LIMITS OF LIMIT SETS I

MAHAN MJ AND CAROLINE SERIES

ABSTRACT. We show that for a strongly convergent sequence of geometrically finite Kleinian groups with geometrically finite limit, the Cannon-Thurston maps of limit sets converge uniformly. If however the algebraic and geometric limits differ, as in the well known examples due to Kerckhoff and Thurston, then provided the geometric limit is geometrically finite, the maps on limit sets converge pointwise but not uniformly.

MSC classification: 30F40; 57M50

1. INTRODUCTION

Hausdorff convergence of limit sets under algebraic and geometric limits has been studied by several authors, see for example [15] p. 203 and Theorem 1.1 below. In this paper and its companion [22], we study convergence of limit sets as *the convergence of a sequence of continuous maps* from a fixed compact set, namely the limit set of a fixed geometrically finite group, to the sphere.

Given an isomorphism $\rho \colon \Gamma \to G$ between two geometrically finite Kleinian groups, Floyd [8] showed that there is a continuous equivariant map from a certain completion of the Cayley graph of Γ to the limit set Λ_G. As long as ρ is weakly type preserving (meaning that the image of every parabolic element of Γ is also parabolic in G), this map factors through the limit set Λ_Γ, giving a continuous equivariant map $\Lambda_\Gamma \to \Lambda_G$ between the limit sets. This result was extended by the remarkable work of Cannon-Thurston [4] who showed that such a map may still exist even when G is totally degenerate, by giving examples with Γ a surface group so that $\Lambda_\Gamma = S^1$, and a continuous surjective equivariant map to $\Lambda_G = S^2$. In fact maps between limit sets of isomorphic groups have a long history prior to Floyd's paper as so-called *boundary mappings*, going back to Nielsen [23], see also [7] Section 25.2 and for example [24, 12]. We will nevertheless stick with what is now well established terminology and call an equivariant continuous map between the limit sets $\Lambda_\Gamma, \Lambda_G$ of two isomorphic Kleinian groups, a Cannon-Thurston or CT-map.

Suppose now that we have a sequence of isomorphisms $\rho_n \colon \Gamma \to G_n$. This paper is the first of two which studies the pointwise behaviour of the CT-maps as the sequence G_n converges to a limiting Kleinian group G_∞. In this paper

we confine ourselves to the case in which both algebraic and geometric limits of (G_n) are geometrically finite; in the second [22] we will use the geometric models given by Minsky's ending lamination theorem [18] and results of the first author on existence of Cannon-Thurston maps [21] under very general hypotheses, to study the situation in which group G_∞ is geometrically infinite.

The results in this paper are the following:

Theorem A. *Let Γ be a fixed geometrically finite Kleinian group and $\rho_n \colon \Gamma \to G_n$ be a sequence of weakly type-preserving isomorphisms to geometrically finite Kleinian groups G_n which converge strongly to a geometrically finite Kleinian group $G_\infty = \rho_\infty(\Gamma)$. Then the sequence of CT maps $\hat{i}_n : \Lambda_\Gamma \to \Lambda_{G_n}$ converges uniformly to $\hat{i}_\infty : \Lambda_\Gamma \to \Lambda_{G_\infty}$.*

Theorem B. *Let Γ be a fixed geometrically finite Kleinian group and $\rho_n \colon \Gamma \to G_n$ be a sequence of weakly type-preserving isomorphisms to geometrically finite Kleinian groups which converge algebraically to $G_\infty = \rho_\infty(\Gamma)$. Suppose also that the geometric limit of the groups G_n is geometrically finite. Then the sequence of CT-maps $\hat{i}_n : \Lambda_\Gamma \to \Lambda_{G_n}$ converge pointwise to $\hat{i}_\infty : \Lambda_\Gamma \to \Lambda_{G_\infty}$.*

We remark that if the geometric limit of geometrically finite groups is geometrically finite, so is the algebraic limit, see [15] Theorem 4.6.1 and [13]. However geometrically finiteness of the algebraic limit does not in general imply the same for the geometric limit, as can be seen for example by examining the many geometric limits of once punctured torus groups constructed in [17].

If the convergence is algebraic but not strong, then uniform convergence necessarily fails. This is a consequence of the following result, which in the generality below is due to Evans.

Theorem 1.1 ([5], [6]). *Suppose $\rho_n \colon \Gamma \to G_n$ is a sequence of weakly type-preserving isomorphisms from a geometrically finite group Γ to Kleinian groups G_n with limit sets Λ_n, such that the sequence converges algebraically to $\rho_\infty \colon \Gamma \to G_\infty$ and geometrically to H. Let Λ_∞ and Λ_H denote the limit sets of G_∞ and H respectively. Then $\Lambda_n \to \Lambda_H$ in the Hausdorff metric. If in addition Γ is non-elementary, the sequence converges strongly if and only if $\Lambda_n \to \Lambda_\infty$ in the Hausdorff metric.*

For the purposes of this paper, in which the geometric limit is geometrically finite with non-empty regular set, then we are in the easier situation of [13] Proposition 4.2, see also [15] Theorem 4.5.4.

Since uniform convergence implies diagonal convergence of limit points and hence Hausdorff convergence, Theorem 1.1 shows that if Λ_∞ and Λ_H differ, then uniform convergence is impossible.

Theorem B is illustrated by the Kerckhoff-Thurston examples [11] of a sequence G_m of quasi-Fuchsian groups converging geometrically to a geometrically finite

group G with a rank 2-cusp. Our proof will make clear how it is that the CT-maps converge pointwise, but not uniformly. The lack of uniform convergence in these examples has also been noted in a remark due to Souto, see Section 9 of [9].

As a special case of Theorem A one obtains an alternative proof of the following well-known application of the λ-lemma from complex dynamics [14]. Let $A \subset \mathbb{C}$ be a connected open set. A family of quasi-Fuchsian groups $G_z, z \in A$ is said to be *holomorphic* if there are maps $\rho_z : \Gamma \to G_z, z \in A$ from a fixed (finitely generated) Fuchsian group Γ such that $z \mapsto \rho_z(g)$ is holomorphic for each $g \in \Gamma$. If the maps ρ_z are all type preserving, then in the formula for the fixed points, the attracting fixed point (or unique parabolic fixed point) is defined by a fixed branch of the square root, so that the maps $i_z : z \mapsto g^+(z)$ are holomorphic. A straightforward application of the λ-lemma allows one to extend i_z to the entire limit set Λ_Γ:

Theorem 1.2. *Let $z \to G_z$ be a holomorphic family of Kleinian groups such that the map $\rho_z : \Gamma \to G_z$ is an isomorphism for $z \in A$ and such that the map $i_z : \Lambda_G^+ \to \hat{\mathbb{C}}$ is injective. Then the natural embedding $i_z : \Lambda_G^+ \to \hat{\mathbb{C}}$ extends to a continuous equivariant homeomorphism $\hat{i}_z : \Lambda_G \to \hat{\mathbb{C}}$, such that $\hat{i}_z : \Lambda_G \times A \to \hat{\mathbb{C}}$ is jointly continuous and such that the map $z \mapsto \hat{i}_z(\xi)$ is holomorphic on A for each $\xi \in \Lambda_\Gamma$.*

Here Λ_G^+ is the set of attractive fixed points of loxodromic elements in Γ, see Section 2.2. This result applies, for example, in the interior of quasifuchsian deformation space $\mathcal{QF}$. However if $z \in \partial\mathcal{QF}$ then the map i_z is not in general a bijection in a neighborhood of z, so the above result does not give information about what happens as we limit on the boundary $\partial\mathcal{QF}$.

Miyachi [20] proved Theorem A without the restrictions on geometric finiteness, in the case in which Γ is a surface group without parabolics and the injectivity radius is bounded along the whole sequence. The only other convergence result we are aware of is that of Francaviglia [9] Theorem 1.8 in which Γ is a discrete subgroup of $\mathbb{H}^k$ which diverges at its critical exponent (and so only applies in our context to Γ Fuchsian with no parabolics, see [15] Theorem 3.14.3). It is shown that, provided that all groups in question are non-elementary, then the corresponding CT-maps exist and converge almost everywhere with respect to Patterson-Sullivan measure on Λ_Γ.

The results in this paper are all based on the geometry of geometrically finite groups. In the geometrically infinite situation of [22], besides the Minsky model of degenerate Kleinian groups, we introduce techniques from the theory of relative hyperbolic spaces and electric geometry. We will prove in [22] that Theorem A holds in many cases of geometrically infinite G_∞, but show that for certain algebraic limits (such as the examples of Brock [2]), even pointwise convergence may fail.

In Section 2 we set up the background and in Section 3 we prove some important estimates based on Floyd [8] on the embedding of the Cayley graph of a geometrically finite group into $\mathbb{H}^3$. This is a prelude to Section 4 in which we

describe the CT-maps carefully and give a geometrical criterion Theorem 4.1 for their existence. In Section 5 this is extended to a criterion Theorem 5.6 for uniform convergence. In Section 6 we prove Theorem A, the case of strong convergence and easily deduce the application to Theorem 1.2. Finally in Section 7 we formulate the corresponding criterion for pointwise convergence and prove Theorem B. Some hyperbolic geometry estimates we need are given in the Appendix.

2. Preliminaries

2.1. **Kleinian groups.** A *Kleinian group* G is a discrete subgroup of $PSL_2(\mathbb{C})$. As such it acts as a properly discontinuous group of isometries of hyperbolic 3-space $\mathbb{H}^3$, whose boundary we identify with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. All groups considered in this paper will be finitely generated and torsion free, so that $M = \mathbb{H}^3/G$ is a hyperbolic 3-manifold. An excellent reference for the background we need is [15].

A Kleinian group is *geometrically finite* if it has a fundamental polyhedron in $\mathbb{H}^3$ with finitely many faces. The limit set $\Lambda_G \subset \hat{\mathbb{C}}$ is the closure of the set of its non-elliptic fixed points. It can also be defined as the set of accumulation points of any G-orbit.

Let $\mathcal{N}$ be the hyperbolic convex hull of Λ_G in $\mathbb{H}^3$. This projects to the convex core of M, that is, the smallest closed convex subset containing all closed geodesics. An alternative characterisation of being geometrically finite is that a δ-neighbourhood of $\mathcal{N}/G$ in $\mathbb{H}^3$ has finite volume for some $\delta > 0$. Note that if G is geometrically finite, then $\mathcal{N}/G$ is compact if and only if G contains no parabolics. Such groups are called *convex cocompact*.

The *thin part* $M_{thin}(\epsilon)$ is by definition that part of M where the injectivity radius is at most ϵ. The Margulis constant $\epsilon_\mathcal{M} > 0$ is such that if $\epsilon \leq \epsilon_\mathcal{M}$, then $M_{thin}(\epsilon)$ is a disjoint union of horocyclic neighbourhoods of cusps and Margulis tubes around short geodesics. It is well known that the δ-neighbourhood of $\mathcal{N}/G$ has finite volume if and only if the thick part of $(\mathcal{N}/G)\backslash M_{thin}(\epsilon)$ has finite diameter. (Consider a maximal collection of disjoint embedded balls of small radius in $(\mathcal{N}/G)\backslash M_{thin}(\epsilon)$.)

We denote by $\mathcal{H} = \mathcal{H}_\epsilon = \mathcal{H}_{\epsilon;G}$ the set of lifts of components of $M_{thin}(\epsilon)$ to $\mathbb{H}^3$. Thus $\mathcal{H}$ is a G-invariant collection of horoballs based at the parabolic fixed points of G, together with equidistant tubes about geodesic axes whose associated loxodromics are short. If we are dealing with a single geometrically finite group G, then by reducing ϵ we can assume that all elements of $\mathcal{H}$ are horoballs, see for example [15] Theorem 3.3.4. We will denote by $\mathcal{V} = \mathcal{V}_\epsilon = \mathcal{V}_{\epsilon;G}$ the ϵ-thick part $\mathcal{N}$ relative to G, that is, $\mathcal{V}$ is the closure of $\mathcal{N} \setminus (\cup_{H \in \mathcal{H}} H)$.

Let Γ be a fixed Kleinian group. A representation $\rho : \Gamma \to PSL_2(\mathbb{C})$ is said to be *type preserving* if it maps loxodromics to loxodromics and parabolics to parabolics. It is *weakly type preserving* if the image of every parabolic element of Γ is also parabolic in G.

Let $\rho_n \colon \Gamma \to PSL_2(\mathbb{C}), n = 1, 2 \ldots$ be a sequence of group isomorphisms. The representations ρ_n are said to converge to the representation $\rho_\infty \colon \Gamma \to PSL_2(\mathbb{C})$ *algebraically* if for each $g \in \Gamma$, $\rho_n(g) \to \rho_\infty(g)$ as elements of $PSL_2(\mathbb{C})$. They are said to converge *geometrically* if $(G_n = \rho_n(G))$ converges as a sequence of closed subsets of $PSL_2(\mathbb{C})$ to $H \subset PSL_2(\mathbb{C})$. Then H is a Kleinian group called the *geometric limit* of (G_n). The sequence (G_n) converges *strongly* to $\rho_\infty(G)$ if $\rho_\infty(G) = H$ and the convergence is both geometric and algebraic.

2.2. Cannon-Thurston Maps.

Let Γ be a Kleinian group and let $\rho \colon \Gamma \to PSL_2(\mathbb{C})$ with $\rho(\Gamma) = G$. Let $\Lambda_\Gamma, \Lambda_G$ be the corresponding limit sets. A *Cannon-Thurston* or *CT*-map is an equivariant continuous map $\hat{i} \colon \Lambda_\Gamma \to \Lambda_G$, that is, a map such that

$$\hat{i}(g \cdot \xi) = \rho(g)\hat{i}(\xi) \quad \text{for all} \quad g \in \Gamma, \xi \in \Lambda_\Gamma.$$

For a loxodromic $A \in PSL_2(\mathbb{C})$, denote by A^+ its attracting fixed point. For simplicity of notation, if A is parabolic, we denote the single fixed point in the same way. From the equivariance it easily follows that a Cannon-Thurston map preserves fixed points, namely

$$\hat{i}(g^+) = \rho(g)^+ \quad \text{for all} \quad g \in \Gamma.$$

Notice that in this we do not necessarily assume that ρ is type preserving, but that weakly type preserving is clearly a necessary condition for the existence of an *CT*-map. Denoting by Λ_Γ^+ the subset of attracting fixed points, we see that the *CT*-map, if it exists, is the continuous extension of the above obvious map $\Lambda_\Gamma^+ \to \Lambda_G^+$ to the whole of Λ_Γ.

Here is an alternative view on the construction of *CT*-maps. The Cayley graph of Γ is naturally embedded in $\mathbb{H}^3$ by a map $j_\Gamma : \mathcal{G}\Gamma \to \mathbb{H}^3$ which sends the vertex $g \in \mathcal{G}\Gamma$ to $g \cdot O$ where $O = O_G \in \mathbb{H}^3$ is a fixed base point and extends in the obvious way to edges, see Section 3 for details. Suppose that $\rho \colon \Gamma \to G$ is weakly type preserving. Then define $i \colon j_\Gamma(\mathcal{G}\Gamma) \to \mathbb{H}^3$ by setting $i(j_\Gamma(g)) = j_G(g)$ on vertices and again extending in the obvious way. Clearly i is equivariant in the sense that $i(hg \cdot O) = \rho(h)i(g \cdot O)$. If the map i extends (with respect to the Euclidean metric in the ball model) to a continuous map $\hat{i} : \Lambda_\Gamma \to \Lambda_G$, it follows easily from the equivariance and continuity that $\hat{i}(g^+) = \rho(g)^+$ for all $g \in \Gamma$, so that $\hat{i}$ is an *CT*-map as defined above. (A necessary condition for this extension to work is that ρ be weakly type preserving.)

The original interest in the Cannon-Thurston maps applied to the case in which Γ is a surface group and G a doubly degenerate group which is the cyclic cover of a 3-manifolds fibering over the circle with pseudo-Anosov monodromy, [4], in which

case of course Λ_Γ is a round circle and $\Lambda_G = \hat{\mathbb{C}}$. However we can also put results of Floyd [8] in this context. Floyd constructed the completion $\bar{\Gamma}$ of the Cayley graph of Γ with a suitable metric and showed:

Theorem 2.1 ([8]). *Let Γ be a geometrically finite Kleinian group with group completion $\bar{\Gamma}$. Then there is an equivariant continuous map $\bar{\Gamma} \setminus \Gamma \to \Lambda_\Gamma$ which is $2 - 1$ on parabolic points and injective elsewhere.*

Floyd's map gives a continuous extension of the embedding $i : j_\Gamma(\mathcal{G}\Gamma) \to \mathbb{H}^3$ to a map $\bar{\Gamma} \to \mathbb{H}^3 \cup \partial\mathbb{H}^3$. In the special case in which G contains no parabolics, $j_\Gamma(\mathcal{G}\Gamma)$ is a hyperbolic metric space and $\bar{G}$ can be identified both with its Gromov boundary and with Λ_Γ. For precise details of the connection between Floyd's result and maps of limit sets, see [25].

In Section 4 we give a criterion for the existence of a CT-map in the case in which Γ is any geometrically finite Kleinian group and $\rho : \Gamma \to G$ is a weakly type preserving isomorphism. It is not hard to deduce the above result of Floyd. In subsequent sections the criterion is extended to deal with converging sequences of groups.

2.3. **Notation.** We denote the hyperbolic metric on $\mathbb{H}^3$ by $d_\mathbb{H}$; sometimes we explicitly use the ball model $\mathbb{B}$ with centre O and denote by $d_\mathbb{E}$ the Euclidean metric on $\mathbb{B} \cup \hat{\mathbb{C}}$. For $P \in \mathbb{H}^3$, write $B_\mathbb{H}(P; R)$ for the hyperbolic ball centre P and radius R, with a similar definition for Euclidean balls $B_\mathbb{E}$. Let β be a path in $\mathbb{H}^3$ with endpoints X, Y. We denote its hyperbolic length by $\ell(\beta)$ and write $[\beta]$ or $[X, Y]$ for the $\mathbb{H}^3$-geodesic from X to Y (so that $\ell(\beta) \geq \ell([\beta])$ with equality if and only if β is itself geodesic).

We write $X \prec Y$ (resp. $X \succ Y$) to mean there is a constant $c > 0$ such that $X < cY$ (resp. $X > cY$) and $X \asymp Y$ to mean $X \prec Y$ and $Y \prec X$. We also write $X \overset{+}{\prec} Y$ to indicate an inequality up to an additive constant, thus $X \overset{+}{\prec} Y$ means there is a constant $c > 0$ such that $c > 0$ such that $X < Y + c$; the notation $Y \overset{+}{\succ} X$ is defined similarly.

