LIMITS OF LIMIT SETS III: THE GENERAL CASE

MAHAN MJ

ABSTRACT. We show that for a strongly convergent sequence of finitely generated Kleinian groups, Cannon-Thurston maps converge uniformly.

Contents

1. Introduction

This is the last of three papers following [MS13a, MS13b] dealing with convergence of Cannon-Thurston maps. The main question we address is:

Question 1.1. *Does strong convergence of finitely generated Kleinian imply uniform convergence of Cannon-Thurston maps?*

We conclude in this paper that the question has a positive answer in full generality. This gives a complete answer to a question of Thurston (Problem 14 of [Thu82]).

Theorem 2.8 *Let* Γ *be a fixed group and* $\rho_n(\Gamma) = G_n$ *be a sequence of finitely generated Kleinian groups converging strongly to a Kleinian group* G*. Further assume that the convergence is strictly type-preserving. Let* M_n *be the corresponding hyperbolic manifolds converging strongly*

Date: November 22, 2013.
2000 *Mathematics Subject Classification.* 57M50.

to M. Let K be a fixed geometrically finite manifold with fundamental group Γ. Consider strictly type-preserving embeddings $\phi_n : K \to M_n, n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies. Then Cannon-Thurston maps for $\widetilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.

We shall freely borrow from [MS13b] and [Mj14] and refer the reader to these papers for background. In [MS13b], we proved Theorem 2.8 for closed surface groups. The purpose of this paper is to indicate the changes necessary to the work in [MS13b] to obtain Theorem 2.8 for arbitrary (finitely generated) Kleinian groups.

Acknowledgments: I would like to thank Caroline Series for a most fruitful collaboration leading to [MS13a, MS13b] and this paper. Without her posing the critical questions she did, this project would never have taken off. However, I claim exclusive credit for the defects of the present paper.

1.1. **Cannon-Thurston maps for relatively hyperbolic spaces.** We give a criterion for the existence of Cannon-Thurston maps for relatively hyperbolic spaces. Let Y, X be hyperbolic rel. $\mathcal{Y}, \mathcal{X}$ respectively. Let $Y^h = \mathcal{G}(Y, \mathcal{Y}), \widehat{Y} = \mathcal{E}(Y, \mathcal{Y})$ and $X^h = \mathcal{G}(X, \mathcal{X}), \widehat{X} = \mathcal{E}(X, \mathcal{X})$. Recall that $B_R^h(Z) \subset X^h$ denotes the R-neighborhood of Z in (X^h, d_h). Also by the Definition of $\mathcal{G}(X, \mathcal{H})$, recall that distances in (X^h, d_h) are proper functions of distances in (X, d). Similarly for Y and Y^h.

Lemma 1.2. [MP11] *A Cannon-Thurston map for $i \colon Y \to X$ exists if and only if there exists a non-negative function $M(N)$ with $M(N) \to \infty$ as $N \to \infty$ such that the following holds:*
Suppose $y_0 \in Y$, and $\hat{\lambda}$ in $\widehat{Y}$ is an electric quasigeodesic segment starting and ending outside horospheres. If $\lambda^b = \hat{\lambda} \setminus \bigcup_{K \in \mathcal{Y}} K$ lies outside an $B_N(y_0) \subset Y$, then for any electric quasigeodesic $\hat{\beta}$ joining the end points of $\hat{i}(\hat{\lambda})$ in $\widehat{X}$, $\beta^b = \hat{\beta} \setminus \bigcup_{H \in \mathcal{X}} H$ lies outside $B_{M(N)}(i(y_0)) \subset X$.

We shall describe this informally as follows:
If λ lies outside a large ball **modulo horoballs** *then so does any geodesic in X joining its endpoints.*

As an immediate consequence we have:

Lemma 1.3. *Let $\rho : \Gamma \to G$ be a weakly type preserving isomorphism of finitely generated Kleinian groups and suppose that Γ is geometrically finite. The Cannon-Thurston map from $\Lambda_\Gamma \to \Lambda_G$ exists if and only if given basepoints O_Γ, O_G, there exists a non-negative function $M(N)$,* *such that $M(N) \to \infty$ as $N \to \infty$ with the following property. Suppose that λ is a d_Γ-geodesic segment lying outside $B_\Gamma(O_\Gamma; N)$ in $\mathcal{G}\Gamma$. Then the hyperbolic geodesic in $\mathbb{H}^3$ joining the end-points of $i(j_\Gamma(\lambda))$ lies outside $B_{\mathbb{H}^3}(O_G; M(N))$ in $\mathbb{H}^3$.*

A Lemma we shall need in this paper is the following (see [Mj14] for terminology):

Lemma 1.4 ([Mj11] Lemma 3.7)**.** *Let X be a hyperbolic metric space and let $\mathcal{H}$ be a collection of ϵ-neighborhoods of mutually cobounded quasiconvex sets. An electro-ambient quasigeodesic lies within bounded distance of any X geodesic with the same endpoints (a tracking geodesic). Moreover it is a quasigeodesic in X.*

1.2. **Uniform convergence criterion.** We recall from [MS13a, MS13b] criteria for convergence of Cannon-Thurston maps. We shall discuss this in the context of Kleinian groups, but it is clear that the same notion goes through in the more general context of a sequence of actions of a fixed hyperbolic (or relatively hyperbolic) group H on a hyperbolic metric space X.

