

# A CRASH COURSE ON KNOTS

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ABSTRACT. We give a quick introduction to Knot Theory following standard sources in the subject. These notes form the basis of lectures given at a workshop on **Topology and Condensed Matter Physics** held at S.N. Bose Centre for Basic Sciences, Kolkata, in November-December, 2015.

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## 1. INTRODUCTION: EQUIVALENCE BETWEEN KNOTS

We shall be mainly using Dale Rolfsen's *Knots and Links* as the primary source below.

**Definition 1.1.** *An embedded copy of the circle  $S^1$  in Euclidean 3-space  $\mathbb{R}^3$  or the 3-sphere  $S^3$  is called a **knot**.*

*The union of finitely many copies of  $S^1$  in Euclidean 3-space  $\mathbb{R}^3$  or the 3-sphere  $S^3$  is called a **link**.*

Knot theory mainly attempts to answer the question:

**Question 1.2.** *Given two knots  $K_1, K_2$  in  $\mathbb{R}^3$  are they equivalent?*

To answer this question, we need to come up with appropriate notions of equivalence. Usually two equivalent notions are used:

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**Definition 1.3.**  $K_1$  and  $K_2$  are said to be **equivalent** if there is an orientation-preserving homeomorphism  $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $h(K_1) = K_2$ .

$K_1, K_2$  are said to be **isotopic** if there is a one-parameter family  $h_t$  of embeddings ( $t \in [0, 1]$ ) of  $S^1$  in  $\mathbb{R}^3$  such that  $h_0(S^1) = K_1$  and  $h_1(S^1) = K_2$ .

An isotopy  $h_t$  is called an ambient isotopy if  $h_t$  can be extended to a one-parameter family of diffeomorphisms of  $\mathbb{R}^3$ .

The **Isotopy Extension Theorem** shows that two knots are equivalent if and only if they are isotopic.

A weaker equivalence between knots is obtained by demanding only that their complements in  $\mathbb{R}^3$  (equipped with an orientation) are homeomorphic via an orientation-preserving homeomorphism.

In a famous paper *Knots are determined by their complements*. *J. Amer. Math. Soc.* 2 (1989), no. 2, 371-415, by Cameron Gordon and John Luecke, the authors showed that if the complements of two tame knots are homeomorphic via an orientation-preserving homeomorphism, then the knots are equivalent. This is referred to cryptically as:

**Theorem 1.4.** *Knots are determined by their complements.*

## 2. KNOT INVARIANTS

It therefore suffices to study **algebraic invariants** of the knot complement  $\mathbb{R}^3 \setminus K$ . Here, by an algebraic invariant of a space  $X$ , we mean a natural way of associating to  $X$  an algebraic gadget, e.g. a group or a module, or a polynomial  $A(X)$ , such that if  $X$  and  $Y$  are homeomorphic, then  $A(X)$  and  $A(Y)$  are isomorphic.

Examples include:

- (1) The fundamental group  $\pi_1(X)$ ,
- (2) More generally, higher homotopy groups  $\pi_n(X)$ ,
- (3) Homology groups  $H_n(X)$ ,
- (4) (The dual) Cohomology groups  $H^n(X)$ .

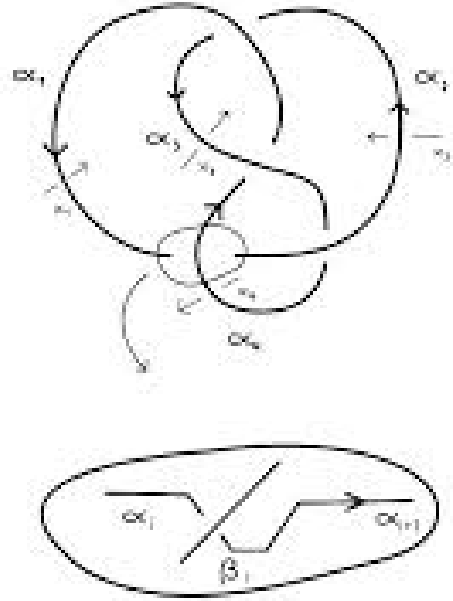
These invariants will make their appearance at different stages of this workshop. Some have appeared already.

## 3. THE KNOT GROUP

In general, the conceptually simplest invariant is the knot group  $\pi_1(\mathbb{R}^3 \setminus K)$ . This is quite a sensitive invariant and distinguishes between most distinct knots. It is also easy to compute given a planar projection of the knot as we show below. The problem with the invariant is that if the same knot has two different planar projections, then they give two different presentations of the same group; and in general it is hard to decide if two presentations give isomorphic groups or not (this is the so-called **Isomorphism Problem**).