3. Embedding the Cayley graph

Let G be a finitely generated Kleinian group with generating set $G^* = \{e_1, \ldots, e_k\}$. Its Cayley graph $\mathcal{G}G$ is the graph whose vertices are elements $g \in G$ and which has an edge between g, g' whenever $g^{-1}g' \in G^*$. The graph metric d_G is defined as the edge length of the shortest path between vertices so that $d_G(1, e_i) = 1$ for all i, where 1 is the unit element of G. Let $|g|$ (or where needed for clarity $|g|_{G^*}$) denote the word length of $g \in G$ with respect to G^*, so that $|g| = d_G(1, g)$. For $X \in \mathcal{G}G$, we denote by $B_G(X; R) \subset \mathcal{G}G$ the d_G-ball centre X and radius R.

Choose a base point $O = O_G \in \mathbb{H}^3$ which is not a fixed point of any element of G. One may if desired assume the basepoint is the centre of the ball model $\mathbb{B}$. For

simplicity, we do this throughout the paper unless indicated otherwise. Then $\mathcal{G}G$ is embedded in $\mathbb{H}^3$ by the map j_G which sends $g \in G$ to $j_G(g) = g \cdot O$, and which sends the edge joining g, g' to the $\mathbb{H}^3$-geodesic joining $j_G(g), j_G(g')$. In particular, $j_G(1) = O$. Note that using the ball model of $\mathbb{H}^3$, the limit set Λ_G may be regarded as the completion of $j_G(\mathcal{G}G)$ in the Euclidean metric $d_{\mathbb{E}}$ on $\mathbb{B} \cup \hat{\mathbb{C}}$.

It will be important for us to understand the relationship between geodesic segments lying outside large balls in $\mathcal{G}G$ and in $\mathbb{H}^3$. For geometrically finite groups, the main facts we need are encapsulated in the following theorem. Recall that for a path $\beta \subset \mathbb{H}^3$, $[\beta]$ denotes the hyperbolic geodesic joining its endpoints.

Theorem 3.1. *Let G be a finitely generated geometrically finite Kleinian group. There exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(N) \to \infty$ as $N \to \infty$ and such that if λ is a d_G-geodesic segment in $\mathcal{G}G$ which lies outside $B_G(1; N)$, then both $j_G(\lambda)$ and $[j_G(\lambda)]$ lie outside $B_{\mathbb{H}}(O; f(N))$ in $\mathbb{H}^3$.*

Our proof is based on the following result of Floyd whose proof we recall as it is fundamental in what follows.

Proposition 3.2 ([8] p. 216)**.** *Let G be a finitely generated geometrically finite Kleinian group with generating set G^*, and let $O \in \mathbb{H}^3$ be a fixed base point. Then there exist constants $a, b, k > 0$ such that if G contains no parabolics then*

$$(1) \qquad\qquad b|g| \leq d_{\mathbb{H}}(O, g \cdot O) \leq a|g| \qquad \forall g \in G,$$

while if G contains parabolics then

$$(2) \qquad\qquad 2log|g| - k \leq d_{\mathbb{H}}(O, g \cdot O) \leq a|g| \qquad \forall g \in G.$$

Proof. The right hand inequality is easy: $d_{\mathbb{H}}(O, g \cdot O) \leq a|g|$ where $a = \max\{d_{\mathbb{H}}(O, e_i \cdot O) \colon e_i \in G^*\}$.

Now for the left hand inequality. Assume first that G contains no parabolics and thus is convex cocompact. Let $\mathcal{D}$ be a finite sided fundamental polyhedron for G, let $\mathcal{N} \subset \mathbb{H}^3$ be the hyperbolic convex hull of the limit set Λ_G, and let $\mathcal{D}' = \mathcal{N} \cap \mathcal{D}$. We may as well assume that $O \in \mathcal{D}'$ and note that by convexity the geodesic from O to $g \cdot O$ is in $\mathcal{N}$ for all $g \in G$.

Let $d = \operatorname{diam} \mathcal{D}'$ and let $C = \max\{|g| : d_{\mathbb{H}}(O, g \cdot O) \leq 1 + 2d\}$. Let $g \in G$. Divide the geodesic between O to $g \cdot O$ into intervals of length 1 (with one shorter interval) and connect each division point to the closest point in $G \cdot O$. This gives the estimate $|g| \leq 1 + C d_{\mathbb{H}}(O, g \cdot O)$.

Now suppose that G contains parabolics. Choose $\epsilon \leq \epsilon_{\mathcal{M}}$ such that $\mathcal{H}_\epsilon$ is an invariant set of disjoint horoballs around the cusps. Let $\mathcal{V} = \mathcal{V}(\epsilon) = \mathcal{N} \setminus \cup_{H \in \mathcal{H}} H$. Then $\mathcal{D}'' = \mathcal{D}' \cap \mathcal{V}$ is compact and so has finite diameter. By reducing ϵ if necessary, we can arrange that the horoballs are small enough so that the geodesic from O to $e_i \cdot O$ is in $\mathcal{V}$ for all the generators $e_i \in G^*$. Define a metric $d'_{\mathbb{H}}$ on $\mathcal{V}$ by setting $d'_{\mathbb{H}}(x, y)$ to be the length of the shortest path in $\mathcal{V}$ between x and y in the induced

metric on $\mathcal{V}$. Then just as above we obtain the estimate $|g| \leq 1 + Cd'_{\mathbb{H}}(O, g \cdot O)$. Now use Lemma A.3 which says that for $H \in \mathcal{H}$ and points $x, y \in \partial H$, $d'_{\mathbb{H}}(x, y) \leq exp\, d_{\mathbb{H}}(x, y)/2$. This leads to the left hand inequality in (2). $\qquad\square$

A path $\beta \subset \mathbb{H}$ is a *K-quasi-geodesic* if for any subsegment $\alpha \subset \beta$,

$$\ell([\alpha])/K - K \leq \ell(\alpha) \leq K\ell([\alpha]) + K$$

where as usual $\ell(\alpha)$ denotes the hyperbolic length of the path α and $[\alpha]$ is the hyperbolic geodesic with the same endpoints as α. We use heavily the important fact that a K-quasi-geodesic is at bounded distance from the geodesic with the same endpoints, with constants depending only on K, see for example [10].

To deal with the thin parts of the manifold $\mathbb{H}^3/G$, we use the following extension of this definition due to McMullen [16].

Definition 3.3. *Let $V \subset \mathbb{H}^3$ be a Riemannian manifold. A path $\beta : [0, 1] \to V$ is an* ambient *K-quasi-geodesic if*

$$\ell(\alpha) \leq K\ell(\gamma) + K$$

for any subsegment $\alpha \subset \beta$ and any path $\gamma \subset V$ with the same endpoints as α.

As in Section 2.1, fix $\epsilon \leq \epsilon_{\mathcal{M}}$ and let $\mathcal{H} = \mathcal{H}_\epsilon$ denote the union of the lifts of the ϵ-thin parts to $\mathbb{H}^3$. We state McMullen's result as applied to $\mathcal{V} = \mathcal{N} \setminus \bigcup_{H \in \mathcal{H}} H$.

Proposition 3.4 ([16] Theorem 8.1)**.** *Let $\beta : [0, 1] \to \mathcal{V}$ be an ambient K-quasi-geodesic. Then β lies within a bounded distance $R(K)$ of $[\beta] \cup \mathcal{H}([\beta])$, where $\mathcal{H}([\beta])$ is the union of those $H \in \mathcal{H}$ which meet $[\beta]$.*

Remark 3.5. Although in this section we only need this result when all elements of $\mathcal{H}_\epsilon$ are horoballs, the result also holds when $\mathcal{H}_\epsilon$ contains Margulis tubes. Moreover the constants involved depend only on ϵ and are independent of the group G.

Here are two ways of constructing ambient quasi-geodesics.

Lemma 3.6. *Let $\gamma \subset \mathcal{G}G$ be a path from 1 to g constructed as in the proof of Proposition 3.2. Then γ is an ambient quasi-geodesic in $\mathcal{V}$.*

Proof. By the construction, $|\gamma| \asymp \ell(\mu)$, where μ is the shortest path from O to $g \cdot O$ in $\mathcal{V}$. Since $\ell(j_G(\gamma)) \prec |\gamma|$, it follows that γ is also an ambient quasi-geodesic in $\mathcal{V}$. $\qquad\square$

Lemma 3.7. *Suppose that G is geometrically finite and that λ is a geodesic in $(\mathcal{G}G, d_G)$. Then $j_G(\lambda)$ is an ambient quasi-geodesic in $\mathcal{V} = \mathcal{N} \setminus \bigcup_{H \in \mathcal{H}} H$.*

Proof. If G is convex cocompact, the result follows immediately from Equation (1), which says that any d_G-shortest path is a quasi-geodesic in $\mathbb{H}^3$.

In the general case, consider any subsegment $\lambda_1 \subset \lambda$ and let $x, y \in \mathcal{G}G$ be its endpoints. Let μ_1 be the shortest path in $\mathcal{V}$ joining $j_G(x), j_G(y)$. As in the proof of Proposition 3.2, there exists a path in $\mathcal{G}G$ from x to y of length L say, such that

$$\ell(\mu_1) \asymp L.$$

Since λ_1 is a shortest d_G-path from x to y we have $|\lambda_1| \leq L$ where $|\lambda_1|$ denotes the length of the geodesic λ_1 in $\mathcal{G}G$. Thus

$$\ell(j_G(\lambda_1)) \prec |\lambda_1| \leq L \prec \ell(\mu_1)$$

where the first inequality follows as usual from the right hand inequality of (2). This shows that $j_G(\lambda)$ is an ambient quasi-geodesic in $\mathcal{V}$ as claimed. $\square$

Corollary 3.8. *Suppose that G is geometrically finite and that λ is a geodesic in $(\mathcal{G}G, d_G)$. Then $j_G(\lambda)$ lies within bounded distance of $[j_G(\lambda)] \cup \mathcal{H}([j_G(\lambda)])$.*

Proof. This follows immediately from Proposition 3.4 and Lemma 3.7. $\square$

Proof of Theorem 3.1. First assume that G contains no parabolics, equivalently, that G is convex cocompact. Suppose that λ is a d_G-geodesic segment in $\mathcal{G}G$ which lies outside $B_G(1; N)$. Then inequality (1) in Proposition 3.2 gives $d_{\mathbb{H}}(O, h \cdot O) \geq b|h| \geq bN$ for all $h \in j_G(\lambda)$. Hence $j_G(\lambda)$ lies outside $B_{\mathbb{H}}(O; bN)$.

Equation (1) says that $j_G(\lambda)$ is a K-quasi-geodesic in $\mathbb{H}^3$ with $K = \max\{b, 1/a\}$. Thus $j_G(\lambda)$ is at bounded distance from the hyperbolic geodesic $[j_G(\lambda)]$ with the same endpoints, with constants depending only on K. Since $j_G(\lambda)$ lies outside $B_{\mathbb{H}}(O; bN)$, it follows that $[j_G(\lambda)]$ lies outside $B_{\mathbb{H}}(O; bN - c)$ for some $c > 0$ depending only on a, b. This completes the proof in the convex cocompact case.

Now suppose that G contains parabolics. The left hand inequality of (2) shows that $j_G(\lambda)$ lies outside $B_{\mathbb{H}}(O; f(N))$ where $f(N) \overset{+}{\succ} 2 \log N$. Without loss of generality, we may as usual assume that the basepoint $O \in \mathbb{H}^3$ lies outside $\bigcup_{H \in \mathcal{H}} H$. It remains to show that $\alpha = [j_G(\lambda)]$ lies outside some ball $B_{\mathbb{H}}(O; f(N))$ for some $f(N) \to \infty$ as $N \to \infty$.

By Lemma 3.7, $j_G(\lambda)$ is an ambient quasi-geodesic in $\mathcal{V}$. Let P_1, P_2 be the entry and exit points of α to some horoball $H \in \mathcal{H}(\alpha)$. By Proposition 3.4, P_1 and P_2 lie within bounded distance of points on $j_G(\lambda)$, hence by (2) outside $B_{\mathbb{H}}(O; 2 \log N - c')$ for some $c' > 0$. Thus by Lemma A.5, the segment $[P_1, P_2]$, and hence α, lies outside $B_{\mathbb{H}}(O; \log N/2 - c'')$ for some $c'' > 0$. This completes the proof. $\square$

4. Existence of CT-maps

In this section we state and prove a criterion for the existence of CT-maps, Theorem 4.1. Variants will be used later to prove our main results.

In [19], the existence of CT-maps is discussed in the context of hyperbolic metric spaces. Suppose that $i : X \to Y$ is an inclusion of such spaces. A *Cannon-Thurston map* in this context is by definition a continuous extension of i to a map

$\hat{i} \colon \hat{X} \to \hat{Y}$, where for a hyperbolic space Z, ∂Z denotes the Gromov boundary and $\hat{Z} = Z \cup \partial Z$ carries the natural topology obtained by extending the Gromov product to the boundary, see [1]. Lemma 2.1 of [19] asserts that a Cannon-Thurston map exists if and only if for all $M > 0$ and $x \in X$, there exists $N > 0$ such that if a geodesic λ in X lies outside an N-ball around x in X, then any geodesic in Y joining the endpoints of λ lies outside the M-ball around $i(x)$ in Y.

If now $\rho \colon \Gamma \to G$ is a weakly type preserving isomorphism of Kleinian groups, then as explained in Section 2.2, the CT-map $\hat{i} \colon \Lambda_\Gamma \to \Lambda_G$ is the continuous extension, if it exists, of the embedding $i \colon j_\Gamma(\mathcal{G}\Gamma) \to \mathbb{H}^3$, $i(j_\Gamma(g)) = j_G \rho(g), g \in \Gamma$. It is well known that if H is a finitely generated convex cocompact Kleinian group, then its Cayley graph $\mathcal{G}H$ with the word metric is a hyperbolic space and the metrics d_H on $\mathcal{G}H$ and the induced hyperbolic metric on $j_H(\mathcal{G}H)$ are equivalent. Moreover the limit set Λ_H may be naturally identified with the Gromov boundary of $\mathcal{G}H$, see [10]. Thus if both groups Γ, G are convex cocompact, the above result is a criterion for the existence of the CT-map $\hat{i} \colon \Lambda_\Gamma \to \Lambda_G$. The main result of this section is the following theorem which extends this to a criterion which applies without the assumption of cocompactness on either Γ or G. Notice that the hypothesis does not require that the image group G be geometrically finite.

Theorem 4.1. *Let $\rho \colon \Gamma \to G$ be a weakly type preserving isomorphism of finitely generated Kleinian groups and suppose that Γ is geometrically finite. The* CT-*map $\Lambda_\Gamma \to \Lambda_G$ exists if and only if there exists a non-negative function $f \colon \mathbb{N} \to \mathbb{N}$, such that $f(N) \to \infty$ as $N \to \infty$, and such that whenever λ is a d_Γ-geodesic segment lying outside $B_\Gamma(1; N)$ in $\mathcal{G}\Gamma$, the $\mathbb{H}^3$-geodesic joining the endpoints of $i(j_\Gamma(\lambda))$ lies outside $B_\mathbb{H}(O_G; f(N))$ in $\mathbb{H}^3$.*

Proof. Since the result is clearly independent of the choice of basepoints for Γ and G, for simplicity we take $O_\Gamma = O_G = O \in \mathbb{B}$. As explained in Section 2.2, the existence of the CT-map is equivalent to the statement that i extends to a continuous map $\hat{i} \colon (\Lambda_\Gamma \cup j_\Gamma(\mathcal{G}\Gamma), d_\mathbb{E}) \to (\Lambda_G \cup j_G(\mathcal{G}G), d_\mathbb{E})$.

Suppose first that i extends continuously. For each $N \in \mathbb{N}$, let $\phi(N) = \sup\{L\}$, where the sup is taken over all $L \geq 0$ with the property that for all d_Γ-geodesic segments λ lying outside $B_\Gamma(1; N) \subset \mathcal{G}\Gamma$, the $\mathbb{H}^3$-geodesic $[i(j_\Gamma(\lambda))]$ joining the endpoints of $i(j_\Gamma(\lambda))$ lies outside $B_\mathbb{H}(O; L)$. (Here $\phi(N) = 0$ is possible.) Suppose that $\sup_N\{\phi(N)\} \leq K < \infty$. Then we can find a sequence of d_Γ-geodesic segments $\lambda_N \subset \mathcal{G}\Gamma$ such that λ_N lies outside the ball $B_\Gamma(1; N)$ in $\mathcal{G}\Gamma$ while $\beta_N = [i(j_\Gamma(\lambda_N))]$ meets the ball $B_\mathbb{H}(O; K + 1)$. Thus passing to a subsequence, the endpoints of $i(j_\Gamma(\lambda_N))$ converge to distinct points in Λ_G. However by Theorem 3.1, there exists $f_1(N)$ such that the geodesic $[j_\Gamma(\lambda_N)]$ lies outside $B_\mathbb{H}(O; f_1(N))$, and such that $f_1(N) \to \infty$ as $N \to \infty$. It follows that after passing to a further subsequence, the endpoints of $j_\Gamma(\lambda_N)$ converge to the same point in Λ_Γ. This contradiction shows that $\phi(N) \to \infty$ with N and so the criterion is satisfied with $f = \phi$.

Now we show that the condition is sufficient. First we need to define $\hat{i} \colon \Lambda_\Gamma \to \Lambda_G$. If $\xi \in \Lambda_\Gamma$ is a parabolic point corresponding to a parabolic $p \in \Gamma$, using the hypothesis that ρ is weakly type preserving we define $\hat{i}(\xi)$ to be the unique fixed point of $\rho(p)$.

Now assume that ξ is not a parabolic fixed point. Note that $i \colon j_\Gamma(\mathcal{G}\Gamma) \to \mathbb{H}^3$ is proper (with respect to the hyperbolic metrics), for if not, there exist points $g_n \cdot O$ converging to Λ_Γ such that $\{i(g_n \cdot O)\}$ lie in a compact set in $\mathbb{H}^3$, which contradicts our hypotheses. By definition, if $\xi \in \Lambda_\Gamma$, then there exists a sequence $g_m \in \Gamma$ such that $j_\Gamma(g_m) \to \xi$ in the Euclidean metric $d_\mathbb{E}$. Since i is proper, $i(j_\Gamma(g_m))$ has a subsequence which converges to a point $\eta \in \Lambda_G$. We want to define $\hat{i}(\xi) = \eta$, so we need to see that η depends only on ξ and not on the sequence g_m. So suppose that $g'_m \in G$ and $j_\Gamma(g'_m) \to \xi$ in $(\mathbb{B} \cup \hat{\mathbb{C}}, d_\mathbb{E})$, but that $i(j_\Gamma(g'_m)) \to \eta' \in \Lambda_G$ where $\eta \neq \eta'$.