Let Γ be a fixed geometrically finite Kleinian group and suppose that $\rho_n : \Gamma \to PSL_2(\mathbb{C})$ is a sequence of discrete faithful representations converging (algebraically or strongly) to $\rho_\infty : G \to PSL_2(\mathbb{C})$. Let $G_n = \rho_n(\Gamma), n = 1, 2, \ldots, \infty$. Assuming they exist, we shall say that Cannon-Thurston maps $\hat{i}_n : \Lambda_\Gamma \to \Lambda_{G_n}$ converge uniformly (resp. pointwise) to $\hat{i}_\infty$ if they do so as maps from Λ_Γ to $\hat{\mathbb{C}} = \mathbb{S}^2_\infty$. However, we need to *normalize carefully* to make this notion precise.

Definition 1.5. *Suppose that a sequence of abstractly isomorphic Kleinian groups $G_n \to G_\infty$ algebraically. Let $M_n = (\mathbb{H}^3/G_n), n = 1, \cdots, \infty$. A sequence of embeddings $\phi_n : K \to M_n, n = 1, \cdots, \infty$ for a 2-complex K is said to be a sequence of* **coarsely equivalent homotopy equivalences** *if*
a) ϕ_n is a homotopy equivalence for $n = 1, \cdots, \infty$.
b) There exists $L_0 \geq 1$, compact subsets $K_n \subset M_n$ and L_0-bi-Lipschitz homeomorphisms $\psi_{mn} : K_m \to K_n$ such that $\psi_n(K) \subset K_n \subset M_n$.
c) For all m and n, ϕ_n and $\psi_{mn} \circ \phi_m$ are homotopic to each other by uniformly bounded homotopies, i.e. lengths of tracks of homotopies are uniformly bounded independent of $n = 1, \cdots, \infty$. We shall say that ϕ_n's are homotopic to each other by **uniformly bounded homotopies** *if such a collection of ψ_{mn}'s exist.*

We shall say that $G_n \to G_\infty$ **semi-strongly** *if there exists such a sequence ϕ_n of coarsely equivalent homotopy equivalences.*

Note: It is clear that strong convergence implies semi-strong convergence. Also, semi-strong convergence necessarily implies that the algebraic limit is contained in the geometric limit.
Caveat: We have introduced the term *coarsely equivalent homotopy equivalences* to avoid confusion by overuse of the term *uniform*, which occurs in two senses:
a) Embeddings $\phi_n : K \to M_n, n = 1, \cdots, \infty$.
b) Convergence of Cannon-Thurston maps (see below).

Remark 1.6. Relative Hyperbolic Generalization: Suppose G is (strongly) hyperbolic relative to a collection $\mathcal{H}$. Let (K, K_0) be a simplicial complex pair such that $\pi_1(K) = G$ and the fundamental groups of the components of K_0 correspond to the elements of $\mathcal{H}$. Coarsely equivalent homotopy equivalences can now be defined as above for pairs of maps $(\phi_n, \partial\phi_n) : (K, K_0) \to (M_n, \partial M_n)$, where the pair $(M_n, \partial M_n)$ denotes now a *pared manifold pair*, i.e. the quotient manifold *minus* cusps and its boundary.

Normalization:
Suppose first that a sequence of abstractly isomorphic finitely presented Kleinian groups $G_n \to G_\infty$ semi-strongly. Let $M_n = (\mathbb{H}^3/G_n), n = 1, \cdots, \infty$ and K, K_n, ϕ_n be as above.

We normalize as follows. Fix $0 \in \widetilde{K}$, $o \in \mathbb{H}^3$, $L > 0$ such that $\widetilde{\phi_n}(0) \in N_L(0)$ for all n. Let $\partial\phi_n$ be the Cannon-Thurston maps for $\widetilde{\phi_n}$ from $\partial\widetilde{K}$ to $\mathbb{S}^2_\infty$ (assuming they exist). We shall say that the Cannon-Thurston maps for $\widetilde{\phi_n}$ converge uniformly (resp. pointwise) if they do so as maps from $\partial\widetilde{K}$ to $\mathbb{S}^2_\infty$. We state below a criterion for the **uniform convergence of Cannon-Thurston maps.**

Proposition 1.7. *Let G be a fixed (relatively) hyperbolic group. Suppose that the following holds.*
$\rho_n : G \to PSL_2(\mathcal{C})$ is a sequence of discrete faithful representations converging to $\rho_\infty : G \to PSL_2(\mathcal{C})$ **semi-strongly**. *Let $G_n = \rho_n(G), n = 1, \cdots, \infty$ and $M_n = (\mathbb{H}^3/G_n)$.*
Semi-strong convergence ensures the existence of a compact 2-complex K with $\pi_1(K) = G$ and coarsely equivalent homotopy equivalences $\phi_n : K \to M_n, n = 1, \cdots, \infty$. Equip the universal cover $\widetilde{K}$ with a simplicial metric, by assigning length one to each edge. Let $\widetilde{\phi}_n : \widetilde{K} \to \mathbb{H}^3$ be lifts of ϕ_n and $o \in \widetilde{K}, 0 \in \mathbb{H}^3$ be fixed points. There exists L such that $\phi_n(o) \in N_L(0), n = 1, \cdots, \infty$.

