H_1–SEMISTABILITY FOR PROJECTIVE GROUPS

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Abstract. We initiate the study of the asymptotic topology of groups that can be realized as fundamental groups of smooth complex projective varieties with holomorphically convex universal covers (these are called here as holomorphically convex groups). We prove the H_1-semistability conjecture of Geoghegan for holomorphically convex groups. In view of a theorem of Eyssidieux, Katzarkov, Pantev and Ramachandran [EKPR], this implies that linear projective groups satisfy the H_1-semistability conjecture.

Contents

1. Introduction

1.1. **Statement of results.** The homological semistability conjecture formulated by Geoghegan, [Gui, Conjecture 5, Section 6.4], is equivalent to the statement that $H^2(G, \mathbb{Z}G)$ is free abelian for every one-ended finitely presented group [Ge, Section 13.7], [GeMi1, GeMi2]. (Geoghegan's conjecture was formulated originally as a question in 1979 [Gui].) This conjecture has been established (in a stronger form) for several special classes of groups arising naturally in the context of geometric group theory: One-relator groups, free products of semistable groups with amalgamation along infinite groups, extensions of infinite groups by infinite groups, (Gromov) hyperbolic groups, Coxeter groups, Artin groups

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and so on [Mi1, Mi2, Mi3, MT1, MT2]. In this paper we establish this conjecture for groups coming from a completely different geometric source: fundamental groups of smooth projective varieties with holomorphically convex universal cover. For convenience we call them as holomorphically convex groups.

Theorem 1.1 (See Theorem 4.7)**.** *Let $G = \pi_1(X)$ be a torsion-free holomorphically convex group. Then $H^2(G, \mathbb{Z}G)$ is a free abelian group.*

A group is called linear if it is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some n. If X is a smooth complex projective variety such that $\pi_1(X)$ is linear, then the universal cover X is holomorphically convex [EKPR]. Therefore, Theorem 1.1 has the following corollary:

Corollary 1.2 (See Corollary 4.10)**.** *If G is a linear torsion-free projective group, then $H^2(G, \mathbb{Z}G)$ is a free abelian group.*

It is an open question whether the dualizing module of a duality group G is a free abelian group or not [Br, p. 224]. It follows from Theorem 1.1 that this is indeed the case if G is holomorphically convex of cohomological dimension two.

Proposition 1.3 (See Proposition 4.12)**.** *Let G be a holomorphically convex group of dimension two. Then G is a duality group with free dualizing module.*

The key ingredients in the proofs of Theorem 1.1, Corollary 1.2 and Proposition 1.3 include

(1) topology (especially second homotopy group) of smooth complex projective surfaces with holomorphically convex universal cover (Section 3),
(2) a spectral sequence argument for computing group cohomology with local coefficients (Section 4), which was inspired in part by an argument of Klingler [Kl],
(3) homological group theory of duality, inverse duality and Poincaré duality groups (Section 4.3), and
(4) a theorem of Eyssidieux, Katzarkov, Pantev and Ramachandran [EKPR] showing that complex projective manifolds with linear fundamental group have holomorphically convex universal cover.

2. Preliminaries

2.1. **Holomorphic convexity.** A connected complex manifold M is called **holomorphically convex** if for every sequence of points $\{x_i\}_{i=1}^{\infty}$ of M without any accumulation point, there is a holomorphic function f on M such that the sequence of nonnegative numbers $\{|f(x_i)|\}_{i=1}^{\infty}$ is unbounded. In this paper, we initiate the study of a natural subclass of projective groups, namely groups that can be realized as fundamental groups of smooth complex projective varieties, of dimension at least two, with holomorphically convex universal covers. We shall call such groups **holomorphically convex groups**. So any holomorphically convex group is the fundamental group of a smooth complex projective surface with holomorphically convex universal cover (see Proposition 3.1). A conjecture of Shafarevich asserts that all smooth projective varieties have holomorphically convex universal covers.

Let $\mathcal{K}$, $\mathcal{P}$, $\mathcal{HC}$ denote respectively the class of Kähler groups, projective groups and holomorphically convex groups.

It is clear that $\mathcal{HC} \subset \mathcal{P} \subset \mathcal{K}$. The question of reversing the last inclusion is a well-known open problem. The following test question asks explicitly if the first inclusion can be reversed. It can be thought of as a group-theoretic version of the Shafarevich conjecture.

Question 2.1. (Group-theoretic Shafarevich conjecture) Is $\mathcal{HC} = \mathcal{P}$?

The (usual) Shafarevich conjecture certainly implies a positive answer to Conjecture 2.1 via the Lefschetz hyperplane Theorem.

2.2. **Semistability.** A subcomplex A of a CW complex X is called **full** if it is the largest subcomplex of X among all subcomplexes with the property that the 0-skeleton coincides with the 0-skeleton A_0 of A. The full subcomplex of X generated by the vertices of $X^0 \setminus A^0$ is called the CW **complement** of A, and it is denoted by $(X \setminus A)_{CW}$.

