Geometry and Dynamics of Kleinian Groups

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Kleinian Groups: 3 Perspectives

Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}})$ Complex Analysis/Dynamics

Discrete subgroup G of $PSL_2(\mathbb{C})$ – Lie group theoretic.

Discrete subgroup G of group of Isometries: $Isom(\mathbb{H}^3)$ i.e. Fundamental group of a hyperbolic manifold $M^3 = \mathbb{H}^3/G$ Geometry

Discrete subgroup G of group of Mobius transformations $Mob(\Delta) = Mob(\mathbb{H}^2)$. Fuchsian Group, discovered by Poincare.

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Boundary = ideal end-points of geodesic rays.

Topology/metric d_v = angle subtended at $v \in \mathbb{H}^3$.

Geodesics are semicircles meeting the boundary at right angles.

Metric = $ds^2 = \frac{dx^2+dy^2}{y^2}$ on upper half plane.

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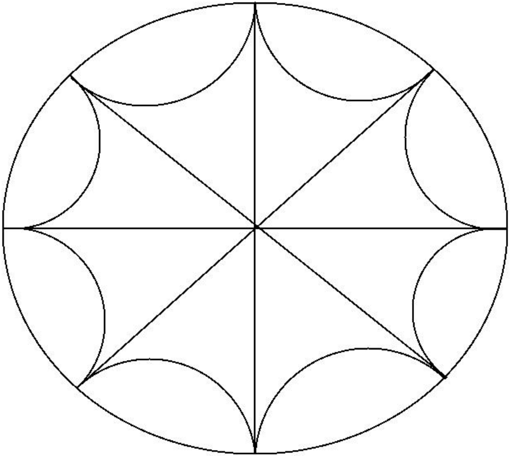

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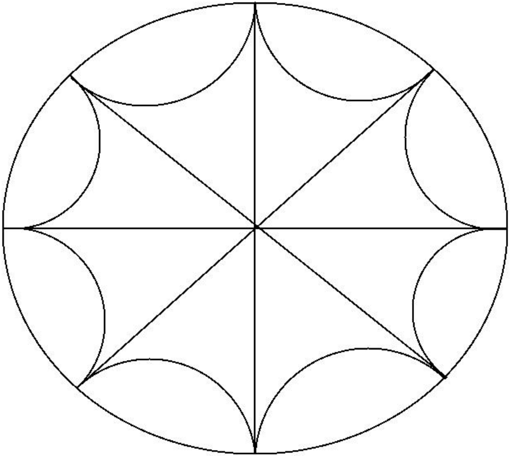

Fuchsian Group as an example of a Kleinian group

Limit set $\Lambda_G =$ Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.

Hence for a Fuchsian group of the kind described above, limit set = round equatorial circle.

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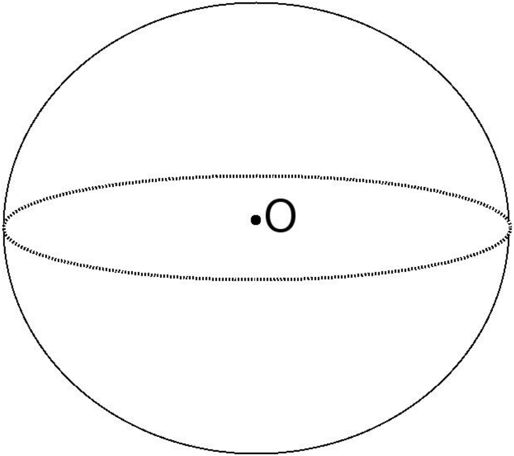
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Complement: Two round open discs.
On each, G acts freely (i.e. without fixed points) properly discontinuously, by conformal automorphisms.
Hence quotient is two copies of the ‘same’ Riemann surface (one dimensional complex analytic manifold.)
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Quasifuchsian groups

Next set of examples of Kleinian groups come from trying to put different conformal structures on the two complementary pieces of the domain of discontinuity.

IMPORTANT NOTE: Conformal Structure on a 2 manifold is EQUIVALENT TO
Constant curvature metric (for us curvature = -1)
which is EQUIVALENT TO structure as a non-singular algebraic curve.
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Ahlfors-Bers simultaneous Uniformization Theorem:
Given any two conformal structures τ_1, τ_2 on a closed topological 2-manifold, there is a discrete subgroup G of $Mob(\widehat{\mathbb{C}})$ whose limit set is *topologically* a circle, and whose domain of discontinuity quotients to two Riemann surfaces τ_1, τ_2.
Limit set is the image under a quasiconformal map of the round circle.
These *(quasi Fuchsian)* groups can be thought of as *deformations* of Fuchsian groups (Lie group theoretically) or quasiconformal deformations (analytically).

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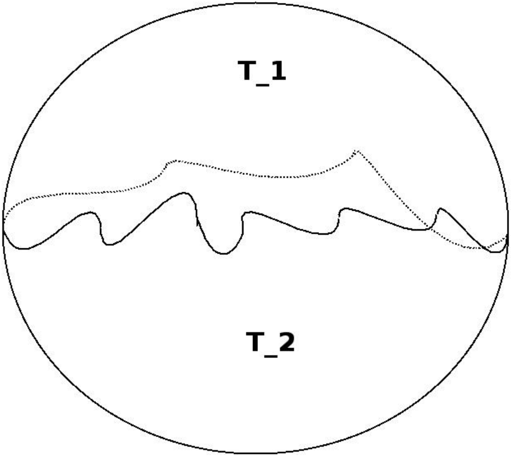
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Complexity of quasi Fuchsian group measured in terms of Hausdorff dimension.

