

CONTROLLED FLOYD SEPARATION AND NON RELATIVELY HYPERBOLIC GROUPS

SHUBHABRATA DAS AND MAHAN MJ

ABSTRACT. We introduce the notion of controlled Floyd separation between geodesic rays starting at the identity in a finitely generated group G . Two such geodesic rays are said to be Floyd separated with respect to quasigeodesics if the (Floyd) length of c -quasigeodesics (for fixed but arbitrary c) joining points on the geodesic rays is asymptotically bounded away from zero. This is always satisfied by Morse geodesics. The main purpose of this paper is to furnish an example of a finitely generated group G such that

- 1) all finitely presented subgroups of G are hyperbolic,
- 2) G has an uncountable family of geodesic rays that are Floyd separated with respect to quasigeodesics,
- 3) G is not hyperbolic relative to any collection of proper subgroups.
- 4) G is a direct limit of hyperbolic CAT(0) cubulated groups.
- 5) G has trivial Floyd boundary in the usual sense.

On the way towards constructing G , we construct a malnormal infinitely generated (and hence non-quasiconvex) subgroup of a free group, giving negative evidence towards a question of Swarup and Gitik.

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1. INTRODUCTION

In this paper, we introduce the notion of controlled Floyd separation between geodesic rays starting at the identity of a finitely generated group G . Here, the term ‘control’ refers in general to a choice of a G -invariant subcollection of paths in a Cayley graph Γ of G with respect to which the Floyd distance is computed (See Definition 2.15). We shall primarily be interested in subcollections given by (κ, κ) -quasigeodesics in Γ . The main purpose of this paper is to furnish an example of a finitely generated group G which satisfies the following properties:

- (A) There are uncountably many geodesic rays starting at the identity in G , which are Floyd separated with respect to (κ, κ) -quasigeodesics for any $\kappa > 1$. See Theorem 7.1. These rays actually lie in a subgroup that is quasiconvex in a strong sense: all geodesics in the subgroup are ‘uniformly Morse’ (Corollary 7.3).
- (B) G is not hyperbolic relative to any collection of proper subgroups. See Theorem 4.1.

This provides a construction that may be used as negative evidence towards a question of Olshanskii, Osin and Sapir [OOS09, Problem 7.11].

The group G we construct is the double of a free group along an infinitely generated malnormal subgroup K : $G = F *_K F$. Here $K = \cup_n K_n$ is an ascending union of malnormal quasiconvex subgroups K_n of F . The group G has a number of other features:

- (1) G is a graded small cancellation (and hence lacunary hyperbolic) group in the sense of Olshanskii, Osin and Sapir [OOS09]. See Theorem 5.7.
- (2) G is a direct limit of cubulated hyperbolic groups G_n : each G_n is hyperbolic and admits a geometric action on a non-positively curved square complex (in fact a \mathcal{VH} complex in the sense of Wise [Wis12]). See Proposition 3.10.
- (3) Every finitely presented subgroup of G is hyperbolic (Proposition 3.11). In particular, G does not contain any Baumslag-Solitar group $BS(m, n)$.
- (4) Each K_n is quasiconvex in a strong sense (‘uniformly Morse’) in G (Corollary 7.3).
- (5) The limit set of $K (= \cup_n K_n)$ in F is all of the Gromov boundary of F (Proposition 3.3).
- (6) In spite of the features (A) and (B) above, G has trivial Floyd boundary in the usual sense. See Theorem 8.1.

The group G and its features above illustrate that the source of triviality of the Floyd boundary of a group G is quite subtle and known sources (e.g. existence of $\mathbb{Z} \oplus \mathbb{Z} \subset G$, wideness of G , etc.) are inadequate to detect this.

On the way towards constructing G , we construct (Proposition 3.2) a malnormal infinitely generated (and hence non-quasiconvex) subgroup of a free group, giving negative evidence towards a question of Swarup and Gitik [Bes04, Question 1.8].

A range of tools is used in proving the properties of G :

- (1) The theory of relative hyperbolicity and relative quasiconvexity [Far98, Bow12, Hru10].
- (2) Graphical small cancellation theory [Wis01, Wis02].
- (3) Graded small cancellation theory [OOS09, Ols91].
- (4) Tree graded spaces [DS05].
- (5) Asymptotic cones and lacunary hyperbolic groups [OOS09, Gro93].

1.1. History of the Problem. The notion of a Floyd function and Floyd boundary were introduced by Floyd in [Flo80] and generalized by Gromov in [Gro93] under the name of conformal boundary of a group. Gromov [Gro93, p. 264] and Karlsson [Kar03] showed that the action of a group on its Floyd boundary is a convergence action (in fact a geometric convergence action in the sense of Gerasimov) provided the boundary is non-trivial, i.e. if it contains more than two (and hence infinitely many) points. In [OOS09, Proposition 4.28], Olshanskii, Osin and Sapir showed that if G is a finitely generated group whose Floyd boundary consists of at least 2 points, then all asymptotic cones of G have cut points. These two facts may be regarded as evidence towards a positive answer to the following question:

Question 1.1. [OOS09, Problem 7.11] (*see also [ACT14, p.18].*) *Suppose that a finitely generated group G has a nontrivial Floyd boundary. Is G hyperbolic relative to a collection of proper subgroups?*

Theorem 7.1 shows, in particular, that the following weaker question has a negative answer (see Section 2.2.1 for precise definitions).

Question 1.2. *Suppose that a finitely generated group G has a pair of geodesic rays which are Floyd-separated with respect to quasigeodesics. Is G hyperbolic relative to a collection of proper subgroups?*

A very general and detailed treatment of relative hyperbolicity and its characterization in topological and dynamic terms has been carried out recently by Gerasimov and Potyagailo in a sequence of papers [Ger09, Ger12, GP11, GP13]. The notion of a controlled Floyd separation is related to, but different from the notion of a Karlsson function introduced in [GP11].

1.2. Notation.

- (1) H will be a subgroup of the free group F_2 and h, k will denote elements (typically of F_2).
- (2) $(h, k)_1$ will denote the Gromov inner product of h, k with respect to 1.
- (3) $|h| = d(h, 1)$
- (4) $(h, H)_1 = \max_{k \in H} (h, k)_1$
- (5) $Rose(F_2)$ will denote the rose on two petals labeled by the generators of F_2 .
- (6) K_n will be a free subgroup of F_2 of rank n . $Rose(K_n)$ will denote the rose on n petals labeled by the generators of K_n .

- (7) If H is a subgroup of a (relatively) hyperbolic group G , the limit set of H is denoted Λ_H .
- (8) $T(n)$ denotes the tower function of height n with base 2, i.e. $T(n) = 2^{2^{2^{\dots}}}$, where the power is taken n times.
- (9) $U(n)(a, b)$ is the word (element) in the free group F_2 on generators a, b given by

$$U(n)(a, b) = [a^{(T(n)+1)}b^{(T(n)+1)}][a^{(T(n)+2)}b^{(T(n)+2)}] \dots [a^{T(n+1)}b^{T(n+1)}].$$

2. PRELIMINARIES

We shall be using two equivalent definitions regarding quasigeodesics in this paper, which we elucidate here at the outset.

Definition 2.1. *Let (X, d) be a geodesic metric space.*

- 1) A path $\sigma : I \rightarrow X$ is a (K, ϵ) -quasigeodesic with $K \geq 1, \epsilon \geq 0$ if for all $t_1, t_2 \in I$,

$$\frac{1}{K}|t_1 - t_2| - \epsilon \leq d(\sigma(t_1), \sigma(t_2)) \leq K|t_1 - t_2| + \epsilon.$$

- 2) Alternately [OOS09][Section 2.4], $\sigma : I \rightarrow X$ is a (λ, ϵ) -quasigeodesic with $\lambda \in (0, 1], \epsilon \geq 0$ if for all $t_1, t_2 \in I$,

$$d(\sigma(t_1), \sigma(t_2)) \geq \lambda l(\sigma([t_1, t_2])) - \epsilon,$$

where $l(\sigma([t_1, t_2]))$ denotes the length of the subpath from $\sigma(t_1)$ to $\sigma(t_2)$.

We use the first definition when we want $K \geq 1$ and the second one when we want $\lambda \in (0, 1]$. The second one is used primarily in Section 5.

2.1. Relative Hyperbolicity. We quickly recall the notion of relative hyperbolicity introduced by Gromov [Gro85]. An equivalent notion was introduced by Farb [Far98] and the equivalence of the two proven by Bowditch [Bow12]. Equivalent definitions and closely related work can also be found in the work of a number of authors including [Bow12, Osi06, Dah03, GM08, Hru10].

Let (X, d) be a path metric space. A collection of closed subsets $\mathcal{H} = \{H_\alpha\}$ of X will be said to be **uniformly separated** if there exists $\epsilon > 0$ such that $d(H_1, H_2) \geq \epsilon$ for all distinct $H_1, H_2 \in \mathcal{H}$.

Definition 2.2. [Gro85] *For any geodesic metric space (H, d) , the hyperbolic cone H^h is the metric space $H \times [0, \infty) = H^h$ equipped with the path metric d_h obtained as follows:*

- 1) $d_{h,t}((x, t), (y, t)) = 2^{-t}d_H(x, y)$, where $d_{h,t}$ is the induced path metric on $H \times \{t\}$. Paths joining $(x, t), (y, t)$ and lying on $H \times \{t\}$ are called horizontal paths.
- 2) $d_h((x, t), (x, s)) = |t - s|$ for all $x \in H$ and for all $t, s \in [0, \infty)$, and the corresponding paths are called vertical paths.
- 3) for all $x, y \in H^h$, $d_h(x, y)$ is the path metric induced by the collection of horizontal and vertical paths.