An inverse sequence $\{G_n\}_{n\in\mathbb{N}}$ of groups is called **semistable** if for each n there exists an integer $\phi(n) \geq n$ such that for all $k \geq \phi(n)$, we have image$(f_n^{\phi(n)})$ = image(f_n^k), where $f_j^i : G_i \longrightarrow G_j$ is the homomorphism in the inverse sequence.

A CW complex X satisfies H_1–**semistability** if the sequence $\{H_1((\widetilde{X}\setminus K_n)_{CW},\, \mathbb{Z})\}_{n\in\mathbb{N}}$ is semistable. A finitely presented group G satisfies H_1–**semistability** if some (hence any) finite complex X with $\pi_1(X) = G$ satisfies H_1–semistability.

It follows from work of Geoghegan and Mihalik [Ge, Section 13.7], [GeMi1, GeMi2] that G satisfies H_1–semistability if and only if $H^2(G,\, \mathbb{Z}G)$ is free abelian.

In a certain sense, the notion of semistability was motivated by a Theorem of Farrell [Fa] which may be thought of as a $H^2(G,\, \mathbb{Z}G)$ (or two-dimensional) version of the fact that a finitely presented group has 0, 1, 2 or infinitely many ends. Geoghegan has given examples of *finitely generated* (but not finitely presented) groups G such that $H^2(G, \mathbb{Z}G)$ is not a free abelian group (see [Ge, p. 321, Ex. 1]).

We shall refer to the integral cohomological dimension of a group G simply as its dimension. Note therefore that any group of finite dimension is *torsion-free*.

3. Holomorphically convex universal cover

A complex analytic manifold M is holomorphically convex if and only if it admits a proper holomorphic map $\Pi : M \longrightarrow Q$ to a Stein space Q such that $\Pi_*\mathcal{O}_M = \mathcal{O}_Q$. The Stein space Q is referred to as the Cartan–Remmert reduction of M [Re].

Proposition 3.1. *Let G be a holomorphically convex group, meaning $G = \pi_1(X)$, where X is a smooth complex projective variety with holomorphically convex universal cover and* $\dim X \geq 2$. *Then there is a smooth complex projective surface S with holomorphically convex universal cover such that $G = \pi_1(S)$.*

Proof. Assume that $\dim X > 2$. Let $\widetilde{X}$ be the universal cover of X. Fix an embedding $X \hookrightarrow \mathbb{CP}^n$. By the Lefschetz hyperplane theorem, the inclusion

$$\iota : Y := X \cap H \hookrightarrow X$$

induces an isomorphism of fundamental groups, where H is a suitable hyperplane. Hence ι lifts to a proper holomorphic embedding $\widetilde{\iota} : \widetilde{Y} \longrightarrow \widetilde{X}$, where $\widetilde{Y}$ is the universal cover of Y. Therefore, $\widetilde{Y}$ is a complex analytic submanifold of $\widetilde{X}$.

Since the embedding of $\widetilde{Y}$ in $\widetilde{X}$ is proper, it follows that if $\{x_n\}$ is a sequence of points of Y without any accumulation point in Y, then $\{x_n\}$ does not have have any accumulation point in X. By holomorphic convexity of $\widetilde{X}$, there is a holomorphic function f on $\widetilde{X}$ such that the sequence $\{|f(x_n)|\}$ is unbounded. Considering the function $f \circ \iota$ we conclude that $\widetilde{Y}$ is holomorphically convex. Now the proposition is deduced inductively. □

3.1. Higher cohomology groups.

In the rest of this section, we assume X to be a smooth complex projective surface such that the universal cover $\widetilde{X}$ of X is noncompact and holomorphically convex. In particular, $\pi_1(X)$ is an infinite group. The Cartan–Remmert reduction of $\widetilde{X}$ will be denoted by $\widetilde{Y}$. We note that $\widetilde{Y}$ is not a point because $\widetilde{X}$ is noncompact.

Narasimhan, [Na], and Goresky–MacPherson, [GoMa], gave restrictions on the topology of $\widetilde{Y}$.

Theorem 3.2 ([Na, GoMa])**.** *Let M be a (not necessarily smooth) connected complex projective surface with Stein universal cover $\widetilde{M}$. Then $H^i(\widetilde{M},\, \mathbb{Z}) = 0$ for $i \geq 3$. Moreover, $\widetilde{M}$ is homotopy equivalent to a 2-dimensional* CW*-complex.*

3.2. The second homotopy group.

Let X be a smooth complex projective surface, and let $f : X \longrightarrow X'$ be the minimal model. Then $f_* : \pi_1(X) \longrightarrow \pi_1(X')$ is an isomorphism; this is because X is obtained by successive blow-up of points starting with X'. Therefore, we may, and we will, assume that the surface X under consideration is minimal.

We use the notation $H_i(M)$ (respectively, $H^i(M)$) to denote $H_i(M,\, \mathbb{Z})$ (respectively, $H^i(M,\, \mathbb{Z})$).

We recall a theorem of Andreotti and Narasimhan [AN].