**3.1. Wirtinger presentation.** We borrow the picture below from Martin Søndergaard Christensen's Bachelor thesis to illustrate our point. Instead of giving

a general description of the algorithm to compute the Wirtinger presentation, we illustrate it by this example.



The generators of the group are  $x_1, \dots, x_4$ . In the general situation, we consider an oriented loop starting from a point  $*$  very far away coming down to the plane on which the knot is projected, go below a strand and then go back to  $*$ . For convenience we only give an orientation on the piece of this loop that crosses the knot below a strand of the knot. In the picture above  $x_1$  crosses the strand  $\alpha_1$  from below. Proceeding anticlockwise, we come to the encircled region, where there is a knot crossing. Then  $x_2$  crosses the strand  $\alpha_2$  from below. Note that the loop indicated by  $x_1$  cannot be homotoped to the loop indicated by  $x_2$  as the strand indicated by  $\alpha_4$  comes in the way. Thus all maximal strands not disconnected by another strand crossing from the top gives rise to a generator of the knot.

**Relations:** Now we compute the relations:

At the encircled crossing in the diagram, there are four of the  $x$ 's.  $x_4$  crosses from right to left both below and above the crossing,  $x_1$  crosses from bottom to top on the left of the crossing and  $x_2$  crosses from bottom to top on the right of the crossing. This gives rise to the relation

$$x_4x_1 = x_2x_4.$$

Note that this could also be written as

$$x_4x_1x_4^{-1} = x_2.$$

Similarly for the other crossings, we get

- (1)  $x_2x_3 = x_4x_2$
- (2)  $x_1x_3 = x_2x_1$
- (3)  $x_4x_3 = x_3x_1$

This gives us a full presentation of the complement of the knot described above, which is also called the figure 8 knot.

It turns out that (any) one of the relations can always be dropped as it is a consequence of the remaining ones.

**3.2. The first homology.** The first homology  $H_1(X)$  is the abelianization of the fundamental group, i.e. it is the group obtained by declaring that all the generators commute.

**Theorem 3.1.**  $H_1(\mathbb{R}^3 \setminus K) = \mathbb{Z}$  for any knot  $K$ .

Again we illustrate this in the figure 8 knot complement case. We need to abelianize the 4 relations above. These give, respectively,

- (1)  $x_1 = x_2$
- (2)  $x_3 = x_4$
- (3)  $x_3 = x_2$
- (4)  $x_4 = x_1$

Thus we have that

$$H_1(\mathbb{R}^3 \setminus K) = \langle x_1, x_2, x_3, x_4 : x_1 = x_2 = x_3 = x_4 \rangle = \mathbb{Z}.$$

#### 4. TORSION

Since all knots have the same first homology,  $H_1(\mathbb{R}^3 \setminus K)$  is of no use as a knot invariant. However, it can be used to extract finer invariants by passing to finite index subgroups of  $\pi_1(\mathbb{R}^3 \setminus K)$ .

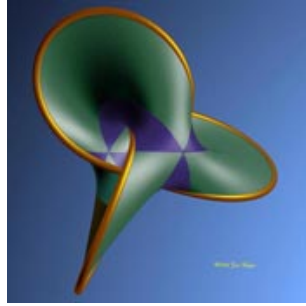
Let  $X = S^3 \setminus K$  and let  $X_j$  be the  $j$ -fold cyclic cover of  $X$ . The torsion-part of  $H_1(X_k)$  is called the  $j$ -th torsion invariant of  $K$ .

This can be defined purely algebraically as follows.

$\pi_1(S^3 \setminus K) \rightarrow H_1(S^3 \setminus K) = \mathbb{Z}$  be the abelianization map. Compose this with the map  $\mathbb{Z} \rightarrow \mathbb{Z}/k$ . Let  $N_k$  be the kernel of this map. Then the abelianization of  $N_k$  is  $H_1(X_k)$  and its torsion part is the  $j$ -th torsion invariant of  $K$ .

#### 5. SEIFERT SURFACES

A Seifert surface for a knot or link  $K$  is a connected bicollared compact surface  $\Sigma$  with  $\partial\Sigma = K$ . Any oriented knot or link  $K$  has an oriented Seifert surface bounding it.



A Seifert surface for  $K$  is constructed as follows. Take a planar projection of  $K$ . Near each crossing point, delete the over- and undercrossings and replace them by 'short-cut' arcs preserving orientation. This gives rise to a disjoint collection of oriented simple closed curves. They bound disks, which may be pushed slightly off each other if necessary to make them disjoint. Finally we connect these disks together at the original crossings using half-twisted strips. The result is a surface with boundary  $K$ .