Then
a) Cannon-Thurston maps for $\widetilde{\phi}_n : \widetilde{K} \to \mathbb{H}^3$ *exist iff there exist non-negative functions* $f(n,N)$*, such that* $f(n,N) \to \infty$ *as* $N \to \infty$ *and for all geodesic segments* λ *lying outside an* N*-ball (modulo horoballs) around* $o \in \widetilde{K}$ *any geodesic segment in* $\mathbb{H}^3$ *joining the end-points of* $\widetilde{\phi}_n(\lambda)$ *lies outside the* $f(n,N)$*-ball around* $0 \in \mathbb{H}^3$ *(modulo horoballs).*
b) Further, Cannon-Thurston maps for $\widetilde{\phi}_n$ *converge uniformly to the Cannon-Thurston map for* $\widetilde{\phi}_\infty$ *iff there exists* $g(N)$ *such that*
(i) $g(N) \to \infty$ *as* $N \to \infty$ *and* $f(n,N) \geq g(N), n = 1, \cdots, \infty$.
*(ii)***Visually small tails of sequences** *For all* g *lying outside an* N*-ball around* $o \in \widetilde{K}$ *and* $m, n \geq N$*, any geodesic segment in* $\mathbb{H}^3$ *joining* $\widetilde{\phi}_m(g), \widetilde{\phi}_n(g)$ *lies outside the* $g(N)$*-ball around* $0 \in \mathbb{H}^3$.

Note: The criterion in $b(ii)$ above ensures that orbits of points lying outside a large ball eventually have small visual diameter. Hence the Cannon Thurston maps ensured by a and $b(i)$ above are uniformly close.

The hypothesis on *visually small tails of sequences* follows from strong convergence and Conclusion (a) of Proposition 1.7.

Proposition 1.8. *Let* G *be a fixed (relatively) hyperbolic group. Suppose that* $\rho_n : G \to PSL_2(\mathcal{C})$ *is a sequence of discrete faithful representations converging to* $\rho_\infty : G \to PSL_2(\mathcal{C})$ **strongly**. *Further suppose that there exists a non-negative functions* $f(N)$*, such that* $f(N) \to \infty$ *as* $N \to \infty$ *and for all geodesic segments* λ *lying outside an* N*-ball (modulo horoballs) around* $o \in \Gamma_G$ *any geodesic segment in* $\mathbb{H}^3$ *joining the end-points of* $\widetilde{\phi}_n(\lambda)$ *lies outside the* $f(N)$*-ball around* $0 \in \mathbb{H}^3$ *(modulo horoballs).*
Then there exists $g(N)$ *such that*
(i) $g(N) \to \infty$ *as* $N \to \infty$
(ii) for all g *lying outside an* N*-ball around* $o \in \Gamma_G$ *and* $m, n \geq N$*, any geodesic segment in* $\mathbb{H}^3$ *joining* $\widetilde{\phi}_m(g), \widetilde{\phi}_n(g)$ *lies outside the* $g(N)$*-ball around* $0 \in \mathbb{H}^3$.

We recall a couple of notions from [MS13a].

Definition 1.9. *Let* Γ *be a fixed finitely generated group and* $\rho_n(\Gamma) = G_n$ *be a sequence of Kleinian groups converging algebraically to* $G_\infty = \rho_\infty(\Gamma)$.

The sequence (ρ_n) *is said to satisfy* **UEP (Uniform Embedding of Points)** *if there exists a non-negative function* $f(N)$*, with* $f(N) \to \infty$ *as* $N \to \infty$*, such that for all* $g \in \Gamma$ *and* $0 \in \mathbb{H}^3$*,* $d_\Gamma(1,g) \geq N$ *implies*

$d_{\mathbb{H}}(\rho_n(g)(0), 0) \geq f(N)$ for all $n = 1, \ldots, \infty$. Here d_Γ represents the distance in the Cayley graph of $\mathcal{G}(\Gamma)$ and $d_{\mathbb{H}}$ denotes the distance in $\mathbb{H}^3$.

The sequence (ρ_n) is said to satisfy **UEPP (Uniform Embedding of Pairs of Points)** *if there exists a non-negative function $f(N)$, with $f(N) \to \infty$ as $N \to \infty$, such that for all $g, h \in \Gamma$ and $0 \in \mathbb{H}^3$, $d_\Gamma(1, [g,h]_\Gamma) \geq N$ implies $d_{\mathbb{H}}([\rho_n(g)(0), \rho_n(h)(0)], 0) \geq f(N)$ for all $n = 1, \ldots, \infty$, where $[g,h]_\Gamma$ denotes a geodesic in Γ joining g, h and $[\rho_n(g)(0), \rho_n(h)(0)]$ denotes a geodesic in $\mathbb{H}^3$ joining $\rho_n(g)(0), \rho_n(h)(0)$.*

Property UEP is used in [MS13a] to give a criterion under which algebraic convergence is also geometric. Property UEPP is used to give the following alternate criterion for proving uniform convergence of Cannon-Thurston maps.