Theorem 3.3 ([AN])**.** *Let M be a (not necessarily smooth) complex projective surface whose universal cover $\widetilde{M}$ is Stein. Then the second homotopy group $\pi_2(M)$ is free abelian.*

Proposition 3.4. *Let X be a smooth complex projective minimal surface whose universal cover $\widetilde{X}$ is holomorphically convex. Then the second homotopy group $\pi_2(X)$ is free abelian.*

In [Gur], Gurjar proved that the second homotopy group of X is torsion-free if the universal cover of X is holomorphically convex. Gurjar has communicated to us an observation due to Deligne that the proof in [Gur] leads to Proposition 3.4. He also showed us his notes on [Gur]. Since Proposition 3.4 is not available in print, we supply here a proof based on notes of Gurjar on [Gur].

Let

$$\phi : \widetilde{X} \longrightarrow \widetilde{Y} \tag{3.1}$$

be the Cartan–Remmert reduction of $\widetilde{X}$. There are two cases to consider:

(1) The Cartan–Remmert reduction $\widetilde{Y}$ is a Riemann surface. In this case, Proposition 3.4 has been proved by Gurjar [Gur].
(2) The Stein space $\widetilde{Y}$ is of complex dimension two. Andreotti and Narasimhan, [AN], have proved that in this case $H_2(\widetilde{X})$ is torsion-free. They further show that if $\widetilde{X}$ has only finitely many compact analytic subvarieties, then $H_2(\widetilde{X})$ is free abelian.

In the rest of this subsection we assume that the complex dimension of $\widetilde{Y}$ is two.

The space $\widetilde{Y}$ is normal. There is a discrete set P of points of $\widetilde{Y}$ such that

(a) P consists of the singularities of $\widetilde{Y}$ and some smooth points, and
(b) $\widetilde{X}$ is obtained by resolving singularities of $\widetilde{Y}$ and blowing up the smooth points in P.

Since X is minimal, we know that X, hence $\widetilde{X}$, does not contain any smooth rational curve that can be contracted. Hence P consists of singularities alone.

It follows that every non-trivial fiber (meaning the fiber is not a single point) F_0 of ϕ in (3.1) is a finite union of irreducible projective curves, and $H_2(F_0)$ is a free abelian group, generated freely by the homology classes of the irreducible components of F_0 [Gur]. Let F denote the union of all nontrivial fibers of ϕ. Hence $H_2(F)$ is a free abelian group, generated freely by the homology classes of all the irreducible components of all the non-trivial fibers of ϕ. Also $\phi|_{\widetilde{X}\setminus F} : \widetilde{X} \setminus F \longrightarrow \widetilde{Y} \setminus P$ is a homeomorphism.

Lemma 3.5. *Let $\iota : F \longrightarrow \widetilde{X}$ be the inclusion map. Then*

$$\iota_* : H_2(F) \longrightarrow H_2(\widetilde{X})$$

is an injection. Also $H_3(\widetilde{X}) = 0$.

Proof. First observe that $H_3(\widetilde{X};\, F) = H_3(\widetilde{Y};\, P)$. Indeed, this follows from the fact that the spaces obtained from $\widetilde{X}$ and $\widetilde{Y}$ by coning off F and P respectively, are homotopy equivalent. Since P is a discrete set of points, we have $H_3(\widetilde{Y}) = H_3(\widetilde{Y};\, P)$ by the relative homology exact sequence for the pair $(\widetilde{Y}\,, P)$. As $\widetilde{Y}$ is Stein, we have $H_3(\widetilde{Y}) = 0$ by Theorem 3.2. Hence $H_3(\widetilde{X};\, F) = 0$. Now the relative homology exact sequence for the pair $(\widetilde{X}, F)$

$$H_3(\widetilde{X};\, F) \longrightarrow H_2(F) \longrightarrow H_2(\widetilde{X})$$

yields the first statement of the lemma.

Since $H_3(\widetilde{X};\, F) = 0 = H_3(F)$, the relative homology exact sequence for the pair $(\widetilde{X};\, F)$

$$H_3(F) \longrightarrow H_3(\widetilde{X}) \longrightarrow H_3(\widetilde{X};\, F)$$

yields the second statement. □

Proof of Proposition 3.4: Let $N \subset \widetilde{Y}$ be a disjoint union of compact contractible neighborhoods of points in P. Let

$$\widetilde{N} = \phi^{-1}(N) \subset \widetilde{X}\,.$$

Then $\widetilde{N} \cup (\widetilde{X} \setminus F) = \widetilde{X}$ and $N \cup (\widetilde{Y} \setminus P) = \widetilde{Y}$. We have the following commutative diagram from the Mayer–Vietoris sequences for these two spaces:

$$\begin{array}{ccccccccc}
H_3(\widetilde{X}) & \longrightarrow & H_2(\partial N) & \longrightarrow & H_2(\widetilde{X}\setminus F)\oplus H_2(\widetilde{N}) & \xrightarrow{i_1} & H_2(\widetilde{X}) & \xrightarrow{j_1} & \\
 & & \Big\downarrow {\scriptstyle \phi_*}\,\wr & & \Big\downarrow {\scriptstyle \phi_*}\,\wr \qquad \Big\downarrow & & \Big\downarrow \phi_* & & \\
H_3(\widetilde{Y})\,(=0) & \longrightarrow & H_2(\partial N) & \longrightarrow & H_2(\widetilde{Y}\setminus P)\oplus \{0\} & \xrightarrow{i_2} & H_2(\widetilde{Y}) & \xrightarrow{j_2} &
\end{array} \tag{3.2}$$