## 6. ALEXANDER POLYNOMIAL

To compute torsion invariants we used finite cyclic covers. The Alexander polynomial is computed using an infinite cyclic cover corresponding to the map  $\pi_1(S^3 \setminus K) \rightarrow H_1(S^3 \setminus K) = \mathbb{Z}$ . Let  $X$  denote the knot complement and  $S$  a Seifert surface. Cut  $X$  open along  $S$  and attach infinitely many copies end to end to obtain the infinite cyclic cover  $Y$ . Let  $t$  be the generators of the deck transformation group (isomorphic to  $H_1(S^3 \setminus K) = \mathbb{Z}$ ). Thus  $H_1(Y)$  can be regarded as a  $\mathbb{Z}[t, t^{-1}]$ -module called the **Alexander module**. The presentation matrix for the Alexander module is called the **Alexander matrix**. When the number of generators,  $k$ , is less than or equal to the number of relations,  $s$ , then the ideal generated by all  $k \times k$  minors of the Alexander matrix is called the **Alexander ideal**. When the Alexander ideal is principal, its generator is called an **Alexander polynomial** of the knot.

## 7. SKEIN RELATIONS

Skein relations are of the form

$$F(L_0, L_+, L_-) = 0.$$

An example is given by the following (from Mina Aganagic's article: String Theory and Math: Why This Marriage Can Last, Mathematics and Dualities of Quantum Physics)

Finding an  $F$  which produces polynomials independent of the sequences of crossings used in a recursion is not easy. Jones uncovered an underlying structure of skein relations when he discovered planar algebras. A skein relation can be thought of as defining the kernel of a quotient map from the planar algebra of tangles. Such a map gives rise to a knot polynomial if all closed diagrams are taken to some (polynomial) multiple of the image of the empty diagram.

The diagram shows a skein relation for the Jones polynomial. On the left, there is a crossing of two strands with arrows pointing upwards. The crossing is labeled with  $q^{-1}$  on the left and  $-q$  on the right. This is followed by an equals sign and a term  $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$  multiplied by a diagram of two parallel strands with arrows pointing upwards.

Skein relation for the Jones polynomial

**7.1. Alexander polynomial.** We have given a geometric description of the Alexander polynomial above. Conway discovered the following skein relation that computes the Alexander polynomial.

$$A_{K_+} - A_{K_-} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_{K_0}.$$

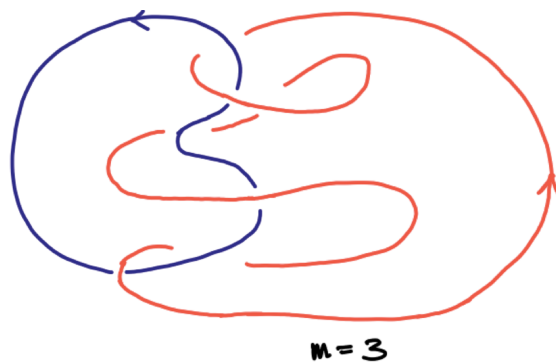
**7.2. Jones polynomial.** The Jones polynomial was discovered by Vaughan Jones in 1984. It is a Laurent polynomial in  $q^{\frac{1}{2}}$ . The figure given above actually gives us the way to compute the Jones polynomial by furnishing the skein relation the Jones polynomial satisfies:

$$q^{-1}J_{K_+} - qJ_{K_-} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_{K_0}.$$

together with specifying its value for the unknot.

## 8. LINKING NUMBER

The linking number is an invariant of a link having two components,  $K_1$  and  $K_2$ . In a sense this was the oldest knot or link invariant. It was discovered by Gauss. Choose a planar projection of the link onto a plane and define the linking number to be half the number of crossings counted with sign (using the right hand thumb rule after orienting the link). See the following figure (from Mina Aganagic's article: String Theory and Math: Why This Marriage Can Last, Mathematics and Dualities of Quantum Physics)



$$m(K_1, K_2) = \frac{1}{2} \sum_{\text{crossings}(K_1, K_2)} \text{sign}(\text{crossing}).$$

Gauss's discovery of the linking number came from his study of electrostatics, and he gave the following formula describing the same topological invariant:

$$m(K_1, K_2) = \frac{1}{2\pi} \oint_{K_1} \oint_{K_2} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \cdot (d\vec{x}_1 \times d\vec{x}_2).$$

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