A similar construction of a ‘combinatorial’ horoball was carried out by Groves-Manning in [GM08].

Definition 2.3. *Let Γ be the Cayley graph of a group G and let \mathcal{H}_0 be a finite collection of subgroups of G . let \mathcal{H} be the collection of all left cosets of elements of \mathcal{H}_0 . G is said to be hyperbolic relative to \mathcal{H} in the sense of Gromov, if $\mathcal{G}(\Gamma, \mathcal{H})$, obtained by attaching the*

hyperbolic cones aH^h to $aH \in \mathcal{H}$ by identifying $(z, 0)$ with z for all $aH \in \mathcal{H}$ and $z \in aH$, is a complete hyperbolic metric space. The collection $\{aH^h : H \in \mathcal{H}\}$ is denoted as \mathcal{H}^h . The induced path metric is denoted as d_h .

The boundary $\partial\mathcal{G}(\Gamma, \mathcal{H})$ is called the **Bowditch boundary**.

The following Theorem of Bowditch [Bow12] shall be useful (see also [Mj08]).

Theorem 2.4. [Bow12] *Let G be a hyperbolic group without torsion. Then G is hyperbolic relative to \mathcal{P} if and only if*

- (1) *Each P_i is quasiconvex in G .*
- (2) *for all $P_i, P_j \in \mathcal{P}$ and $g \in G$, $gP_i g^{-1} \cap P_j = \{1\}$ unless $i = j$ and $g \in P_i$.*

In particular if \mathcal{P} has exactly one subgroup P_1 , and G is hyperbolic relative to P_1 , the latter is malnormal quasiconvex in G .

2.1.1. *Relative Quasiconvexity.* In [Hru10] Hruska gives a number of equivalent criteria for relative quasiconvexity for subgroups of relatively hyperbolic groups. We give one of these below to fix notions.

Definition 2.5. [Hru10] *Let G be hyperbolic relative to a collection \mathcal{P} of parabolic subgroups. A subgroup $H \subset G$ is relatively quasiconvex if the following holds. Let \mathcal{S} be some (any) finite relative generating set for (G, \mathcal{P}) , and let \mathcal{E} be the set containing all the elements of $P_i \in \mathcal{P}$ (for all i). Let Γ be the Cayley graph of G with respect to the generating set $\mathcal{S} \cup \mathcal{E}$ with all edges of length one. Let d be some (any) proper, left invariant metric on G . Then there is a constant $D_0 = D_0(\mathcal{S}, d)$ such that for each geodesic \bar{c} in Γ connecting two points of H , every vertex of \bar{c} lies within a d -distance D_0 of H .*

It is shown in [Hru10] that the above definition is independent of the choice of finite relative generating set \mathcal{S} and the choice of proper metric d .

Theorem 2.6. [Hru10][Theorem 1.2] *Let G be a countable group that is relatively hyperbolic with respect to a finite family of subgroups $\mathcal{P} = \{P_1, \dots, P_n\}$.*

- (1) *If $H \subset G$ is relatively quasiconvex, then H is relatively hyperbolic with respect to a natural induced collection of subgroups.*
- (2) *If $H_1, H_2 \subset G$ are relatively quasiconvex, then $H_1 \cap H_2$ is also relatively quasiconvex.*

Theorem 2.7. [Hru10][Theorem 1.5] *Let G be a finitely generated relatively hyperbolic group and let H be a finitely generated subgroup. If H is undistorted in G , then H is relatively quasiconvex.*

Combining the equivalence of characterizations in [Hru10] with a Lemma of Yang we get

Lemma 2.8. [Yan12][Lemma 2.6] *Let H be relatively quasiconvex in a relatively hyperbolic group G such that $|\Lambda(H)| \geq 2$. Then for any subgroup $H \subset J \subset G$ satisfying $\Lambda(H) = \Lambda(J)$, we have that H is of finite index in J . In particular, J is relatively quasiconvex.*

2.2. Floyd Boundary and Controlled Floyd Separation.

Definition 2.9. *A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is a **Floyd scaling function** if*

- (1) $\sum_{n \geq 0} f_n < \infty$,
- (2) *There exists a positive λ such that $1 \geq f_{n+1}/f_n \geq \lambda$ for all $n \in \mathbb{N}$.*

In this paper we shall refer to a Floyd scaling function simply as a **scaling function**.

Let G be a group with symmetric generating set S and $\Gamma = \Gamma(G, S)$ denote the Cayley graph of G with respect to S and let d denote the word-metric.

Definition 2.10. *Let f be a scaling function. Given an edge $[x, y]$ of Γ , define its Floyd length to be $f(d([x, y], 1))$. The resulting **path metric** on Γ is called the **Floyd metric** with respect to the scaling function f and is denoted as d_f .*

*A geodesic in (Γ, d_f) is called a **Floyd geodesic**.*

*The metric completion $\overline{(\Gamma, d_f)}$ is called the **Floyd completion** and $\partial_f \Gamma = \overline{(\Gamma, d_f)} \setminus \Gamma$ is called the **Floyd boundary**. For $f(n) = \lambda^n$ with $\lambda \in (0, 1)$, d_f will also be denoted as d_λ and $\partial_f \Gamma$ will also be denoted as $\partial_\lambda \Gamma$.*

We give a somewhat different but equivalent description of the Floyd boundary that will be better adapted to the applications we have in mind.

Lemma 2.11. *Let f be a (Floyd) scaling function for Γ . Let r be a geodesic ray in Γ starting at $1 \in \Gamma$. Then r is a Floyd geodesic in (Γ, d_f) .*

Proof. Let p_m, p_n be points on r such that $d(1, p_m) = m$ and $d(1, p_n) = n$. Let r_{mn} be the subsegment of r joining p_m, p_n . Then any path σ from p_m to p_n must have, for $m \leq i < n$, at least one edge E_i joining a point q_i to a point q_{i+1} where $d(q_i, 1) = i$ and $d(q_{i+1}, 1) = i + 1$ since any such path must travel between the i - and $(i + 1)$ -spheres at least once. Hence $l_f(\sigma) \geq \sum_m^{n-1} l_f(E_i) = l_f(r_{mn})$. \square

Definition 2.12. *The set of geodesic rays in Γ starting at the identity will be called the **preboundary** of Γ .*

Since geodesic rays starting at the identity in Γ are also Floyd geodesics, we shall denote the preboundary as $Pre(\partial)_f \Gamma$, emphasizing the scaling function f .

Remark 2.13. *A sequence of points tending to infinity along a geodesic ray r starting at the identity is necessarily Cauchy in the Floyd metric and hence $Pre(\partial)_f \Gamma$ can be identified with the ideal points of these rays. If p denote such an ideal point, we shall also denote the ray from 1 corresponding to p as $[1, p)$. It follows from the definition of Floyd boundary that the Floyd boundary of Γ is a singleton set if and only if for any pair $p, q \in Pre(\partial)_f \Gamma$, and sequences $p_n \rightarrow p, q_n \rightarrow q$, there exist paths σ_n in Γ joining p_n, q_n such that $l_f(\sigma_n) \rightarrow 0$ as $n \rightarrow \infty$.*

The following Theorem will be used later.

Theorem 2.14. [Ger12, Proposition 3.4.6] *Let G be relatively hyperbolic with respect to a collection \mathcal{P} of subgroups. Then there exists $\lambda \in (0, 1)$ such that the identity map $G \rightarrow G$ extends to a continuous equivariant map F from the Floyd completion $\overline{(\Gamma, d_f)}$ with respect to $f(n) = \lambda^n$ to the Bowditch completion of G with respect to \mathcal{P} . In particular, the Floyd boundary of G is non-trivial.*

2.2.1. Controlled Floyd Separation. It is at this stage that we introduce an essentially new ingredient.

Definition 2.15. *Let f be a scaling function and let $l(\sigma)$ denote the Floyd length of a path σ with respect to the scaling function f . Let Λ denote a G -invariant collection of paths in*

Γ such that for every $p, q \in \Gamma$, the geodesic joining p, q belongs to Λ . Let $\Lambda(p, q)$ denote the subcollection of paths in Λ joining p, q . We let $d_{f,\Lambda}(p, q) = \inf_{\sigma \in \Lambda(p, q)} l(\sigma)$.

If Λ consists of all (κ, κ) -quasigeodesics then we also denote $d_{f,\Lambda}(p, q)$ by $d_{f,\kappa}(p, q)$. For $p, q \in \text{Pre}(\partial)\Gamma$ we define

$$d_{f,\Lambda}(p, q) := \liminf_{p_n \rightarrow p, q_n \rightarrow q} d_{f,\Lambda}(p_n, q_n),$$

where $p_n \in [1, p)$ and $q_n \in [1, q)$ are Cauchy sequences converging to p, q respectively.

Note that for $p, q \in \text{Pre}(\partial)\Gamma$, the quantity $d_{f,\Lambda}(p, q)$ is independent of the Cauchy sequences $\{p_n\}, \{q_n\}$. If Λ consists of (κ, κ) -quasigeodesics and $p, q \in \text{Pre}(\partial)\Gamma$, then $d_{f,\Lambda}(p, q)$ is also denoted as $d_{f,\kappa}(p, q)$.