$$\begin{array}{ccccccc}
\xrightarrow{j_1} & H_1(\partial N) & \longrightarrow & H_1(\widetilde{X}\setminus F)\oplus H_1(\widetilde{N}) & \longrightarrow & 0 \\
 & \Big\downarrow {\scriptstyle \phi_*}\,\wr & & \Big\downarrow {\scriptstyle \phi_*}\,\wr \qquad \Big\downarrow & & \\
\xrightarrow{j_2} & H_1(\partial N) & \longrightarrow & H_1(\widetilde{Y}\setminus P)\oplus \{0\} & \longrightarrow & H_1(\widetilde{Y})
\end{array}$$

In the above diagram, $H_3(\widetilde{Y}) = 0$ by Theorem 3.2. Also $i_1|_{H_2(\widetilde{N})}$ is injective by Lemma 3.5. Hence by (a slight modification of the proof of) the 5-lemma, the homomorphism ϕ_* induces an isomorphism between $H_2(\widetilde{X})/i_1(H_2(\widetilde{N}))$ and a subgroup of $H_2(\widetilde{Y})$.

Now, $H_2(\widetilde{Y})$ is a free abelian group, because $\widetilde{Y}$ is Stein (see Theorem 3.3). Hence the the homomorphism ϕ_* induces an isomorphism between $H_2(\widetilde{X})/i_1(H_2(\widetilde{N}))$ and a free abelian group. It follows that $H_2(\widetilde{X})$ is isomorphic to $H_2(\widetilde{N}) \oplus H_2(\widetilde{Y})$ as $i_1|_{H_2(\widetilde{N})}$ is injective. But $H_2(\widetilde{N}) = H_2(F)$ is free. Hence $H_2(\widetilde{X})$ is free.

Finally, since $\widetilde{X}$ is simply connected, it follows from the Hurewicz theorem that $\pi_2(X) = \pi_2(\widetilde{X}) = H_2(\widetilde{X})$ is a free abelian group. □

3.3. The second (co)homology groups.

Corollary 3.6. *Let X be a smooth complex projective minimal surface whose universal cover $\widetilde{X}$ is holomorphically convex. Then the second homology group $H_2(\widetilde{X}, \mathbb{Z})$ is free abelian. Also, the second cohomology group is isomorphic to* $\mathrm{Hom}(H_2(\widetilde{X}, \mathbb{Z}), \mathbb{Z})$*, and it is a direct product of copies of $\mathbb{Z}$.*

Proof. The first statement follows from the Hurewicz theorem as $\widetilde{X}$ is simply connected.

The second statement follows from the universal coefficient theorem. Since $H_1(\widetilde{X}, \mathbb{Z}) = 0$, it follows that $H^2(\widetilde{X}, \mathbb{Z})$ is torsion-free. Hence $H^2(\widetilde{X}, \mathbb{Z})$ is isomorphic to $\mathrm{Hom}(H_2(\widetilde{X}, \mathbb{Z}), \mathbb{Z})$, which is a direct product of copies of $\mathbb{Z}$, because $H_2(\widetilde{X}, \mathbb{Z})$ is a free abelian group as per the first statement. □

4. H_1–SEMISTABILITY

4.1. Second cohomology with local coefficients.

Lemma 4.1. *Let X be a smooth complex projective surface with holomorphically convex universal cover $\widetilde{X}$. Assume that $G := \pi_1(X)$ is torsion-free. Let $\phi : \widetilde{X} \longrightarrow \widetilde{Y}$ be the Cartan–Remmert reduction. Assume that $\widetilde{Y}$ is an open Riemann surface. Then G is isomorphic to the fundamental group of a compact Riemann surface.*

Proof. The Riemann surface $\widetilde{Y}$ must be biholomorphic to either the complex plane or the upper half plane [Gur, p. 703]. Since G is torsion-free, and ϕ is a proper G–equivariant map, it follows that G acts on Y freely properly discontinuously and cocompactly by holomorphic automorphisms of $\widetilde{Y}$. Hence $\widetilde{Y}/G$ must be a closed Riemann surface. □

Lemma 4.2. *Let X be a smooth complex projective minimal surface with holomorphically convex universal cover $\widetilde{X}$. Let $\phi : \widetilde{X} \longrightarrow \widetilde{Y}$ be the Cartan–Remmert reduction. Assume that $\widetilde{Y}$ is a complex surface. Also, assume that $G := \pi_1(X)$ is torsion-free. Then*

$$\phi^* : H^4_c(\widetilde{X}) \longrightarrow H^4_c(\widetilde{Y})$$

is an isomorphism.