Proposition 1.10. [MS13a] *Let Γ be a geometrically finite Kleinian group and let $\rho_n : \Gamma \to G_n$ be weakly type preserving isomorphisms to Kleinian groups. Suppose that ρ_n converges algebraically to a representation ρ_∞. Then if (ρ_n) satisfies UEPP, the* CT*-maps converge uniformly. If Γ is non-elementary, the converse also holds.*

2. Strong Convergence

2.1. **Models of Ends.** We recall some of the material from [Mj14] and [Mj10] that we shall need. For convenience of exposition, we shall deal primarily with the case of Kleinian groups without parabolics. Modifications necessary in the presence of parabolics will be indicated from time to time. The main theorem of [Mj14] is the following.

Theorem 2.1. [Mj14] *Cannon-Thurston maps exist for hyperbolic 3-manifolds corresponding to simply or doubly degenerate surface Kleinian groups.*

Theorem 2.1 is generalized in [Mj10] to arbitrary finitely generated Kleinian groups.

Theorem 2.2. *Let Γ be a fixed group and $\rho(\Gamma) = G$ be a Kleinian group and M the associated hyperbolic 3-manifold. Let K be a geometrically finite hyperbolic manifold with fundamental group Γ. Consider a strictly type-preserving homotopy equivalence $\phi : K \to M$. Then a Cannon-Thurston map for $\widetilde{\phi}$ exists.*

We also proved the following stronger consequence (Theorem 2.4 below) of the *proof* of Theorem 2.1 in [Mj14], which will be useful in dealing with sequences of manifolds. The notion of a Scott core we shall use here is slightly different from the one existing in literature.

Definition 2.3. *A Scott core K of a complete hyperbolic 3-manifold M_0 with ends $E_1, \cdots, E_N$ is the convex core K of a geometrically finite manifold N and a κ-bi-Lipschitz embedding $\phi : K \to M_0$ such that $M_0 \setminus \phi(K) = \bigcup_i E_i$.*

We shall identify K with $\phi(K) \subset M_0$. We shall find it convenient to fix a common Scott core K for a collection of manifolds (denoted by a generic M for convenience).

Theorem 2.4. [Mj14] *Let K be a Scott core of M_0 with ends $E_1, \cdots, E_N$ all of which are simply degenerate. Let $S_1, \cdots, S_N$ be the boundary components of K.*

Then for all compact $E_i^1 \subset E_i$ homeomorphic to $S_i \times [0,1]$ with $E_i^1 \cap K = S_i$, $i = 1 \cdots N$, there exist
1) Compact $E_i^2 \subset E_i$ homeomorphic to $S_i \times [0,1]$ with $E_i^2 \cap K = S_i$, $i = 1 \cdots N$ and $E_i^1 \subset E_i^2$
2) and a function $N_1(N) \to \infty$ as $N \to \infty$ such that the following holds.
Let $M^{(j)} = K \bigcup_i E_i^j$, $j = 1, 2$ glued along the boundary components $S_1, \cdots, S_N$.
Let M be any hyperbolic 3-manifold such that
a) there exists a $2-$ bi-Lipschitz embedding $i : M^{(2)} \to M$, and
b) $M \setminus i(M^{(2)})$ consists of ends (geometrically finite or infinite).
Then for all geodesic segments λ lying outside an N-ball around $o \in \widetilde{K}$ (the lift of K to the universal cover) and any geodesic segment μ in $\widetilde{M}$ joining the end-points of λ, $\mu \cap \widetilde{i(M^{(1)})}$ lies outside the $N_1(N)$-ball around $o \in \widetilde{M}$.

Remark 2.5. Theorem 2.4 above is a paraphrasing of Corollary 6.13 of [Mj14]. We give some specific references. The existence of E_i^2 for E_i^1 is a consequence of the Minsky model [Min10] for a degenerate end and the selection of split level surfaces carried out in Section 4 of [Mj14]. The selection process chooses a sequence $\{\Sigma_{n,i}\}$ of split level surfaces exiting every degenerate end E_i. In [Mj14] we proved the existence of an integer D (termed the *graph quasiconvexity constant of split components*) such that if E_i^1 is the product region between $\Sigma_{1,i}$ and $\Sigma_{m,i}$, then E_i^2 can be taken to be the product region between $\Sigma_{1,i}\}$ and $\Sigma_{m+D,i}$.

Remark 2.6. Modification in the presence of punctures: The analogous statement in the presence of parabolics is a small modification of Proposition 2.4 above. The hypothesis is exactly the same. The conclusion is modified as follows:

For all geodesic segments λ lying outside an N-ball around $o \in \widetilde{K}$ modulo horoballs, and any geodesic segment μ in $\widetilde{M}$ joining the end-points of λ, $\mu \cap i(\widetilde{M^{(1)}})$ lies outside the $N_1(N)$-ball around $o \in \widetilde{M}$ modulo horoballs.

Remark 2.7. Modification in the presence of geometrically finite ends: If some end(s) E_k of M_0 are geometrically finite in Proposition 2.4 or Remark 2.6 then the same conclusion holds provided
a) the index i runs over the ends of M_0 that are degenerate.
b) the ends of M corresponding to the geometrically finite ends of M_0 are required to be **uniformly** bi-Lipschitz to them.