Let λ_m be the d_Γ-geodesic joining g_m and g'_m. Since $g_m \cdot O \to \xi$ and $g'_m \cdot O \to \xi$, the $\mathbb{H}^3$-geodesic $\alpha_m = [j_\Gamma(\lambda_m)]$ joining $j_\Gamma(g_m)$ and $j_\Gamma(g'_m)$ lies outside $B_\mathbb{H}(O, N_m)$, where $N_m \to \infty$ as $m \to \infty$. We claim that $j_\Gamma(\lambda_m)$ also lies outside some ball $B_\mathbb{H}(O, M_m)$ in $\mathbb{H}^3$, where $M_m \to \infty$ as $m \to \infty$.

Let $\mathcal{H} = \mathcal{H}_{\epsilon;\Gamma}$ denote the set of lifts to $\mathbb{H}^3$ of the thin parts of $\mathbb{H}^3/\Gamma$, where $\epsilon \leq \epsilon_\mathcal{M}$ is chosen so that all elements of $\mathcal{H}$ are horoballs. By Corollary 3.8, $j_\Gamma(\lambda_m)$ is at uniformly bounded distance to $\alpha_m \cup \mathcal{H}(\alpha_m)$ where $\mathcal{H}(\alpha_m) \subset \mathcal{H}$ is the union of those thin parts traversed by α_m. By Lemma A.5, if the entry and exit points P, P' of α_m to a component H of $\mathcal{H}(\alpha_m)$ are at distance at least N to the base point O, then the segment $[P, P'] \subset \alpha_m$ is at distance at least $N/4 - c$ from O for some universal $c > 0$. It follows that $j_\Gamma(\lambda_m)$ is outside a large ball $B_\mathbb{H}(O, N'_m)$ where $N'_m \succ N_m$, unless there is an infinite subsequence of the α_m each of which contains a segment $[P_m, P'_m]$ contained in a horoball $H_m \in \mathcal{H}(\alpha_m)$, such that $P_m \in B_\mathbb{H}(O, K)$ for some $K > 0$ independent of m. Since there are only finitely many horoballs which meet $B_\mathbb{H}(O, K)$, up to passing to a subsequence we may assume that all the α_m pass through a fixed horoball H. Since ξ is not a parabolic point, it is not the basepoint of H. Hence by taking m large enough, we can clearly find $g_m \cdot O$ and $g'_m \cdot O$ close enough to ξ so that α_m does not intersect H. This contradiction proves that $j_\Gamma(\lambda_m)$ lies outside some ball $B_\mathbb{H}(O, M_m)$ in $\mathbb{H}^3$ as claimed. (Note that if ξ is a parabolic fixed point the above discussion fails. For then we can find sequences $g_m \cdot O, g'_m \cdot O$ which converge to ξ while the d_Γ-geodesic λ_m joining g_m and g'_m is such that $j_\Gamma(\lambda_m)$ follows around the horoball $H \in \mathcal{H}$ based at ξ and hence always penetrates a hyperbolic ball $B_\mathbb{H}(O, K)$ for fixed $K > 0$.)

If $j_\Gamma(\lambda_m)$ lies outside some ball $B_\mathbb{H}(O, M_m)$, it follows immediately from the inequalities in Proposition 3.2 that λ_m lies outside some ball $B_\Gamma(1, M'_m) \subset \mathcal{G}\Gamma$, where $M'_m \to \infty$ as $m \to \infty$. On the other hand since $\eta \neq \eta'$, there exists $R > 0$ such that the $\mathbb{H}^3$-geodesic joining η to η' has to pass through $B_\mathbb{H}(O; R)$. Hence

there exist constants $c > 0$ and $m_0 \in \mathbb{N}$ such that for all $m > m_0$, the $\mathbb{H}^3$-geodesic $[i(j_\Gamma(\lambda_m))]$ joining $j_G(g_m)$ and $j_G(g'_m)$ passes through $B_\mathbb{H}(O; R + c)$. Since $R + c$ does not depending on the index m, this contradicts the hypothesis of the theorem, so $\eta = \eta'$ and we can define $\hat{i}(\xi) = \lim_{m\to\infty} j_G(g_m)$ for any sequence $j_\Gamma(g_m) \to \xi$. This completes our justification of the definition of the map $\hat{i}$.

Now we turn to the continuity of $\hat{i}$. If $\hat{i}$ is not continuous, there exist sequences $x_m, x'_m \in j_\Gamma(\mathcal{G}\Gamma) \cup \Lambda_\Gamma$ such that $d_\mathbb{E}(x_m, x'_m) \to 0$ but so that $\hat{i}(x_m), \hat{i}(x'_m)$ converge to distinct points in Λ_G. Replacing the points x_m, x'_m by points in $j_\Gamma(\mathcal{G}\Gamma)$ if needed, it suffices to show that for every sequence $g_m \cdot O \in j_\Gamma(\mathcal{G}\Gamma)$ with $g_m \cdot O \to \xi$, we have $j_G(g_m) \to \hat{i}(\xi)$. This is of course exactly what we have already done, except in the case in which ξ is the fixed point p^* of a parabolic $p \in \Gamma$.

So suppose that $x_m = g_m \cdot O \to p^*$ but $i(x_m)$ does not converge to $\rho(p)^*$. Also let $u_m = p^m \cdot O$ so that $i(u_m) = \rho(p)^m \cdot O \to \rho(p^*)$. Let λ_m be a d_Γ-geodesic from g_m to p^m in $\mathcal{G}\Gamma$ and consider the $\mathbb{H}^3$-geodesic $\gamma_m = [j_\Gamma(\lambda_m)]$. If $\ell(\gamma_m)$ is bounded independent of m then by Lemma A.3 $|\lambda_m|_\Gamma$ must also be bounded. It follows that $i(u_m)$ must be at bounded distance from $i(x_m)$ and so $i(x_m) \to \rho(p)^*$ contrary to our assumption.

Otherwise, $\ell(\gamma_m) \to \infty$ as $m \to \infty$. Let H be the horoball based at p^*. If all but bounded length initial and final segments of γ_m are in H, then x_m and u_m are at bounded distance to H. By Lemma 4.3 below, we may assume that $g_m = hp^m$ where $h \in \mathrm{Stab}_\Gamma H$. In this case $i(x_m) = \rho(h)\rho(p)^m \cdot O$ and clearly $|hp^m|_\Gamma \to \infty$ since $\ell(\gamma_m) \to \infty$. Thus $i(x_m) \to \rho(p)^*$ again contrary to our assumption.

We have thus reduced to the case in which γ_m contains a segment γ'_m outside H such that $\ell(\gamma'_m) \to \infty$ with m. Let $\lambda'_m = j_\Gamma^{-1}\pi^{-1}(\gamma'_m)$ where π is the orthogonal projection from $j_\Gamma(\lambda_m)$ to γ_m. The projection of $j_\Gamma(\lambda'_m)$ to γ'_m is at bounded distance to $\gamma'_m \cup \mathcal{H}(\gamma'_m)$ and clearly $H \notin \mathcal{H}(\gamma'_m)$. Thus by the same arguments as above, $j_\Gamma(\lambda'_m)$ lies outside a fixed ball $B_\mathbb{H}(O; K)$ for all m and hence λ'_m lies outside a ball of fixed radius in $\mathcal{G}\Gamma$.

On the other hand, the endpoints of $ij_\Gamma(\lambda'_m)$ converge to distinct points and so $ij_\Gamma(\lambda'_m)$ always meets a fixed ball $B_\mathbb{H}(O; K')$ for some $K' > 0$. The sequence λ'_m thus violates the hypothesis of the theorem and we have proved that whenever $x_m \to p^*$, $i(x_m)$ converges to $\rho(p)^*$. In view of our previous discussion, this completes the proof of continuity of $\hat{i}$. $\square$

We immediately deduce our first main result which can be viewed as the convergence of a constant sequence of CT-maps.

Theorem 4.2. *Let Γ, G be finitely generated geometrically finite groups and let $\phi : \Gamma \to G$ be weakly type preserving isomorphism. Then the CT-map $\hat{i} : \Lambda_\Gamma \to \Lambda_G$ exists. Moreover if $\xi \in \Lambda_\Gamma$ and $g_r \in \Gamma, g_r \cdot O \to \xi$, then $\hat{i}(\xi) = \lim_{r\to\infty} \rho(g_r) \cdot O$.*

Proof. This follows immediately from Theorems 4.1 and 3.1. $\square$

Here is the lemma used in the proof of Theorem 4.1.

Lemma 4.3. *Let Γ be a geometrically finite Kleinian group. There exists $c = c(\Gamma) > 0$ with the following property. Let λ be a geodesic segment in $\mathcal{G}\Gamma$, and let π denote projection from $j_\Gamma(\lambda)$ to $[j_\Gamma(\lambda)]$. Suppose that for some $g, h \in \Gamma$, the segment of $j_\Gamma(\lambda)$ from $j_\Gamma(g)$ to $j_\Gamma(h)$ projects to an arc contained in a single component $T \in \mathcal{H}_{\epsilon,\Gamma}$ of the lift to $\mathbb{H}^3$ of the ϵ-thin part of $\mathbb{H}^3/\Gamma$ for some $\epsilon \leq \epsilon_{\mathcal{M}}$. Then $g^{-1}h = g^{-1}zgy$ where $z \in Stab_\Gamma T$ and $d_\Gamma(1, y) \leq c$.*

Proof. For $X \in j_\Gamma(\lambda)$ let $\Pi(X)$ be the point on ∂T at which the perpendicular from X to $[j_\Gamma(\lambda)]$ meets ∂T. By Corollary 3.8, there exists $D = D(\Gamma) > 0$ so that $d_{\mathbb{H}}(X, \Pi(X)) \leq D$ for all $X \in j_\Gamma(\lambda)$.

Let $\mathcal{D}$ be the Dirichlet domain for Γ with centre O. There exists $c > 0$ such that any polyhedron which meets the D-neighbourhood of $\mathcal{D}$ must be of the form $x\mathcal{D}$ where $d_\Gamma(1, x) \leq c(D)$. Thus since $g \cdot O \in g\mathcal{D}$ and since $\Pi(g \cdot O)$ is a bounded distance away from $g \cdot O$, we have $\Pi(g \cdot O) \in gx_1\mathcal{D}$ where $x_1 \in B_\Gamma(1, c(D))$. Likewise $\Pi(h \cdot O) \in hx_2\mathcal{D}$ with $x_2 \in B_\Gamma(1, c(D))$. Since $\Pi(g \cdot O), \Pi(h \cdot O) \in \partial T$, there exists $z \in Stab_\Gamma T$ such that $zgx_1\mathcal{D} = hx_2\mathcal{D}$ from which we get $zgx_1 = hx_2$ and so $g^{-1}h = g^{-1}zgx_1x_2^{-1}$ which gives the result. $\square$

5. The criterion for uniform convergence

In this section we prove our criterion Theorem 5.6 for the uniform convergence of a sequence of CT-maps corresponding to a converging sequence of representations ρ_n.

Let Γ be a fixed geometrically finite Kleinian group and suppose that $\rho_n \colon \Gamma \to PSL_2(\mathbb{C})$ is a sequence of discrete faithful weakly type preserving representations converging algebraically to $\rho_\infty \colon \Gamma \to PSL_2(\mathbb{C})$. Let $G_n = \rho_n(\Gamma), n = 1, 2, \ldots, \infty$ and write Λ_n for Λ_{G_n}. To normalize, we embed all the Cayley graphs with the same base point $O = O_{G_n}$ for all n and set $j_n(g) = j_{G_n}(g) = \rho_n(g) \cdot O, g \in \mathcal{G}\Gamma$. Let $i_n \colon j_\Gamma(\mathcal{G}\Gamma) \to j_n(\mathcal{G}G_n)$ be the obvious extension to $\mathcal{G}\Gamma$ of the map $j_\Gamma(g) \mapsto j_n(g), g \in \mathcal{G}\Gamma$.

By Theorem 4.1, $\hat{i}_n \colon \Lambda_\Gamma \to \Lambda_n$ exists if and only if there exists a function $f_n \colon \mathbb{N} \to \mathbb{N}$, such that $f_n(N) \to \infty$ as $N \to \infty$, and such that whenever λ is a d_Γ-geodesic segment lying outside $B_\Gamma(1; N)$ in $\mathcal{G}\Gamma$, the $\mathbb{H}^3$-geodesic $[i_n(j_\Gamma(\lambda))]$ lies outside $B_{\mathbb{H}}(O; f_n(N))$ in $\mathbb{H}^3$. Assuming they exist, we shall say that the CT-maps $\hat{i}_n \colon \Lambda_\Gamma \to \Lambda_n$ converge uniformly (resp. pointwise) to $\hat{i}_\infty$ if they do so as maps from Λ_Γ to $\hat{\mathbb{C}}$.

Before stating the convergence criterion, we introduce a property UEP (Uniform Embedding of Points) of the sequence (ρ_n).

Definition 5.1. *Let $\rho_n \colon \Gamma \to G_n$ be a sequence of weakly type preserving isomorphisms of Kleinian groups. Then (ρ_n) is said to satisfy UEP if there exists a*

non-negative function $f \colon \mathbb{N} \to \mathbb{N}$, *with* $f(N) \to \infty$ *as* $N \to \infty$, *such that for all* $g \in \Gamma$, $d_\Gamma(1, g) \geq N$ *implies* $d_{\mathbb{H}}(\rho_n(g) \cdot O, O) \geq f(N)$ *for all* $n \in \mathbb{N}$.

Here is an alternative characterisation of UEP.

Lemma 5.2. *For* $N \in \mathbb{N}$, *define*

$$u_N = \inf\{t > 0 : d_{\mathbb{H}}(\rho_n(g) \cdot O, O) > t\},$$

where the infimum is taken over all $g \in \Gamma$ *with* $|g| > N$ *and all* n. *Then* (ρ_n) *satisfies UEP if and only if* $u_N \to \infty$ *as* $N \to \infty$.

Proof. Clearly $\ldots u_N \leq u_{N+1}$ for all N. If $u_N \to \infty$ as $N \to \infty$, then (ρ_n) satisfies UEP with $f(N) = u_N$.

Now suppose there exists $K > 0$ so that $u_N \leq K$ for all N. Suppose (ρ_n) satisfies UEP with the function f. Choose N so that $f(N) > K + 1$. From the definition of u_N, there exist g, n, such that $|g| > N$ while $j_n(g) \in B_{\mathbb{H}}(O, K)$. On the other hand, by UEP $j_n(g)$ is outside $B_{\mathbb{H}}(O, f(N))$. This is impossible. $\square$

Proposition 5.3. *Suppose that a sequence of discrete faithful weakly type preserving representations* $(\rho_n \colon \Gamma \to PSL_2(\mathbb{C}))$ *converges algebraically to* ρ_∞. *Then* (ρ_n) *converges strongly if and only if it satisfies UEP.*

Proof. Suppose that the sequence of representations $(\rho_n(\Gamma))$ converges algebraically and satisfies UEP with a function f. If the convergence is not strong, then there exists a sequence $(\rho_n(g_{m_n}))$ with $|g_{m_n}| \to \infty$ and $n \to \infty$ which converges in $PSL_2(\mathbb{C})$, and hence for which $d_{\mathbb{H}}(\rho_n(g_{m_n}) \cdot O, O) < M$ for some $M > 0$.

Choose $N \in \mathbb{N}$ such that $f(N) > M$. Then $d_\Gamma(1, g) \geq N$ implies $d_{\mathbb{H}}(\rho_n(g) \cdot O, O) \geq f(N) > M$. So for any n, $d_{\mathbb{H}}(\rho_n(g) \cdot O, O) \leq M$ implies $d_\Gamma(1, g) \leq N$. Since the ball $B_\Gamma(1; N) \subset \mathcal{G}\Gamma$ is finite, we must have $g_m = g_*$ for infinitely many m and some $g_* \in B_\Gamma(1; N)$. This contradicts $|g_m| \to \infty$.

Conversely if UEP fails, by Lemma 5.2 there exists $K > 0$ such that with u_N defined as in that lemma, $u_N \leq K$ for all N. Thus for all N there exist $g_N \in \Gamma$ and $n_N \in \mathbb{N}$ such that $d_{\mathbb{H}}(\rho_{n_N}(g_N) \cdot O, O) \leq K + 1$ and $|g_N| > N$. Hence $(\rho_{n_N}(g_N))$ has a convergent subsequence while $|g_N| \to \infty$, which is impossible by the strong convergence. $\square$

Now we introduce a further property UEPP (Uniform Embedding of Pairs of Points) of the sequence (ρ_n).

Definition 5.4. *Let* $\rho_n \colon \Gamma \to G_n$ *be a sequence of weakly type preserving isomorphisms of Kleinian groups. Then* (ρ_n) *satisfies UEPP if there exists a function* $f_1 \colon \mathbb{N} \to \mathbb{N}$, *such that* $f_1(N) \to \infty$ *as* $N \to \infty$, *and such that whenever* λ *is a* d_Γ-*geodesic segment lying outside* $B_\Gamma(1; N)$ *in* $\mathcal{G}\Gamma$, *the* $\mathbb{H}^3$-*geodesic* $[j_n(\lambda)]$ *lies outside* $B_{\mathbb{H}}(O; f_1(N))$ *for all* $n \in \mathbb{N}$.

Note that if a sequence of representations (ρ_n) satisfies UEPP, then it automatically satisfies UEP and so by Proposition 5.3, if it converges algebraically then it also converges strongly. We also remark that the condition of UEPP is just the statement that the second condition of Theorem 3.1 holds uniformly in n.

Here is an alternative characterisation of UEPP, whose proof is essentially identical to that of Lemma 5.5.

Lemma 5.5. *For $N \in \mathbb{N}$, define*

$$v_N = \inf\{t > 0 : d_{\mathbb{H}}([j_n(\lambda)], O) > t\},$$

where the infimum is taken over all $\mathcal{G}\Gamma$ geodesics λ which are outside $B_\Gamma(1, N)$ and all n. Then ρ_n satisfies UEPP if and only if $v_N \to \infty$ as $N \to \infty$.