Proof. As before, P denotes the singular locus of $\widetilde{Y}$, which is a discrete subset of points because $\widetilde{Y}$ is normal. Let

$$B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots$$

be a (relatively) compact exhaustion of $\widetilde{Y}$ such that $\partial B_n \cap P = \emptyset$ for all n. Then

$$H^4(\widetilde{Y};\, (\widetilde{Y} \setminus B_n) \cup P) = H^4(\widetilde{Y};\, (\widetilde{Y} \setminus B_n)),$$

because $P \cap B_n$ is a finite set of points.

Let $\widetilde{B}_n := \phi^{-1}(B_n)$. Then

$$\widetilde{B}_1 \subset \widetilde{B}_2 \subset \cdots \subset \widetilde{B}_n \subset \cdots$$

is a (relatively) compact exhaustion of $\widetilde{X}$. Also, if $F = \phi^{-1}(P)$, then

$$H^4(\widetilde{X};\, (\widetilde{X} \setminus \widetilde{B}_n) \cup F) = H^4(\widetilde{X};\, (\widetilde{X} \setminus \widetilde{B}_n)),$$

because F is a union of curves (its real dimension is two).

Further, since $\partial B_n \cap P = \emptyset$, it can be deduced that

$$H^4(\widetilde{X};\, (\widetilde{X} \setminus \widetilde{B}_n) \cup F) = H^4(\widetilde{Y};\, (\widetilde{Y} \setminus B_n) \cup P).$$

Indeed, this is easily seen from the fact that the space obtained by coning off $(\widetilde{X} \setminus \widetilde{B}_n) \cup F$ in $\widetilde{X}$ is homotopy equivalent to the space obtained by coning off $(\widetilde{Y} \setminus B_n) \cup P$ in $\widetilde{Y}$.

Hence ϕ^* induces an isomorphism between $H^4(\widetilde{X};\, (\widetilde{X} \setminus \widetilde{B}_n)$ and $H^4(\widetilde{Y};\, (\widetilde{Y} \setminus B_n))$ for all n. Now the lemma follows by taking limits. □

A Mayer–Vietoris argument gives the following:

Lemma 4.3. *Let X be a smooth complex projective minimal surface with holomorphically convex universal cover $\widetilde{X}$. Let $\phi : \widetilde{X} \longrightarrow \widetilde{Y}$ be the Cartan–Remmert reduction. Assume that the complex dimension of $\widetilde{Y}$ is two. Also assume that $G := \pi_1(X)$ is torsion-free. Define $Y := \widetilde{Y}/G$. Then $H^4(Y) = \mathbb{Z}$.*

Proof. Since ϕ is equivariant with respect to the actions of G on $\widetilde{X}$ and $\widetilde{Y}$, it follows that Y is obtained from X by collapsing some finitely many complex curves (possibly singular) in X to points. Let $A_1, \cdots, A_n$ be the connected components that are thus collapsed to points. Then Y is homotopy equivalent to X with cones cA_i attached to A_i, $1 \leq i \leq n$. The cohomological Mayer-Vietoris sequence gives

$$H^3(\bigcup_{i=1}^n A_i) \longrightarrow H^4(Y) \longrightarrow H^4(X) \oplus H^4(\bigcup_{i=1}^n cA_i) \longrightarrow H^4(\bigcup_{i=1}^n A_i).$$

Since each A_i is 2-dimensional, and each cA_i is contractible, it follows that $H^4(Y) = H^4(X) = \mathbb{Z}$. □

Remark 4.4. Thus we are reduced to looking at smooth complex projective minimal surfaces X such that the Cartan–Remmert reduction $\widetilde{Y}$ of $\widetilde{X}$ is a complex surface. We collect together the topological facts proven so far:

(1) $H^3(\widetilde{X}) = H^3(\widetilde{Y}) = 0$ (Lemma 3.5).
(2) $H^4(X) = H^4(Y) = \mathbb{Z}$ (Lemma 4.3).
(3) $\phi^* : \mathbb{Z} = H^4_c(\widetilde{X}) \longrightarrow H^4_c(\widetilde{Y})$ is an isomorphism (Lemma 4.2).
(4) $\pi_2(X)$, $\pi_2(Y)$, $H_2(\widetilde{X})$, $H_2(\widetilde{Y})$, are free abelian groups (Theorem 3.3, Proposition 3.4 and Corollary 3.6). Also $H^2(\widetilde{X})$ and $H^2(\widetilde{Y})$ are direct products of copies of $\mathbb{Z}$ by Corollary 3.6.

4.2. **Proof of semistability for holomorphically convex groups.** We now set up a Leray–Serre cohomology spectral sequence for the classifying maps

$$Y := \widetilde{Y}/G \longrightarrow K(G,1) \quad \text{and} \quad X \longrightarrow K(G,1)$$

for the principal G–bundles $\widetilde{X} \longrightarrow X$ and $\widetilde{Y} \longrightarrow Y$ respectively. (See [Hu, p. 286], [Dy, Theorem 2.2] and [Kl, Proposition 1] for closely related arguments and [BMP][Section 6] for a somewhat more general statement.)

