What is Hyperbolic Geometry?

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History

Euclid's Axioms

1. Any two points in a plane may be joined by a straight line.
2. A finite straight line may be extended continuously in a straight line.
3. A circle may be constructed with any centre and radius.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

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5th postulate is equivalent to

Through a point not on a straight line there is one and only one straight line through the point parallel to the given straight line.

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In the nineteenth century, hyperbolic geometry was extensively explored by the Hungarian mathematician Janos Bolyai and the Russian mathematician Nikolai Ivanovich Lobachevsky, after whom it is sometimes named. Lobachevsky published a paper entitled *On the principles of geometry* in 1829-30, while Bolyai discovered hyperbolic geometry and published his independent account of non-Euclidean geometry in the paper *The absolute science of space* in 1832.

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The Problem

Parallel Postulate

Given a straight line L in a plane P and a point x on the plane P lying outside the line L, there exists a unique straight line L' lying on P passing through x and parallel to L.

Problem

Prove the Parallel Postulate from the other axioms of Euclidean geometry.

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Euclidean Geometry Revisited

Answers for Euclidean Geometry:

The Euclidean plane is $\mathbb{R}^2$ equipped with the **metric**

$$ds^2 = dx^2 + dy^2.$$

(Infinitesimal Pythagoras)

Meaning: Lengths of curves σ (= *smooth* maps of $[0,1]$ into $\mathbb{R}^2$) are computed as per the formula

$$l(\sigma) = \int_0^1 ds = \int_0^1 [(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2]^{\frac{1}{2}} dt \qquad (A)$$

for some parametrization $x = x(t), y = y(t)$ of the curve σ.

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Theorem

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Definition

Two bi-infinite straight lines are said to be parallel if they do not intersect.

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General Geometries

Answers for a slightly more general geometry: Metric on $U \subset \mathbb{R}^2$:

$$ds^2 = f(x,y)dx^2 + g(x,y)dy^2.$$

Here, $l(\sigma) = \int_0^1 ds = \int_0^1 [f(x(t),y(t))(\frac{dx}{dt})^2 + g(x(t),y(t))(\frac{dy}{dt})^2]^{\frac{1}{2}} dt....(B)$

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Definitions:

1) Given two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, the **geodesic** between (x_1, y_1) and (x_2, y_2) is the unique path that realizes the shortest distance (as per formula B) between them.

2) An **isometry** I is a map that preserves the metric, i.e. if $I((x, y)) = (x_1, y_1)$ then

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Hyperbolic Geometry

Answers for hyperbolic geometry:

A Model for hyperbolic geometry is the upper half plane $\mathbf{H} = (x, y) \in \mathbb{R}^2, y > 0$ equipped with the metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$.

Where does this come from? Another story.

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Vertical straight lines in **H** are geodesics. In fact, the vertical segment between a, b is the **unique** geodesic between a, b.

Theorem 2:

1) *Translations:* Define $f : \mathbf{H} \to \mathbf{H}$ by $f(x, y) = (x + a, y)$ for some fixed $a \in \mathbb{R}$.

2) *Inversions about semicircles:* Define $g : \mathbf{H} \to \mathbf{H}$ by $g(z) = \frac{R^2}{\overline{z}}$ for some $R > 0$, where $\overline{z}$ denotes the complex conjugate of z.

Then f, g are isometries of **H**.

Observation 3:

Image of a geodesic under an isometry is another geodesic. Hence images of vertical geodesics under inversions are geodesics.

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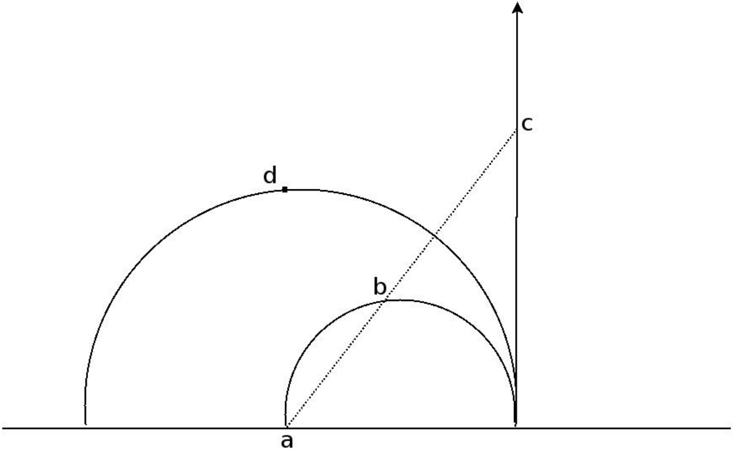
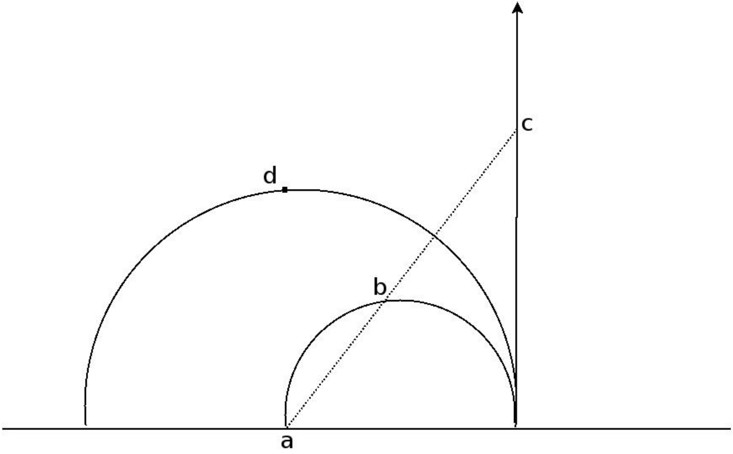


Inversions about semicircles

$l(\overline{ab}) \cdot l(\overline{ac}) = l(\overline{ad})^2.$



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