Definition 2.16. If $d_{f,\Lambda}(p, q) > 0$ for $p, q \in \text{Pre}(\partial)\Gamma$, we shall say that p, q are **Floyd separated with respect to Λ** . If $d_{f,\kappa}(p, q) > 0$ for $p, q \in \text{Pre}(\partial)\Gamma$ and for all κ , we shall say that p, q are **Floyd separated with respect to quasigeodesics**.

Remark 2.17. (Gerasimov) Note that $d_{f,\Lambda}(p, q)$ is not necessarily a metric as the triangle inequality need not be satisfied.

In future, if we want to emphasize that the Floyd length of a path σ is being computed with respect to a Floyd function f and Λ consists of (κ, κ) -quasigeodesics, we shall have occasion to use the suggestive notation $l_f^\kappa(\sigma)$ in place of $l_f(\sigma)$ only to remind ourselves that Λ consists of all (κ, κ) -quasigeodesics and hence σ itself is a (κ, κ) -quasigeodesic.

Remark 2.18. Let γ be a bi-infinite Morse geodesic in the Cayley graph Γ of a group (or more generally a space X). Then the end-points of γ in $\text{Pre}(\partial)\Gamma$ will necessarily be Floyd separated with respect to quasigeodesics. In particular, hyperbolically embedded subgroups [DGO11] of mapping class groups, $\text{Out}(F_n)$ and so on will have boundary points that are Floyd separated with respect to quasigeodesics.

Floyd separation of $p, q \in \text{Pre}(\partial)\Gamma$ with respect to quasigeodesics is, in fact, weaker than the existence of a Morse geodesic joining p, q . To see this, let G be hyperbolic relative to a $\mathbb{Z} \oplus \mathbb{Z}$, e.g. the fundamental group of a finite volume hyperbolic 3-manifold with one cusp. Let γ be a bi-infinite geodesic that is a union of two rays γ_1, γ_2 such that γ_1 is Morse and γ_2 lies close to a peripheral flat. Then γ is not Morse but $p, q \in \text{Pre}(\partial)\Gamma$ are Floyd-separated with respect to quasigeodesics. However, the proof of Floyd separation with respect to quasigeodesics that we provide in this paper (Theorem 7.1) actually furnishes the stronger conclusion of existence of a large collection of Morse (quasi)geodesics (Corollary 7.3).

Further, we believe that the source of Floyd separation in this paper comes from a subgroup ($K_2 \subset G$ in the example of Section 3) which is *not* hyperbolically embedded in G (See Question 8.2 at the end of the paper).

2.3. Graphical Graded Small Cancellation. In this Subsection we give a discussion of graphical Graded Small Cancellation following Wise [Wis01, Wis02]. Our definitions are equivalent to those occurring in [Wis01, Wis02]. Let B be a metric graph with edges of length one. Let A be a (non-metric) topological graph (i.e. equipped with only a CW complex structure). A map $\phi : A \rightarrow B$ is *combinatorial* if

- (1) ϕ takes vertices of A to vertices of B .
- (2) ϕ restricts to an immersion on the interior of every edge of A (where an immersion is a local injection).

A combinatorial map ϕ induces a metric graph structure A_ϕ on A after subdividing A (e.g. by pulling back the CW structure on B via ϕ) making ϕ a simplicial isometry restricted to the interior of every edge of A_ϕ .

A path $\phi : P \rightarrow B$ is a combinatorial map where P is a real interval. $|P|$ will denote the length of P , i.e. the number of edges in the graph P_ϕ . When defined, $PQ \rightarrow B$ will denote the concatenation of $P \rightarrow B$ and $Q \rightarrow B$. Let P^{-1} denote the inverse of P . In this paper, there will be a correspondence between combinatorial paths and words, and so the notation $|W|$ for the length of a word, agrees with the notation $|W|$ for the length of the corresponding path.

A (non-metric) graph A is **graded** if it is equipped with a function $g : \text{Edges}(A) \rightarrow \mathbb{N}$, where $g(J)$ is called the grade of J . Let $\phi : A \rightarrow B$ be a combinatorial map where A is a graded graph and B is a metric graph. Let A_ϕ denote the induced metric graph structure on A . Let J be a 1-cell of the (non-metric) graph A and J_ϕ the induced metric graph structure on it. i_J denotes the inclusion of J in A .

Definition 2.19. A combinatorial path $\alpha : P \rightarrow J$ is a piece of J of **grade** n provided the following hold:

- (1) There exists a path $\beta : P \rightarrow J'$ where J' is an edge of A with $g(J') = n$ and n is the least such natural number.
- (2) $\alpha : P \rightarrow J$ and $\beta : P \rightarrow J'$ represent distinct paths in A , i.e. $i_J(\alpha(P))$ and $i_{J'}(\beta(P))$ are distinct edge-paths in A_ϕ .
- (3) $\alpha : P \rightarrow J$ and $\beta : P \rightarrow J'$ project to the same path in B , i.e. $\phi(i_J(\alpha(P)))$ and $\phi(i_{J'}(\beta(P)))$ give the same edge path in B .
- (4) P is maximal with respect to the above conditions.

Definition 2.20. [Wis02][Definition 2.2] Let $\phi : A \rightarrow B$ be a combinatorial map from a (non-metric) graded graph A to a metric graph B . ϕ satisfies the **graded $c(p)$ condition** provided that for each edge J of the (non-metric) graded graph A , and for any expression of J_ϕ as the concatenation P_1, \dots, P_r of pieces of J of grade $\leq g(J)$, at least p of these pieces have grade equal to $g(J)$.

Theorem 2.21. [Wis02][Theorem 1.7] Let $\phi : A \rightarrow B$ be a combinatorial map between a graded (non-metric) graph A and a metric graph B . If ϕ satisfies the graded $c(3)$ small-cancellation condition then ϕ is π_1 -injective. If ϕ satisfies the graded $c(5)$ small-cancellation condition then $\phi_*(\pi_1 A) \subset \pi_1 B$ is malnormal.

3. THE MAIN EXAMPLE

The main example is constructed in two steps:

- (1) We first construct a malnormal infinitely generated subgroup $K = K_\infty$ of F_2 satisfying a number of properties.
- (2) Then we double F_2 along K to get $G(\infty) = F_2 *_{K_\infty} F_2$.

3.1. The Construction of K . We shall first construct a sequence of malnormal subgroups $\{K_n\} \subset F_2 = F_2(a, b)$ by choosing a sequence of elements g_n satisfying a suitable small cancellation condition such that

- (1) No proper power of a or b is an element of K_n and hence K_n is a proper infinite index subgroup.
- (2) $K_{n+1} = \langle K_n, g_{n+1} \rangle$ where $g_{n+1} \in F_2 \setminus K_n$.
- (3) $(g_{n+1}, K_n)_1 = \min_{h \in (F_2 \setminus K_n)} (h, K_n)_1$.
- (4) g_{n+1} is cyclically reduced.

We start the induction with $K_1 := \langle g_1 \rangle$ – a malnormal cyclic subgroup of F_2 ; in particular g_1 is not a proper power. We shall define $K = K_\infty = \bigcup_n K_n$ and ensure that K is of infinite index in F_2 . The choice of g_n satisfying a suitable small cancellation condition will ensure a few things:

- (1) $K = K_\infty$ is malnormal in F_2 .
- (2) The limit set ΛK of K in ∂F_2 is all of ∂F_2 .
- (3) K is of infinite index in F_2 .

Recall from the Introduction that

- (1) $T(n)$ denotes the tower function of height n with base 2, i.e. $T(n) = 2^{2^{2^{\dots}}}$, where the power is taken n times.
- (2) $U(n)(a, b)$ is the word (element) in the free group F_2 on generators a, b given by

$$U(n)(a, b) = [a^{T(n+1)}b^{T(n+1)}][a^{T(n+2)}b^{T(n+2)}] \dots [a^{T(n+1)}b^{T(n+1)}].$$

We choose g_{n+1} inductively to be of the form $g_{n+1} = h_{n+1}W_{n+1}$, where h_{n+1} satisfies the following:

- (1) $(h_{n+1}, K_n)_1 = \min(r, K_n)_1$, where the minimum is taken over geodesic rays r in F_2 starting at 1 and h_{n+1} is an initial segment of r .
- (2) $|h_{n+1}| = (h_{n+1}, K_n)_1 + 1$,

and W_{n+1} is of the form $W_{n+1} = c_{n+1}U(N(n))d_{n+1}$ where

- (A) $N(n)$ is at least twice the maximum value of m for which a^m or b^m is a subword of some freely reduced word $w_i \in K_n$.
- (B) $N(n) > |h_{n+1}|$.
- (C) c_{n+1}, d_{n+1} are chosen from $\{a, b, a^{-1}, a^{-1}\}$ to ensure that g_{n+1} is cyclically reduced.

Define

$$K = K_\infty = \bigcup_n K_n.$$

By the choice of g_i , it follows that K is of infinite index in F_2 since K is infinitely generated free.

3.2. K is malnormal.

Lemma 3.1. $K_n \subset F_2$ is malnormal.

Proof. It suffices by Theorem 2.21 to show that the map $\phi_n : \text{Rose}(K_n) \rightarrow \text{Rose}(F_2)$ between the roses corresponding to K_n, F_2 satisfies the graded $c(5)$ condition.

The i th petal (labeled by the generator g_i) of $\text{Rose}(K_n)$ is assigned grade i , ($i = 1 \dots n$). Then by Conditions (A) and (B) in the construction of W_{n+1} above, pieces of grade n are of the form $a^m b^m$ or $b^m a^{m+1}$. Since the number of such pieces is at least 5, the map $\phi_n : \text{Rose}(K_n) \rightarrow \text{Rose}(F_2)$ satisfies the graded $c(5)$ condition. \square

Proposition 3.2. $K = K_\infty \subset F_2$ is malnormal.

