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Height in splittings of hyperbolic groups

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Abstract. Suppose H is a hyperbolic subgroup of a hyperbolic group G. Assume there exists $n > 0$ such that the intersection of n essentially distinct conjugates of H is always finite. Further assume G splits over H with hyperbolic vertex and edge groups and the two inclusions of H are quasi-isometric embeddings. Then H is quasiconvex in G. This answers a question of Swarup and provides a partial converse to the main theorem of [23].

Keywords. Hyperbolic groups; quasi-isometric embeddings; splittings of groups.

1. Introduction

Let G be a hyperbolic group in the sense of Gromov [24]. Let H be a hyperbolic subgroup of G. We choose a finite symmetric generating set for H and extend it to a finite symmetric generating set for G. Let Γ_H and Γ_G denote the Cayley graphs of H, G respectively with respect to these generating sets.

If H is not quasiconvex in G, we would like to understand the group theoretic (or algebraic) mechanism contributing to the distortion of H in G. The first examples of distorted hyperbolic subgroups of hyperbolic groups were fiber subgroups of fundamental groups

Editors' comments

We thank the referee for the following comments:

This paper is an unedited publication of Mahan Mitra's 1997 preprint. The paper has been used or referred to in the following papers and book:

G A Swarup, Proof of a weak hyperbolization theorem, *Quart. J. Math.* **51** (2000) 529–533

M Kapovich and B Kleiner, Hyperbolic groups with low-dimensional boundary, *Ann. Sci. de ENS Paris*, t. **33** (2000) 647–669

and the Monograph:

M Kapovich, Hyperbolic manifolds and discrete groups, Volume 183 in the series Progress in Mathematics (Boston, Basel, London: Burkhauser) (2001).

Ilya Kapovich has published a (slightly weaker) version of the Main Theorem 4.6 of Mitra's paper in "The combination theorem and quasiconvexity", *Int. J. Algebra Comput.* **11** (2001) 185–216. The stronger form is used in Swarup's paper referred to above and is concerned with the notion of acylindricity or 'no long annuli'. In §5 of Mitra's paper, he mentions that his main argument does not generalize to graphs of groups. But as observed in the above papers, the extension to graphs of groups is straightforward. It is not clear what Mitra had in mind. Since the subsequent questions he raises are still interesting and unsolved, the presentation has not been changed.

Editors

of closed hyperbolic 3-manifolds fibering over the circle. The extrinsic geometry in this case was studied in detail by Cannon and Thurston [15] and later by the author [36,37]. General examples of normal hyperbolic subgroups of hyperbolic groups have been studied in [5,41]. A substantially larger class of examples arise from the combination theorem of Bestvina and Feighn [3]. In fact almost all examples of distorted hyperbolic subgroups of hyperbolic groups use the combination theorem in an essential way (see [13,38] however). It is natural to wonder if there are any other methods of building distorted hyperbolic subgroups. To get a handle on this issue one needs the notion of height of a subgroup [23].

DEFINITION.

Let H be a subgroup of a group G. We say that the elements $\{g_i|1 \leq i \leq n\}$ of G are essentially distinct if $Hg_i \neq Hg_j$ for $i \neq j$. Conjugates of H by essentially distinct elements are called essentially distinct conjugates.

Note that we are abusing notation slightly here, as a conjugate of H by an element belonging to the normalizer of H but not belonging to H is still essentially distinct from H. Thus in this context a conjugate of H records (implicitly) the conjugating element.

DEFINITION.

We say that the height of an infinite subgroup H in G is n if there exists a collection of n essentially distinct conjugates of H such that the intersection of all the elements of the collection is infinite and n is maximal possible. We define the height of a finite subgroup to be 0.

The following question of Swarup [9] formulates the problem we would like to address in this paper:

Question. Suppose H is a finitely presented subgroup of a hyperbolic group G. If H has finite height, is H quasiconvex in G? A special case to be considered is when G splits over H and the inclusions are quasi-isometric embeddings.

We shall answer the above question affirmatively in the special case mentioned.

Theorem 4.6. *Let G be a hyperbolic group splitting over H (i.e. $G = G_1 *_H G_2$ or $G = G_1 *_H$) with hyperbolic vertex and edge groups. Further, assume the two inclusions of H are quasi-isometric embeddings. Then H is of finite height in G if and only if it is quasiconvex in G.*

The main theorem of [23] states:

Theorem 1.1. *If H is a quasiconvex subgroup of a hyperbolic group G, then H has finite height.*

Thus the purpose of this paper is to prove the converse direction.

Certain group theoretic analogs of Thurston's combination theorems [30] were deduced in [3]. Extending the analogy with [30], in this paper we prove quasiconvexity of certain surface subgroups.

PROPOSITION 5.1.

Let $G = G_1 *_H G_2$ be a hyperbolic group such that G_1, G_2, H are hyperbolic and the two inclusions of H are quasi-isometric embeddings. If H is malnormal in one of G_1 or G_2 then H is quasiconvex in G.

The following corollary is a group-theoretic analog of a theorem of Thurston's [30].

COROLLARY 5.3.