Our main criterion for uniform convergence of CT-maps is the following:

Theorem 5.6. *Let Γ be a geometrically finite Kleinian group and let $\rho_n : \Gamma \to G_n$ be weakly type preserving isomorphisms to Kleinian groups. Suppose that ρ_n converges algebraically to a representation ρ_∞. Then if (ρ_n) satisfies UEPP, the CT-maps $\hat{i}_n \colon \Lambda_\Gamma \to \Lambda_n$ converge uniformly to $\hat{i}_\infty$. If Γ is non-elementary, the converse also holds.*

Remark 5.7. The converse result is not needed for the proof of Theorem A but we include it for completeness. To see that the converse fails if Γ is elementary, consider the sequence of groups $< A_n >$ where A_n is a single loxodromic converging to a parabolic A_∞ in such a way that a subsequence of powers $A_n^{m_n}$ has geometric limit B where B is a parabolic and $< B, A_\infty > = \mathbb{Z}^2$. For the detailed construction of such an example, see [15] Section 4.9.

To prove Theorem 5.6 we need a lemma which ensures that d_Γ-geodesic paths lying outside a large ball eventually have small visual diameter.

Lemma 5.8. *Let $\rho \colon \Gamma \to G_n$ be weakly type preserving isomorphisms of finitely generated Kleinian groups. Suppose that G_n converges algebraically to a group G_∞, normalised as above. Suppose that (ρ_n) satisfies UEPP. Then there exists $f_2 \colon \mathbb{N} \to \mathbb{N}$ such that $f_2(N) \to \infty$ as $N \to \infty$, and such that for all $g \in \mathcal{G}\Gamma$ lying outside $B_\Gamma(1; N)$ and $m, n \geq f_2(N)$, the $\mathbb{H}^3$-geodesic $[j_m(g), j_n(g)]$ lies outside $B_{\mathbb{H}}(O; f_2(N))$.*

Proof. Given $N \in \mathbb{N}$, by the algebraic convergence we can choose N_0 such that for all $m, n \geq N_0$ and $g \in B_\Gamma(1; N + 1)$ we have $d_{\mathbb{H}}(j_m(g), j_n(g)) \leq 1$. Now, let $h \in \Gamma$ be such that $d_\Gamma(h, 1) \geq N + 1$. For $g, g' \in \mathcal{G}\Gamma$, denote by $[g, g']_\Gamma$ a d_Γ-geodesic segment in $\mathcal{G}G$ from g to g'. Let $h_0 \in [1, h]_\Gamma$ such that $d_\Gamma(1, h_0) = N$. By hypothesis, the $\mathbb{H}^3$-geodesic segments $\gamma_m = [j_m([h_0, h]_\Gamma)]$ and $\gamma_n = [j_n([h_0, h]_\Gamma)]$ lie outside $B_{\mathbb{H}}(O; f_1(N))$. Let δ be a hyperbolicity constant for $\mathbb{H}^3$, in the sense that any side of a triangle is contained in a δ-neighbourhood of the other two. Then the

$\mathbb{H}^3$-geodesic segment $[j_m(h), j_n(h)]$ is within distance 2δ of $\gamma_m \cup \gamma_n \cup [j_m(h_0), j_n(h_0)]$ and hence lies outside the $(f_1(N) - 1 - 2\delta)$-ball around $O \in \mathbb{H}^3$. Choosing $f_2(N) = \max\{N_0, f_1(N) - 1 - 2\delta\}$ gives the result. $\qquad\square$

Proof of Theorem 5.6. Given $\xi \in \Lambda_\Gamma$, choose $g_r \in \Gamma, g_r \cdot O \to \xi$. Since Γ acts properly discontinuously on $\mathbb{H}^3$, $|g_r| \to \infty$ as $r \to \infty$. By Theorem 4.2, $\hat{i}_n(\xi) = \lim_{r\to\infty} \rho_n(g_r \cdot O)$ for $n = 1, 2, \ldots, \infty$. We first show that if (ρ_n) satisfies UEPP, then $\lim_{r\to\infty} \rho_n(g_r) \cdot O$ converges to $\lim_{r\to\infty} \rho_\infty(g_r) \cdot O$ uniformly in ξ as $n \to \infty$.

By UEPP, given $N \in \mathbb{N}$, there exists $f_1(N)$ such that for all n and ξ, $i_n(g_r) = \rho_n(g_r) \cdot O$ is outside $B_\mathbb{H}(O; f_1(N))$ whenever $|g_r| > N$. Thus by Lemma 5.8, the $\mathbb{H}^3$-geodesic $[i_n(g_r), i_m(g_r)]$ is outside $B_{\mathbb{H}^3}(O; f_2(N))$ whenever $|g_r| \geq N$ and $n, m \geq f_2(N)$, where $f_2(N)$ is determined from $f_1(N)$ as in that lemma. Thus working in the ball model with $d_\mathbb{E}$ denoting Euclidean distance, from Lemma A.1 we have $d_\mathbb{E}(i_n(g_r), i_m(g_r)) \prec e^{-f_2(N)}$ whenever $|g_r| > N$ and $n, m \geq f_2(N)$.

Now $i_n(g_r) \to \hat{i}_n(\xi)$ as $r \to \infty$. Moreover by UEPP and Lemma A.1, we have $d_\mathbb{E}(i_n(g_r), i_n(g_s)) \prec e^{-f_1(N)}$ whenever $|g_r|, |g_s| > N$. Thus $d_\mathbb{E}(i_n(g_r), \hat{i}_n(\xi)) \prec e^{-f_1(N)}$ whenever $|g_r| > N$. So $d_\mathbb{E}(\hat{i}_n(\xi), \hat{i}_m(\xi)) \prec (e^{-f_2(N)} + e^{-f_1(N)})$ which gives uniform convergence.

To prove the converse, suppose the convergence is uniform and that Γ is non-elementary. Uniform convergence implies diagonal convergence and hence that the limit sets of the geometric and algebraic limits are the same. Now Theorem 1.1, together with the assumption that Γ is non-elementary, gives that the convergence is strong. By Proposition 5.3, this implies UEP.

Suppose that UEPP fails. By Lemma 5.5, the sequence $v_N = \inf\{t > 0 : d([j_n(\lambda)], O) > t\}$ is bounded above by K say, so for all N there exists a $\mathcal{G}\Gamma$ geodesic λ_N and $n_N \in \mathbb{N}$ such that λ_N is outside $B_\Gamma(1, N)$ while $[j_{n_N}(\lambda_N)]$ intersects the ball $B_\mathbb{H}(O, K)$.

We claim that $n_N \to \infty$ as $N \to \infty$. If not, there exists $L > 0$ such that $n_N \leq L$ for all N. By Theorem 3.1, for each r there is a function $f_r \colon \mathbb{N} \to \mathbb{N}$ with $f_r(N) \to \infty$ with N, and such that if λ is outside $B_\Gamma(1, N)$ then $[j_r(\lambda)]$ is outside $B_\mathbb{H}(O, f_r(N))$. Thus if N is large enough that $\min\{f_1(N), f_2(N), \ldots, f_L(N)\} > K$ then $[j_{n_N}(\lambda_N)]$ is outside $B_\Gamma(1, K)$, contrary to the choice of λ_N.

Thus the sequence n_N is unbounded, so that we can choose a sequence $n_r \to \infty$ and $\mathcal{G}\Gamma$ geodesics λ_r such that λ_r is outside $B_\Gamma(1, r)$ while $[j_{n_r}(\lambda_r)]$ meets $B_\Gamma(1, K)$ for all r.

Suppose that λ_r has endpoints $g_r, h_r \in \mathcal{G}\Gamma$. By Theorem 3.1, after passing to a subsequence we may assume that the points $g_r \cdot O$ and $h_r \cdot O$ converge to the same point $\xi \in \Lambda_\Gamma$. It follows from Lemma A.1 combined with UEP and the uniform convergence that $i_{n_r}(g_r)$ and $i_{n_r}(h_r)$ both limit on $\hat{i}_\infty(\xi) \in \Lambda_\infty$, contradicting the fact that $[j_{n_r}(\lambda_r)]$ meets $B_\mathbb{H}(O, K)$ for all r. $\qquad\square$

Proof of Theorem 1.2. Theorem A gives an alternative proof of Theorem 1.2. With the notation of the statement in the Introduction, by Theorem 4.2, if Γ is a finitely generated Fuchsian group and $\rho\colon \Gamma \to G_z$ is a type preserving isomorphism to a quasi-Fuchsian group G_z, the natural map $i_z\colon \Lambda_\Gamma^+ \to \Lambda_G^+$ extends to a continuous map $\hat{i}_z\colon \Lambda_\Gamma \to \Lambda_G$.

Now replace the parameter $z \in A$ by a sequence $(z_n) \to z_\infty$ and write G_n for G_{z_n} etc. Suppose that Γ, G_n, G_∞ are all quasifuchsian. If (G_n) converges algebraically to G_∞, it automatically converges strongly. Hence by Theorem A the CT-maps $\hat{i}_n : \Lambda_\Gamma \to \Lambda_{G_n}$ converge uniformly to $\hat{i}_\infty$. Uniform convergence gives diagonal convergence. Since this argument applies to any sequence $z_n \in A$ with limit $z_\infty \in A$, we easily deduce joint continuity of the map $\hat{i}_z$.

Let $\xi \in \Lambda_\Gamma$ and pick a sequence of attractive fixed points $\xi_m = g_m^+ \to \xi$ where $g_m \in \Gamma$ are hyperbolic. The maps $z \mapsto \rho_z(g_m)^+ = \hat{i}_z(\xi_m)$ are holomorphic for each m and by our result $\hat{i}_z(\xi_m) \to \hat{i}_z(\xi)$ for each z. Moreover the family of maps $z \mapsto \hat{i}_z(\xi_m)$ is uniformly bounded and hence normal. It follows that $z \mapsto \hat{i}_z(\xi)$ is holomorphic as claimed. $\square$

6. Strong convergence

In this section we prove Theorem A, that if a geometrically finite group G_∞ is a strong limit of a sequence G_n, then the corresponding CT-maps converge uniformly. To do this, it is sufficient in view of Theorem 5.6 to check the criterion UEPP. The main ingredient is the following uniform bound on the diameters of the thick parts of the convex cores. Note that as long as the geometric limit is geometrically finite, the hypothesis only requires algebraic rather than strong convergence; this will be important when we come to the proof of Theorem B.

Proposition 6.1. *Suppose given a sequence of geometrically finite Kleinian groups $G_n = \rho_n(\Gamma)$ where Γ is geometrically finite and the representations ρ_n are faithful and weakly type preserving. Suppose that the sequence (G_n) converges geometrically to a geometrically finite group H. Then the algebraic limit G_∞ is geometrically finite. Moreover the thick parts of the convex cores have uniformly bounded diameters.*

Proof. For the first statement, see [15] Theorem 4.6.1. By the same result, the limit sets Λ_n converge to Λ_H in the Hausdorff topology and the ordinary sets Ω_n converge to Ω_H in the sense of Carathéodory.

Let $\mathcal{D}_H$ be a fundamental domain for H and let $\mathcal{N}_H$ be the hyperbolic convex hull of the limit set Λ_H. Denote by $\mathcal{V}_H = \mathcal{V}_{\epsilon,H}$ the ϵ-thick part of $\mathcal{N}_H$ relative to H and define $\mathcal{D}_n, \mathcal{N}_n, \mathcal{V}_n$ similarly.

Since H is geometrically finite, we can find a hyperbolic ball $B_r = B_\mathbb{H}(O; r) \subset \mathbb{H}^3$ which contains $\mathcal{V}_H \cap \mathcal{D}_H$. By geometric convergence we have uniform convergence of $\mathcal{D}_n$ to $\mathcal{D}_H$ inside B_r. Moreover the Hausdorff convergence of Λ_n to Λ_H implies

that inside B_r, $\mathcal{N}_n$ is eventually contained in a bounded neighbourhood of $\mathcal{N}_H$. This does not however automatically give a uniform bound on the diameters of $\mathcal{N}_n$, as it says nothing about what happens far outside B_r.

To understand the problem outside B_r, consider the following toy example. Let $\mathcal{Z} = \{(x,y)\colon x \in [1,\infty), 0 \le y \le 1/x\} \subset \mathbb{R}^2$. Let $\mathcal{Z}_n$ be the part of $\mathcal{Z}$ with $x \in [0, 2n]$. Let $C_n \subset \mathbb{R}^2$ be the disk of radius n and centre $2n \in \mathbb{R}$ and let $Q_n = \mathcal{Z}_n \cup C_n$. Then the sets Q_n converge uniformly on compact sets in the Hausdorff metric to $\mathcal{Z}$ and $\mathcal{Z}$ has finite area, but $\mathrm{diam}\, Q_n \to \infty$.

To resolve this problem it we shall prove the following claim: there exists $A > 0$ such that if $\mathcal{V}_n \cap \mathcal{D}_n$ contains points outside B_{r+2+A}, then there are points of $\mathcal{V}_n \cap \mathcal{D}_n$ in the shell between B_{r+1} and B_{r+2}.

Suppose the claim holds. Suppose that (up to a subsequence) there are points $X_n \in \mathcal{V}_n \cap \mathcal{D}_n$ with $d_{\mathbb{H}}(O, X_n) > r + 2 + A$. Use the claim to choose points $Y_n \in \mathcal{V}_n \cap \mathcal{D}_n$ with $r + 1 \le d_{\mathbb{H}}(O, Y_n) \le r + 2$. By compactness we may assume $Y_n \to Y \in \mathcal{N}_H$, and by the geometric convergence of $\mathcal{D}_n$ to $\mathcal{D}_H$ inside B_{r+2}, we have $Y \in \overline{\mathcal{D}}_H$. Since Y is outside B_r it must be in the complement of $\mathcal{V}_H \cap \mathcal{D}_H$, that is, in the thin part of $\mathcal{N}_H \cap \mathcal{D}_H$. So there exist $h \in H$ and a neighbourhood U of Y such that such that $d_{\mathbb{H}}(hY', Y') < \epsilon$ for all $Y' \in U$. Choose $g_n \in \Gamma$ with $\rho_n(g_n) \to h$. We have $d_{\mathbb{H}}(\rho_n(g_n)Y', Y') < \epsilon$ for all large enough n and hence $d_{\mathbb{H}}(\rho_n(g_n)Y_n, Y_n) < \epsilon$ as $n \to \infty$. This contradicts the choice of Y_n. We conclude that eventually $\mathcal{V}_n \cap \mathcal{D}_n$ is contained in B_{r+2+A} so that the sequence of diameters $\mathcal{V}_n \cap \mathcal{D}_n$ is uniformly bounded above.

Now we prove the claim, which can be seen as a very rudimentary form of Canary's filling theorem [3]. Since all groups G_n are isomorphic and geometrically finite, the boundaries $\partial \mathcal{N}_n / G_n$ are homeomorphic and the thick parts of $\partial \mathcal{N}_n / G_n$ have uniformly bounded diameter A say (depending only on the maximum genus of the hyperbolic components of $\partial \mathcal{N}_\Gamma / \Gamma$). If there are points of $\mathcal{V}_n \cap \mathcal{D}_n$ outside B_{r+2+A}, then there are also points of $\partial \mathcal{V}_n \cap \mathcal{D}_n$ outside B_{r+2+A}. Then any component $S_n \subset \partial \mathcal{N}_n / G_n$ containing such points is outside B_{r+2}. Since $\partial \mathcal{N}_n / G_n$ has finitely many components, passing to a subsequence, we may fix one such component which is homeomorphic to some fixed hyperbolic surface S, and whose thick part lies outside B_{r+2}. (Since we arranged that $\mathcal{V}_n \cap \mathcal{D}_n \subset B_r$), we do not have to worry about components of $\partial \mathcal{N}_n / G_n$ on the boundary of horoballs or tubes.)

Fix a lift $\tilde{S} \subset \partial \mathcal{N}_n$ and let $W \subset \hat{\mathbb{C}}$ denote the component of the regular set Ω_Γ of Γ corresponding to $\tilde{S}$. Let $K = \mathrm{Stab}\, W \subset \Gamma$. Since $K \subset \Gamma$ is non-elementary (because $\tilde{S}/K$ is a hyperbolic component of $\partial \mathcal{N}_\Gamma / \Gamma$), we can choose a pair of non-commuting elements $\alpha, \beta \in K$, both of which are non-trivial in Γ and hence in H. (For the fact that Γ injects into H, see Lemma 4.4.1 in [15].) Since by geometric finiteness only finitely many elements of Γ are parabolic or short in H, we may

assume in addition that neither α nor β is parabolic or the core of a large Margulis tube in H.

Set $M_n = \mathbb{H}^3/G_n$ and $M_H = \mathbb{H}^3/H$ and let O_n^*, O_H^* be the projections of the basepoint $O \in \mathbb{H}^3$ to M_n, M_H respectively. Let d_n, d_H denote distance in M_n, M_H respectively. Note that no pair of loops in the homotopy classes $[\alpha], [\beta]$ of α, β can be contained in the same component of a thin part of M_n, since the fundamental group of any such component is abelian. Choose loops $\alpha_n \in [\alpha], \beta_n \in [\beta]$ on $S \subset \partial \mathcal{N}_n/G_n$. By construction the distance from each of α_n, β_n to O_n^* is at least $r + 2$. On the other hand, in M_H, the geodesic representatives $\rho_\infty(\alpha)^*, \rho_\infty(\beta)^*$ are contained inside $\mathcal{N}_H/H$ and hence, since by construction neither is parabolic in H and additionally neither is in the thin part of M_H, $d_H(\rho_\infty(\alpha)^*, O^*) \le r$ and similarly for β. By algebraic convergence, we can find lifts to $\mathbb{H}^3$ of the geodesic representatives $\rho_n(\alpha)^*, \rho_n(\beta)^*$ of $[\alpha], [\beta]$ in M_n near to corresponding $\rho_\infty(\alpha)^*, \rho_\infty(\beta)^*$ in $\mathbb{H}^3$. Hence $d_n(\rho_n(\alpha)^*, O_n^*) \le r + 1$ for sufficiently large n, and similarly for β.

Choose homotopies H_α^n, H_β^n between $\rho_n(\alpha)^*$ and α_n and between $\rho_n(\beta)^*$ and β_n. More precisely, let H_α^n be the image of a continuous family of maps $f_t \colon [0,1] \to M_n, t \in [0,1]$ with $f_0([0,1]) = \rho_n(\alpha)^*$ and $f_1([0,1]) = \alpha_n$, and similarly for H_β^n.