Proof. First observe that a nested union of malnormal subgroups is malnormal. Suppose $H = \bigcup_i H_i$, where $H_i \subset H_{i+1}$ and each H_i is malnormal in G . For some $h \in H, h \neq 1$ and $g \in G$ suppose that $ghg^{-1} = h_1 \in H$. Then there exists n such that $h, h_1 \in H_n$ forcing $g \in H_n$ since H_n is malnormal in G . Hence $g \in H$ and H is malnormal in G .

Since $K = K_\infty = \bigcup_n K_n$, and each K_n is malnormal in F_2 by Lemma 3.1, it follows that K is malnormal in F_2 . \square

3.3. K has full limit set.

Proposition 3.3. $\Lambda K = \partial F_2$.

Proof. This shall follow from the fact that in our construction, $(g_{n+1}, K_n)_1 = (h_{n+1}, K_n)_1 = \min_{h \in (F_2 \setminus K_n)} (h, K_n)_1$.

It suffices to show that for every $g \in F_2$, there exists n and $h \in K_n$ such that $g \in [1, h]$, i.e. g lies on the geodesic from 1 to h . Suppose not. Then $g \notin K_n$ for all n . Further, $(g, K_n)_1 \leq |g|$ for all n . Hence $(h_i, K_n)_1 \leq |g|$ for all $i \geq 2$. Hence $|h_i| \leq |g| + 1$ for all $i \geq 2$. This forces h_i to range in a finite set – a contradiction. \square

3.4. Doubling. Define

- (1) $G(n) = F_2(a, b) *_{K_n(a,b)=K_n(x,y)} F_2(x, y)$.
- (2) $G = G(\infty) = F_2(a, b) *_{K_\infty(a,b)=K_\infty(x,y)} F_2(x, y)$.

In Section 5 we shall need to reindex $G(n)$ and define $G_n := G(n + j_0)$ for a fixed j_0 ; hence we ask the reader's indulgence in this slight incongruity of notation.

We shall need the following fact from [BF92, Mit04]:

Theorem 3.4. *Let G_0 be a hyperbolic group and H a malnormal quasiconvex subgroup. Then $G = G_0 *_H G_0$ is hyperbolic and H is quasiconvex in it.*

The following is an immediate consequence:

Proposition 3.5. $G(n)$ is hyperbolic and K_n is quasiconvex in it.

Proof. It suffices by Theorem 3.4 to show that K_n is malnormal quasiconvex. Malnormality is the content of Lemma 3.1. Since K_n is finitely generated in F_2 , it is quasiconvex. \square

3.4.1. *Geodesics in G .* Fix a symmetric generating set $\{a, b, x, y, a^{-1}, b^{-1}, x^{-1}, y^{-1}\}$ of $G(n), G$. We first fix notation. Let a, b denote the generators of the first copy of F_2 and let x, y denote the generators of the second copy of F_2 . Let Γ denote the Cayley graph of G with respect to (the symmetrization of) a, b, x, y . Let Γ_1 denote the (sub) Cayley graph of the subgroup $F_2(a, b) \subset G$. Similarly let Γ_2 denote the (sub) Cayley graph of the subgroup $F_2(x, y) \subset G$. Let i_1, i_2 denote the inclusion maps of Γ_1, Γ_2 respectively. There exists natural projection maps $\Pi_j : \Gamma \rightarrow \Gamma_j$ identifying the generators a, b with x, y respectively.

Lemma 3.6. $F_2(a, b)$ and $F_2(x, y)$ are isometrically embedded in $G(n)$ and G .

Proof. Clearly $d(\Pi_j(u), \Pi_j(v)) \leq d(u, v)$. Since $\Pi_j \circ i_j$ is the identity map (on the first or second F_2 respectively) it follows that $F_2(a, b)$ and $F_2(x, y)$ are isometrically embedded in G . \square

We shall need the following normal form Lemma (cf. [LS, p. 285-6]) for free products with amalgamation. For the group $B *_A C$, let i, j denote the inclusion maps of A into B, C respectively.

Lemma 3.7. *Every element g of $B *_A C$ which is not in the image of A can be written in the **normal form** $v_1 v_2 \cdots v_n$ where the terms v_k lie in $B - i(A)$ or $C - j(A)$ and alternate between these two sets. The length n , called the **normal length** of the group element, is uniquely determined and two such expressions $v_1 v_2 \cdots v_n$ and $w_1 w_2 \cdots w_n$ give the same*

element of $B *_A C$ iff there are elements $a_1, \dots, a_{n1} \in A$ so that $w_k = a_{k-1}v_k a_k^{-1}$. (where $a_0 = a_n = 1$).

Definition 3.8. We shall say that a normal form word $g = v_1 v_2 \dots v_n$ is in **reduced normal form** if for each i , the concatenation $v_i v_{i+1}$ is a geodesic word, i.e. amongst all representatives of the form $(v_i a_i)(a_i^{-1} v_{i+1})$ with each of $(v_i a_i)$ and $(a_i^{-1} v_{i+1})$ replaced by geodesic words, $v_i v_{i+1}$ is shortest.

Proposition 3.9. Given an element $g \in G(n)$ there exists a normal form geodesic word $v_1 v_2 \dots v_n$ representing g .

Proof. Let T denote the Bass-Serre tree of the splitting. Observe first that a normal form word as defined above projects to a path in T that does not return to a vertex after leaving it.

Now, let u be any word that projects to a path in T having a non-trivial subpath σ that starts and ends at the same vertex of T . Let u' be the subword of u corresponding to σ . Then u' corresponds to an element of either $F_2(a, b)$ or $F_2(x, y)$. Replace u' by the geodesic word in $F_2(a, b)$ or $F_2(x, y)$ representing it. By Lemma 3.6, it follows that the resulting word is at most as long as u . Continuing this process we finally obtain a normal form word whose length is at most that of u . \square

3.4.2. *Cyclic Geodesics in $G(n)$.* We shall now carefully choose a geodesic word R_{n+1} in $G(n)$ denoting the new relator which when added to the relator set of $G(n)$ gives $G(n+1)$. R_{n+1} will satisfy the condition that all its cyclic conjugates are geodesic words in $G(n)$.

We proceed inductively to define R_n of the form $w^{-1}(x, y)w(a, b)$. Recall that $g_{n+1} = h_{n+1}c_{n+1}U(N(n))d_{n+1}$, where h_{n+1} is a geodesic word in F_2 . Let $R_n^0 = g_n^{-1}(x, y)g_n(a, b)$. R_n will be obtained from R_n^0 after cyclic reduction.

We shall define w_{n+1} to be a reduced normal form geodesic representative of $g_{n+1}^{-1}(x, y)g_{n+1}(a, b)$ in $G(n)$ (of normal length 2 as per Lemma 3.7) obtained by replacing a maximal subword w_0 of the form $w^{-1}(x, y)w(a, b)$ by a subword w' of the same form using the structure of $G(n)$ as a double along K_n . By Lemma 3.7 any normal form word representing $g_{n+1}^{-1}(x, y)g_{n+1}(a, b)$ is of the form $g_{n+1}^{-1}u^{-1}u(x, y)g_{n+1}(a, b)$, where $u \in K_n$. In particular, the shortest normal form representative of $g_{n+1}^{-1}(x, y)g_{n+1}(a, b)$ is of the form $w^{-1}(x, y)w(a, b)$ where $w(a, b) \in (F_2(a, b) \setminus K_n(a, b))$ and $w(x, y) \in (F_2(x, y) \setminus K_n(x, y))$. Let R_{n+1} denote such a representative.

Since Item(A) of the defining condition on $U(N(n))$ demands that $N(n)$ is at least twice the maximum value of m for which a^m or b^m is a subword of some freely reduced word in K_n , it follows that after any reduction as above, $w(a, b)$ contains a subword of the form $V(n)(a, b) = [a^{\lfloor \frac{T(n)+1}{2} \rfloor} b^{T(n)+1}] [a^{T(n)+2} b^{T(n)+2}] \dots [a^{T(n)+1} b^{T(n)+1}]$.

Next, $g_{n+1}(a, b)g_{n+1}^{-1}(x, y)$ is already a shortest normal form representative as it contains $b^{T(n)+1} y^{-T(n)+1}$ as a subword which cannot be shortened further, again by Item(A) of the defining condition on $U(N(n))$. It follows by the same reason that every cyclic conjugate of R_{n+1} is a geodesic in $G(n)$.

Thus R_{n+1} satisfies the following properties:

Property 1: All cyclic conjugates of R_{n+1} are geodesics in $G(n)$.

Property 2: R_{n+1} is of the form $w_n^{-1}(x, y)w_n(a, b)$ where $w(a, b)$ contains a subword of the form $V(n)(a, b) = [a^{\lfloor \frac{T(n)+1}{2} \rfloor} b^{T(n)+1}] [a^{T(n)+2} b^{T(n)+2}] \dots [a^{T(n)+1} b^{T(n)+1}]$.

Property 3: $|R_{n+1}| = O(|V(n)(a, b)|) = O(T(n+1)^2)$

Property 4: The largest piece of R_{n+1} is of the form $b^{(T(n+1)-1)}a^{(T(n+1))}$ which has length $2T(n+1) - 1$. (since $N(n) > |h_{n+1}|$ by choice, initial segments cannot be maximal pieces).