Let M_1 be a hyperbolic atoroidal acylindrical 3-manifold and S_1 an incompressible surface in its boundary. Let M_2 be a hyperbolic atoroidal 3-manifold and S_2 an incompressible surface in its boundary. If S_1 and S_2 are homeomorphic then gluing M_1 and M_2 along this common boundary $S(= S_1 = S_2)$ one obtains a 3-manifold M such that

1. *$\pi_1(M)$ is hyperbolic.*
2. *$\pi_1(S)$ is quasiconvex in $\pi_1(M)$.*

2. Preliminaries

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [24]. For details, see [16,22]. Let (X, d) be a hyperbolic metric space.

DEFINITION.

A subset Z of X is said to be *k-quasiconvex* if any geodesic joining $a, b \in Z$ lies in a k-neighborhood of Z. A subset Z is *quasiconvex* if it is k-quasiconvex for some k. A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a (K, ε)-*quasi-isometric embedding* if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \varepsilon \leq d_Z(f(y_1), f(y_2)) \leq K d_Y(y_1, y_2) + \varepsilon.$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a *quasi-isometry*. A (K, ε)-quasi-isometric embedding that is a quasi-isometry will be called a (K, ε)-quasi-isometry.

A (K, ε)-*quasigeodesic* is a (K, ε)-quasi-isometric embedding of a closed interval in $\mathbb{R}$. A $(K, 0)$-quasigeodesic will also be called a K-quasigeodesic.

DEFINITION. [17,25]

If $i : \Gamma_H \to \Gamma_G$ be an embedding of the Cayley graph of H into that of G, then the distortion function is given by

$$\text{disto}(R) = \text{Diam}_{\Gamma_H}(\Gamma_H \cap B(R)),$$

where $B(R)$ is the ball of radius R around $1 \in \Gamma_G$.

If H is quasiconvex in G the distortion function is linear and we shall refer to H as an undistorted subgroup. Else, H will be termed distorted. Note that the above definition makes sense for metric spaces and their subspaces too.

3. Trees of hyperbolic metric spaces

For a general discussion of graphs of groups, see [47]. In this paper we will deal with graphs of hyperbolic groups satisfying the quasi-isometrically embedded condition of [3]. We will need some results from [38].

DEFINITION.

A tree (T) of hyperbolic metric spaces satisfying the q(uasi) i(sometrically) embedded condition is a metric space (X,d) admitting a map $P : X \to T$ onto a simplicial tree T, such that there exist δ,ε and $K > 0$ satisfying the following:

1. For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ with the induced path metric d_v is a δ-hyperbolic metric space. Further, the inclusions $i_v : X_v \to X$ are uniformly proper, i.e. for all $M > 0$, $v \in T$ and $x,y \in X_v$, there exists $N > 0$ such that $d(i_v(x),i_v(y)) \leq M$ implies $d_v(x,y) \leq N$.
2. Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point of e. Then X_e with the induced path metric is δ-hyperbolic.
3. There exist maps $f_e : X_e \times [0,1] \to X$, such that $f_e|_{X_e\times(0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric.
4. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ are (K,ε)-quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ will occasionally be referred to as f_{v_1} and f_{v_2} respectively.

d_v and d_e will denote path metrics on X_v and X_e respectively. i_v, i_e will denote inclusion of X_v, X_e respectively into X.

We shall need a construction used in [38]. For convenience of exposition, T shall be assumed to be rooted, i.e. equipped with a base vertex v_0. We shall refer to X_{v_0} as Y. Let $v \neq v_0$ be a vertex of T. Let v_- be the penultimate vertex on the geodesic edge path from v_0 to v. Let e denote the directed edge from v_- to v. Define $\phi_v : f_{e_-}(X_{e_-}\times\{0\}) \to f_{e_-}(X_{e_-}\times\{1\})$ as follows:

If $p \in f_{e_-}(X_e\times\{0\}) \subset X_{v_-}$, choose $x \in X_e$ such that $p = f_{e_-}(x\times\{0\})$ and define

$$\phi_v(p) = f_{e_-}(x\times\{1\}).$$

Note that in the above definition, x is chosen from a set of bounded diameter.

Let μ be a geodesic in X_{v_-}, joining $a,b \in f_{e_-}(X_{e_-}\times\{0\})$. $\Phi_v(\mu)$ will denote a geodesic in X_v joining $\phi_v(a)$ and $\phi_v(b)$. Let $X_{v_0} = Y$ and $i = i_{v_0}$.

The next lemma follows easily from the fact that local quasigeodesics in a hyperbolic metric space are quasigeodesics [22]. If x,y are points in a hyperbolic metric space, $[x,y]$ will denote a geodesic joining them.

Lemma 3.1. *Given $\delta > 0$, there exist D,C_1 such that if a,b,c,d are vertices of a δ-hyperbolic metric space (Z,d), with $d(a,[b,c]) = d(a,b)$, $d(d,[b,c]) = d(c,d)$ and $d(b,c) \geq D$ then $[a,b]\cup[b,c]\cup[c,d]$ lies in a C_1-neighborhood of any geodesic joining a,d.*

Given a geodesic segment $\lambda \subset Y$, we now recall from [38] the construction of a quasiconvex set $B_\lambda \subset X$ containing $i(\lambda)$.

Construction of quasiconvex sets

Choose $C_2 \geq 0$ such that for all $e \in T$, $f_e(X_e \times \{0\})$ and $f_e(X_e \times \{1\})$ are C_2-quasiconvex in the appropriate vertex spaces. Let $C = C_1 + C_2$, where C_1 is as in Lemma 3.1.