Let $\partial B(O_n^*; R)$ denote the boundary of the ball of radius R centre O_n^* in M_n. By construction, α_n, β_n lie on a component of $\partial \mathcal{N}_n/G_n$ outside $B(O_n^*; r+2)$, so H_α^n, H_β^n must both intersect the shell between $B(O_n^*; r+1)$ and $B(O_n^*; r+2)$, and in particular the surface $\partial B(O_n^*; r+3/2)$. (Note that $\partial B(O_n^*; R)$ is just the projection of the boundary of the ball of radius R in $\mathbb{H}^3$ and hence an immersed 2-manifold in M_n.) Hence we can find paths $\alpha_1 \in [\alpha], \beta_1 \in [\beta]$ in $\partial B(O_n^*; r+3/2)$. In detail, by adjusting H_α^n slightly if needed, we can arrange that $\partial B(O_n^*; r+3/2)$ and H_α^n are transverse, so that the intersection $\partial B(O_n^*; r+3/2) \cap H_\alpha^n$ consists of a finite number of closed loops. Adjust H_α^n to 'push off' any components which are homotopically trivial. Since H_α^n is a cylinder whose two boundary components are separated by $\partial B(O_n^*; r+3/2)$, $\partial B(O_n^*; r+3/2) \cap H_\alpha^n$ must contain at least one homotopically non-trivial component which we take to be α_1. The construction of β_1 is similar.

As discussed above, α_1 and β_1 cannot both be contained in the thin part of $\mathcal{N}_n/G_n$. Lifting to $\mathbb{H}^3$, this produces the points Y_n as required.

Thus $\mathcal{V}_n \cap \mathcal{D}_n$ is eventually contained in a compact ball B_r, which immediately gives the required bound.

$\square$

It is now easy to prove Theorem A.

Proof of Theorem A. By Theorem 5.6 it is sufficient to check that the sequence (ρ_n) satisfies UEPP. In other words, we have to check that the second condition of Theorem 3.1 holds uniformly in all the groups G_n. To do this, we go through that proof of Floyd's result Proposition 3.2 with this requirement in mind. (Notice that ρ_∞ is automatically weakly type preserving.)

First consider the right hand inequality of (1): $d_{\mathbb{H}}(O, g \cdot O) \leq a|g|$ where $a = \max_i d(O, e_i \cdot O)$ and $\{e_1, \ldots, e_k\}$ is a finite set of generators for Γ. Algebraic convergence implies that $\rho_n(e_i)$ converges to $\rho_\infty(e_i)$ for each i, so we have $\max_i d(O, \rho_n(e_i) \cdot O) \leq A$ for some uniform A independent of n. Thus the right hand side of (1) holds uniformly, precisely:

$$d_{\mathbb{H}}(O, \rho_n(g) \cdot O) \leq A|\rho_n(g)| = A|g| \text{ for all } g \in \Gamma \text{ and } n \in \mathbb{N}.$$

We now need to prove the left hand inequality. The constants involved are the diameter d of the truncated fundamental domain $\mathcal{D} \cap \mathcal{V}$, and the constant $C = \max\{|g| : d_{\mathbb{H}}(O, g \cdot O) \leq 1 + 2d\}$ in the proof of Proposition 3.2.

The diameters of the truncated fundamental domain $\mathcal{D}_n \cap \mathcal{V}_n$ are uniformly bounded by Proposition 6.1. That we can choose the constant $C_n = \max\{|g| : d_{\mathbb{H}}(O, g \cdot O) \leq 1 + 2d\}$ with a uniform bound $C_n \leq C$ for all n follows immediately from UEP which has already been proved in Proposition 5.3.

The proof now follows exactly as in Theorem 3.1, using in addition the distortion Lemmas A.4 and A.6 for Margulis tubes.   □

7. Algebraic limits

7.1. **Pointwise convergence criterion.** In this section we prove Theorem B. We will use conditions similar to UEP and UEPP, relaxed so as to allow for dependence on the limit point ξ. We call these new conditions $EP(\xi)$ (Embedding of Points) and $EPP(\xi)$ (Embedding of Pairs of Points).

Let G be a geometrically finite Kleinian group with generators $G^* = \{e_1, \ldots, e_k\}$, and let $\xi \in \Lambda_G$. Suppose given a sequence $e_{i_1}, e_{i_2} \ldots \in G^*$ such that setting $g_r = e_{i_1} e_{i_2} \ldots e_{i_r}$ we have $g_n \cdot O \to \xi$ in $\mathbb{H}^3 \cup \partial \mathbb{H}^3$. We denote the infinite path joining vertices $1, g_1, g_2, g_3, \ldots$ in $\mathcal{G}G$ by $[1, \xi)$ and write $\hat{j}_G(e_{i_r} e_{i_2} \ldots e_{i_s})$ for the piecewise geodesic path in $\mathbb{H}^3$ joining in order the points $j_G(g_r), j_G(g_{r+1}), \ldots, j_G(g_s)$.

Definition 7.1. *Let Γ be a fixed finitely generated Kleinian group and $\rho_n \colon \Gamma \to G_n$ be a sequence of isomorphisms to Kleinian groups G_n. Let $\xi \in \Lambda_\Gamma$ and let $(g_r) = [1, \xi)$ be any infinite path as above.*

(1) *The pair $((\rho_n), [1, \xi))$ is said to satisfy $EP(\xi)$ if there exist functions $f_\xi \colon \mathbb{N} \to \mathbb{N}$ and $M_\xi \colon \mathbb{N} \to \mathbb{N}$, with $f_\xi(N) \to \infty$ as $N \to \infty$, such that for all $g \in [1, \xi)$, $d_\Gamma(1, g) \geq N$ implies $d_{\mathbb{H}}(\rho_n(g) \cdot O, O) \geq f_\xi(N)$ for all $n \geq M_\xi(N)$.*

(2) *The pair $((\rho_n), [1, \xi))$ satisfies $EPP(\xi)$ if there exists a function $f_{1,\xi}(N) \colon \mathbb{N} \to \mathbb{N}$ such that $f_{1,\xi}(N) \to \infty$ as $N \to \infty$ such that for any subsegment $e_{i_r} e_{i_{r+1}} \ldots e_{i_s}$ of $[1, \xi)$ lying outside $B_\Gamma(1; N)$ in $\mathcal{G}\Gamma$, the geodesic $[j_n(g_{i_r}), j_n(g_{i_s})]$ lies outside $B_{\mathbb{H}}(O; f_{1,\xi}(N))$ for all $n \geq M_\xi(N)$, where M_ξ is as in (1).*

Remark 7.2. Although *a priori* the conditions $EP(\xi), EPP(\xi)$ depend on the choice of sequence $e_{i_1} e_{i_2} \ldots$, it will follow from our proof below that (in the case of geometrically finite limits) provided we choose suitable paths $[1, \xi)$, they depend

only on ξ. To do this we will restrict the class of paths used, so that $[1, \xi)$ is quasi-geodesic and satisfies an additional hypothesis which ensures that it tracks shortest Euclidean paths across the boundaries of rank 2-horoballs. We call such paths *standard*. It is not completely obvious that such paths $[1, \xi)$ exist; we prove this in Proposition 7.5 below. We will show that for a sequence (ρ_n) of representations as in the hypotheses of Theorem B, the conditions $EP(\xi)$ and $EPP(\xi)$ hold for any standard quasi-geodesic $[1, \xi)$.

Although clearly, $EPP(\xi)$ implies $EP(\xi)$, we shall first prove $EP(\xi)$ and deduce $EPP(\xi)$.

Theorem 7.3. *Suppose that $\rho_n : G \to PSL_2(\mathbb{C})$ is a sequence of discrete faithful representations converging algebraically to $\rho_\infty \colon G \to PSL_2(\mathbb{C})$, and suppose the corresponding CT maps $\hat{i}_n \colon \Lambda_\Gamma \to \Lambda_{G_n}$ exist, $n = 1, 2 \ldots, \infty$. Let $\xi \in \Lambda_\Gamma$. Then $\hat{i}_n(\xi) \to \hat{i}_\infty(\xi)$ as $n \to \infty$ if $((\rho_n); \xi)$ satisfies $EPP(\xi)$.*

Proof. The proof is the same as that of Theorem 5.6. Lemma 5.8 works just as before with the condition $EPP(\xi)$ replacing $UEPP$. Notice that in both of these proofs, it is sufficient to require that the conditions (1) and (2) hold only for all $n \geq M(N)$. $\qquad\square$

Contrary to the case of strong convergence, the converse to Theorem 7.3, that if $\hat{i}_n(\xi) \to \hat{i}_\infty(\xi)$ then $((\rho_n); \xi)$ satisfies $EPP(\xi)$, is false. In fact if ξ is the fixed point of a loxodromic element p which becomes parabolic so as to give rise to a $\mathbb{Z}^2$- cusp in the geometric limit, then $\hat{i}_n(\xi) \to \hat{i}_\infty(\xi)$ by the algebraic convergence but $EP(\xi)$ fails, because we have a sequence $\rho_n(p^{m_n})$ where $m_n \to \infty$ with $d_{\mathbb{H}}(\rho_n(p^{m_n}) \cdot O, O)$ remaining bounded. This is discussed further below.

7.2. **Standard quasi-geodesics.** Recall from Proposition 3.4 that if G is geometrically finite, then any ambient quasi-geodesic β in the thick part of the convex core $\mathcal{V}_G$ is a bounded distance from $[\beta] \cup \mathcal{H}([\beta])$. In what follows, we would like to assert further that any two ambient quasi-geodesics in $\mathcal{V}_G$ are a bounded distance apart, with constants depending only on G. However this may not be true if G contains rank two cusps. Indeed let $H \subset \mathbb{H}^3$ be a horoball corresponding to a such a cusp and let $x, y \in \partial H$. Since the induced metric on ∂H is Euclidean, there can be no bound on the distance between quasi-geodesic paths from x to y independent of $d_{\mathbb{H}}(x, y)$. To deal with this, if H is a rank two horoball and $x, y \in \partial H$, we say a path from x to y is *D-standard* (resp. *standard*) if it is within bounded distance D (resp. bounded distance depending only on G) of the Euclidean geodesic from x to y on ∂H.

Let β be an ambient quasi-geodesic β in $\mathcal{V} = \mathcal{V}_G$. For a component $H \in \mathcal{H}([\beta])$, let $\beta_H \subset \beta$ be the segment which projects to $[\beta] \cap H$ and for $x \in \beta_H$ denote by $\Pi_H(x) \in \partial H$ the point where the perpendicular from x to $[\beta]$ meets ∂H. We say β

is *standard* if for any component H of $\mathcal{H}([\beta])$ corresponding to a rank two cusp, the path $\Pi_H(\beta)$ given by $x \mapsto \Pi_H(x); x \in \beta_H$ across ∂H joining the initial and final points of $[\beta] \cap H$ is standard. Finally we call a path $\lambda \subset \mathcal{G}G$ *standard* if its image $j_G(\lambda)$ in $\mathcal{V}_G$ is standard.

Lemma 7.4. *Let $K > 0$. The image of a standard K-quasi-geodesic in $(\mathcal{G}G, d_G)$ is a standard ambient quasi-geodesic in $\mathcal{V}_G$. Moreover any two standard K-quasi-geodesics in $(\mathcal{G}G, d_G)$ with the same endpoints are a bounded distance apart in $\mathbb{H}^3$ with a constant which depends only on K and (G, G^*).*

Proof. Let γ be a standard K-quasi-geodesic in $\mathcal{G}G$. That $j_\Gamma(\gamma)$ is an ambient K'-quasi-geodesic in $\mathcal{V}_G$ for suitable $K' = K'(K, G)$ follows from the same argument as in Lemma 3.7. By definition its image in $\mathcal{V}_G$ is standard.

Now let β, β' be two standard ambient quasi-geodesics in $\mathcal{V}$ with the same endpoints so that $[\beta] = [\beta']$ is the $\mathbb{H}^3$-geodesic joining their common endpoints. Adjust the constant ϵ so that the only components of $M_{thin}(\epsilon)$ of $\mathbb{H}^3/G$ are horoballs. By Proposition 3.4 both β, β' are within bounded distance of $[\beta] \cup \mathcal{H}([\beta])$ with constants which depend only on K and (G, G^*).

We have to show that the projections of β, β' onto $[\beta] \cup \mathcal{H}([\beta])$ are bounded distance apart. This is certainly true of the projections onto segments of $[\beta]$ outside $\mathcal{H}([\beta])$. Suppose $H \in \mathcal{H}([\beta])$ is a horoball corresponding to a parabolic $p \in G$. Let η be the Euclidean shortest path in $\partial H \cap \mathcal{V}$ joining the entry and exit points of $[\beta]$ to H. If $H \in \mathcal{H}([\beta])$ corresponds to a rank one parabolic, then since G is geometrically finite, $\partial H \cap \mathcal{V}$ is a strip of bounded width containing η and which also contains the paths $\Pi_H(\beta), \Pi_H(\beta')$, so the result is clear. If H corresponds to a rank two parabolic, then the condition that β, β' be standard again ensures that $\Pi_H(\beta), \Pi_H(\beta')$ are within bounded distance of η and the result follows. $\qquad \square$

Proposition 7.5. *Let Γ be a fixed geometrically finite Kleinian group with Cayley graph $\mathcal{G}\Gamma$ relative to a fixed set of generators $\Gamma^* = \{e_1, \dots, e_k\}$, and let $\xi \in \Lambda_\Gamma$. Then there exists a sequence $e_{i_1}, e_{i_2} \dots \in \Gamma^*$ such that the path $\hat{j}_\Gamma(e_{i_1} e_{i_2} \dots)$ is a standard quasi-geodesic in $\mathcal{G}\Gamma$ and such that $g_n \cdot O \to \xi$ in $\mathbb{H}^3 \cup \partial \mathbb{H}^3$.*

Proof. Let $\alpha \subset \mathbb{H}^3$ be the infinite hyperbolic geodesic from O to ξ and as usual let $\mathcal{H}(\alpha)$ be the set of thin parts traversed by α. Exactly as in the proof of Proposition 3.2 we can construct a path in $\mathcal{G}\Gamma$ whose image under j_Γ tracks $\alpha \cup \mathcal{H}(\alpha)$ and which is ambient quasi-geodesic in the thick part $\mathcal{V}_\Gamma$ of $\mathcal{N}_\Gamma$. We can clearly ensure that in addition, segments of the path which track the boundary of rank two horoballs are standard. The result follows. $\qquad \square$

7.3. **Proof of Theorem B.** Suppose now that we are in the situation of Theorem B, that is, Γ is a fixed geometrically finite Kleinian group and $\rho_n \colon \Gamma \to PSL_2(\mathbb{C})$ is a sequence of discrete faithful weakly type preserving representations

converging algebraically to $\rho_\infty \colon \Gamma \to PSL_2(\mathbb{C})$ and geometrically to H, and such that H is geometrically finite. (As noted in the introduction, this implies G_∞ is also geometrically finite.) Let $G_n = \rho_n(\Gamma), n = 1, 2, \ldots, \infty$. In this section we prove $EP(\xi)$ and $EPP(\xi)$, from which Theorem B follows.

The example to keep in mind is that of a loxodromic $\rho_n(p), p \in \Gamma$ converging to a parabolic $\rho_\infty(p)$, in such a way that suitable powers of the loxodromic also converge to another parabolic $q = q(p) \in H \setminus G_\infty$ which together with $\rho_\infty(p)$ generates a $\mathbb{Z}^2$-subgroup. This process is explained in detail in [15] §4.9 and is what drives the well known Kerckhoff-Thurston examples [11] of groups whose algebraic and geometric limits differ. The main point is, that as $n \to \infty$ the translation length and rotation angle of $\rho_n(p)$ are related in such a way that for suitable powers $\rho_n(p^{m_n}), m_n \to \infty$, the sequence $\rho_n(p^{m_n})$ converges to q. In particular, $d_{\mathbb{H}}(O, \rho_n(p^{m_n}) \cdot O)$ is bounded while the word length $|p^{m_n}| \to \infty$. This of course violates UEP. When H is geometrically finite, this is the worst that can happen:

Theorem 7.6 ([13, 15] Theorem 4.6.1). *Suppose that Γ is geometrically finite and that $\rho_n \colon \Gamma \to PSL_2(\mathbb{C})$ are discrete, geometrically finite, faithful representations converging algebraically to $\rho_\infty \colon G \to PSL_2(\mathbb{C})$ and geometrically to H. If H is also geometrically finite, then it is generated by $\{\Gamma^*, q_1, \ldots, q_s\}$ where Γ^* is a generating set for Γ and q_i are the 'extra' parabolics which arise in the limit as a result of the phenomenon above.*

In the proof of Theorem A, we used the crucial fact (Lemma 3.7) that the image in $\mathbb{H}^3$ of a geodesic segment in $\mathcal{G}\Gamma$ is uniformly an ambient quasi-geodesic in $\mathcal{V}_n = \mathcal{V}_{\epsilon, G_n}$. However in the above situation, the elements $\rho_n(p^{m_n})$ in the approximating groups G_n give paths which track a Margulis tube for distance $O(m_n)$, but whose initial and final points are distance $O(1)$ apart. Clearly such paths cannot be ambient quasi-geodesics with uniform constants independent of n. To get around this, we first approximate by taking a new set of generators G_n^* for G_n constructed so as to be close to those of the geometric limit H, and then substitute back in for the original generators of Γ. Before doing this, however, we pause to discuss parabolic elements in Γ, G_n and G_∞.

7.4. **Parabolic blocks.** The group G_∞ contains a finite number of conjugacy classes of parabolic subgroups. Choose $\mathcal{P} = \{p_1, \ldots, p_t\} \subset \Gamma$ so that $\{\rho_\infty(p) \colon p \in \mathcal{P}\}$ contains one representative of each class, such that the horoball T_∞^p based at the fixed point of $\rho_\infty(p)$ intersects the Dirichlet domain $\mathcal{D}_\infty$, and such that $\rho_\infty(p)$ generates $Stab_{G_\infty}(T_\infty^p)$. (We changed notation here from H for a horoball to T for a tube, because a thin component which is the quotient of a horoball in some $\mathbb{H}^3/G_n$ may be a tube in another. From now on we shall use the notation T for both horoballs and equidistant tubes, the latter being the lift of a Margulis tube to $\mathbb{H}^3$, see immediately before the proof of Lemma A.3 for a precise definition.)

By the algebraic convergence, there is a group G_{n_0} such that if $p \in \mathcal{P}$ then $\rho_n(p)$ is short in G_n for all $n \geq n_0$, in the sense that its length is less than the constant $\epsilon < \epsilon_\mathcal{M}$ chosen above. Denote by T_n^p the lift to $\mathbb{H}^3$ of the thin part $M_{thin}(\epsilon)$ of $M = \mathbb{H}^3/G_n$ whose stabiliser is generated by $\rho_n(p)$. Note that while by construction T_∞^p is a horoball, T_n^p may be either a horoball if $\rho_n(p)$ is parabolic or an equidistant tube from the short loxodromic $\rho_n(p)$ otherwise. (We do not exclude the possibility that $\mathbb{H}^3/G_n$ may contain other thin parts or even horoballs not associated to elements of $\mathcal{P}$.) By slightly reducing the choice of ϵ if necessary, we can choose $a > 0$ such that the distance between any two thin parts of $\mathcal{N}_n$ is at least a for any $n \in \mathbb{N} \cup \infty$.