Proposition 4.5. *Let X be a smooth complex projective minimal surface with holomorphically convex universal cover $\widetilde{X}$. Assume that $G := \pi_1(X)$ is torsion-free, and the Cartan–Remmert reduction $\widetilde{Y}$ of $\widetilde{X}$ is a complex surface. Let R be one of the following left $\mathbb{Z}G$–modules:*

1. *the trivial $\mathbb{Z}G$–module $\mathbb{Z}$, or*
2. *$\mathbb{Z}G$ itself.*

Then

$$H^{p+3}(G,\, R) \,=\, H^p(G,\, H^2(\widetilde{M},\, R))$$

for all $p \geq 3$.

There is an exact sequence of G-modules

$$0 \longrightarrow H^2(G,\, R) \longrightarrow H^2(M,\, R) \longrightarrow (H^2(\widetilde{M},\, R))^G \longrightarrow H^3(G,\, R) \longrightarrow H^3(M,\, R) \longrightarrow$$
$$H^1(G,\, H^2(\widetilde{M},\, R)) \longrightarrow H^4(G,\, R) \longrightarrow H^4(M,\, R) \longrightarrow H^2(G,\, H^2(\widetilde{M},\, R)) \longrightarrow H^5(G,\, R) \longrightarrow 0\,.$$

Here M is X or $Y := \widetilde{Y}/G$ and $\widetilde{M}$ is $\widetilde{X}$ or $\widetilde{Y}$ respectively.

Proof. Let $\widetilde{M} \longrightarrow M$ be a principal G–bundle. Take a $K(G,1)$ space K; its universal cover $\widetilde{K}$ is contractible. Let $f : M \longrightarrow K$ be a classifying map. Let

$$\widetilde{g} : \widetilde{M} \times \widetilde{K} \longrightarrow \widetilde{K}$$

be the natural projection. The group G acts on $\widetilde{K}$ through deck transformations, and it acts on $\widetilde{M} \times \widetilde{K}$ via the diagonal action. Since $\widetilde{g}$ is equivariant with respect to these actions, it induces a map

$$g : W := (\widetilde{M} \times \widetilde{K})/G \longrightarrow \widetilde{K}/G = K\,.$$

The fibers of g are homotopy equivalent to $\widetilde{M}$ (see [Hu, pp. 285–286] for more details).

Note that $H^3(\widetilde{Y}) = 0 = H^4(\widetilde{Y})$ by Theorem 3.2. Also, $\pi_1(\widetilde{Y}) = H_1(\widetilde{Y}) = 0$ and so $H^1(\widetilde{Y}) = 0$ by the universal coefficient theorem. Hence $H^i(\widetilde{Y}) = 0$ for $i \neq 0, 2$. Since $\mathbb{Z}G$ is a free abelian group, $H^i(\widetilde{Y}, \mathbb{Z}G) = 0$ for $i \neq 0, 2$ by the universal coefficient theorem.

The Leray–Serre cohomology spectral sequence for the above fibration with local coefficients R gives

$$H^p(K,\, (H^q(\widetilde{M},\, R)) \Longrightarrow H^{p+q}(M,\, R)\,,$$

and hence

$$H^p(G,\, (H^q(\widetilde{M},\, R)) \Longrightarrow H^{p+q}(M,\, R)$$

since K is a $K(G,1)$ space.

As $H^i(\widetilde{Y}) = 0$, $i \neq 0,\, 2$, it follows that $E_2^{p,0} = H^p(G,\, R)$ and $E_2^{p,2} = H^p(G,\, H^2(\widetilde{M},\, R))$ are the only (possibly) non-zero $E_2^{p,q}$ terms. Since $E_2^{p,1} = 0$, the differential $d_2 = 0$. Also, the differentials d_i are zero for $i > 3$. Thus d_3 is the only (possibly) non-zero differential. Hence

$$E_3 = E_2 \quad \text{and} \quad E_4 = E_5 = \cdots = E_\infty\,,$$

and also

- $E_\infty^{0,0} = H^0(G,\, H^0(\widetilde{M},\, R)) = H^0(G,\, R) = R^G$ (see [Br, p. 58] for instance),
- $E_\infty^{1,0} = H^1(G,\, H^0(\widetilde{M},\, R)) = H^1(G,\, R)$,
- $E_\infty^{2,0} = H^2(G,\, H^0(\widetilde{M},\, R)) = H^2(G,\, R)$,
- $E_\infty^{p,q} = H^p(G,\, H^q(\widetilde{M},\, R)) = H^p(G,\, 0) = 0$, for $q \neq 0, 2$.

Further, we have the following exact sequences:

- $0 \longrightarrow E_\infty^{0,2} \longrightarrow H^2(\widetilde{M},\, R)^G \xrightarrow{d_3} H^3(G,\, R)) \longrightarrow 0$,
- $0 \longrightarrow H^{p-3}(G,\, H^2(\widetilde{M},\, R)) \xrightarrow{d_3} H^p(G,\, R) \longrightarrow E_\infty^{p,0} \longrightarrow 0$, for all $p \geq 3$, and
- $0 \longrightarrow E_\infty^{p,2} \longrightarrow H^p(G,\, H^2(\widetilde{M},\, R)) \xrightarrow{d_3} H^{p+3}(G,\, R) \longrightarrow 0$, for all $p \geq 1$.