For $Z \subset X_v$, let $N_C(Z)$ denote the C-neighborhood of Z, that is the set of points at distance less than or equal to C from Z.

Step 1. Let $\mu \subset X_v$ be a geodesic segment in (X_v, d_v). Then $P(\mu) = v$. For each edge e incident on v, but not lying on the geodesic (in T) from v_0 to v, choose p_e, $q_e \in N_C(\mu) \cap f_v(X_e)$ such that $d_v(p_e, q_e)$ is maximal. Let $v_1, \ldots, v_n$ be terminal vertices of edges e_i for which $d_v(p_{e_i}, q_{e_i}) > D$, where D is as in Lemma 3.1 above. Observe that there are only finitely many v_i's as μ is finite. Define

$$B^1(\mu) = i_v(\mu) \cup \bigcup_{k=1\ldots n} \Phi_{v_i}(\mu_i),$$

where μ_i is a geodesic in X_v joining p_{e_i}, q_{e_i}.

Note that $P(B^1(\mu)) \subset T$ is a finite tree.

The reason for insisting that the edges e do not lie on the geodesic from v_0 to v is to prevent 'backtracking' in Step 2 below.

Step 2. Step 1 above constructs $B^1(\lambda)$ in particular. We proceed inductively. Suppose that $B^m(\lambda)$ has been constructed such that the convex hull of $P(B^m(\lambda)) \subset T$ is a finite tree. Let $\{w_1, \ldots, w_n\} = P(B^m(\lambda)) \backslash P(B^{m-1}(\lambda))$. (Note that n may depend on m, but we avoid repeated indices for notational convenience.) Assume further that $P^{-1}(v_k) \cap B^m(\lambda)$ is a path of the form $i_{v_k}(\lambda_k)$, where λ_k is a geodesic in (X_{v_k}, d_{v_k}). Define

$$B^{m+1}(\lambda) = B^m(\lambda) \cup \bigcup_{k=1\ldots n} (B^1(\lambda_k)),$$

where $B^1(\lambda_k)$ is defined in Step 1 above.

Since each λ_k is a finite geodesic segment in Γ_H, the convex hull of $P(B^{m+1}\lambda)$ is a finite subtree of T. Further, $P^{-1}(v) \cap B^{m+1}(\lambda)$ is of the form $i_v(\lambda_v)$ for all $v \in P(B^{m+1}(\lambda))$. This enables us to continue inductively. Define

$$B(\lambda) = \cup_{m \geq 0} B^m \lambda.$$

Note that the convex hull of $P(B(\lambda))$ in T is a locally finite tree T_1. Further $B(\lambda) \cap P^{-1}(v)$ is a geodesic in X_v for $v \in T_1$ and is empty otherwise.

Construction of retraction

One of the main theorems of [38] states that $B(\lambda)$ constructed above is uniformly quasiconvex. To do this we constructed a retraction Π_λ from (the vertex set of) X onto B_λ and showed that there exists $C_0 \geq 0$ such that $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d_X(x, y)$. Recall this construction from [38]. Let $\pi_v : X_v \to \lambda_v$ be a nearest point projection of X_v onto λ_v. Π_λ is defined on $\bigcup_{v \in T_1} X_v$ by

$$\Pi_\lambda(x) = i_v \cdot \pi_v(x) \quad \text{for} \quad x \in X_v.$$

If $x \in P^{-1}(T \setminus T_1)$ choose $x_1 \in P^{-1}(T_1)$ such that $d(x, x_1) = d(x, P^{-1}(T_1))$ and define $\Pi'_\lambda(x) = x_1$. Next define $\Pi_\lambda(x) = \Pi_\lambda \cdot \Pi'_\lambda(x)$.

Theorem 3.2 [38]. *There exists $C_0 \geq 0$ such that $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d(x,y)$ for x, y vertices of X. Further, $B(\lambda)$ is C_0-quasiconvex.*

We need one final lemma from [38]. Let $i : Y \to X$ denote inclusion.

Lemma 3.3. *There exists $A > 0$, such that if $a \in P^{-1}(v) \cap B(\lambda)$ for some $v \in T_1$ then there exists $b \in i(\lambda) = P^{-1}(v_0) \cap B(\lambda)$ with $d(a,b) \leq A d_T(Pa, Pb)$. Further, let $v_0, v_1, \ldots, v_n = v$ be the sequence of vertices on a geodesic in T connecting the root vertex v_0 to v. There exists a sequence $b = a_0, a_1, \ldots, a_n = a$ with $a_i \in P^{-1}(v_i) \cap B(\lambda)$ such that $d(a_i, a_j) \leq A d_T(Pa_i, Pa_j) = A d_T(v_i, v_j)$.*

The above lemma says that we can construct a quasi-isometric section of a geodesic segment $[v_0, v]$ ending at a.

DEFINITION.

An *A-quasi-isometric section* of $[v_0, v]$ ending at $a \in P^{-1}(v) \cap B(\lambda)$ is a sequence of points in X satisfying the conclusions of Lemma 3.3 above.

Note that the quasi-isometric sections considered are all images of $[v_0, v]$ where v_0 is the root vertex of T. Abusing notation slightly we will refer to the map or its image as a quasi-isometric section.

So far we have considered a tree of hyperbolic metric spaces. It is time to introduce the relevant groups.

Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let $\mathcal{G}$ denote the quotient graph T/G. The metric on T will be denoted by d_T. Assume $\mathcal{G}$ has only one edge and H is the stabilizer of this edge. This is the situation when G splits over H.

Suppose H is a vertex or edge subgroup. Further, suppose H is distorted in G. We would like to show that H has infinite height. Here is a brief sketch of the proof of the main theorem of this paper:

Since H is distorted, there exist geodesics $\lambda_i \subset \Gamma_H$ such that geodesics in Γ_G joining the end points of λ_i leave larger and larger neighborhoods of Γ_H. From the construction of $B(\lambda)$ it follows that the diameters $\mathrm{dia}(P(B(\lambda_i))) \to \infty$ as $i \to \infty$. The edges of T can be lifted to Γ_G and one can after a pigeon-hole principle argument look upon these lifts as conjugating elements. The geodesics in $B(\lambda_i) \cap P^{-1}(v)$ can be thought of as elements of H. Thus as $i \to \infty$ one obtains a sequence of elements $g_i \in G$ such that $\cap g_i^{-1} H g_i \neq 1$. This proves that H has infinite height. The next section is devoted to making this rigorous.

4. Proof of Main Theorem

We start our discussion with a basic lemma.

Lemma 4.1. *If $X_{v_0} = Y$ is distorted in X, there exist a sequence of geodesics λ_i in Y such that $\mathrm{dia}(P(B(\lambda_i))) \to \infty$ as $i \to \infty$, where the diameter is calculated with respect to the metric d_T.*

Proof. It follows from Lemma 3.3 that $B(\lambda_i)$ lies in an $A\ \mathrm{dia}(P(B(\lambda_i)))$ neighborhood of $i(\lambda_i)$ and hence of Y. Further from Theorem 3.2 a geodesic in X joining the end points of $i(\lambda_i)$ lies in a (uniform) C_0-neighborhood of $B(\lambda_i)$.

Since Y is distorted in X, there exist $\lambda_i \subset Y$ such that geodesics in X joining end points of λ_i leave an i-neighborhood of Y for $i = 1, 2, \ldots$.

Hence $i \leq A$ dia $(P(B(\lambda_i))) + C$.

The lemma follows. □

Construction of hallways

We would like to construct certain special subsets of $B(\lambda)$ closely related to the essential hallways of Bestvina and Feighn [3]. We retain the terminology.

DEFINITION.

A disk $f : [0,m] \times I \to X$ is a hallway of length m if it satisfies:

1. $f^{-1}(\cup X_v : v \in T) = \{0, 1, \ldots, m\} \times I$.
2. f maps $i \times I$ to a geodesic in X_v for some vertex space.
3. $(P \circ f) : [0,m] \times I \to T$ factors through the canonical retraction to $[0,m]$ and an isometry of $[0,m]$ to T.

DEFINITION.

A hallway is ρ-thin if $d(f(i,t), f(i+1,t)) \leq \rho$ for all i, t.

We will now construct A-thin hallways using the quasi-isometric sections of Lemma 3.3. The arguments are carried out for trees of metric spaces.

Given λ and $x \in B(\lambda)$ let Σ^x_λ be an A-quasi-isometric section of $[v_0, P(x)]$ into $B(\lambda)$ ending at x. From Lemma 3.3 such quasi-isometric sections exist. Further, if $a \in \Sigma^x_\lambda$ then define $\sigma^x_\lambda(a)$ to be a point $i(\lambda) \cap \Sigma^x_\lambda$. The choice involved in the definition of $\sigma^x_\lambda(a)$ is bounded purely in terms of A.

Lemma 4.2. *Suppose $Y = X_{v_0}$ is distorted in X. Then there exist geodesics $\lambda_i \subset Y, a_i, b_i, x_i, y_i \in B(\lambda_i)$ such that*

1. $d(x_i, y_i) \leq 1$.
2. $P(x_i) = P(y_i)$.
3. *μ_i is a geodesic subsegment of λ_i in Y joining $\sigma^{a_i}_{\lambda_i}(x_i)$ and $\sigma^{b_i}_{\lambda_i}(y_i)$ with length of μ_i greater than or equal to i.*

Proof. Suppose not. Then there exists $C \geq 0$ such that for all geodesics λ_i in Y and all $a_i, b_i, x_i, y_i \in B(\lambda_i)$ satisfying

1. $a_i, b_i, x_i, y_i \in B(\lambda_i)$.
2. $d(x_i, y_i) \leq 1$.
3. $P(x_i) = P(y_i)$.
4. μ_i is a geodesic subsegment of λ_i in Y joining $\sigma^{a_i}_{\lambda_i}(x_i)$ and $\sigma^{b_i}_{\lambda_i}(y_i)$.

We have length of μ_i less than or equal to C. For all $x \in B(\lambda_i)$ choose $a \in B(\lambda_i)$ such that $x \in \Sigma^a_{\lambda_i}$ and define

$$\pi(x) = \sigma^a_{\lambda_i}(x).$$

Recall that $\pi(x)$ is chosen from a set of (uniformly) bounded diameter. Thus we might as well take $a = x$. Note that π defines a retraction of $B(\lambda_i)$ onto λ_i.