The set $\mathcal{P}$ is divided into two subsets, the set $\mathcal{P}_0 = \{p_1, \ldots, p_s\}$ which commute with the 'extra' parabolics $q_1, \ldots, q_s$ of Theorem 7.6, and the remaining set $\mathcal{P}_1 = \mathcal{P} \setminus \mathcal{P}_0$. By assumption the maps ρ_n, ρ_∞ are weakly type preserving, in other words every parabolic element $g \in \Gamma$ is also parabolic in $G_n, n = 1, 2 \ldots, \infty$. Note also that a parabolic element in a Kleinian group lies in a $\mathbb{Z}^2$-subgroup if and only if it stabilises a rank two cusp. Since the groups Γ, G_n, G_∞ are all abstractly isomorphic, this means that rank two parabolics $\rho_\infty(p), p \in \mathcal{P}$ are 'persistent', so that necessarily $p \in \Gamma$ and $\rho_n(p) \in G_n$ are also parabolic and moreover $p \in \mathcal{P}_1$.

We can choose $n_0 \in \mathbb{N}$ such that if $p \in \mathcal{P}$ then $\rho_n(p)$ is short in G_n for all $n \geq n_0$, in the sense that its length is less than the Margulis constant $\epsilon_\mathcal{M}$. We would like to renumber so that G_{n_0} becomes Γ. However this causes a minor technical difficulty in the case in which G_{n_0} contains parabolic elements which are not parabolic in either Γ or G_∞, so that we cannot replace the maps $\hat{i}_n \colon \Lambda_\Gamma \to \Lambda_{G_n}$ by a map $\Lambda_{G_{n_0}} \to \Lambda_{G_n}$. To simplify notation we renumber so that $G_{n_0} = G_0, j_{n_0} = j_0$ etc, and live with the minor annoyance of two distinct groups Γ and G_0.

To make the next definition, we use the following generalisation of Lemma 4.3, whose proof is identical to the earlier version. We write $X =_\Gamma Y$ to indicate that X is equal to Y as elements in Γ, but not necessarily as words in Γ^*.

Lemma 7.7. *Let Γ, G_0 be as above. There exists $c > 0$ with the following property. Let λ be a quasi-geodesic segment in $\mathcal{G}\Gamma$, and let π denote projection from $j_0(\lambda)$ to $[j_0(\lambda)]$. Suppose that for some $g, h \in \Gamma$, the segment of $j_0(\lambda)$ from $j_0(g)$ to $j_0(h)$ projects to an arc contained in a single component $T \in \mathcal{H}_{\epsilon, G_0}$. Then $g^{-1}h =_\Gamma g^{-1}zgy$ where $z \in Stab_{G_0} T$ and $d_\Gamma(1, y) \leq c$.*

Definition 7.8. *A geodesic segment $e_{r+1} \ldots e_s$ in $\mathcal{G}\Gamma$ is called a* parabolic block *relative to $p \in \mathcal{P}$ if $e_{r+1} \ldots e_s =_\Gamma p^k y$ for some y with $d_\Gamma(1, y) \leq c$. We call $|k|$ the* length *of the block.*

Remark 7.9. It is worth clarifying exactly how this definition relates to the lemma. Setting $g_r = e_{i_1} e_{i_2} \ldots e_{i_r}$, the segment $\hat{j}_0(e_{r+1} \ldots e_s)$ runs from $j_0(g_r)$ to $j_0(g_s)$. Set $g = g_r$ and $h = g_s$ in the lemma. Then $e_{r+1} \ldots e_s = g_r^{-1} g_s = g_r^{-1} z g_r y$ where

$\rho_0(z) \in \operatorname{Stab} T$. Thus $\rho_0(g_r^{-1} z g_r) \in \operatorname{Stab} \rho_0(g_r)^{-1} T$ and $\rho_0(g_r)^{-1} T = T_0^p$ for some $p \in \mathcal{P}$.

Lemma 7.10. *Given $D > 0$, there exists $k_0 = k_0(D) > 0$ such that given $k_1 > k_0$, there exists $M = M(k_1, D) \in \mathbb{N}$ such that if $p^k y, p \in \mathcal{P}_0$ is a parabolic block of length $k \in [k_0, k_1]$, then any geodesic segment whose endpoints are at distance at most D from $j_n(1)$ and $j_n(p^k y)$ respectively penetrates the thin part T_n^p for all $n > M$. If $p \in \mathcal{P}_1$ then the same result is true for all $k \geq k_0$.*

Proof. Since y is in a bounded neighbourhood of $1 \in \mathcal{G}\Gamma$, using the algebraic convergence we get a uniform bound $d_{\mathbb{H}}(O, j_n(y)) \leq D_1$ for all n. Hence

$$
\begin{aligned}
d_{\mathbb{H}}(j_n(p^k y), T_n^p) &\leq d_{\mathbb{H}}(j_n(p^k), T_n^p) + d_{\mathbb{H}}(j_n(p^k y), j_n(p^k)) \\
&= d_{\mathbb{H}}(j_n(p^k), T_n^p) + d_{\mathbb{H}}(j_n(y), O) \\
&= d_{\mathbb{H}}(O, \rho_n(p^{-k}) T_n^p) + d_{\mathbb{H}}(j_n(y), O) = d_{\mathbb{H}}(O, T_n^p) + d_{\mathbb{H}}(j_n(y), O).
\end{aligned}
$$

Since by Proposition 6.1, O is a uniformly bounded distance from T_n^p, this gives a uniform bound $d_{\mathbb{H}}(j_n(p^k y), T_n^p) \leq D_2$ say.

Now let A, B be points at distance at most D to $O = j_n(1)$ and $j_n(p^k y)$ respectively, and let X, Y denote their projections onto T_n^p. Then A, B are at distance at most $D + D_2$ to T_n^p. Thus there exists a uniform constant $c > 0$ such that if the distance between X and Y along ∂T_n^p is at least c, then $[A, B]$ penetrates T_n^p.

If $\rho_n(p)$ is parabolic, since $Y = \rho_n(p)^k \cdot X$ and since $\rho_n(p)$ translates a definite distance along ∂T (because the injectivity radius of $\mathbb{H}^3/G_n$ at points on $\partial T/G_n$ is some fixed $\epsilon > 0$), the result is now straightforward. However if $\rho_n(p)$ is loxodromic, we have to take care that the multiplier is not such that $d_{\mathbb{H}}(X, \rho_n^k(p) \cdot X) = O(1)$.

Let G be a Kleinian group and let $S \in G$ be a loxodromic transformation with multiplier bounded in modulus by some fixed θ_0. Let $T \subset \mathbb{H}^3$ be the equidistant tube corresponding to S, such that the injectivity radius of $\mathbb{H}^3/G$ at points on the boundary of the image of T in $\mathbb{H}^3/G$ is $\epsilon > 0$. Then there exists a constant $c' > 0$ such that provided $r\theta_0 < \pi/4$, the distance between $Q \in \partial T$ and $S^r(Q) \in \partial T$ along ∂T is at least $c' r \epsilon$. In particular, since the T_n^p are all components of $\mathcal{H}_{\epsilon; G_n}$ for fixed ϵ, this discussion applies to our present situation with $S = \rho_n(p)$.

Choose $k_0 \in \mathbb{N}$ so that $c' k \epsilon > c$ whenever $k \geq k_0$. Since $\rho_n(p) \to \rho_\infty(p)$, the multiplier λ_n of $\rho_n(p)$ converges to 1. Thus by the above discussion, given $k_1 > k_0$ we can find $M = M(k_1)$, so that for all $n \geq M$, the argument of the multiplier of $\rho_n(p)$ is bounded in modulus by $\pi/4k_1$ and hence so that the argument of the multiplier of $\rho_n(p^r)$ is bounded in modulus by $\pi/4$ for all $|r| \leq k_1$.

It follows that if $X \in \partial T_n^p$ then, provided $k_0 \leq k \leq k_1$, the distance from X to $\rho_n(p^k) \cdot X$ along ∂T_n^p is at least c and so if A, B are points at distance at most D to $O = j_n(1)$ and $j_n(p^k y)$ respectively, then the geodesic segment $[A, B]$ penetrates T_n^p.

Note that if $p \in \mathcal{P}_0$ it is crucial here to have the fixed upper bound $k \leq k_1$; in fact $d(X, \rho_n(p^k) \cdot X) \asymp 1$ for values of k such that $\rho_n(p^k)$ is close to $q \in H$. If $p \in \mathcal{P}_1$ this restriction is unnecessary. $\qquad \square$

7.5. Correspondence of generators.

We now make a precise correspondence between a set of generators of G_n and those of the geometric limit H.

Lemma 7.11. *There exists $\delta_0 > 0$ such that for any $h \in H$, there exists $m_0 = m_0(h)$ such that there is a unique $g \in \Gamma$ with $j_n(g) \in B_{\mathbb{H}}(h \cdot O, \delta_0)$ whenever $n \geq m_0$.*

Proof. Let $\mathcal{D}_n, \mathcal{D}_H$ be the Dirichlet domains for G_n, H centre O respectively. As in the proof of Proposition 6.1, the thick part of $\mathcal{N}_H \cap \mathcal{D}_H$ is contained in a ball B_r of finite diameter in $\mathbb{H}^3$. The groups G_n converge geometrically, and hence polyhedrally, to H (meaning that the faces of $\mathcal{D}_n$ converge to those of $\mathcal{D}_H$ uniformly on compact subsets of $\mathbb{H}^3$). By the universal ball property, see [15] Lemma 4.3.11, there exists $\delta > 0$ such that $\mathcal{D}_n \cap B_r$ contains the ball $B_{\mathbb{H}}(O, \delta)$ for all n. Suppose that $g, g' \in \Gamma$ are such that $j_n(g), j_n(g') \in B_{\mathbb{H}}(h \cdot O, \delta/2)$ for some $h \in H$. Then $d(O, j_n(g^{-1}g')) < \delta$ from which it follows that $g = g'$. That $B_{\mathbb{H}}(h \cdot O, \delta/2) \cap G_n \cdot O \neq \emptyset$ for sufficiently large n follows from the geometric convergence, proving the lemma with $\delta_0 = \delta/2$. $\qquad \square$

Given any finite set $A \subset H$, Lemma 7.11 allows us to make a bijective correspondence between A and a corresponding subset $A_n \subset G_n$ for all sufficiently large n. Choose a set of generators Γ^* for Γ and set $H^* = \{\Gamma^*, q_1, \ldots, q_s\}$ with q_i chosen as in Theorem 7.6. We define G_n^* to be the corresponding set of elements in G_n. This is well defined for any $n > \max\{m_0(g) : g \in \Gamma^*\}$, with $m_0(g)$ as in Lemma 7.11. According to that lemma, each element of G_n^* is either already in Γ^*, or is close to q_i for one of the additional generators as in Theorem 7.6. Since each such q_i is the limit of a sequence of the form $\rho_n(p^{m_n})$ where $p \in \mathcal{P}_0$ and $m_n \to \infty$, we may suppose that n_0 in Section 7.4 above is also chosen so that the additional elements in G_n^* are all of this form. Once again, we renumber so that G_{n_0} becomes G_0.

7.6. Proof of Theorem B.

Now we turn to the main part of the proof of Theorem B.

For $g \in \Gamma$, we use $|g|_\Gamma$ or $|g|$ to denote word length relative to the generating set Γ^* and $|g|_n$ to denote word length relative to the generating set G_n^*. We shall need to distinguish between equality as group elements and equality as words (meaning that all letters are identical). We write $W = AB$ to mean that all letters of W are exactly the same as those in the juxtaposed strings of letters A, B and $W =_G AB$ to mean that W is equal to AB as elements in a group G.

Definition 7.12. *Let G be a Kleinian group with generators G^* and let $|\cdot|_G$ denote the word metric. We say a path $e_{i_1} e_{i_2} \ldots e_{i_s}, e_{i_j} \in G^*$ is L-quasi-geodesic with respect to $(\mathcal{G}G, |\cdot|_G)$ if $(b-a)/L \le |e_{i_a} \ldots e_{i_{b-1}}|_G \le L(b-a)$ for any $1 \le a < b \le s$.*

Lemma 7.13. *Suppose that $W = w_{i_1} w_{i_2} \ldots w_{i_s}, w_{i_j} \in \Gamma^*$ is L-quasi-geodesic in $(\mathcal{G}G_n, |.|_n)$ for some $n \ge 0$. Then W is also L-quasi-geodesic in $(\mathcal{G}\Gamma, |.|_\Gamma)$.*

Proof. Let $V = w_{i_a} \ldots w_{i_{b-1}}$ be a subsegment of W. Then $b - a < L|V|_n \le L|V|_\Gamma$ where the last inequality follows since any expression in Γ^* is also one in G_n^*. Since $w_{i_j} \in \Gamma^*$ for all j, the inequality $|V|_\Gamma \le b - a$ is obvious. $\square$

The key step in the proof of Theorem B is the following proposition, which says that although the image $j_n(\lambda)$ of a quasi-geodesic $\lambda \subset \mathcal{G}\Gamma$ may not be an ambient quasi-geodesic in $\mathcal{V}_n$ (see the discussion following the statement of Theorem 7.6), it is still within uniformly bounded distance of $[j_n(\lambda)] \cup \mathcal{H}([j_n(\lambda)])$. The idea is first to track $[j_n(\lambda)]$ with uniform bounds by word paths W_n in the generators G_n^*, and then to study carefully how these word paths look when rewritten in terms of the generators Γ^*. This second step is the content of Proposition 7.17, which is needed in the proof of Proposition 7.14.

Proposition 7.14. *Suppose that λ is a standard K-quasi-geodesic in $\mathcal{G}\Gamma$. Then there exists $n_1 \in \mathbb{N}$ such that $j_n(\lambda)$ is at uniformly bounded distance to $[j_n(\lambda)] \cup \mathcal{H}([j_n(\lambda)])$ for all $n \ge n_1$, with a bound D that depends on K but is independent of n.*

Proof. To simplify, we may as well assume that the initial point of λ is at $1 \in \mathcal{G}\Gamma$. Denote the final point by $g_\lambda \in \Gamma$. Floyd's construction in Proposition 3.2 allows us to find a standard word path W in the generators G_n^* so that $W =_\Gamma g_\lambda$, such that $j_n(\lambda)$ is at bounded distance K_n say to $[j_n(\lambda)] \cup \mathcal{H}[(j_n(\lambda))]$ and whose length is comparable to $|\lambda|_n$, the length of λ relative to the generators G_n^*. The constants involved in determining K_n are: the diameter d_n of the thick part of the truncated Dirichlet domain $\mathcal{D}_n \cap \mathcal{V}_n$, $a_n = \max\{d(O, g \cdot O) : g \in G_n^*\}$ and $C_n = \max\{|g|_n : d(O, g \cdot O) \le D\}$ for some uniform $D > 0$. (It is crucial here that word length $|\cdot|_n$ is defined using the generating set G_n^*.) Then d_n is uniformly bounded by Proposition 6.1, a_n is uniformly bounded by the *geometric* convergence since so is $\max\{d(O, h \cdot O) : g \in H^*\}$, see Lemma 7.11, and finally, again in virtue of Lemma 7.11, C_n is uniformly bounded by $C_H = \max\{|h|_H : h \in H : d(O, h \cdot O) \le 1 + 2d_n + \delta\}$, where δ is as in Lemma 7.11. Thus all constants involved are uniform in n and hence there exists $L > 0$ such that the path W (which is written in terms of the generators G_n^*) is L-quasi-geodesic in $(\mathcal{G}G_n, |\cdot|_n)$ for all n.

The only letters in W which are not also generators of Γ^* correspond to elements in $G_n^* \setminus \Gamma^*$ and are therefore of the form $u = p^k$ for some $p \in \mathcal{P}_0$. Rewriting W in terms of the generators Γ^* by substituting these terms we find

$W =_\Gamma V_1 U_1 V_2 U_2 \ldots V_s$ where all of the letters in V_i belong to Γ^*, and where $U_i = u_i^{a_i} = p^{a_i m_i}$ for some $p \in \mathcal{P}_0$. In this expression we may have $V_i = \emptyset$ but in this case (by combining adjacent terms belonging to the same parabolic) we assume that U_i, U_{i+1} are associated to distinct elements of $\mathcal{P}_0$. We define W_Γ to be the word $V_1 U_1 V_2 U_2 \ldots V_s$ in Γ^* and emphasize once again that W and W_Γ both depend heavily on n.

Since V_i is by construction L-quasi-geodesic in $(\mathcal{G}G_n, |\cdot|_n)$, by Lemma 7.13 it is also L-quasi-geodesic in $(\mathcal{G}\Gamma, |\cdot|_\Gamma)$. We would like to claim that W_Γ is quasi-geodesic in $(\mathcal{G}\Gamma, |\cdot|_\Gamma)$. However this may not be the case because, as illustrated in Figure 1, there could be a segment which is a power of p_i at the end of V_i which cancels into U_i, and likewise a segment at the beginning of V_i which cancels into U_{i-1}. To remedy this we will show in Proposition 7.17 below that it is possible to split V_i into blocks $V_i = X_i Y_i Z_i$ where X_i cancels into U_{i-1} and Y_i cancels into U_i, in such a way that $Z_i U_i X_{i+1} =_\Gamma \hat{U}_i$ and $j_0(Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s)$ is a standard ambient L'-quasi-geodesic in $\mathcal{V}_0$, for some L' depending on L but independent of n. The essential idea is illustrated in Figure 1. Notice that following through the definitions, $g_\lambda =_\Gamma Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s$.

Assuming this we proceed as follows. By hypothesis, λ is a standard quasi-geodesic in $\mathcal{G}\Gamma$ with initial point O and endpoint $g_\lambda \cdot O$. Thus by construction $\hat{j}_0(\lambda)$ has the same endpoints as $\hat{j}_0(Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s)$. Hence by Lemma 7.4, the paths $\hat{j}_0(Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s)$, $\hat{j}_0(\lambda)$ are a bounded distance apart. Since $j_n j_0^{-1} \colon \mathcal{G}\Gamma \to \mathbb{H}^3$ is uniformly Lipschitz because of the algebraic convergence, $\hat{j}_n(\lambda)$ is a uniformly bounded distance from $\hat{j}_n(Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s)$, with a bound that depends on K and L'.