The group G therefore admits a presentation $G = \langle S | \mathcal{R} \rangle$, where

- (1) $S = \{a, b, x, y, a^{-1}, b^{-1}, x^{-1}, y^{-1}\}$.
- (2) $\mathcal{R} = \bigcup_i \{R_i\}$.

A more general statement is true. We learnt the proof of the following from Dani Wise. We refer the reader to [Wis12] for details on non-positively curved (npc) complexes.

Proposition 3.10. *$G = F *_A F$ is a double of a finitely generated free group F along a finitely generated subgroup A , with generating set given by the union of the generating sets of the two copies of F . Then G has cohomological dimension 2.*

Then any reduced normal form word g of the form uvw (i.e. of normal length 3), where u, w are geodesic words in the first factor and v is a geodesic in the second factor are geodesics in G .

More generally any reduced normal form word is a geodesic.

Proof. By Theorem 7.2 of [Wis12], G can be represented as the fundamental group of a non-positively curved (\mathcal{VH}) square complex formed by taking two graphs \mathcal{G} (with F as fundamental group) as vertex spaces, and an edge space that is a $\mathcal{G} \times I$ that is attached by a local isometry on each side. It immediately follows that G has cohomological dimension 2.

Then g corresponds to a path σ labeled uvw consisting of three vertical paths labeled u, v, w joined by two horizontal edges, one from the terminal point of u to the initial point of v and one from the terminal point of v to the initial point of w .

Consider a disk diagram D between σ and an arbitrary edge path η with the same end-points. The dual curves emanating from v cannot end on u or w because of the hypothesis that uv and vw are geodesics. The dual curves emanating from u cannot end on the w since then the middle path v must have dual curves that start and end on itself, contradicting that it is itself reduced. Thus all dual curves emanating from σ must end on η . Thus η has at least as many vertical edges as σ , forcing σ to be a geodesic. This proves the first statement.

The proof of the last statement is now completed by a straightforward induction: We repeat the above argument after combining the first $(n-1)$ words into a single geodesic. \square

3.5. Subgroups of G .

Proposition 3.11. *All finitely presented subgroups H of G are hyperbolic.*

Proof. Let $H = \langle S_1 | R_1 \rangle$ be a finite presentation. Let $S_1 = \{s_1, \dots, s_n\}; R_1 = \{r_1, \dots, r_m\}$. Let H_0 be the subgroup of $G(0)$ generated by S_1 . Then there exists n such that each r_i is trivial in $G(n)$ and hence in $G(m)$ for all $m \geq n$. Since H is a subgroup of G , it follows that H is a subgroup of $G(m)$ for all $m \geq n$. By the first part of Proposition 3.10, G has cohomological dimension 2.

Hence H is a finitely presented subgroup of a hyperbolic group $G(n)$ of cohomological dimension 2. It now follows from [Ger96] that H is hyperbolic. \square

4. NON RELATIVE HYPERBOLICITY

In this Section we show that:

Theorem 4.1. $G = F_2 *_K F_2$ is not hyperbolic relative to any proper collection of parabolic subgroups \mathcal{P} .

Proof. Suppose not, i.e. G is hyperbolic relative to a proper collection of parabolic subgroups \mathcal{P} .

By Theorem 2.7, $F_2(a, b)$ and $F_2(x, y)$ are relatively quasiconvex in G and hence are relatively hyperbolic with respect to $\mathcal{P} \cap F_2(a, b)$, $\mathcal{P} \cap F_2(x, y)$ respectively. Since $K = F_2(a, b) \cap F_2(x, y) (\subset G)$, it follows that K is also relatively quasiconvex in G by Theorem 2.6. But $K \subset F_2(a, b) \subset G$. Hence K is also relatively quasiconvex in $F_2(a, b)$, $F_2(x, y)$.

By Proposition 3.3, K has full limit set in $F_2(a, b)$ as well as in $F_2(x, y)$, i.e. $\Lambda K = \partial F_2(a, b)$ and $\Lambda K = \partial F_2(x, y)$. By Lemma 2.8, it follows that K is of finite index in $F_2(a, b)$. This contradicts the construction of K as an infinite index subgroup.

This final contradiction yields the Theorem. \square

Theorem 4 will also follow from Theorems 8.1 and 2.14, but the above proof is direct and simple.

5. GRADED SMALL CANCELLATION STRUCTURE OF G

In this section our primary goal is to show that the group $G = F_2(a, b) *_K F_2(x, y)$ constructed in Section 3 satisfies a certain (technical) graded small cancellation condition [OOS09, Ols91]. This will give (as usual with small cancellation theory) a number of geometric consequences.

5.1. Definitions and Preliminaries on Graded Small Cancellation.

Definition 5.1. A set \mathcal{R} of words in an alphabet is **symmetrized** if for any $R \in \mathcal{R}$, all cyclic shifts of $R^{\pm 1}$ are contained in \mathcal{R} and each R is reduced. (Note that the feature that each R is **reduced** is incorporated as part of the definition.)

Definition 5.2. [OOS09, Definition 4.2] Let G be a group with a finite symmetric generating set S . Let \mathcal{R} be a symmetrized set of words in S . Given $\epsilon > 0$, a subword U of a word $R \in \mathcal{R}$ is called an ϵ -piece if there exists $R' \in \mathcal{R}$ such that:

- (1) $R = UV, R' = U'V'$, for some words V, U', V' ,
- (2) $U' = YUZ$ for words Y, Z satisfying $\max\{|Y|, |Z|\} \leq \epsilon$.
- (3) $YRY^{-1} \neq R'$.

In the above definition ' $=$ ' is to be interpreted as equality as elements of G . In the following definition we rename condition SC of [OOS09] as the relative small cancellation condition.

Definition 5.3. [OOS09, Definition 4.3] Let $\epsilon \geq 0$, $\mu \in (0, 1)$, and $\rho > 0$. A symmetrized set \mathcal{R} of words in a symmetrized generating set S for a group G satisfies the **relative small cancellation condition** $C(\epsilon, \mu, \rho)$ if

- (RSC₁) The words in \mathcal{R} represent geodesics in G .
- (RSC₂) $|R| \geq \rho$ for any $R \in \mathcal{R}$.
- (RSC₃) for any ϵ -piece U and any $R \in \mathcal{R}$, with $U \subset R$, $|U| \leq \mu|R|$.

Definition 5.4. [OOS09, Definitions 4.12, 4.14] Fix $\alpha = 10^{-2}, K = 10^6$. The presentation

$$(1) \quad G = \langle S | \mathcal{R} \rangle = \left\langle S \left| \bigcup_{i=0}^{\infty} \mathcal{R}_i \right. \right\rangle$$

is a **graded small cancellation presentation** if there exist sequences $\epsilon = (\epsilon_n), \mu = (\mu_n)$, and $\rho = (\rho_n)$ of positive real numbers ($n = 1, 2, \dots$) satisfying:

(GSC₀) The group $G_0 = \langle S | \mathcal{R}_0 \rangle$ is δ_0 -hyperbolic for some δ_0 .

(GSC₁) $\epsilon_{n+1} > 8 \max\{|R|, R \in \mathcal{R}_n\} = O(\rho_n)$.

(GSC₂) $\mu_n \rightarrow 0$ as $n \rightarrow \infty$, $\mu_n \leq \alpha$, and $\mu_n \rho_n > K \epsilon_n$ for any $n \geq 1$.

(GSC₃) For all $n \geq 1$, \mathcal{R}_n satisfies $C(\epsilon_n, \mu_n, \rho_n)$ over $G_{n-1} = \left\langle S \left| \bigcup_{i=0}^{n-1} \mathcal{R}_i \right. \right\rangle$.

Remark 5.5. Though we have fixed $\alpha = 10^{-2}, K = 10^6$ following [OOS09] above, any sufficiently large K and sufficiently small α suffices.

We shall need the following:

Lemma 5.6. [OOS09, Lemmas 4.13, 4.18] Let

$$(2) \quad G = \langle S | \mathcal{R} \rangle = \left\langle S \left| \bigcup_0^{\infty} \mathcal{R}_i \right. \right\rangle$$

be a **graded small cancellation presentation**. Then G_n is δ_n -hyperbolic, where $\delta_n \leq 4 \max_{R \in \mathcal{R}_n} |R|$.

Any subword of any word $R_n \in \mathcal{R}_n$ of length at most $|R_n|/2$ is a $(1-o(1), 0)$ -quasigeodesic in the Cayley graph of G with respect to the generating set S .

5.2. Graded Small Cancellation for G .

Theorem 5.7. The group $G = F_2 *_K F_2$ equipped with the presentation $G = \langle S | \mathcal{R} \rangle$, where $S = \{a, b, x, y, a^{-1}, b^{-1}, x^{-1}, y^{-1}\}$ and $\mathcal{R} = \bigcup_i \{R_i\}$ (cf. Section 3.4.2 and Proposition 3.10) is a graded small cancellation group and the presentation is a graded small cancellation presentation.

Proof: The proof will proceed by checking the properties in Definitions 5.4 and 5.3.

Proof of Condition GSC₀: By Theorem 3.4, each $G(k) = F_2 *_K F_2$ is hyperbolic and hence any $G(k)$ can be taken as the starting group G_0 . k will be taken large enough to ensure that the parameter values $\alpha = 10^{-2}, K = 10^6$ are satisfied and will be determined during the course of the proof. We reindex if necessary by setting $G_i = G(k+i)$ for all i .