For any $x, y \in B(\lambda_i)$ such that $P(x) = P(y)$ we have $d(\pi(x), \pi(y)) \leq Cd(x, y)$.

Next suppose $x, y \in B(\lambda_i)$, $d(P(x), P(y)) = 1$ and $d(x, y) \leq A$. Assume without loss of generality $d(P(x), v_0) < d(P(y), v_0)$. Then by Lemma 3.3 there exists $z \in B(\lambda_i) \cap \Sigma^y_{\lambda_i}$ such that $P(x) = P(z)$, $d(x, z) \leq 2A$ and hence $d(\pi(x), \pi(y)) \leq 2AC + C$.

Hence there exists C' such that for any λ_i and $x, y \in \lambda_i$, $d(\pi(x), \pi(y)) \leq C'd(x, y)$. Thus λ_i is uniformly quasiconvex in $B(\lambda_i)$ and hence (by Theorem 3.2) in X.

Therefore Y is quasiconvex in X, contradicting the hypothesis. $\square$

DEFINITION.

An A-thin hallway $\mathscr{H}$ with ends μ_0, μ_n trapped by A-quasi-isometric sections Σ_1 and Σ_2 is a collection of geodesics $\mu_i \subset X_{v_i}$, $i = 0, \ldots, n$ such that

1. $v_0, \ldots, v_n$ are successive vertices on a geodesic $[v_0, v_n]$ in T.
2. μ_i joins $\Sigma_1(v_i)$ to $\Sigma_2(v_i)$.

As before n is called the length of the hallway.

Note that the geodesics are allowed to have length 0.

COROLLARY 4.3. Existence of hallways

Suppose Y is distorted in X. Then there exist geodesics $\lambda_i \subset Y$ and A-thin hallways $\mathscr{H}_i$ with ends λ_i, η_i trapped by quasi-isometric sections Σ_{1i}, Σ_{2i} such that the lengths of λ_i and the hallway $\mathscr{H}_i$ are greater than i.

Proof. From Lemma 4.2 there exist geodesics $\lambda_i \subset Y$, $a_i, b_i, x_i, y_i \in B(\lambda_i)$ such that

1. $d(x_i, y_i) \leq 1$.
2. $P(x_i) = P(y_i)$.
3. μ_i is a geodesic subsegment of λ_i in Y joining $\sigma^{a_i}_{\lambda_i}(x_i)$ and $\sigma^{b_i}_{\lambda_i}(y_i)$ with length of μ_i greater than i.

Take $\Sigma_{1i} = \Sigma^{a_i}_{\lambda_i}$ $\Sigma_{2i} = \Sigma^{b_i}_{\lambda_i}$ and rename μ_i as λ_i (we are abusing notation slightly here).

Passing to a subsequence if necessary and arguing as in Lemma 4.1 we can assume that the length of $\mathscr{H}_i$ is greater than i.

The corollary follows. $\square$

Construction of annuli

The discussion so far has not entailed the use of group actions. We would like to establish a dictionary between the geometric objects constructed above and elements of a group G acting on T.

Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let $\mathscr{G}$ denote the quotient graph T/G. The metric on T will be denoted by d_T. Assume $\mathscr{G}$ has only one edge and H is the stabilizer of this edge. This is the situation when G splits over H. That is $G = G_1 *_H G_2$

or $G = G_1 *_H$. Then by the restrictions on the G-action on T, the inclusions of H into G_i are quasi-isometric embeddings.

The stabilizers of edges of T are conjugates of H. We can take $\Gamma_H = X_{v_0} = Y$, $\Gamma_G = X$ and $i : Y \to X$ the natural inclusion. Let $\lambda \subset Y$ be a geodesic.

Recall the construction of $B(\lambda)$ from the previous section. $B(\lambda)$ was constructed as the union of certain geodesics $\lambda_i \subset X_{v_i}$. Further, each λ_i was in the image of an edge space. Therefore if λ_i has a_i, b_i as its end points, then $a_i^{-1} b_i \in H$.

We need to now examine the hallways constructed above. Let $\mathscr{H} = \cup_{i=0,\dots,n} \mu_i$ be an A-thin hallway trapped between quasi-isometric sections Σ_1 and Σ_2 with ends μ_0 and μ_n. Note that each μ_i is a geodesic subsegment of some λ_i joining a_i, b_i and $a_i^{-1} b_i \in H$.

Since edge spaces are (uniformly) quasi-isometrically embedded in vertex spaces, there exists a constant D_1 such that if μ_i joins c_i, d_i then ${c_i}^{-1} d_i = u_i h_i v_i$, where $h_i \in H$, $|u_i| \leq \frac{D_1}{2}$ and $|v_i| \leq \frac{D_1}{2}$. ($|.|$ denotes length.) Also, from the definition of A-thin hallways trapped between quasi-isometric sections, we have

$$\Sigma_1(i) = c_i,$$
$$\Sigma_2(i) = d_i,$$
$$|\Sigma_1(i)^{-1} \Sigma_2(i)| \leq \frac{D_1}{2} \ \text{ for all } \ i.$$

DEFINITION.

An $(A + D_1)$-thin H-hallway $\mathscr{H}$ with ends μ_0, μ_n trapped by $(A + D_1)$-quasi-isometric sections Σ_1 and Σ_2 is a collection of geodesics $\mu_i \subset X_{v_i}$, $i = 0, \dots, n$ such that

1. $v_0, \dots, v_n$ are successive vertices on a geodesic $[v_0, v_n]$ in T.
2. μ_i joins $\Sigma_1(v_i) = c_i$ to $\Sigma_2(v_i) = d_i$.
3. $c_i^{-1} d_i \in H$.