How about geometric picture of these groups?
Convex hull CH_G of limit set Λ_G = smallest closed convex subset of $\mathbb{H}^3$ invariant under G.
Can be constructed by joining all pairs of points on limit set by bi-infinite geodesics and iterating this construction.
Quotient of CH_G by G is homeomorphic to $S \times I$, where $\pi_1(S)$ is isomorphic to G.
Called *Convex core* $CC(M)$ of $M = \mathbb{H}^3/G$.
Thickness (= 'length' of the I direction) of CH_G/G is a geometric measure of the complexity of the group G.

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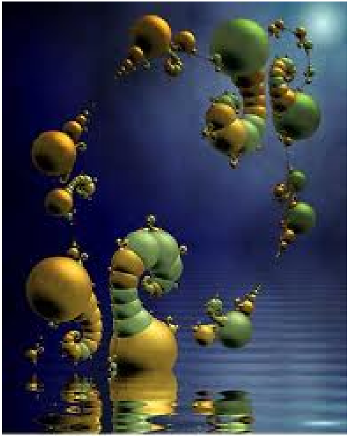
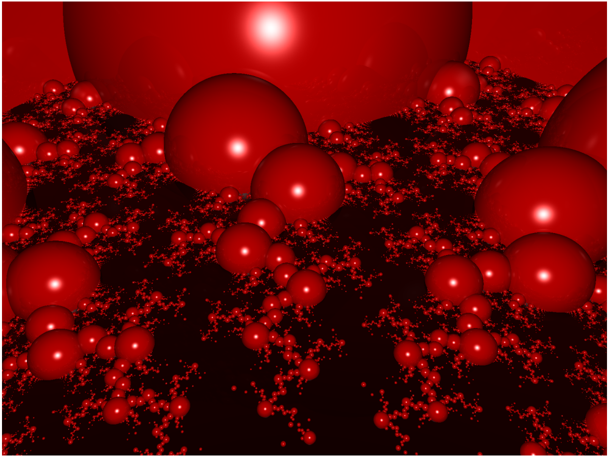


"Indra Family," by Jos Leys



Limits of quasiFuchsian groups:

Thickness of Convex core $CC(M)$ tends to infinity.

2 possibilities: Degenerate only τ_1. Degenerate both τ_1, τ_2.

i.e. $I \to [0, \infty)$ **(simply degenerate)**

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Lipman Bers:

- Examples exist.
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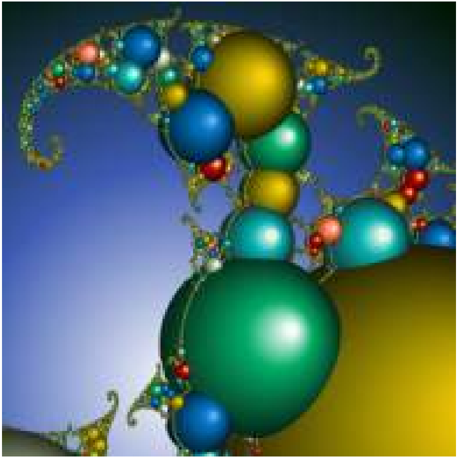

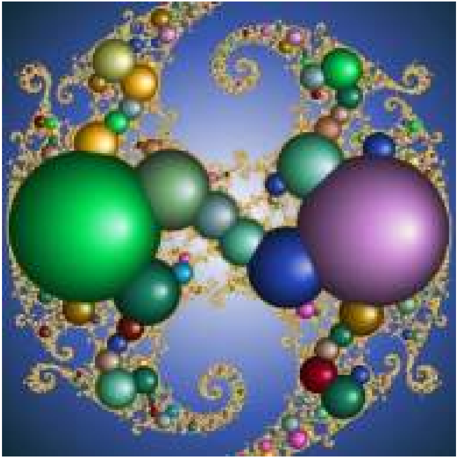

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1) **Ending Lamination Conjecture** (Proved by Minsky, Brock-Canary-Minsky): Ending laminations (pair of these in the doubly degenerate case) along with conformal structure on quotient of domain of discontinuity (in the simply degenerate case) is a complete invariant of the isometry type of M.

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Limit set = quotient of $S^1 = \partial\mathbb{H}^2 = \partial\widetilde{S}$ by relation identifying end-points of bi-infinite leaves of ending laminations.

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Totally Degenerate Surface Groups

Consequences:

- Connected limit sets of f.g. (3d) Kleinian groups are locally connected
- There exist continuous boundary extensions: If Γ is the Cayley graph of a f.g. Kleinian group G, then (fixing a base point $0 \in \mathbb{H}^3$) the natural map $i : \Gamma \to \mathbb{H}^3$ extends continuously to a map $\hat{i} : \widehat{\Gamma} \to \widehat{\mathbb{H}^3}$ between the compactifications.
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Application

Question: (Shalom) If Γ is a Zariski dense, infinite covolume, discrete subgroup of a semi-simple Lie group L, is $\mathrm{Comm}(\Gamma)$ discrete?

Answer: (M–) Yes, if
a) The limit set $\Lambda_\Gamma \subset \partial_F G$ (=Furstenberg boundary) is not invariant under a simple factor, OR
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