2.2. **Strong Convergence with Geometrically Infinite Limit.** Let G_m be a sequence of finitely generated Kleinian groups converging strongly to a Kleinian group G. Further, assume that the limit is strongly type-preserving. Let M_n be the corresponding hyperbolic manifolds converging strongly to M. Also, let K be the convex core of a geometrically finite hyperbolic 3-manifold admitting proper homotopy equivalences $\phi_m : K \to M_m, n = 1, \cdots, \infty$ such that the maps ϕ_m are homotopic to each other by uniformly bounded homotopies. We shall now show that the Cannon-Thurston maps for $\widetilde{\phi_m}$ exist and converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.

Theorem 2.8. *Let Γ be a fixed group and $\rho_n(\Gamma) = G_n$ be a sequence of finitely generated Kleinian groups converging strongly to a Kleinian group G. Further assume that the convergence is strictly type-preserving. Let M_n be the corresponding hyperbolic manifolds converging strongly to M. Let K be a fixed geometrically finite manifold with fundamental group Γ. Consider strictly type-preserving embeddings $\phi_n : K \to M_n, n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies. Then Cannon-Thurston maps for $\widetilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.*

Proof. Since G_m converges to $G = G_\infty$, we can fix a base codimension zero submanifold $\phi_m : K \subset M_m$ for all $m = 1 \cdots \infty$. Also, by geometric convergence of M_m to $M = M_\infty$, we may identify all the copies of $\phi_m(K) \subset M_m$ with each other. Choose the lift of a base point on $K \subset M_m$ to lie in a uniformly bounded neighborhood of $0 \in \mathbb{H}^3$. Thus the hypotheses of Proposition 1.7 (by choosing the complex K of Proposition 1.7 to be the codimension zero submanifold K) are satisfied.

We need to check:
4) For $\widetilde{\phi_m} : \widetilde{K} \to \mathbb{H}^3$, there exist non-negative functions $f(m, N)$, such

that $f(m,N) \rightarrow \infty$ as $N \rightarrow \infty$ and for all geodesic segments λ lying outside an N-ball around $o \in \widetilde{K}$ any geodesic segment in $\mathbb{H}^3$ joining the end-points of $\widetilde{\phi_m}(\lambda)$ lies outside the $f(m,N)$-ball around $0 \in \mathbb{H}^3$. For each $m = 1, 2, \cdots$ or ∞, this is the content of Theorems 2.1 and 2.2. Hence Cannon-Thurston maps exist for each $m = 1, \cdots, \infty$.
5) To show that Cannon-Thurston maps for $\widetilde{\phi_m}$ converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$ it remains to show that there exists $g(N)$ such that $g(N) \rightarrow \infty$ as $N \rightarrow \infty$ and $f(m,N) \geq g(N), n = 1, \cdots, \infty$.

The rest of the discussion is devoted to proving criterion 5. We shall prove the following equivalent statement:
$5'$) Given $L \geq 0$, there exists $N \geq 0$ such that if λ lies outside an N-ball around $o \in \widetilde{K}$, then any geodesic segment in $\mathbb{H}^3$ joining the end-points of $\widetilde{\phi_m}(\lambda)$ lies outside the L-ball around $o \in \mathbb{H}^3$ for all $= 1, \cdots, \infty$.

For ease of exposition, we assume that there are no parabolics in M.
Step 1) Let E_i, $i = 1 \cdots N$ be the ends of M. Given L, there exist compact submanifolds $E_i^1 \subset E_i$ homeomorphic to $S_i \times [0,1]$ such that $d_M((E_i \setminus E_i^1), K) \geq 2L$.

Step 2) By Theorem 2.4 there exist compact submanifolds E_i^2 with $E_i^1 \subset E_i^2$ homeomorphic to $S_i \times [0,1]$ and an integer N_1 such that the following holds.
Let $M^{(j)} = K \bigcup_i E_i^j$, $j = 1, 2$ glued along the boundary components $S_1, \cdots, S_N$.
Let W be any hyperbolic 3-manifold such that
a) there exists a $2-$ bi-Lipschitz embedding $i : M^{(2)} \rightarrow W$, and

b) $W \setminus i(M^{(2)})$ consists of ends (geometrically finite or infinite).
For all geodesic segments λ lying outside an N_1-ball around $o \in \widetilde{K}$ and any geodesic segment μ in $\widetilde{W}$ joining the end-points of λ, $\mu \cap \widetilde{i(M^{(1)})}$ lies outside the L-ball around $o \in \widetilde{W}$.

Hence by Step 1, and since i is 2-biLipschitz, μ lies outside the L-ball around $o \in \widetilde{W}$.

Step 3) By strong convergence, there exists a natural number $C_0 = C_0(L)$ such that for all $k \geq C_0$, there exists a $2-$ bi-Lipschitz embedding $i : M^{(2)} \rightarrow M_k$ and hence by Step 2, $\mu (= \mu(k))$ lies outside the L-ball around $o \in \widetilde{M_k}$ whenever λ lies outside an N_1-ball around $o \in \widetilde{K}$.

Step 4) By Theorem 2.2, there exist $N'_1, \cdots, N'_{C_0}$ such that if λ lies outside an N'_k-ball around $o \in \widetilde{K}$, then $\mu(= \mu(k))$ lies outside the L-ball around $o \in \widetilde{M_k}$ for $k = 1 \cdots C_0$.