The following lemma is the group-theoretic counterpart of Corollary 4.3 and follows from the discussion above.

Lemma 4.4. *Suppose $Y (= X_{v_0} = \Gamma_H)$ is distorted in $X (= \Gamma_G)$. Then there exist geodesics $\lambda_i \subset Y$ and $(A + D_1)$-thin H-hallways $\mathscr{H}_i$ with ends λ_i, η_i trapped by $A + D_1$-quasi-isometric sections Σ_{1i}, Σ_{2i} such that the lengths of λ_i and the hallway $\mathscr{H}_i$ are greater than i.*

We would now like to paste two of these H-hallways together along a common bounding quasi-isometric section.

Given $n > 0$ consider $(A + D_1)$-thin hallways $\mathscr{H}_i$ with one end $\lambda_i \subset Y = \Gamma_H$ of length n. Clearly there exist infinitely many distinct such from Lemma 4.4 (taking a long enough hallway with one end in Y and truncating it to one of length n gives such a hallway).

DEFINITION.

The ordered boundary $\Delta_{\mathscr{H}}$ of an H-hallway $\mathscr{H}$ of length n trapped by quasi-isometric sections Σ_1, Σ_2 is given by

$$\Delta_{\mathscr{H}} = \{\Sigma_1(v_{j-1})^{-1} \Sigma_1(v_j), \Sigma_2(v_{j-1})^{-1} \Sigma_1(v_j), : j = 1 \dots n\},$$

where $[v_0, v_n] \subset T$ is the geodesic in T to which $\mathscr{H}$ maps under P.

The ith element of the above set will be denoted by $\Delta_{\mathscr{H}}(i)$.

If the hallway is $A+D_1$-thin, then $|\Sigma_i(v_{j-1})^{-1}\Sigma_i(v_j)| \leq A+D_1$.

Since there exist infinitely many distinct $(A+D_1)$-thin H-hallways of length n and only finitely many words in G of length less than or equal to $(A+D_1)$, there exist (by the pigeon-hole principle) infinitely many distinct H-hallways of length n with the same ordered boundary Δ.

Choose two such hallways and glue one to the 'reflection' of the other. More precisely, let $\mathscr{H}_i = \cup_{j=1\ldots n}\mu_{ij}$ for $i = 1,2$ be two such hallways. Let μ_{ij} have $a_{ij}, b_{ij} \in X_{v_j} \subset \Gamma_G$ as its end points.

Then since $\mathscr{H}_i$ are $(A+D_1)$-thin H-hallways with the same ordered boundary, we have

$$a_{ij}^{-1}b_{ij} \in H,$$

$$a_{1j}^{-1}a_{1,j+1} = a_{2j}^{-1}a_{2,j+1},$$

$$b_{1j}^{-1}b_{1,j+1} = b_{2j}^{-1}b_{2,j+1}.$$

Let η_j denote a geodesic in X_{v_j} joining a_{1j} and $c_{1j} = b_{1j}b_{2j}^{-1}a_{2j}$. Then $\mathscr{H} = \cup_{j=1\ldots n}\eta_j$ is an $(A+D_1)$-thin H-hallway. If Δ be its ordered boundary, then it follows from the above equations that $\Delta(2i) = \Delta(2i-1)$ for $i = 1\ldots n$.

DEFINITION.

An H-hallway of length n with ordered boundary Δ is called an H-annulus if $\Delta(2i) = \Delta(2i-1)$ for $i = 1\ldots n$.

The above definition is related to the annuli of Bestvina and Feighn [3].

From the above discussion and Lemma 4.4 the following crucial theorem follows:

Theorem 4.5. *Suppose $Y(=X_{v_0} = \Gamma_H)$ is distorted in $X(=\Gamma_G)$. Then there exist geodesics $\lambda_i \subset Y$ and $(A+D_1)$-thin H-annuli $\mathscr{H}_i$ with ends λ_i, η_i trapped by $(A+D_1)$-quasi-isometric sections Σ_{1i}, Σ_{2i} such that the lengths of λ_i and the hallway $\mathscr{H}_i$ are greater than i. In fact there exist infinitely many distinct such H-annuli with the same ordered boundary.*

The main theorem of this paper follows from Theorem 4.5 by unravelling definitions. We state this below.

Theorem 4.6. *Let G be a hyperbolic group splitting over H (i.e. $G = G_1 *_H G_2$ or $G = G_1 *_H$) with hyperbolic vertex and edge groups. Further, assume the two inclusions of H are quasi-isometric embeddings. Then H is of finite height in G if and only if it is quasiconvex in G.*

Proof. Suppose H is distorted in G. Then from Theorem 4.5 there exists an H-annulus $\mathscr{H} = \cup_{i=0\ldots n}\lambda_i$ of length n such that $|\lambda_0| > n$. (In fact there are infinitely many distinct such. However, we start off with one in the interests of notation.)