Proof of Condition GSC₁:

- (1) $\rho_n = |R_{n+k}|$.
- (2) $\epsilon_{n+1} = 10\rho_n$.

Condition GSC₁ immediately follows.

Proof of Condition GSC₂: It suffices to choose $\mu_n > \frac{10^7 \rho_{n-1}}{\rho_n}$. Recall from Property 3 satisfied by R_{n+k} that $\rho_n = O((T(n+k))^2)$. Since $\frac{(T(n+k-1))^2}{(T(n+k))^2} \rightarrow 0$ as $n \rightarrow \infty$ very fast (more than exponentially), it will be easy to make a specific choice for μ_n during the proof of GSC₃ below.

5.2.1. *Proof of GSC₃*: This is the crucial step where we shall prove that the set $\mathcal{R}(n+k)$ of cyclic conjugates of R_{n+k} satisfies the **relative small cancellation condition** $C(\epsilon_n, \mu_n, \rho_n)$ (Definition 5.3) over $G_{n-1} = \langle S | \mathcal{R} = \bigcup_{i=0}^{n+k-1} \mathcal{R}(i) \rangle$, for suitable choices of ϵ_n, μ_n and ρ_n , for all n .

We observe that, by Proposition 3.5 all the groups $G(i)$ are δ_i -hyperbolic. We need to therefore choose parameters ϵ_n, μ_n and ρ_n for $\mathcal{R}(n+k)$ satisfying $C(\epsilon_n, \mu_n, \rho_n)$ over $G_{n-1} = \langle S | \mathcal{R} = \bigcup_{i=0}^{n+k-1} \mathcal{R}(i) \rangle$.

Proof of RSC1: $\mathcal{R}(n+k)$ consists of cyclic conjugates of exactly one relator R_{n+k} all of which are geodesics by their construction (Section 3.4.2 and Proposition 3.10).

Proof of RSC2: Since all cyclic conjugates of R_{n+k} are geodesics (Section 3.4.2 and Proposition 3.10), all of them have the same length ρ_n .

Proof of RSC3: We will show that for the choice of $\epsilon_n = 10\rho_{n-1}$ as in the Proof of Condition GSC₁, the length of an ϵ_n -piece in $\mathcal{R}(n+k)$ has length less than $\mu_n |R_{n+k}|$ for suitably chosen μ_n .

Let U be any ϵ_n -piece. By the form of the words R_{n+k} , a subword W with length at least half that of U is a geodesic lying in one of the factors $F(a, b)$ or $F(x, y)$. Further, since U, U' are geodesics starting and ending at most ϵ_n -apart, it follows that there exists a geodesic subword W' of U' such that the initial and final points of W, W' are at most $(\epsilon_n + 4\delta_{n-1})$ apart. Hence assume without loss of generality that U is an $(\epsilon_n + 4\delta_{n-1})$ -piece lying entirely in $F_2(a, b)$. It follows from Definition 5.2 that

- (1) $R = UV, R' = U'V'$, for some words V, U', V' and $R, R' \in \mathcal{R}(n+k)$,
- (2) $U' = YUZ$ for words Y, Z satisfying $\max\{|Y|, |Z|\} \leq (\epsilon_n + 4\delta_{n-1})$.
- (3) $YRY^{-1} \neq R'$.

After reducing $Y^{-1}U'Z^{-1}$ to reduced normal form, we obtain therefore a piece U_0 (in the usual small cancellation sense) of length at least $|U| - 2(\epsilon_n + 4\delta_{n-1})$. Recall that by Property 4 of R_{n+k} , a maximal piece (in the usual small cancellation sense) has length at most $2T(n+k)$. Hence $|U_0| < 2T(n+k)$ and so $|U| < 2T(n+k) + 2(\epsilon_n + 4\delta_{n-1})$. **Choice of μ_n :** It therefore suffices to choose μ_n such that $2T(n+k) + 2(\epsilon_n + 4\delta_{n-1}) \leq \mu_n \rho_n$. Define $\mu_n = 2 \max\{\frac{2T(n+k) + 20\rho_{n-1}}{\rho_n}, \frac{10^7 \rho_{n-1}}{\rho_n}\}$. Since $\rho_n = O((T(n+k))^2)$, and $\delta_{n-1} = O(\rho_n)$, it follows that $\mu_n \rightarrow 0$ as $n \rightarrow \infty$.

Choice of k : It remains to choose k . The choice of k is dictated by the requirement that $\mu_n \leq \alpha$ for all n . Since $\mu_m \rightarrow 0$ as $m \rightarrow \infty$ it follows that there exists $k > 0$ such that $\mu_m \leq \alpha$ for all $m \geq k$. This decides the choice of k and completes the proof of Theorem 5.7. \square

6. ASYMPTOTIC CONE OF $G = F_2 *_K F_2$

We refer the reader to [DS05] for details on tree-graded spaces and [Gro93, OOS09] for details on asymptotic cones.

6.1. Tree-graded Spaces.

Definition 6.1. *Let X be a complete geodesic metric space and let \mathcal{P} be a collection of closed geodesic subsets. Then X is said to be tree-graded with respect to \mathcal{P} if the following hold:*

- (1) *Any two elements of \mathcal{P} have at most one common point.*

- (2) Every simple geodesic triangle (i.e. a simple loop composed of three geodesics) in X is contained in exactly one element of \mathcal{P} .

Definition 6.2. [OOS09] If X is tree-graded with respect to \mathcal{P} such that each element of \mathcal{P} is a circle of radius uniformly bounded away from 0 and ∞ , then X is called a circle-tree.

6.2. Asymptotic Cones of Graded Small Cancellation Groups. Let ω be a non-principal ultrafilter. We shall say that $a_i = O_\omega(b_i)$ if the ω -limit of $\{\frac{a_i}{b_i}\}$ is finite and non-zero. Similarly, $a_i = o_\omega(b_i)$ if the ω -limit of $\{\frac{a_i}{b_i}\}$ is zero.

Definition 6.3. [OOS09, Definitions 4.16,4.19] Given an ultrafilter ω and a scaling sequence $d = (d_n)$, a sequence of real numbers $f = (f_n)$ is said to be (ω, d) -**visible** or simply **asymptotically visible**, if there exists a subsequence (f_{n_i}) of f such that $f_{n_i} = O_\omega(d_i)$.

For a group G and its Cayley graph Γ , we shall say that a sequence of loops $\{\sigma_i\} \subset \Gamma$ is **asymptotically visible** if the sequences $\{d(1, \sigma_i)\}$ and $\{|\sigma_i|\}$ are asymptotically visible.

The following is one of the main Theorems of [OOS09].

Theorem 6.4. [OOS09][Theorem 4.17] Let G be a group having a graded small cancellation presentation with symmetrized relator set $\mathcal{R} = \cup_i \{R_i\}$, where $\{R_i\}$ is the grade i symmetrized relator set. For any ultrafilter ω , and any sequence of scaling constants $d = (d_n)$, the asymptotic cone $\text{Con}^\omega(G, d)$ is a circle-tree. $\text{Con}^\omega(G, d)$ is an \mathbb{R} -tree if and only if the sequence (ρ_n) from Definition 5.4 is not (ω, d) -visible.

Further any simple loop C in $\text{Con}^\omega(G, d)$ can be realized as an ω -limit of translates of $\{R_i\}$'s.

The last statement follows from Lemma 4.23 of [OOS09].

6.3. Properties of the Asymptotic Cone of G .

Proposition 6.5. $\text{Con}^\omega(G)$ is a circle-tree.

Proof. This follows from Theorems 5.7 and 6.4. □

Let \mathcal{G} denote $\text{Con}^\omega(G)$.

Proposition 6.6. $i : K_2 \rightarrow G$ induces a bi-Lipschitz embedding $\hat{i} : \text{Con}^\omega(K_2) \hookrightarrow \mathcal{G}$.

Proof. Since $K_2 \hookrightarrow F(a, b)$ is a quasi-isometric embedding and $F(a, b) \hookrightarrow G$ is an isometric embedding by Lemma 3.6, it follows that $i : K_2 \rightarrow G$ is a quasi-isometric embedding. The result follows. □

6.3.1. Limit Circles in $\text{Con}^\omega(G)$ intersect $\text{Con}^\omega(K_2)$ trivially. The asymptotic cone $C^\omega(G, \{d_n\})$ of the graded small cancellation group G is a circle-tree with limit circles of uniformly bounded radius. Let \mathcal{C} denote the collection of circles in $C^\omega(G, \{d_n\})$.

The asymptotic cone $C^\omega(K_2, \{d_n\})$ of the free group K_2 (constructed at the second stage of the inductive construction of K) is an \mathbb{R} -tree \mathcal{T} . The next Theorem is crucial in showing that K_2 is quasiconvex in G in a strong sense. This, in turn will lead to the conclusion that any pair of distinct geodesic rays in K_2 starting at the origin are Floyd separated with respect to quasigeodesics.

Theorem 6.7. For any $C \in \mathcal{C}$, the intersection $C \cap \mathcal{T}$ is either empty or contains a single point.

Proof. Suppose not. Since \mathcal{G} is a circle-tree by Theorem 6.4, it follows that there exists a bigon σ consisting of the union of two arcs:

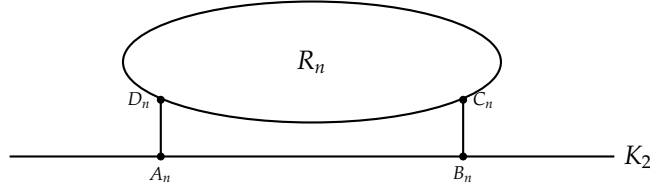


FIGURE 1.