Now we show that any point $Q \in \hat{j}_n(Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s)$ is at uniformly bounded distance from $[j_n(\lambda)] \cup \mathcal{H}([j_n(\lambda)])$. We have $Q = j_n(A)$ where A is a subsegment of $Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s$ starting at 1. Those subwords starting at 1 and ending in a letter in some $\hat{U}_i$ have images under $\hat{j}_n$ within uniformly bounded distance of the corresponding thin part $T_i \in \mathcal{H}([j_n(\lambda)])$, while the image of points corresponding to subsegments starting from 1 and ending at a letter contained in a segment Y_i are within uniformly bounded distance of $[j_n(\lambda)] \cup \mathcal{H}([j_n(\lambda)])$ since any word of the form $Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_r'$ with Y_r' an initial subsegment of $Y_r, r \leq s$, is by construction equal to a subword of W.

We have shown that $\hat{j}_n(\lambda)$ is a uniformly bounded distance from $\hat{j}_n(Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s)$ and that any point in $\hat{j}_n(Y_1 \hat{U}_1 Y_2 \hat{U}_2 \ldots Y_s)$ is at uniformly bounded distance from $[j_n(\lambda)] \cup \mathcal{H}([j_n(\lambda)])$. The result follows. $\square$

To prove Proposition 7.17 we need two easy lemmas.

Lemma 7.15. *There exists $\ell_0 > 0$ with the following property. Suppose that $T_i, i = 1, \ldots, s$ is a sequence of thin parts $T_i \in \mathcal{H}_{\epsilon, G_0}$ such that $T_i \neq T_{i+1}$ for all i. Suppose that $[Q_i P_{i+1}]$ is the common perpendicular to T_i, T_{i+1} and that μ is*

the piecewise geodesic arc joining points $Q_1, P_2, Q_2, \ldots, Q_{s-1}, P_s$, and suppose also that the geodesic segment $[P_iQ_i]$ has length at least $\ell_0, i = 2, \ldots, s-1$. Then μ is a quasi-geodesic in $\mathbb{H}^3$.

Proof. The angle between the segments $[P_iQ_i]$ and $[Q_iP_{i+1}]$ is at least $\pi/2$ and by the choice of ϵ (see the discussion in Section 7.4), $[Q_iP_{i+1}]$ has length at least $a > 0$. The result is now standard. $\square$

Lemma 7.16. *Suppose that $D > 0$ is given and that $T, T' \in \mathcal{H}_{\epsilon, G_0}$. Let P, P' be points within distance D of T, T' respectively such that the geodesic segment $[P, P']$ is disjoint from both T and T'. Then $[P, P']$ is within bounded distance of the common perpendicular to T and T' with a bound that depends only on D and ϵ.*

Proof. Let X, X' be the nearest points to P, P' on T, T' respectively and let $Q \in T, Q' \in T'$ be the endpoints of the common perpendicular to T and T'. Consider the quadrilateral with vertices X, Q, Q', X'. Since the angles at Q, Q' are at least $\pi/2$ and since $d_{\mathbb{H}}(Q, Q') > a$, the geodesic segment $[XX']$ is at bounded distance to the union of the arcs $[XQ], [QQ'], [X'Q']$. Since $[XQ]$ penetrates T, so does $[PP']$ unless $[XQ]$ has bounded length; likewise with $[X'Q']$. $\square$

Proposition 7.17. *Given $L > 0$, there exist $L' = L'(L) > 0$ and $n_1 \in \mathbb{N}$ with the following property. Suppose given a word $V_1U_1V_2U_2 \ldots V_s$ in Γ^* as in the proof of Proposition 7.14, that is, so that V_i is either empty or is L-quasi-geodesic in $(\mathcal{G}\Gamma, |\cdot|_\Gamma)$, and where $U_i = p_{j_i}^{a_i m_{j_i}}$ for some $p_{j_i} \in \mathcal{P}_0, a_i \in \mathbb{Z}, a_i \neq 0$, and so that if $V_i = \emptyset$ then $p_{j_i} \neq p_{j_{i+1}}$ for all i. Then we can split V_i into blocks as $V_i = X_iY_iZ_i$ (where any of X_i, Y_i or Z_i may be empty) in such a way that there exists a word $\hat{U}_i =_\Gamma Z_iU_iX_{i+1}$ in the generators Γ^* which is a parabolic block relative to p_{j_i} and such that $\hat{j}_0(Y_1\hat{U}_1Y_2\hat{U}_2 \ldots Y_s)$ is a standard ambient L'-quasi-geodesic in $\mathcal{V}_0$, for all $n \geq n_1$.*

Proof. We continue with the notation and discussion of Proposition 7.14. The statement and the idea of the proof are illustrated in Figure 1. Throughout the proof, when we say that various distances are 'bounded', we mean they are uniformly bounded in n and independent of the choices of λ and words W.

Let $g_i = \rho_0(V_1U_1V_2U_2 \ldots U_{i-1})$ and $g'_i = \rho_0(V_1U_1V_2U_2 \ldots V_i)$. For $i = 1, \ldots, s$, let $\beta_i = [j_0(g_i), j_0(g'_i)]$ and $\gamma_i = [j_0(g'_i), j_0(g_{i+1})]$. Thus $\beta_i = g_i([j_0(V_i)])$ and $\gamma_i = g'_i([j_0(U_i)])$. Recall that each word U_i corresponds to a thin part $T_i \in \mathcal{H}_{\epsilon, G_n}$.

Temporarily let us drop the subscript i and write $V = V_i$ etc. For simplicity we will work with $\beta = [j_0(V)] = g_i^{-1}(\beta_i)$ rather than β_i. Let A, D be the initial and final points of β.

Let T^1, T^2 denote respectively the (distinct) horoballs $g_i^{-1}(T_{i-1}), g_i^{-1}(T_i)$. It follows from the construction that A is a bounded distance from T^1 and D is a bounded distance from T^2. Consider the projection π from $\hat{j}_\Gamma(V)$ to β. Set
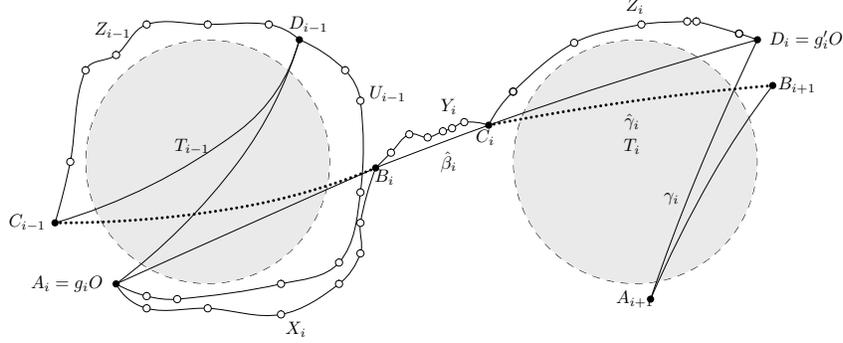
FIGURE 1. A segment of the path $j_0(W)$ showing the segment V_i from A_i to D_i split as $V_i = X_i Y_i Z_i$. Note the cancellation in the path $U_{i-1} V_i$ from D_{i-1} to A_i to D_i. We shorten $\beta_i = [A_i, D_i]$ to $\hat{\beta}_i = [B_i, C_i]$ and replace $[D_{i-1}, A_i] = \gamma_i$ by $[C_{i-1}, B_i] = \hat{\gamma}_i$. The key point is to see $\ell(\hat{\gamma}_i) \to \infty$ independent of the choices made.

$\beta^j = \beta \cap T^j, j = 1, 2$. Now define orbit points B, C on $\hat{j}_\Gamma(V)$ as follows: if $\beta^1 = \emptyset$ then $B = A$, otherwise B is a nearest orbit point to the final point of $\pi^{-1}(\beta^1)$; likewise if $\beta^2 = \emptyset$ then $C = D$, otherwise C is a nearest orbit point to the first point on $\pi^{-1}(\beta^2)$. (We remark that it is possible that C precedes B in order along $\hat{j}_\Gamma(V)$. In this case by Lemma 7.10 there is a bounded distance between B and C. Thus up to changing constants by a bounded amount, we can replace C by the point B.)

Denote the segments of V from A to B, from B to C, and from C to D by X, Y, Z respectively. Thus $V = XYZ$. By Lemma 7.7, since A and B are at a bounded distance from T^1, the segment X is a parabolic block relative to T^1, and similarly for Z relative to T^2.

Now let $A_i, B_i, C_i, D_i, X_i, Y_i, Z_i$ be the images of A, B, C, D and $j_0(X), j_0(Y)$ and $j_0(Z)$ under g_i. Let $\hat{\gamma}_i$ denote the geodesic from C_{i-1} to B_i. We claim that (independent of all the many choices made) its hyperbolic length $\ell(\hat{\gamma}_i) \to \infty$ uniformly with n. In particular, given $\ell_0 > 0$ we can choose n_1 so that $\ell(\hat{\gamma}_i) \geq \ell_0$, whenever $n \geq n_1$. To prove this, it is clearly enough to show that $|C_{i-1} B_i|_\Gamma \to \infty$ as $n \to \infty$. We have

$$|C_{i-1} B_i|_\Gamma \geq |C_{i-1} B_i|_n \succ$$
$$|C_{i-1} D_{i-1}|_n + |D_{i-1} A_i|_n + |A_i B_i|_n = (|C_{i-1} D_{i-1}|_\Gamma + |A_i B_i|_\Gamma) + |U_{i-1}|_n,$$

where the second inequality is because the path $C_{i-1} D_{i-1} A_i B_i$ is by definition L-quasi-geodesic in G_n^* and the final equality follows since by definition the words Z_{i-1}

corresponding to the path $C_{i-1}D_{i-1}$ and X_i corresponding to the path $D_{i-1}A_i$ are quasi-geodesic words in G_n^* which happen to be expressed entirely by generators in Γ^*. Since

$$|C_{i-1}D_{i-1}|_\Gamma + |A_iB_i|_\Gamma \geq |D_{i-1}A_i|_\Gamma - |C_{i-1}B_i|_\Gamma$$

we have

$$2|C_{i-1}B_i|_\Gamma \succ |D_{i-1}A_i|_\Gamma + |U_{i-1}|_n = m_{j_{i-1}}|U_{i-1}|_n + |U_{i-1}|_n,$$

where we used $|D_{i-1}A_i|_\Gamma = |U_{i-1}|_\Gamma = m_{j_{i-1}}|U_{i-1}|_n$, as in the proof of Proposition 7.14. Now by definition m_{j_i} is the exponent such that $p_{j_i}^{m_{j_i}} \to q_{j_i}$ in the geometric limit, so $m_{j_i} \to \infty$ with n. Hence $\ell(\hat{\gamma}_i) \to \infty$ with n as claimed.

Now let $\hat{\beta}_i$ denote the geodesic from B_i to C_i. Note that $\hat{\beta}_i$ has endpoints within bounded distance of the segment of β_i which is outside both T_{i-1} and T_i. We claim that for sufficiently large ℓ_0, the path β obtained by concatenating $\hat{\beta}_1, \hat{\gamma}_1, \hat{\beta}_2, \ldots, \hat{\gamma}_s$ is quasi-geodesic in $\mathbb{H}^3$, whenever $n \geq n_1$. By construction, the endpoints of $\hat{\beta}_i$ are within bounded distance of T_{i-1} and T_i respectively. Moreover by construction the segment $\hat{\beta}_i$ is outside both T_{i-1} and T_i. Thus we are in the situation of Lemma 7.16 so that $\hat{\beta}_i$ is within bounded distance of the common perpendicular to T_{i-1} and T_i. Adjusting the endpoints of each γ_i by at most a uniformly bounded amount, we see we are in the situation of Lemma 7.15, and the result follows.

From the construction, the path $\hat{j}_0(Y_i)$ tracks $\hat{\beta}_i$ at bounded distance. Now consider the segment $\hat{j}_0(Z_iU_iX_{i+1})$ from C_i to B_{i+1}. We claim that this can be replaced by a path $\hat{j}_0(\hat{U}_i)$ with the same initial and final points, and where $\hat{U}_i =_\Gamma Z_iU_iX_{i+1}$ is a parabolic block relative to p_{j_i}. By construction the initial and final points C_i to B_{i+1} are at bounded distance to T_i. Hence by the method of Lemma 7.7, if $C_i = h_iO$ to $B_{i+1} = h'_{i+1}O$ then $(h'_{i+1})^{-1}h_i$ is a parabolic block relative to p_{j_i}, proving the claim.

Now $j_0(Y_i)$ tracks $\hat{\beta}_i$ and $j_0(\hat{U}_i)$ tracks the shortest path from C_i to B_{i+1} on ∂T_i at bounded distance, $\hat{\gamma}_i$ being the geodesic with the same endpoints. Since the concatenation β of $\hat{\beta}_1, \hat{\gamma}_1, \ldots, \hat{\beta}_s, \hat{\gamma}_s$ is quasi-geodesic for $n \geq n_1$ it follows that $\hat{j}_0(Y_1\hat{U}_1Y_2\hat{U}_2\ldots Y_s)$ is an ambient quasi-geodesic in $\mathbb{H}^3$. All constants involved are independent of the various choices made and of n. This completes the proof. $\quad\square$

We are finally ready to prove Theorem B. Let $e_{i_1}e_{i_2}\ldots$ be a standard quasi-geodesic path in $\mathcal{G}\Gamma$ such that $e_{i_1}e_{i_2}\ldots e_{i_k}\cdot O \to \xi$. There are three possibilities:

(1) the length of parabolic blocks in $e_{i_1}e_{i_2}\ldots$ is bounded above;
(2) $e_{i_1}e_{i_2}\ldots$ contains parabolic blocks of arbitrarily long length;
(3) $e_{i_1}e_{i_2}\ldots$ terminates in an infinite parabolic block.

It is straightforward to see that case (3) happens if and only if ξ is a parabolic fixed point, in which case $\hat{i}_n(\xi) \to \hat{i}_\infty(\xi)$ follows immediately from the algebraic convergence. Thus we have only to prove $EP(\xi)$ and $EPP(\xi)$ for cases (1) and (2). We start with $EP(\xi)$. As usual, denote $\hat{j}_\Gamma(e_{i_1}e_{i_2}\ldots)$ by $[1,\xi)$.

Proposition 7.18. *Let $e_{i_1} e_{i_2} \ldots$ be a standard quasi-geodesic $\mathcal{G}\Gamma$ in which there is an upper bound on the length of parabolic blocks. Then $((\rho_n), [1, \xi))$ satisfies $EP(\xi)$.*

Proof. We have to show that given $N \in \mathbb{N}$, there exists $f_\xi(N) \to \infty$ as $N \to \infty$ such that $d(O, j_n(e_{i_1} e_{i_2} \ldots e_{i_N})) \geq f_\xi(N)$ for any $n \in \mathbb{N}$. If the result is false, there exist $A > 0$ and $n_k, N_k \to \infty$ such that $d(O, j_{n_k}(e_{i_1} e_{i_2} \ldots e_{i_{N_k}})) \leq A$. Passing to a subsequence, we may assume that $\rho_{n_k}(e_{i_1} e_{i_2} \ldots e_{i_{N_k}})$ converges geometrically to some $h \in H$. By Theorem 7.6, we must have $h = h_{i_1} \ldots h_{i_p}$ where $h_{i_j} \in H^*$, and since $N_k \to \infty$ we must have $h_{i_j} \in \{q_1, \ldots, q_s\}$ for some j. It follows that $e_{i_1} e_{i_2} \ldots e_{i_{N_k}}$ contains arbitrarily long parabolic blocks contrary to hypothesis. $\square$

Proposition 7.19. *Suppose that $e_{i_1} e_{i_2} \ldots$ contains arbitrarily long parabolic blocks. Then $((\rho_n), [1, \xi))$ satisfies $EP(\xi)$.*

Proof. By Proposition 7.14, $j_n(e_{i_1} e_{i_2} \ldots e_{i_N})$ is at uniformly bounded distance D say to $\alpha_n \cup \mathcal{H}(\alpha_n)$, where $\alpha_n = [j_n(e_{i_1} e_{i_2} \ldots e_{i_N})]$. Choose $k_0 = k_0(D)$ as in Lemma 7.10. Define $f_\xi(N)$ to be the number of parabolic blocks of length at least k_0 in $e_{i_1} e_{i_2} \ldots e_{i_N}$. By our assumption, $f_\xi(N) \to \infty$ as $N \to \infty$.

Let $k_1 = k_1(N; \xi)$ be the maximum length of these $f_\xi(N)$ blocks. By Lemma 7.10, there exists $M = M(k_1, D) = M_\xi(N)$ so that α_n penetrates $\rho_n(g)T_n^p$ provided $n \geq M$. Recall there is a constant $a > 0$ such that the distance between any two components of $\mathcal{H}_{\epsilon, G_n}$ is at least a for any n. It follows that $d(O, j_n(e_{i_1} e_{i_2} \ldots e_{i_r})) \geq Na$ provided $n \geq M_\xi(N)$ and the result follows. $\square$

Proposition 7.20. *The pair $((\rho_n), \xi)$ satisfies $EPP(\xi)$.*

Proof. We have to show that there exists $f_{1, \xi}(N) = f_1(N)$ such that $f_1(N) \to \infty$ as $N \to \infty$ and such that for any geodesic subsegment λ of $[1, \xi)$ lying outside $B_\Gamma(1; N)$, the $\mathbb{H}^3$-geodesic $[j_n(\lambda)]$ lies outside $B_\mathbb{H}(O; f_1(N))$ in $\mathbb{H}^3$.

Use Propositions 7.18 and 7.19 to find $f_1(N)$ such that $d_\Gamma(1, g) \geq N$ implies $d_\mathbb{H}(j_n(g), O) \geq f_1(N)$ for all $n \geq M_\xi(N)$. Then $d_\mathbb{H}(\rho_n(\lambda), O) \geq f_1(N)$ for all $n \geq M_\xi(N)$.