Let Δ be the ordered boundary of $\mathscr{H}$. By definition of H-annulus $\Delta(2i) = \Delta(2i-1)$ for $i = 1\ldots n$. Let c_i, d_i be the endpoints of λ_i such that

$$\Delta(2i-1) = c_{i-1}^{-1}c_i = d_{i-1}^{-1}d_i = \Delta(2i).$$

Also $c_i^{-1}d_i = h_i \in H$. Let $g_i = \Delta(2)\ldots\Delta(2i)$. Reading relations around 'quadrilaterals' we have,

$$h_{i-1} = \Delta(2i)h_i\Delta(2i)^{-1} \text{ for all } i = 1\ldots n.$$

Therefore

$$h_0 = g_i h_i g_i^{-1} \text{ for all } i = 1\ldots n.$$

Recall that $P : \Gamma_G \to T$ is the projection onto T. Since $P(c_0 g_i) \neq P(c_0 g_j)$ for $i \neq j$ we have n essentially distinct conjugates $g_i H g_i^{-1}$ whose intersection contains $h_0 \neq 1$.

Now we need the fact that there are infinitely many distinct H-annuli (Theorem 4.5) with the same ordered boundary. Without loss of generality, let this boundary be Δ above. The above argument then furnishes infinitely many distinct $h \in H \cap_{i=1\ldots n} g_i H g_i^{-1}$.

Thus given any $n > 0$ there exist $n+1$ essentially distinct conjugates of H whose intersection is infinite. Therefore H has infinite height. Along with Theorem 1.1 this proves the Theorem. □

5. Consequences and questions

Malnormality

We deduce a couple of group-theoretic consequences of Theorem 4.6.

DEFINITION.

A subgroup H of a group G is said to be *malnormal* in G if $gHg^{-1} \cap H = 1$ for all $g \notin H$.

PROPOSITION 5.1.

Let $G = G_1 *_H G_2$ be a hyperbolic group such that G_1, G_2, H are hyperbolic and the two inclusions of H are quasi-isometric embeddings. If H is malnormal in one of G_1 or G_2 then H is quasiconvex in G.

Proof. Assume without loss of generality that H is malnormal in G_2. Let $g \in G \backslash H$ and $h, h_1 \in H$ be such that $ghg^{-1} = h_1 \neq 1$. Let $g = a_1 b_1 \ldots a_n b_n$ with $a_i \in G_1$ and $b_i \in G_2$. Then by normal form for free products with amalgamation ([28], p. 178) we have $b_n H b_n^{-1} \in H$ and hence $b_n \in H$ by malnormality of H in G_2. Continuing inductively, we get $a_i \ldots a_n h a_n^{-1} \ldots a_i^{-1}$ and $b_i \in H$ for all $i = 1\ldots n$. In particular $g \in G_1$. Therefore $H \cap gHg^{-1} \neq 1$ implies $g \in G_1$.

Since H is quasi-isometrically embedded in G_1 we have by Theorem 1.1 that H has finite height in G_1. Therefore by the above argument H has finite height in G. Finally by Theorem 4.6, H is quasiconvex in G. □

The above proposition holds good if malnormal is replaced by height zero.

A similar argument using Britton's lemma ([28], p. 178) gives the following:

PROPOSITION 5.2.

Let $G = G_1 *_H$ be a hyperbolic group such that G_1, H are hyperbolic and the two images H_1, H_2 of H are quasiconvex in G_1. If $gH_1g^{-1} \cap H_2$ is finite for all $g \in G_1$ then H is quasiconvex in G.

The hypotheses in the above propositions cannot be relaxed as the following example shows.

Example. Let $G_i = \{a_i, b_{1i}, b_{2i}, c_{1i}, c_{2i} | a_i b_{ji} a_i^{-1} = c_{ji}, j = 1,2\}$ be two copies (for $i = 1,2$) of a group isomorphic to the free group on 3 generators.

Let $H = \{b_1, b_2, c_1, c_2\}$ be the free group on 4 generators. Let $i : H \to G_1$ be given by sending b_i to b_{i1} and c_i to c_{i1} for $i = 1,2$.

Let $j : H \to G_2$ be given by sending b_i to b_{i2} for $i = 1,2$ and c_i to 'long words' u_i in c_{12} and c_{22} such that the 'flare' condition of [3] is satisfied for the free product with amalgamation $G = G_1 *_H G_2$.

In fact one gets

$$G = \langle a_1, a_2, c_1, c_2 | a_1 a_2^{-1} c_i a_2 a_1^{-1} = u_i(c_1, c_2), i = 1,2 \rangle$$

such that this is a small cancellation presentation with G hyperbolic.

It is clear that the subgroup generated by c_1, c_2 is a free group on two generators with infinite height in G. Hence the amalgamating subgroup H above is of infinite height.

In [30] McMullen shows that glueing an acylindrical, atoroidal hyperbolic 3-manifold to another hyperbolic atoroidal 3-manifold along a common incompressible boundary surface S gives a hyperbolic 3-manifold in which S is quasifuchsian. We deduce the following group theoretic version of this from Proposition 5.1 above.

COROLLARY 5.3.