Choosing $N = max(N_1, N'_1, \cdots, N'_{C_0})$ we are through. □

A corollary of Theorem 2.8 is the special case of quasi-Fuchsian *punctured* surface groups. We briefly outline this. The arguments for Theorem 2.8 show that for a sequence of quasi-Fuchsian groups converging to a simply degenerate surface group *without* accidental parabolics, the Cannon-Thurston maps converge uniformly. Thus for a geodesic ray $[0, \xi)$ in the universal cover $\widetilde{S}$ of the punctured surface S, $\rho_n(\xi)$ converges to $\rho_\infty(\xi)$, where (abusing notation slightly) ρ_n (resp. ρ_∞) represents the Cannon-Thurston maps corresponding to the representations ρ_n (resp. ρ_∞). Let $\rho_n([0, \xi))$ (resp. $\rho_\infty([0, \xi)))$ represent the geodesics in $\mathbb{H}^3$ joining 0 to $\rho_n(\xi)$ (resp. $\rho_\infty(\xi)$). Then $\rho_n([0, \xi))$ converges to $\rho_\infty([0, \xi))$ in the compact open topology on $\mathbb{H}^3$. Now, the universal covers $\widetilde{CC(M_n)}$ (resp. $\widetilde{CC(M_\infty)}$) of the convex cores $CC(M_n)$ (resp. $CC(M_\infty)$) of the manifolds M_n (resp. M_∞) corresponding to ρ_n (resp. ρ_∞) are strongly hyperbolic relative to the equivariant collection of horoballs corresponding to lifts of the cusps [Far98]. Let $\rho_n([0, \xi))_{ea}$ (resp. $\rho_\infty([0, \xi))_{ea}$) represent the electro-ambient quasi-geodesics corresponding to $\rho_n([0, \xi))$ (resp. $\rho_\infty([0, \xi)))$ in the appropriate spaces. Then since horoballs in $\widetilde{CC(M_n)}$ converge to those in $\widetilde{CC(M_\infty)}$, we have the following.

Corollary 2.9. *Let G_n be a sequence of quasi-Fuchsian punctured surface groups converging strongly to a simply degenerate punctured surface group without accidental parabolics. Let M_n be the corresponding hyperbolic manifolds. Consider a geodesic ray $[0, \xi)$ in the universal cover $\widetilde{S}$ of the punctured surface S. Let $\rho_n([0, \xi))_{ea}$ and $\rho_\infty([0, \xi))_{ea}$ be as above. Then $\rho_n([0, \xi))_{ea}$ converges to $\rho_\infty([0, \xi))_{ea}$.*

2.3. From Strictly Type-Preserving to Weakly Type-Preserving.

For a given Kleinian group Γ, a representation $\rho : \Gamma \to PSL_2(\mathbb{C})$ is said to be *weakly type preserving* if the image of every parabolic element of Γ is parabolic. As remarked in the Introduction to this paper, the hypothesis that the convergence is strictly type-preserving in Theorem 2.8 can be removed by combining Theorem 2.8 with a Theorem from [MS13a] that allows new parabolics to develop for geometrically finite groups converging strongly to a geometrically finite limit. In this subsection we sketch the argument.

As in Theorem 2.8, let Γ be a fixed group and $\rho_n(\Gamma) = G_n$ be a sequence of finitely generated Kleinian groups converging strongly to a Kleinian group $G = \rho_\infty(\Gamma)$. Suppose that the convergence is *not* strictly type-preserving. Then there exists a finite number $\gamma_1, \cdots, \gamma_k$ of primitive elements in Γ that are not parabolic in infinitely many G_n but are parabolic in G. Without loss of generality we assume that $\gamma_1, \cdots, \gamma_k$ are not parabolic in any G_n (else we pass to subsequences with smaller values of k and prove uniform convergence of Cannon-Thurston maps for these). We shall refer to the γ_i's as **limiting accidental parabolics**. The proof in this general situation boils down to investigating geometric limits of ends. Let $M = \mathbb{H}^3/G$ and $M_n = \mathbb{H}^3/G_n$ as before. All the manifolds M_n have a common Scott core K.

Recall that $\rho_n \to \rho_\infty$ strongly. By [Min10] and [BCM12], the geometrically infinite ends of M are determined by their ending laminations. Thus the limiting accidental parabolics $\gamma_1, \cdots, \gamma_k$ can be identified with simple closed curves lying on the boundary of K. For ease of exposition, let $k = 1$ and $\gamma_1 = \gamma$.

Let $E_n \subset M_n$ and $E \subset M$ be the end containing γ. Two cases arise:
a) E_n's are geometrically finite and they converge geometrically to a geometrically finite $E \subset M$ containing a limiting accidental parabolic corresponding to γ.
b) E_n's converge to E where E is geometrically infinite and contains a limiting accidental parabolic. In this case E minus the rank one cusp corresponding to γ must consist of one or two degenerate ends according as γ is non-separating or separating.