The above descriptions of the $E_\infty^{p,q}$ terms can be assembled to produce the following two exact sequences for the fibration:

$$0 \longrightarrow H^2(G,\, R) \longrightarrow H^2(M,\, R) \longrightarrow (H^2(\widetilde{M},\, R))^G \xrightarrow{d_3} H^3(G,\, R))$$

(assembling $E_\infty^{0,2}$ and $E_\infty^{0,2}$), and

$$H^{p-3}(G,\, H^2(\widetilde{M},\, R)) \xrightarrow{d_3} H^p(G,\, R) \longrightarrow H^p(M,\, R) \longrightarrow H^{p-2}(G,\, H^2(\widetilde{M},\, R))$$
$$\xrightarrow{d_3} H^{p+1}(G,\, R)) \quad \text{for all} \ \ p \geq 3\,.$$

Since $H^p(M, R) = 0$ for all $p > 4$, we immediately get from the above second exact sequence that

$$H^{p+3}(G,\, R) \,=\, H^p(G,\, H^2(\widetilde{M},\, R))$$

for all $p \geq 3$. Also, concatenating the first exact sequence with the second exact sequence for $p = 3, 4, 5$, we get the long exact sequence in the proposition. □

Here, as in what follows, we use the left G–module structure on $\mathbb{Z}G$ to define $H^k(X,\, \mathbb{Z}G)$ and the right G–module structure on $\mathbb{Z}G$ to give its G–module structure.

The following is a consequence of Proposition 4.5.

Corollary 4.6. *Let X be a smooth complex projective minimal surface with holomorphically convex universal cover $\widetilde{X}$. Let the Cartan–Remmert reduction $\widetilde{Y}$ of $\widetilde{X}$ be a complex surface. Assume that $G = \pi_1(X)$ has dimension less than four. Then*

$$H^4(M,\, R) \,=\, H^2(G,\, H^2(\widetilde{M},\, R))\,.$$

In particular,

(1) $\mathbb{Z} = H^4(M, \mathbb{Z}) = H^2(G,\, H^2(\widetilde{M}))$, *and*
(2) $\mathbb{Z} = H_c^4(\widetilde{M},\, \mathbb{Z}) = H^4(M,\, p_!\mathbb{Z}) = H^2(G,\, H^2(\widetilde{M},\, p^*p_!\mathbb{Z}))$.

Here M is X or $Y := \widetilde{Y}/G$ and $\widetilde{M}$ is $\widetilde{X}$ or $\widetilde{Y}$ respectively and $p : \widetilde{M} \longrightarrow M$ denotes the (universal) covering map.

Proof. Lemma 4.3 gives that $H^4(X,\, \mathbb{Z}) = \mathbb{Z} = H^4(Y,\, \mathbb{Z})$, while Lemma 4.2 gives that $H^4(X,\, p_!\mathbb{Z}) = \mathbb{Z} = H^4(Y,\, p_!\mathbb{Z})$.

The rest follows from the exact sequence in Proposition 4.5, putting $H^4(G,\, R) = H^5(G,\, R) = 0$. □

We are now in a position to prove H_1−semistability for holomorphically convex groups.

Theorem 4.7. *Let $G = \pi_1(X)$ be a torsion-free group that is the fundamental group of a smooth complex projective variety X with holomorphically convex universal cover. Then there exists an exact sequence of G-modules*

$$0 \longrightarrow H^2(G,\, \mathbb{Z}G) \longrightarrow \pi_2(X) \longrightarrow \mathrm{Hom}_G(\pi_2(X),\, \mathbb{Z}G) \longrightarrow H^3(G,\, \mathbb{Z}G) \longrightarrow 0\,.$$

It follows that $H^2(G,\, \mathbb{Z}G)$ is a free abelian group.

Proof. By the Lefschetz hyperplane Theorem, we can assume, without loss of generality that X is a surface. From Proposition 4.5, there is an exact sequence of G-modules,

$$0 \longrightarrow H^2(G,\, \mathbb{Z}G) \longrightarrow H^2(X,\, p_!\mathbb{Z}) \longrightarrow (H^2(\widetilde{X},\, p^*p_!\mathbb{Z}))^G \longrightarrow H^3(G,\, \mathbb{Z}G) \longrightarrow H^3(X,\, p_!\mathbb{Z})\,.$$

The first statement of the proposition follows from the following sequence of observations.
1) We have $H^2(X,\, p_!\mathbb{Z}) = H_c^2(\widetilde{X},\, \mathbb{Z}) = H_2(\widetilde{X},\, \mathbb{Z}) = \pi_2(X)$, where the first equality is the standard interpretation for cohomology with $p^*p_!\mathbb{Z}$ (isomorphic to $\mathbb{Z}G$) coefficients (see [Br, p. 209] for instance), the second equality follows from Poincaré duality applied to $\widetilde{X}$, and the third equality follows from the

Hurewicz Theorem.
2) We have $H^3(X,\, p_!\mathbb{Z}) \,=\, H^3_c(\widetilde{X},\, \mathbb{Z}) \,=\, H_1(\widetilde{X},\, \mathbb{Z}) \,=\, 0$, by a similar argument.
3) Next, $H^2(\widetilde{X},\, p^* p_!\mathbb{Z}) \,=\, \mathrm{Hom}(H_2(\widetilde{X}),\, \mathbb{Z}G) \,=\, \mathrm{Hom}(\pi_2(X),\, \mathbb{Z}G)$.
4) Finally, $\mathrm{Hom}(M,\, N)^G \,=\, \mathrm{Hom}_G(M,\, N)$.