- (1) a geodesic \overline{pq} in \mathcal{T} .
- (2) an embedded path γ in \mathcal{G} joining p, q with interior disjoint from \overline{pq} .

Since all elements of \mathcal{C} can be obtained as ω -limits of translates of elements of \mathcal{R} by Theorem 6.4, we can assume that there exists a sequence of translates $\{g_{i_n}R_{i_n}\}$ whose ω -limit is σ . For the purposes of this proof, reindex by setting $g_{i_n}R_{i_n} = R_n$. Geodesic segments in the Cayley graph of G will be denoted by $[E, F]$. Hence there exists a sequence $\{R_n, A_n, B_n, C_n, D_n\}$ such that (see Figure 1)

- (1) $R_n \in \mathcal{R}$
- (2) The scaling sequence $d_n = O(\text{dia}(R_n)) = O(\rho_n) = O(\delta_n)$, where δ_n is the hyperbolicity constant of $G(n)$ (The last equality follows from Lemma 5.6 and the asymptotic visibility of the sequence $\{R_n\}$).
- (3) $[A_n, B_n] \subset K_2$;
- (4) $[D_n, C_n] \subset R_n$
- (5) $[A_n, D_n], [B_n, C_n]$ are geodesics in normal form in G .
- (6) $\|[A_n, D_n]\|, \|[B_n, C_n]\|$ are $o_\omega(d_n)$
- (7) $\|[A_n, B_n]\|, \|[D_n, C_n]\|$ are $O_\omega(d_n)$
- (8) $\lim_\omega(A_n) = \lim_\omega(D_n) = p$
- (9) $\lim_\omega(B_n) = \lim_\omega(C_n) = q$

By cutting off initial and final pieces from $[A_n, B_n], [D_n, C_n]$ of length $o(d_n)$ if necessary and using thinness of quadrilaterals we can assume without loss of generality that $\|[A_n, D_n]\|, \|[B_n, C_n]\|$ are at most $O(d_{n-1}) = O(\delta_{n-1})$. More precisely, choose a subsegment $[A'_n, B'_n]$ such that $d(A_n, A'_n) = d(B_n, B'_n) = 2\max\{\|[A_n, D_n]\|, \|[B_n, C_n]\|\}$. Then there exist points D'_n, C'_n on $[D_n, C_n]$ such that $d(D'_n, A'_n) \leq 2\delta_{n-1}$ and $d(C'_n, B'_n) \leq 2\delta_{n-1}$. Rename the four vertices A'_n, B'_n, C'_n, D'_n as A_n, B_n, C_n, D_n . Then as in the proof of Condition RSC_3 in the proof of Theorem 5.7 we get a genuine (in the usual small cancellation sense) piece of length $O(\|[A_n, B_n]\|)$ as a subword of $[A_n, B_n]$. Further, any maximal piece of R_n is $o(\rho_n)$ and hence $\frac{\|[A_n, B_n]\|}{\rho_n} \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, since $\lim_\omega(A_n) = \lim_\omega(D_n) = p$ and $\lim_\omega(B_n) = \lim_\omega(C_n) = q$, and since $p \neq q$, it follows that $\frac{\|[A_n, B_n]\|}{\rho_n}$ is bounded away from zero. This is a contradiction. \square

7. CONTROLLED FLOYD SEPARATION

The main purpose of this section is to show:

Theorem 7.1. *For any Floyd function f , $c \geq 1$, and $p, q \in \partial K_2 \subset \text{Pre}(\partial)G$, the points p, q are Floyd separated with respect to c -quasigeodesics.*

Proof. Let $p, q \in \partial K_2$ be distinct points. Let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be Cauchy sequences in K_2 , converging to p and q respectively. We shall show that $d_{f,c}(p, q) > 0$.

Suppose not. Then there exists a Floyd function f and a constant $c \geq 1$ such that $d_{f,c}(p, q) = 0$. Hence there exists a sequence of (c, c) -quasi geodesic paths $\{\gamma_n\}$, joining q_n to p_n in G , such that (the c -controlled Floyd lengths) $l_f^c(\gamma_n) \rightarrow 0$.

Let D_n be the Dehn-diagram obtained from the trivial word with label $[p_n, q_n] \cdot \gamma_n$ in G . We will construct a sequence $\{D'_n\}$ of subdiagrams from $\{D_n\}$ and a sequence of numbers M_n such that

- (1) M_n is the maximum possible distance of a point t_n on γ_n , from $[p_n, q_n]$.
- (2) The ultralimit $\lim^\omega(\partial D'_n, \{M_n\})$ contains a circle C with non-zero but finite length in the asymptotic cone $\text{Con}^\omega(G, \{M_n\})$.
- (3) The intersection $C \cap \mathcal{T}$ has nonzero length.

This will contradict Theorem 6.7.

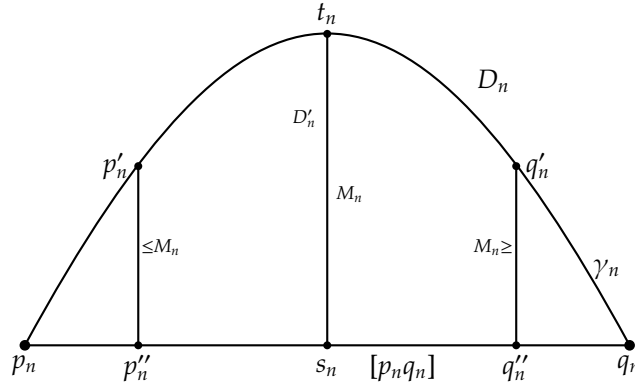


FIGURE 2. D_n and D'_n

Choose $t_n \in \gamma_n$ such that $d(t_n, [p_n, q_n]) = M_n$ is maximal and let $s_n \in [p_n, q_n]$ be such that $d(t_n, s_n) = M_n$. Since $l_f^c(\gamma_n) \rightarrow 0$ as $n \rightarrow \infty$, it follows that $M_n \rightarrow 0$ as $n \rightarrow \infty$. The actual dependence of M_n on f, c is not important as we shall only use the sequence $\{M_n\}_n$ as a sequence of scale factors to extract a limiting asymptotic cone.

The proof now splits into the following cases:

- Case 1: The subarcs of γ_n joining p_n, t_n and t_n, q_n each have length at least $5cM_n$.
- Case 2: The subarcs of γ_n joining p_n, t_n and t_n, q_n each have length less than $5cM_n$.
- Case 1: Exactly one of the subarcs of γ_n joining p_n, t_n and t_n, q_n has length at least $5cM_n$.

We shall prove the Theorem for Cases 1 and 2. We shall only give a sketch of Case 3, which is a hybrid and the proofs of Cases 1 and 2 combine in an obvious way.

Case 1:

Choose p'_n, q'_n to the left and right respectively of t_n such that the subarcs of γ_n joining p'_n, t_n and t_n, q'_n each have length $5cM_n$. The points p'_n, q'_n exist by rectifiability and connectedness of the arc γ_n . It follows that the c -quasi geodesic arc ($c > 1$) joining p'_n and q'_n has length $10cM_n$ and hence $d(p'_n, q'_n) \geq 10M_n - c$.

Let p''_n (resp. q''_n) be points on $[p_n, q_n]$ closet to p'_n (resp. q'_n). By the choice of t_n , $d(p'_n, p''_n) \leq M_n$ and $d(q'_n, q''_n) \leq M_n$. Hence $||[p''_n, q''_n]|| \geq 8M_n - c$.

Let D'_n be the subdiagram of D_n with sides $[p'_n, p''_n]$, $[p''_n, q''_n]$, $[q''_n, q'_n]$ and the subarc of γ_n joining q'_n and p'_n .

Let $D' = \lim^\omega \partial D'_n \subset \text{Con}^\omega(G, \{M_n\})$; $p' = \lim^\omega p'_n$; $q' = \lim^\omega q'_n$; $p'' = \lim^\omega p''_n$; $q'' = \lim^\omega q''_n$; $t = \lim^\omega t_n$. Then $||[p'', q'']|| \geq 8$, $||[p', p'']|| \leq 1$, $||[q', q'']|| \leq 1$. Also the top arc η of D' , joining p', t, q' is an **embedded** (c -biLipschitz) arc of length $10c$ in $\text{Con}^\omega(G, \{M_n\})$. Hence D' contains a loop θ such that $10c - 2 \leq |\theta \cap \eta| \leq 10c$ and $6 \leq ||[p', q']||$. Since $\text{Con}^\omega(G, \{M_n\})$ is a circle tree, and since $\eta, [p'', q'']$ are both embedded arcs, it follows that θ contains a simple loop C (a circle) whose intersection with $[p'', q'']$ has non-zero length. This contradicts Theorem 6.7.

Case 2:

In this case, since $M_n \leq |\gamma_n| + d(p, q)$ (where $|\gamma_n|$ denotes the usual length of γ_n), it follows that both $|\gamma_n|$ and $d(p, q)$ are $O(M_n)$. Next, since γ_n is a (c, c) -quasigeodesic, $\lim^\omega(\gamma_n)$ is a c -biLipschitz path. In particular, $\lim^\omega(\gamma_n)$ has no self-intersections. Hence, $D = \lim^\omega \partial D_n \subset \text{Con}^\omega(G, \{M_n\})$ is a biLipschitz bigon with $p := \lim^\omega p_n$ and $q := \lim^\omega q_n$ as its two vertices. Since $t := \lim^\omega t_n$ lies at distance 1 from $[p, q]$, it follows that there is a nontrivial bigon B with bounding arcs α, β such that

- (1) the interiors of α, β do not intersect,
- (2) α contains t and is a nontrivial subarc of $\lim^\omega \gamma_n$,
- (3) β is a nontrivial subarc of $[p, q]$.