Let M_1 be a hyperbolic atoroidal acylindrical 3-manifold and S_1 an incompressible surface in its boundary. Let M_2 be a hyperbolic atoroidal 3-manifold and S_2 an incompressible surface in its boundary. If S_1 and S_2 are homeomorphic then glueing M_1 and M_2 along this common boundary $S (= S_1 = S_2)$ one obtains a 3-manifold M such that

1. *$\pi_1(M)$ is hyperbolic.*
2. *$\pi_1(S)$ is quasiconvex in $\pi_1(M)$.*

Proof. Hyperbolicity of $\pi_1(M)$ follows from the combination theorem of Bestvina and Feighn [3]. Quasiconvexity follows from Proposition 5.1 above. □

Using Proposition 5.2 one can deduce similar results.

Graphs of hyperbolic groups

The main argument of this paper does not generalize directly to graphs of hyperbolic groups satisfying the quasi-isometrically embedded condition. Given a distorted edge or vertex group $H \subset G$, the pigeon-hole principle argument of the previous section does furnish an edge group H_1 of infinite height in G such that a conjugate of H intersects H_1 in a distorted subgroup of G.

However H and H_1 need not be the same. The basic problem lies in dealing with quasiconvex subgroups of edge (or vertex) groups that are distorted in G. We state the problem explicitly:

Question. Suppose G splits over H satisfying the hypothesis of Theorem 4.6 and H_1 is a quasiconvex subgroup of H. If H_1 has finite height in G is it quasiconvex in G? More generally, if H_1 is an edge group in a hyperbolic graph of hyperbolic groups satisfying the qi-embedded condition, is H quasiconvex in G if and only if it has finite height in G?

The above question is a special case of the general question of Swarup on characterizing quasiconvexity in terms of finiteness of height.

There are two cases where a complete answer to the above question is known. These are extensions of $\mathbb{Z}$ by surface groups [48] or free groups [5,39]. Both these solutions involve a detailed analysis of the ending laminations [37].

Other questions

A closely related problem [9,35] can be formulated in more geometric terms:

Question. Let X_G be a finite 2 complex with fundamental group G. Let X_H be a cover of X_G corresponding to the finitely presented subgroup H. Let $I(x)$ be the injectivity radius of X_H at x.

Does $I(x) \to \infty$ as $x \to \infty$ imply that H is quasi-isometrically embedded in G?

A positive answer to this question for G hyperbolic would provide a positive answer to Swarup's question.

The answer to this question is negative if one allows G to be only finitely generated instead of finitely presented as the following example shows:

Example. Let $F = \{a,b,c,d\}$ denote the free group on four generators. Let $u_i = ab^i$ and $v_i = cd^{f(i)}$ for some function $f : \mathbb{N} \to \mathbb{N}$. Introducing a stable letter t conjugating u_i to v_i one has a finitely generated HNN extension G. The free subgroup generated by a,b provides a negative answer to the question above for suitable choice of f. In fact one only requires that f grows faster than any linear function.

If f is recursive one can embed the resultant G in a finitely presented group by Higman's embedding theorem. But then one might lose malnormality of the free subgroup generated by a,b. A closely related example was shown to the author by Steve Gersten.

A counterexample to the general question of Swarup might provide a means of constructing acyclic non-hyperbolic finitely presented groups without $(Z+Z)$ answering a question of Bestvina and Brady [9]. Suppose H is a malnormal torsion-free hyperbolic subgroup of a hyperbolic torsion-free group G. If H is distorted in G, then doubling G along H (i.e. $G*_H G$) one gets a finitely presented acyclic group which is not hyperbolic, nor does it contain $(Z+Z)$. This was independently observed by Sageev.

On the other hand one might develop an analog of Thurston's theory of pleated surfaces [52] for hyperbolic subgroups H of hyperbolic groups G following Gromov's suggestion about using hyperbolic simplices ([24], §8.3). Let X_G be a finite 2 complex with fundamental group G. Let X_H be a cover of X_G corresponding to the finitely presented subgroup H. Let K be a finite complex with fundamental group H. One needs to consider homotopy equivalences between K and X_H. Then one might try to prove a geometric analog of Paulin's theorem [42] so as to obtain a limiting action of a subgroup of H on a limit metric space (in [42] the limiting object is an $\mathbb{R}$-tree). This would be an approach to answering the above question affirmatively.

The general problem attempted in this paper is one of characterizing quasiconvexity of subgroups H of hyperbolic groups G purely in terms of group theoretic notions. Swarup's question aims at one such characterization. One might like stronger criteria, though this might be over-optimistic. Consider the following conditions:

1. $H \subset G$ is not quasiconvex.
2. H has infinite height in G.
3. H has *strictly infinite height* in G, i.e. there exist infinitely many essentially distinct conjugates $g_iHg_i^{-1}, i = 1, 2, \ldots$ such that $\cap_i g_iHg_i^{-1} \neq \emptyset$.
4. There exists an element $g \in G$ such that $g^i \notin H$ for $i \neq 0$ and $\cap_i g^iHg^{-i} \neq \emptyset$.
5. There exists an element $g \in G$ such that $g^i \notin H$ for $i \neq 0$ and $\cap_i g^iH_1g^{-i} \neq \emptyset$ where H_1 is a subgroup of H isomorphic to a free product of free groups and surface groups.
6. There exists an element $g \in G$ such that $g^i \notin H$ for $i \neq 0$ and $\cap_i g^iH_1g^{-i} \neq \emptyset$ where H_1 is a *quasiconvex* subgroup of H isomorphic to a free product of free groups and surface groups.

It is clear that $(6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ (the last implication follows from [23]). One would like to know if any of these can be reversed.

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