To prevent tedious book-keeping, we shall focus on two representative special cases of surface Kleinian groups to illustrate what needs to be done.
Case 1: $\Gamma = \pi_1(S)$ for a closed surface S. $\rho_n(\Gamma) = G_n$ is a sequence of quasi-Fuchsian Kleinian groups converging to G where $M = \mathbb{H}^3/G$ has one simply degenerate end and one geometrically finite end containing a single limiting accidental parabolic γ.
Case 2: $\Gamma = \pi_1(S)$ for a closed surface S. $\rho_n(\Gamma) = G_n$ is a sequence of quasi-Fuchsian Kleinian groups converging to G where $M = \mathbb{H}^3/G$ has one geometrically finite end without parabolics. The other end E of M contains a single rank one cusp (corresponding to a limiting accidental parabolic γ)

Case 1:

Abusing notation slightly let M_n (resp. M) denote the convex core of $M = \mathbb{H}^3/G_n$ (resp. $M = \mathbb{H}^3/G$). Let $P \subset M$ denote the limiting rank one cusp corresponding to γ and let $P_n \subset M_n$ denote a solid torus neighborhood of the geodesic corresponding to the group element $\rho_n(\gamma)$ in M_n. We may assume that $P_n \cap \partial M_n$ is an annulus whose core curve represents the element γ. Then $M_n \setminus P_n$ converges geometrically to $M \setminus P$. Also $M_n \setminus P_n$ (resp. $M \setminus P$) are uniformly bi-Lipschitz homeomorphic to quotients of $\mathbb{H}^3$ by quasiFuchsian Kleinian groups $\rho'_n(\Gamma)$ (resp. a quotient of $\mathbb{H}^3$ by a simply degenerate Kleinian groups $\rho'(\Gamma)$) such that ρ'_n converges strongly to ρ' and $\rho'(\Gamma)$ has no accidental parabolics. It follows from Theorem 2.8 that the Cannon-Thurston maps for ρ'_n converge uniformly to the Cannon-Thurston map for ρ'.

In fact Theorem 2.4 shows that the property UEPP holds for the collection ρ'_n, i.e. there exists a non-negative function $f(N)$, with $f(N) \to \infty$ as $N \to \infty$, such that for all $g, h \in \Gamma$ and $0 \in \mathbb{H}^3$, $d_\Gamma(1, [g,h]_\Gamma) \geq N$ implies $d_{\mathbb{H}}([\rho'_n(g)(0), \rho'_n(h)(0)], 0) \geq f(N)$ for all $n = 1, \ldots, \infty$, where $[g,h]_\Gamma$ denotes a geodesic in Γ joining g, h and $[\rho'_n(g)(0), \rho'_n(h)(0)]$ denotes a geodesic in $\mathbb{H}^3$ joining $\rho'_n(g)(0), \rho'_n(h)(0)$.

Identify $\Gamma = \pi_1(S) = \pi_1(\partial(M \setminus P))$ with the fundamental group of the component of $\partial(M_n \setminus P_n)$ intersecting P_n non-trivially. In fact we can identify the component of $\partial(M_n \setminus P_n)$ intersecting P_n non-trivially with S (by uniformly bi-Lipschitz homeomorphisms).

Now electrocute $P_n \subset M_n$ and let $(\widetilde{M_n}, d_{n,el})$ be the lifted metric to the universal cover. Similarly, electrocute $P \subset M$ and let $(\widetilde{M}, d_{el})$ be the lifted metric to the universal cover. Also let $(\widetilde{S}, d_e)$ denote the electric metric on $\widetilde{S}$ with the lifts of the geodesic corresponding to γ electrocuted.

Note that electroambient quasigeodesics in each of the spaces $(\widetilde{M_n}, d_{n,el})$, $(\widetilde{M}, d_{el})$, $(\widetilde{S}, d_e)$ are uniform quasigeodesics in $\widetilde{M_n}$, $\widetilde{M}$, $\widetilde{S}$ respectively by Lemma 1.4. Hence

1) there exists $C \geq 0$ such that for all $g, h \in \Gamma$ and $o \in \widetilde{S}$, $d_\Gamma(1, [g,h]_\Gamma) \geq N$ implies $d_{\widetilde{S}}(o, [g,h]_{ea}) \geq N - C$, where $[g,h]_{ea}$ denotes the electro-ambient quasigeodesic joining $g(o), h(o) \in \widetilde{S}$.

2) $d_{\widetilde{S}}(o, [g,h]_{ea}) \geq N - C$ implies that $d_{\mathbb{H}}([\rho'_n(g)(0), \rho'_n(h)(0)], 0) \geq f(N)$ for all $n = 1, \ldots, \infty$ (by Theorem 2.4).

Since $M_n \setminus P_n$ (resp. $M \setminus P$) are uniformly bi-Lipschitz homeomorphic to quotients of $\mathbb{H}^3$ by $\rho'_n(\Gamma)$ (resp. $\rho'(\Gamma)$) it follows that $d_{\mathbb{H}}([\rho_n(g)(0), \rho_n(h)(0)]_{ea}, 0) \geq f(N) - C'$ for some C' by Lemma 1.4 since $[\rho_n(g)(0), \rho_n(h)(0)]_{ea}$ and $[\rho'_n(g)(0), \rho'_n(h)(0)]$ track each other away from the lifts of P_n. In particular, $[\rho_n(g)(0), \rho_n(h)(0)]_{ea}$ enters and leaves a lift of P_n outside a large ($=f(N) - C'$) ball about $0 \in \mathbb{H}^3$.

