Kähler manifold Z of dimension d.

The *elimination of indeterminacies* says that there is a finite sequence of holomorphic maps

$$Z_n \xrightarrow{f_n} Z_{n-1} \xrightarrow{f_{n-1}} Z_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} Z_1 \xrightarrow{f_1} Z_0 = Z$$

such that each (Z_i, f_i), $1 \leq i \leq n$, is a blow-up of a smooth complex submanifold of Z_{i-1}, and the bimeromorphic map

$$\phi \circ f_1 \circ \cdots \circ f_n : Z_n \dashrightarrow X$$

extends to a holomorphic map

$$\widetilde{\phi} : Z_n \longrightarrow X;$$

see [Hi2] and [Hi1] by Hironaka. We refer the reader to [AKMW, p. 539, § 1.2.4] for some explanation of how the above process can be carried out — ensuring, especially, that each successive blow-up is along a smooth center — using [Hi2]. (This process works in the analytic case as well as in the algebraic; the case of complex-analytic manifolds is addressed in the last two paragraphs of [AKMW, § 1.2.4].) The blow-up of a smooth submanifold of a Kähler manifold is Kähler [Bl, p. 202, Théorème II.6]. Since Z_0 is Kähler, we conclude that all Z_i are Kähler.

We set $Y := Z_n$ and $f := \widetilde{\phi}$ to obtain the desired pair (Y, f). □

We now have all the tools needed to prove Theorem 1.1.

Proof of Theorem 1.1. A compact Kähler manifold is in the Fujiki class $\mathcal{C}$. Therefore, a Kähler group is a Fujiki group.

To prove the converse, let X be a compact connected complex manifold in Fujiki class $\mathcal{C}$. Take (Y, f) as in Lemma 2.1. Let $D \subset Y$ be the effective divisor over which f fails to be a submersion. More precisely, consider the top exterior product of the differential of the map f

$$\bigwedge^d df \,\in\, H^0\big(Y,\, \mathrm{Hom}(\bigwedge^d TY\,, f^*\bigwedge^d TX)\big)\,,$$

where d is the complex dimension of X. Then D is the divisor for the section $\bigwedge^d df$.

The complement $X \setminus f(D)$ will be denoted by U. Let

$$\iota\,:\, U\,\hookrightarrow\, X$$

be the inclusion map. Take a point $x_0 \in U$. The homomorphism

$$f_*\,:\, \pi_1(Y,\, f^{-1}(x_0))\,\longrightarrow\, \pi_1(X,\, x_0) \tag{2.1}$$

can be shown to be an isomorphism. For this, first note that the image $f(D) \subset X$ is of complex codimension at least two, because the degree of f is one. Therefore, the induced homomorphism

$$\iota_*\,:\, \pi_1(U,\, x_0)\,\longrightarrow\, \pi_1(X,\, x_0)$$

is an isomorphism. The map

$$f\vert_{f^{-1}(U)}\,:\, f^{-1}(U)\,\longrightarrow\, U$$

is an isomorphism, whence the homomorphism $\pi_1(f^{-1}(U)\,, f^{-1}(x_0)) \longrightarrow \pi_1(X\,, x_0)$ is surjective (since ι_* is surjective). But the latter homomorphism coincides with the composition of the homomorphism

$$\pi_1(f^{-1}(U),\, f^{-1}(x_0))\,\longrightarrow\, \pi_1(Y,\, f^{-1}(x_0)) \tag{2.2}$$

induced by the inclusion $f^{-1}(U) \hookrightarrow Y$ with the homomorphism f_* in (2.1). Since the composition is surjective, we conclude that f_* is surjective.

The homomorphism in (2.2) is also surjective. Therefore, the composition

$$\pi_1(X,\, x_0)\,\stackrel{(\iota_*)^{-1}}{\longrightarrow}\, \pi_1(U,\, x_0)\,=\,\pi_1(f^{-1}(U),\, f^{-1}(x_0))\,\longrightarrow\, \pi_1(Y,\, f^{-1}(x_0))$$

is surjective. We will denote this composition by $\widehat{f}$. It is straightforward to check that $\widehat{f}$ is the inverse of the homomorphism f_* in (2.1).

Therefore, $\pi_1(X,\, x_0) = \pi_1(Y,\, f^{-1}(x_0))$. In particular, $\pi_1(X,\, x_0)$ is a Kähler group. □

3. A criterion for biholomorphism: The proof of Theorem 1.2

Let X and Y be compact connected complex manifolds with $\dim X = \dim Y$, and let

$$\varphi : Y \longrightarrow X$$

be a surjective holomorphic map of degree one. In [BB] the following was proved: if the underlying real manifolds for X and Y are diffeomorphic, and also

$$\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y),$$

then φ is a biholomorphism.

Theorem 1.2 is a variation of the above result.

Proof of Theorem 1.2. The pullback homomorphisms of cohomologies

$$\varphi_i^* : H^i(X, \mathbb{Q}) \longrightarrow H^i(Y, \mathbb{Q})$$

are injective for all i. Therefore, from the given condition that $\dim H^2(X, \mathbb{Q}) = \dim H^2(Y, \mathbb{Q})$ it follows that the homomorphism

$$\varphi_2^* : H^2(X, \mathbb{Q}) \longrightarrow H^2(Y, \mathbb{Q}) \tag{3.1}$$

is an isomorphism.