Again, this contradicts Theorem 6.7.

Case 3:

Without loss of generality, suppose that the subarc of γ_n joining p_n, t_n has length at least $5cM_n$ and the subarc of γ_n joining q_n, t_n has length less than $5cM_n$. Construct p'_n, p''_n as in Case 1 and let D'_n be the subdiagram of D_n bounded by $[p'_n, p''_n]$, $[p''_n, q_n]$ and the subarc of γ_n joining q_n to p''_n . Scaling by M_n and taking limits we again obtain a nontrivial bigon contradicting Theorem 6.7. \square

We isolate now a notion of ‘quasiconvexity with respect to quasigeodesics’ (slightly stronger than usual quasiconvexity) that the *proof* of the above Theorem furnishes for K_2 :

Definition 7.2. Let G be a finitely generated group and Γ a Cayley graph with respect to a finite generating set. For any $c \geq 1$, a finitely generated subgroup $H \subset G$ is said to be **quasiconvex with respect to c -quasigeodesics** if there exists $D > 0$ such that for any $h_1, h_2 \in H$, any (c, c) -quasigeodesic in Γ lies in a D -neighborhood of H .

$H \subset G$ is said to be **quasiconvex with respect to quasigeodesics** if it is quasiconvex with respect to c -quasigeodesics for all c .

Thus a subgroup is quasiconvex with respect to quasigeodesics if all geodesics on it are ‘uniformly Morse’.

Corollary 7.3. $K_2 \subset G$ is quasiconvex with respect to quasigeodesics.

Proof. We continue with the notation used in Theorem 7.1. Note first that in the proof of Theorem 7.1, the final contradiction only requires that the distance $d(t_n, (p, q)) \rightarrow \infty$ as $n \rightarrow \infty$ for some $c \geq 1$.

Suppose that there exists $c \geq 1$ such that $K_2 \subset G$ is not quasiconvex with respect to c -quasigeodesics. Then there exist (c, c) -quasigeodesics γ_n joining $p_n, q_n \in K_2$ such that $d(\gamma_n, (p, q)) \rightarrow \infty$ as $n \rightarrow \infty$ for some $c > 0$. Hence there exists $h_n \in \gamma_n$ such that $d(h_n, K_2) \rightarrow \infty$ as $n \rightarrow \infty$. Translating by an element of K_2 , we can assume that $d(h_n, K_2) = d(h_n, 1)$. The argument of Theorem 7.1 now goes through as before to furnish a circle C with non-zero but finite length in the asymptotic cone $\text{Con}^\omega(G, \{h_n\})$ such that the intersection $C \cap \mathcal{T}$ has nonzero length. This contradicts Theorem 6.7. \square

8. TRIVIALITY OF FLOYD BOUNDARY

For completeness, we show in this section that the usual Floyd boundary $\partial_f G$ consists of a single point and is hence trivial:

Theorem 8.1. *$\partial_f G$ consists of a single point.*

Proof. Let $\partial_f F_2(a, b) \subset \partial_f G$ denote the set of limit points of $F_2(a, b)$ in $\partial_f G$. Let $p, q \in \partial_f F_2(a, b)$ and let (p, q) denote the label of the bi-infinite geodesic path between them. Assume without loss of generality that $1 \in (p, q)$. By the construction of the h_n 's in Section 3, there exist $p_n \rightarrow p, q_n \rightarrow q$ such that

- (1) $[p_n, q_n]$ is centered at 1
- (2) the geodesic path $[p_n, q_n]$ (i.e. the word $p_n^{-1}q_n$) in a, b is a subpath of $[1, h_m]$ (cf. the construction in Section 3) for some $m = m(n)$, surviving in the construction of R_m .

We make condition (2) more precise. Choose h_m and $c_n, d_n \in [1, h_m]$ such that

- (1) $c_n^{-1}d_n = p_n^{-1}q_n$
- (2) Any geodesic $[1, k]$ with $k \in K$, containing $[1, c_n]$ as a subsegment must have length greater than $2|c_n|$, i.e. $|k| > 2|c_n|$. Hence, the reduced normal form of $h_m^{-1}(x, y)h_m(a, b)$ contains $c_n^{-1}(a, b)d_n(a, b)$ (and hence also $c_n^{-1}(x, y)d_n(x, y)$) as a subword.

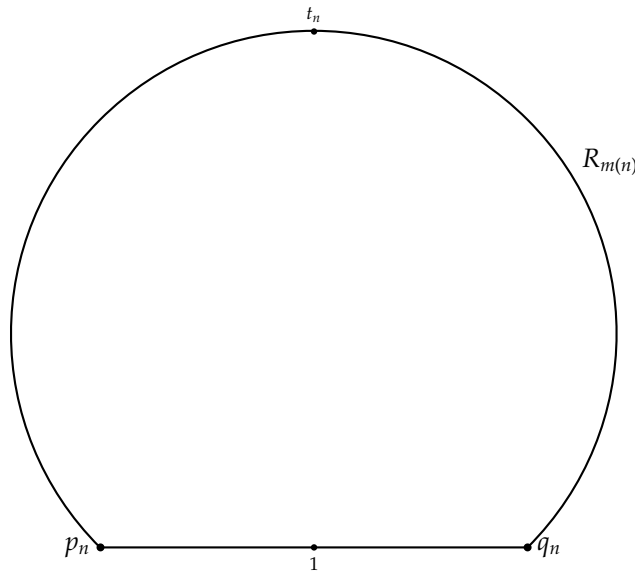
By ensuring the above, we are guaranteed the existence of a relator $R_m (= R_{m(n)}) \in \mathcal{R}$ such that $p_n^{-1}(a, b)q_n(a, b)$ (and hence also $p_n^{-1}(x, y)q_n(x, y)$) is a subword of R_m .

By Lemma 5.6, for n sufficiently large, $R_{m(n)}$ can be described as the union of two (2, 2) quasigeodesics $[1, t_n]_1$ and $[1, t_n]_2$ both starting at 1 and ending at an 'antipodal' point $t_n \in R_{m(n)}$. Further, $p_n \in [1, t_n]_1$ and $q_n \in [1, t_n]_2$ (See Figure 3).

Assume that the Floyd length of the geodesic rays $[1, p)$ (and $[1, q)$) are normalized to one. Hence for all $\epsilon > 0$, there exists n sufficiently large such that the Floyd lengths of the subpaths $[p_n, t_n]_1$ and $[q_n, t_n]_2$ are each less than $\frac{\epsilon}{2}$. It follows that $d_f(p_n, q_n) < \epsilon$.

Since $\epsilon > 0$ is arbitrary, p, q correspond to the same point on the Floyd boundary. Hence $\partial_f F_2(a, b)$ consists of a single point. Similarly, $\partial_f F_2(x, y)$ consists of a single point z .

Next, from the relators R_n of the form $w_n(x, y)w_n^{-1}(a, b)$ centered at 1 one can extract subsequential limits $z_1 \in \partial_f F_2(x, y)$ and $z_2 \in \partial_f F_2(a, b)$ of $w_n(x, y)$ and $w_n(a, b)$ to conclude that $z_1 = z_2$ on $\partial_f G$. Hence $\partial_f F_2(x, y) = \partial_f F_2(a, b) = z$. Since the action of G on $\partial_f G$ is minimal [Kar03, Theorem 2 and Section VI], and since z is invariant under both $F_2(a, b)$ as well as $F_2(x, y)$ and hence under G , it follows that $\partial_f G = \{z\}$. \square

FIGURE 3. $d_f(p_n, q_n)$ is small.

Concluding Remarks: As pointed out in Remark 2.18, Floyd separation of points of $\partial K_2 \subset \text{Pre}(\partial)G$ with respect to quasigeodesics would follow if K_2 was hyperbolically embedded in G . However, this seems quite unlikely for the following reason. The proof of Theorem 4 shows that if G admits an action by isometries on a hyperbolic space X and both $F_2(a, b)$ and $F_2(x, y)$ are qi embedded then the limit sets of $F_2(a, b)$ and $F_2(x, y)$ must coincide. This would mean that we can take X to be quasi-isometric to the Cayley graph of $F_2(a, b)$ (or $F_2(x, y)$). But then all elements of the form $w(x, y)w(a, b)^{-1}$ would lie in the kernel of the action preventing such an action from being acylindrical in any sense. Perhaps this argument can be strengthened to give a negative answer to the following:

Question 8.2. *Is K_2 hyperbolically embedded in G ?*

We should point out however that G *does* admit an acylindrical action on the Bass-Serre tree T corresponding to the splitting along K . Of course K keeps an edge of T fixed.

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SCHOOL OF MATHEMATICAL SCIENCES, DEPARTMENT OF MATHEMATICS, RKM VIVEKANANDA UNIVERSITY,
P.O. BELUR MATH, HOWRAH 711202, INDIA
E-mail address: shubhavrata.gt@gmail.com

SCHOOL OF MATHEMATICAL SCIENCES, DEPARTMENT OF MATHEMATICS, RKM VIVEKANANDA UNIVERSITY,
P.O. BELUR MATH, HOWRAH 711202, INDIA
E-mail address: mahan.mj@gmail.com, mahan@rkmvu.ac.in