Since $\partial P_n \cap \partial(M_n \setminus P_n)$ is an annulus on S under the identification of the component of $\partial(M_n \setminus P_n)$ intersecting P_n non-trivially with S. Hence any piece of $[\rho_n(g)(0), \rho_n(h)(0)]_{ea}$ inside a lift of P_n also lies outside a large ($=f(N) - C'$) ball about $0 \in \mathbb{H}^3$. (This part of the argument is very similar to an argument in the geometrically finite case in [MS13a] where a sequence of representations of a hyperbolic element converges to a parabolic.)

Finally, by Lemma 1.4 again, $[\rho_n(g)(0), \rho_n(h)(0)]$ lies outside a large ($=f(N)-C_1$) ball about $0 \in \mathbb{H}^3$ for all n. Hence the Cannon-Thurston maps for ρ_n converge uniformly to the Cannon-Thurston map for ρ by Proposition 1.10.

Case 2:

As before, $\Gamma = \pi_1(S)$ for a closed surface S. $\rho_n(\Gamma) = G_n$ is a sequence of quasi-Fuchsian Kleinian groups converging to G where $M = \mathbb{H}^3/G$ has one geometrically finite end without parabolics. The other end E of M contains a single rank one cusp corresponding to a limiting accidental parabolic γ. Let $M_n = \mathbb{H}^3/G_n$ and let E_n be the end of M_n such that E_n converges to E strongly.

It follows from strong convergence that E minus the rank one cusp consists of one or two degenerate ends according as γ is non-separating or separating. In either case, from the Minsky model [Min10] of the ends, there exists a Margulis tube $T_n \subset E_n \subset M_n = \mathbb{H}^3/G_n$ with core curve homotopic to γ such that the number of building blocks on either side of T_n in the Minsky model of E_n tends to infinity as $n \to \infty$. To fix notions, suppose γ is non-separating (since we shall lift everything to the universal cover the argument in the separating case is similar).

Let $[a, b]$ be an electro-ambient quasigeodesic in $\widetilde{S}$ with lifts of γ electrocuted. By Lemma 1.4 $[a, b]$ is a (uniform) quasigeodesic in the universal cover $\widetilde{S}$. We can therefore choose finitely many blocks on either side of T_n in M_n such that this number ($m = m(n)$ say) tends to infinity as $n \to \infty$.

The degenerating end E_n^c on the two sides of T_n (there are two in case γ is separating) consisting of the union of the $m(n)$ blocks above now satisfy the hypothesis of Theorem 2.4. The electro-ambient geodesic $[a, b]$ now consists of maximal segments $l_1, \cdots, l_k$ such that the segments l_{2i+1} with odd indices are geodesics in a lift of $S \setminus \{\gamma\}$ to $\widetilde{S}$ whereas the segments l_{2i} with even indices are geodesic subsegments of lifts of γ. Let r_{2i+1} be the geodesic joining the end-points of l_{2i+1} in the corresponding lift of E_n^c. Since E_n^c satisfies the hypothesis of Theorem 2.4, UEPP follows for each segment l_{2i+1} as in the case of simply degenerate ends without accidental parabolics. Also the alternating union of

r_{2i+1}'s with l_{2i}'s gives an electro-ambient quasigeodesic with (lifts of) the Margulis tube T_n electrocuted. Hence, by Lemma 1.4 again UEPP is established and uniform convergence of Cannon-Thurston maps follows as in Theorem 2.8.

References

[BCM12] J. F. Brock, R. D. Canary, and Y. N. Minsky. The Classification of Kleinian surface groups II: The Ending Lamination Conjecture. *Ann. of Math. 176 (1), arXiv:math/0412006*, pages 1–149, 2012.

[Far98] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal. 8*, pages 810–840, 1998.

[Min10] Y. N. Minsky. The Classification of Kleinian surface groups I: Models and bounds. *Annals of Mathematics. Second Series 171 (1)*, pages 1–107, 2010.

[Mj10] M. Mj. Cannon-Thurston Maps for Kleinian Groups. *preprint, arXiv:math 1002.0996*, 2010.

[Mj11] M. Mj. Cannon-Thurston Maps, i-bounded Geometry and a Theorem of McMullen. *Actes du séminaire Théorie spectrale et géométrie 28, Année 2009-10, arXiv:math.GT/0511104*, pages 63–108, 2011.

[Mj14] M. Mj. Cannon-Thurston Maps for Surface Groups. *Ann. of Math. 179 (1)*, pages 1–80, 2014.

[MP11] M. Mj and A. Pal. Relative Hyperbolicity, Trees of Spaces and Cannon-Thurston Maps . *Geometriae Dedicata, Volume 151, Number 1, arXiv:0708.3578*, pages 59–78, 2011.

[MS13a] M. Mj and C. Series. Limits of Limit Sets I: Geometrically Finite Groups . *Geometriae Dedicata 167(1)*, pages 35–67, 2013.

[MS13b] M. Mj and C. Series. Limits of Limit Sets II: Geometrically Infinite Groups . *arXiv:1306.3081*, 2013.

[Thu82] W. P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.*, page 357382, 1982.

mahan@rkmvu.ac.in; mahan.mj@gmail.com
http://maths.rkmvu.ac.in/~mahan/
School of Mathematical Sciences, RKM Vivekananda University
P.O. Belur Math, Dt. Howrah, WB 711202, India