In particular, $H^2(G,\, \mathbb{Z}G)$ injects into $\pi_2(X)$ which is free abelian by Proposition 3.4. The second statement of the proposition follows. ☐

As a consequence of Theorem 4.7 we deduce H_1–semistability as it is usually defined (cf. [Ge]). Theorem 13.3.3 of [Ge] combined with Theorem 4.7 gives:

Corollary 4.8. *Let $G \,=\, \pi_1(X)$ be a torsion-free group that is the fundamental group of a smooth complex projective surface X with holomorphically convex universal cover. Let K_i be a (filtered) CW exhaustion of $\widetilde{X}$. Then the sequence $\{H_1((\widetilde{X} \setminus K_i)_{CW},\, \mathbb{Z})\}_{i\geq 0}$ is semistable.*

The following theorem of Eyssidieux, Katzarkov, Pantev and Ramachandran [EKPR] (see also [KR, Ey]) shall be needed in obtaining a useful Corollary:

Theorem 4.9 ([EKPR])**.** *Let X be a smooth complex projective variety such that its fundamental group $\pi_1(X)$ is linear (meaning a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some n). Then the universal cover $\widetilde{X}$ of X is holomorphically convex.*

Theorem 4.7 and Theorem 4.9 together immediately give the following:

Corollary 4.10. *Let G be a linear torsion-free projective group. Then $H^2(G,\, \mathbb{Z}G)$ is a free abelian group.*

4.3. **Duality groups.** We refer to [BE] (see also [Br, p. 220]) for details on duality groups. In this Section, we deduce (as a consequence of Theorem 4.7) that the dualizing module for a 2-dimensional holomorphically convex group is free abelian as a group. It is unknown [Br][p. 224] if this is always the case for duality groups.

Theorem 4.11 (Bieri–Eckmann)**.** *A group G is a duality group of dimension n if it satisfies one the following two equivalent conditions:*

- *There exists a $\mathbb{Z}G$-module I (called the* **dualizing module***) such that for any $\mathbb{Z}G$-module A, there is an isomorphism induced by cap-product with a fundamental class:*
$$H^i(G,\, A) \,\simeq\, H_{n-i}(G,\, I \otimes_{\mathbb{Z}} A)\,.$$
- *G is of type FP and*
$$H^i(G,\, \mathbb{Z}G) \,=\, \begin{cases} 0, & \text{for } i \neq n\,, \\ I, & \text{for } i = n\,. \end{cases}$$

If the equivalent conditions hold, then I is isomorphic to $H^n(G,\, \mathbb{Z}G)$ as a $\mathbb{Z}G$-module and is torsion-free as an abelian group.

As a consequence of Theorem 4.7 and Theorem 4.11 we have the following:

Proposition 4.12. *Let G be a holomorphically convex group of dimension two. Then G is a duality group with free dualizing module.*

We end with the following simple fact about duality groups.

Proposition 4.13. *Let G be a two-dimensional duality group, and let I be the dualizing module. Then for any $\mathbb{Z}G$-module N, the cohomology $H^2(G,\, N \otimes \mathbb{Z}G)$ is isomorphic to $N \otimes I$.*

Proof. Since G is a two-dimensional duality group, the dualizing module I is isomorphic to $H^2(G,\, \mathbb{Z}G)$ as a $\mathbb{Z}G$–module. Hence, by Theorem 4.11, for any G-module Q, we have $H^2(G,\, Q) \,=\, H_0(G,\, Q \otimes I)$. Further, $H_0(G,\, Q \otimes I) \,=\, (Q \otimes I)_G$ (see [Br, p. 55]).

Next, taking $Q \,=\, N \otimes \mathbb{Z}G$ it follows that $H^2(G,\, N \otimes \mathbb{Z}G)$ is isomorphic to $(N \otimes \mathbb{Z}G \otimes I)_G$, which, in turn, is isomorphic to $(N \otimes I \otimes \mathbb{Z}G)_G$.

Finally, for any $\mathbb{Z}G$–modules B and C, we have $(B \otimes C)_G = B \otimes_{\mathbb{Z}G} C$ [Br, p. 55]. Hence

$$(N \otimes I \otimes \mathbb{Z}G)_G = ((N \otimes I) \otimes \mathbb{Z}G)_G = (N \otimes I) \otimes_{\mathbb{Z}G} \mathbb{Z}G = N \otimes I\,.$$

Thus $H^2(G,\, N \otimes \mathbb{Z}G)$ is isomorphic to $N \otimes I$. $\square$

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