The complex dimension of X will be denoted by d. Let

$$\bigwedge^d d\varphi \in H^0\big(Y, \mathrm{Hom}(\bigwedge^d TY, f^* \bigwedge^d TX)\big)$$

be the exterior product of the differential of φ. The divisor for this homomorphism $\bigwedge^d d\varphi$ will be denoted by D. The map φ is a biholomorphism if D is the zero divisor.

Since the degree of φ is one, the image $\varphi(D)$ is of complex codimension at least two in X. Therefore, if

$$c_D \in H_{2d-2}(Y, \mathbb{Q})$$

is the class of D, then its image $\varphi_*(c_D) \in H_{2d-2}(X, \mathbb{Q})$ is zero. This implies that the Poincaré duality pairing of c_D with $\varphi^*(H^2(X, \mathbb{Q})) \subset H^2(Y, \mathbb{Q})$ vanishes identically. Since φ_2^* in (3.1) is surjective, we now conclude that

$$c_D = 0. \tag{3.2}$$

Let us first assume that Y lies in the Fujiki class $\mathcal{C}$. By Lemma 2.1, there is a compact connected Kähler manifold Z of dimension d and a surjective holomorphic map

$$f : Z \longrightarrow Y$$

of degree one. Consider the effective divisor

$$\widetilde{D} := f^{-1}(D) \subset Z.$$

From (3.2) we know that the class of $\widetilde{D}$ in $H_{2d-2}(Z, \mathbb{Q})$ vanishes. Since Z is Kähler, this implies that $\widetilde{D}$ is the zero divisor. Hence D is the zero divisor. Consequently, φ is a biholomorphism.

Now assume that X lies in the Fujiki class $\mathcal{C}$. Therefore, X is bimeromorphic to a compact connected Kähler manifold Z of dimension d [Va2]; also see [Ba, p. 31, Theorem 10]. Since φ is a bimeromorphic map between X and Y, it follows that Y is bimeromorphic to Z. Hence Y lies in the Fujiki class $\mathcal{C}$. We have already shown that φ is a biholomorphism if Y lies in the Fujiki class $\mathcal{C}$. □

4. An application: Gromov partial order

In a lecture he gave at the Graduate Center CUNY in the spring of 1978, Gromov had introduced a notion of "domination" between smooth manifolds as follows [To, CT]:

> Let X, Y be closed smooth $n-$manifolds. We say that $X \geq Y$ if there is a smooth map of positive degree from X to Y.

Gromov's question was in the context of real hyperbolic manifolds. It is not clear a priori whether "$\geq$" is in fact a partial order or not. We transfer this question to the context of projective and Kähler manifolds and holomorphic maps between them. We rephrase this as follows:

Question 4.1. *Let X, Y be compact projective (respectively, Kähler) manifolds of complex dimension n. We say that $X \geq Y$ if there is a holomorphic map of positive degree from X to Y. We say that $X \geq_1 Y$ if there is a holomorphic map of degree one from X to Y.*
1) If $X \geq Y$ and $Y \geq X$, are X and Y biholomorphic ?
2) If $X \geq_1 Y$ and $Y \geq_1 X$, are X and Y biholomorphic ?

Remark 4.2. Given the conditions on the manifolds X, Y in the discussion above, any positive-degree holomorphic map from one of them to the other is automatically surjective. Thus, whenever we apply Theorem 1.2 to some positive-degree map in the proofs below, we will not remark upon its surjectivity.

As a consequence of Theorem 1.2 we have the following proposition, which is a step towards answering Question 4.1. Proposition 1.3, stated in the introduction forms a part of the following proposition, and Part (1) below provides an answer to Question 4.1(2).

Proposition 4.3. *Let X and Y be compact connected complex manifolds with $\dim X = \dim Y$ such that at least one of X, Y belongs to the Fujiki class $\mathcal{C}$.*

1. *If $X \geq_1 Y$ and $Y \geq_1 X$, then X and Y are biholomorphic.*
2. *Assume that $X \geq Y$ and $Y \geq X$. If X, Y are not biholomorphic, then X admits a self-endomorphism of degree greater than one.*

Proof. We may take X to be in the Fujiki class $\mathcal{C}$. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are degree one maps, then $g \circ f$ is a holomorphic automorphism by Theorem 1.2. Therefore, f is a biholomorphism, which proves Part (1).

We now consider Part (2). Since $X \geq Y$ (respectively, $Y \geq X$), we have $b_2(X) \geq b_2(Y)$ (respectively, $b_2(Y) \geq b_2(X)$) because, by the assumption of positivity of degree of the map from X to Y (respectively, Y to X), the pullback homomorphism of cohomologies is injective. Therefore, $b_2(X) = b_2(Y)$. If X is not biholomorphic to Y, then by Theorem 1.2 we conclude that the degree of any surjective holomorphic map between X and Y is at least two. Now Part (2) now follows by taking composition of two such maps $X \longrightarrow Y$ and $Y \longrightarrow X$. □

The next result summarizes standard facts about non-existence of non-trivial self-endomorphisms (see, for instance, [Fto] and references therein).

Result 4.4. *Let X be a compact connected complex manifold, and let X satisfy one of the following:*

1. *X is of general type.*
2. *X is Kobayashi hyperbolic.*

(3) *X is a rational homogeneous manifold of Picard number 1.*

Then, any self-endomorphism of X of positive degree is an automorphism.

We combine Proposition 4.3 with Result 4.4 to deduce the following:

Theorem 4.5. *Let X and Y be compact connected projective manifolds with* $\dim X = \dim Y$. *Suppose that $X \geq Y$ and $Y \geq X$. Also suppose that at least one of X, Y belongs to one of the three classes listed in Result 4.4. Then X and Y are biholomorphic.*

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