By Proposition 7.14, $j_n(\lambda)$ is at uniformly bounded distance D to $[j_n(\lambda)] \cup \mathcal{H}([j_n(\lambda)])$. Hence the entry and exit points of $[j_n(\lambda)]$ to any thin component $T \in \mathcal{H}([j_n(\lambda)])$ are outside $B_\mathbb{H}(O, f_1(N) - D)$, as is the sub-path of $j_n(\lambda)$ joining them. So by Lemma A.6, $[j_n(\lambda)]$ is outside $B_\mathbb{H}(O, f_1(N)/4 - c)$ for a suitable uniform constant c. The proof follows as in Theorem 3.1. $\square$

Proof of Theorem B. This follows from Proposition 7.20 and Theorem 7.3. $\square$

APPENDIX: HYPERBOLIC GEOMETRY ESTIMATES

Lemma A.1. *In the ball model with O as centre, suppose that $X, Y \in \mathbb{H}^3$ lie outside $B_\mathbb{H}(O; R)$ and that the geodesic $[X, Y]$ joining them is also outside $B_\mathbb{H}(O; R)$. Then $d_\mathbb{E}(X, Y) \prec e^{-R}$.*

Proof. In the ball model $\mathbb{B}$, let ξ, η denote the endpoints of the radial lines from O through X, Y on $\partial \mathbb{H}^3$. Let P be the footpoint of the perpendicular from O to the geodesic segment $[XY]$ and let P' be the endpoint of this ray at ∞. It will be sufficient to show that $d_{\mathbb{E}}(P, X) \prec e^{-R}$.

Let $\theta = \angle XOP$. Then

$$\tan \theta = \frac{\tanh d_{\mathbb{H}}(X, P)}{\sinh d_{\mathbb{H}}(O, P)} < e^{-R}/2,$$

from which it follows that $d_{\mathbb{E}}(\xi, P') < e^{-R}/2$.

Now from $R \leq d_{\mathbb{H}}(O, X)$ we find easily $d_{\mathbb{E}}(X, \xi) \prec e^{-R}$. The result follows. $\square$

The following lemma allows us to replace orbit points by attracting fixed points.

Lemma A.2. *Suppose that $A \in PSL_2(\mathbb{C})$ is loxodromic (resp. parabolic), and that $d_{\mathbb{H}}(O, A \cdot O) > R$, where O is the centre of the ball model $\mathbb{B}$. Let $A^+ \in \partial \mathbb{B}$ be the attracting fixed point (resp. fixed point) of A. Then $d_{\mathbb{E}}(A \cdot O, A^+) \prec e^{-R/2}$.*

Proof. We use a fact which we learned from [15] Lemma 1.5.4: in $\mathbb{B}$, the isometric circle $I_{A^{-1}}$ of A^{-1} is the perpendicular bisector L of the line from O to $A \cdot O$. (To see this, let τ denote inversion in L. Then $A^{-1}\tau$ fixes O and hence is a Euclidean isometry. Using this together with the fact that $|D\tau|_L = 1$, it follows easily from the chain rule that $|DA^{-1}|_L = 1$.)

Now $I_{A^{-1}}$ contains both $A \cdot O$ and the attracting fixed point A^+. Since $d_{\mathbb{H}}(O, A \cdot O) > R$ we have $d_{\mathbb{H}}(O, I_{A^{-1}}) > R/2$, hence the Euclidean diameter of $I_{A^{-1}}$ is $O(e^{-R/2})$ and the result follows. $\square$

We need estimates of the distortion caused by skirting around horoballs or Margulis tubes. If $L \subset \mathbb{H}^3$ is a geodesic we call $T = \{x \in \mathbb{H}^3 : d_{\mathbb{H}}(x, L) \leq R\}$ the *equidistant tube of radius R* around L. If H is a horoball or T is a tube, and if $P_1, P_2 \in \partial H$ (resp. $P_1, P_2 \in \partial T$), we denote by $d_{\partial H}(P_1, P_2)$ (resp. $d_{\partial T}(P_1, P_2)$) the length of the shortest path on ∂H (resp. ∂T) from P_1 to P_2.

Lemma A.3. *Let $H \subset \mathbb{H}^3$ be a horoball and suppose $P_1, P_2 \in \partial H$. Set $l = d_{\partial H}(P_1, P_2)$ and $d = d_{\mathbb{H}}(P_1, P_2)$. Suppose that $d > d_0$ for some fixed $d_0 > 0$. Then $l \asymp e^{d/2}$ with constants depending only on d_0.*

Proof. As is easily checked by explicit computation using the angle of parallelism formula, $l = 2 \sinh d/2$, see for example [8] p. 213. $\square$

One can make a similar estimate for tubes of sufficiently large radius.

Lemma A.4. *Fix positive constants R_0, d_0 and h_0. Let $T \subset \mathbb{H}^3$ be the equidistant tube of radius $R \geq R_0$ around a geodesic $L \subset \mathbb{H}^3$ and let $P_1, P_2 \in \partial T$. Let $l = d_{\partial T}(P_1, P_2)$ and $d = d_{\mathbb{H}}(P_1, P_2)$. Then $l \prec e^{d/2}$ where the constant involved depends only on R_0. If moreover the distance between the projections of P_1, P_2 onto the axis*

L of T is at most h_0 and $d \geq d_0$, then $l \asymp e^{d/2}$ where the constant involved depends only on R_0, h_0 and d_0.

Proof. For convenience we arrange things so that L is the line from the origin $\omega \in \mathbb{C}$ to ∞ in the upper half space model of $\mathbb{H}^3$. Then ∂T is a Euclidean cone with axis L. By the angle of parallelism formula, the angle ψ of the cone with the base plane $\mathbb{C}$ is given by $\cosh R \sin \psi = 1$, alternatively $\sinh R = \cot \psi$.

Let π denote perpendicular projection from $\mathbb{H}^3$ to L, and let $p_i = \pi(P_i), i = 1, 2$, see Figure 2. Let $h + i\phi$ be the complex distance between the perpendiculars $[p_1, P_1]$ and $[p_2, P_2]$ (so that $h = d_{\mathbb{H}}(p_1, p_2)$ and ϕ is the rotation angle).

We estimate l by a line integral on ∂T. Take polar coordinates on ∂T, so that for a point $P = (t, \theta) \in \partial T$, $t > 0$ denotes the Euclidean distance from ω to P and θ is the angle between the plane containing L and ωP and the plane containing L and the real axis in $\mathbb{C}$. Since the hyperbolic metric is conformally the same as the Euclidean metric, the element of hyperbolic arc length on ∂T is

$$(3) \qquad ds^2 = \frac{(t \cos \psi d\theta)^2 + dt^2}{(t \sin \psi)^2} = \sinh^2 R d\theta^2 + \cosh^2 R dt^2 / t^2.$$

For $x, y > 0$ we have $\sqrt{x^2 + y^2} \asymp x + y$. Thus integrating (3) gives

$$(4) \qquad l \asymp \phi \sinh R + h \cosh R \asymp e^R (h + \phi).$$
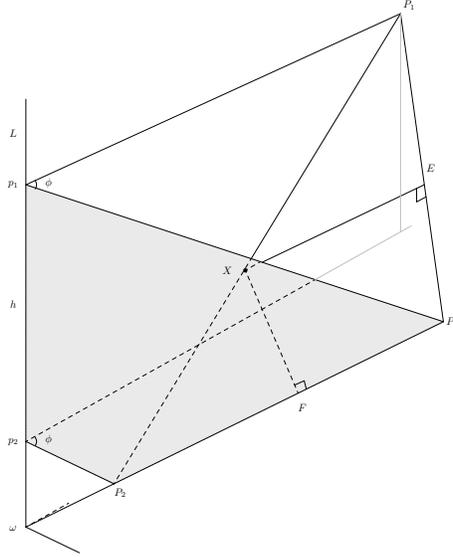
FIGURE 2. The estimate of d in the proof of Lemma A.4

Now we estimate d. Referring to Figure 2, let X be the midpoint of the segment $P_1 P_2$, so that $d/2 = |XP_1|$, where $|AB|$ denotes the hyperbolic length of

the geodesic segment from A to B. Let P_1' be the point in the hyperbolic plane orthogonal to L through p_1 (and thus containing p_1) such that $\angle P_1' p_1 P_1 = \phi$. By symmetry, $|XP_1| = |XP_1'|$, and by construction, P_1', P_2 and L are coplanar. Let E, F be the feet of the perpendiculars from X to the geodesic lines $P_1 P_1'$ and $P_1' P_2$ respectively. Considering the two right angled triangles XEP_1' and XFP_1' we find

$$d/2 = |XP_1'| > |XE| + |EP_1'| - \text{const.} > |EP_1'| - \text{const.}$$

and

$$d/2 = |XP_1'| > |XF| + |FP_1'| - \text{const.} > |FP_1'| - \text{const.}$$

so that

$$e^{d/2} \succ e^{|FP_1'|} \quad \text{and} \quad e^{d/2} \succ e^{|EP_1'|}.$$

Computing in the plane through P_1, P_1', p_1 we find $\sinh |EP_1'| = \sin \phi/2 \sinh R$, while computing in the plane containing L, P_1' and P_2 we have $\sinh |FP_1'| = \sinh h/2 \cosh R$. Thus since R is bounded below by R_0,

$$e^{d/2} \succ e^{|EP_1'|} \succ \phi e^R$$

and likewise

$$e^{d/2} \succ e^{|FP_1'|} \succ h e^R.$$

Hence

$$e^{d/2} \succ \max\{\phi, h\} e^R \asymp (\phi + h) e^R \asymp l$$

which by (4) proves that $e^{d/2} \prec l$.

To prove the inequality in the other direction note that provided that $h \leq h_0$ we have from the above $\sinh |EP_1'| \prec \phi/2 \sinh R$ and $\sinh |FP_1'| \prec h/2 \cosh R$. Now $d \leq |P_1 P_1'| + |P_1' P_2|$ gives

$$d/2 \leq |EP_1'| + |FP_1'| \leq 2 \max(|EP_1'|, |FP_1'|).$$

Since $d \geq d_0$ at least one of $\sinh |EP_1'|, \sinh |FP_1'|$ is bounded away from 0 so that

$$e^{d/2} \leq \max(e^{2|EP_1'|}, e^{2|Fa'|}) \asymp \max(\phi e^R, h e^R) \asymp e^R(\phi + h).$$

The result follows from (4). □

The next two lemmas involve the penetration of geodesics into tubes and horoballs.

Lemma A.5. *Suppose that in the ball model $\mathbb{B}$, H is a horoball such that $O \notin IntH$. Suppose also that points $P_1, P_2 \in \partial H$ lie outside $B_{\mathbb{H}}(O; N)$. Then the geodesic segment $[P_1, P_2]$ lies outside $B_{\mathbb{H}}(O; N/4 - c)$, for some universal $c > 0$.*

Proof. First consider the case in which $O \in \partial H$. Let $d_i = d_{\mathbb{H}}(O, P_i)$ and let $l_i = d_{\partial H}(O, P_i)$, where as above $d_{\partial H}(.,.)$ denotes distance measured along ∂H. Also let $d = d_{\mathbb{H}}(P_1, P_2)$ let $l = d_{\partial H}(P_1, P_2)$. By hypothesis $d_i \geq N, i = 1, 2$. Clearly

we may fix some $d_0 > 0$ and assume that $d \geq d_0$, otherwise the result is trivial. Hence by Lemma A.3, $l_i \asymp e^{d_i/2}$ and $l \asymp e^{d/2}$. Thus

$$e^{d/2} \asymp l \leq l_1 + l_2 \asymp e^{d_1/2} + e^{d_2/2} \leq 2e^{\max(d_1,d_2)/2}$$

so that $d \overset{+}{\prec} \max(d_1, d_2)$.
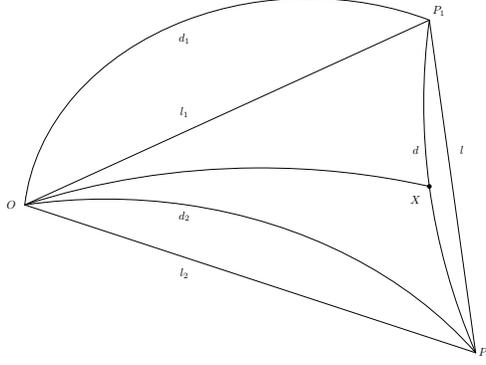
FIGURE 3. Configuration for Lemma A.5. The points O, P_1, P_2 are all on the boundary of a horoball H.

Assume that $d_1 \geq d_2$. Considering the hyperbolic triangle OP_1P_2 and its altitude OX, we have

$$\begin{aligned} d = d_{\mathbb{H}}(X, P_2) + d_{\mathbb{H}}(X, P_1) &\quad \overset{+}{\prec} \quad d_1 \\ d_{\mathbb{H}}(O, X) + d_{\mathbb{H}}(X, P_1) &\quad \overset{+}{\asymp} \quad d_1 \\ d_{\mathbb{H}}(O, X) + d_{\mathbb{H}}(X, P_2) &\quad \overset{+}{\asymp} \quad d_2. \end{aligned}$$

The first two lines give $d_{\mathbb{H}}(X, P_2) \overset{+}{\prec} d_{\mathbb{H}}(O, X)$ and hence by the last line $d_{\mathbb{H}}(O, X) \overset{+}{\succ} d_2/2 \geq N/2$. Since X is the closest point on $[P_1, P_2]$ to O, the result follows.

Finally suppose that $O \notin \partial H$. Let O' be the foot of the perpendicular from O to H. Then for any point $Y \in H$, since the angle between the geodesic segments $[O, O']$ and $[O', Y]$ is at least $\pi/2$,

$$d_{\mathbb{H}}(O, Y) \overset{+}{\asymp} d_{\mathbb{H}}(O, O') + d_{\mathbb{H}}(O', Y).$$

If $d_{\mathbb{H}}(O, O') \geq N/2$ there is nothing to prove since by convexity the nearest point on $[P_1, P_2]$ to O is in H. If $d_{\mathbb{H}}(O, O') < N/2$ then $d_{\mathbb{H}}(O', P_i) \overset{+}{\succ} N/2$. The proof above with $N/2$ in place of N then gives $d_{\mathbb{H}}(O', [P_1, P_2]) \overset{+}{\succ} N/4$ so that $d_{\mathbb{H}}(O, [P_1, P_2]) \overset{+}{\succ} d_{\mathbb{H}}(O, O') + d_{\mathbb{H}}(O', [P_1, P_2]) \overset{+}{\succ} N/4$ as claimed. $\square$

Lemma A.6. *Suppose that $T \subset \mathbb{H}^3$ is an equidistant tube of radius $R \geq R_0$ around a geodesic L in the ball model, and suppose that $O \notin IntT$. Suppose also that $P_1, P_2 \in \partial T$ lie outside $B_{\mathbb{H}}(O; N)$, and that in addition there is a path joining*

P_1, P_2 on ∂T and outside $B_{\mathbb{H}}(O; N)$. Then the geodesic segment of $[P_1, P_2]$ lies outside $B_{\mathbb{H}}(O; N/4 - c)$ for some universal $c > 0$.

Proof. As in the proof of Lemma A.5, it will be enough to show that in the case $O \in \partial T$ that $[P_1, P_2]$ is outside $B_{\mathbb{H}}(O; N/2 - c)$.

Let π denote perpendicular projection from $\mathbb{H}^3$ onto the axis L of T. As in the proof of Lemma A.2, let $p_i = \pi(P_i)$ and write $h + i\phi$ for the complex distance between the perpendiculars $[p_i, P_i], i = 1, 2$, see Figure 2.

First suppose that the distance from $o = \pi(O)$ to the segment $[p_1, p_2]$ is at least 1, and suppose that o is nearer to p_1 than p_2. Let $\rho = d_{\mathbb{H}}(O, p_1)$. Let Π be the plane perpendicular to L through p_1, and let K be the closed half space cut off by Π and not containing o. Then K contains both P_1 and P_2, and hence the segment $[P_1, P_2]$; moreover $d_{\mathbb{H}}(O, P_1) \overset{+}{\asymp} 2R + \rho$ while for any point $X \in K$ we have $d_{\mathbb{H}}(O, X) \overset{+}{\succ} R + \rho$. Since $d_{\mathbb{H}}(O, P_1) \geq N$ it follows that $R + \rho/2 \overset{+}{\succ} N/2$ so that $d_{\mathbb{H}}(O, [P_1, P_2]) \geq R + \rho \geq N/2$.

Now suppose that o is at distance at most 1 to the segment $[p_1, p_2]$, and suppose also that $|h| \geq 1$. The hyperbolic geodesic α from P_1 to P_2 is at distance at most c to the union of the geodesic segments $[P_1, p_1], [p_1, p_2], [p_2, P_2]$. Let Π' be the plane orthogonal to L containing O, so that $o \in \Pi'$. Then Π' separates T and the points P_1, P_2 are in opposite sides of Π'. Hence the projection of any path from P_1 to P_2 on ∂T onto L must contain o. Let $o' \in \pi^{-1}(o) \in \beta$, where β is a path from P_1 to P_2 on ∂T. Note o is the centre of a circle of radius R whose boundary contains both O and o'. Hence $R \geq d_{\mathbb{H}}(O, o')/2$. By hypothesis since $o' \in \beta$ we have $d_{\mathbb{H}}(O, o') \geq N$, so $R \geq N/2$. On the other hand, since α tracks $[P_1, p_1] \cup [p_1, p_2] \cup [p_2, P_2]$ within distance c for some universal $c > 0$, since $o \in [p_1, p_2]$, and since o is the nearest point to O on L, we have that $d_{\mathbb{H}}(O, [P_1, P_2]) \overset{+}{\asymp} d_{\mathbb{H}}(O, o) = R$. Thus $d_{\mathbb{H}}(O, [P_1, P_2]) \overset{+}{\succ} N/2$ as claimed.

Finally suppose that $|h| \leq 1$. Let $d = d_{\mathbb{H}}(P_1, P_2)$, $l = d_{\partial T}(P_1, P_2)$, $d_i = d_{\mathbb{H}}(O, P_i) \geq N$ and $l_i = d_{\partial T}(O, P_i), i = 1, 2$. Picking $h_0 = 2$ in Lemma A.6 we have $l \asymp e^{d/2}$, and in addition, since o is at distance at most 1 to the segment $[p_1, p_2] \subset L$, we have $d_{\mathbb{H}}(o, p_i) = d_{\mathbb{H}}(\pi(O), \pi(P_i)) \leq h_0$ so that $l_i \asymp e^{d_i/2}$. Then exactly the same proof as in Lemma A.5 gives that $d_{\mathbb{H}}(O, [P_1, P_2]) \overset{+}{\succ} N/2$. This completes the proof. $\qquad\square$

Remark A.7. Lemma A.6 required a hypothesis not needed in Lemma A.5, namely that the shortest path on ∂T from P_1 to P_2 is outside $B_{\mathbb{H}}(O; N)$. This is only used in the case in which $\pi(O)$ is near to $[\pi(P_1), \pi(P_2)]$ and $|h| \geq 1$, however here it is crucial. To see this consider the situation in which P_1, P_2 and L are coplanar and O is the midpoint of the path from P_1 to P_2 on ∂T (contrary to the hypothesis under discussion). Then the geodesic $[P_1, P_2]$ tracks $[P_1, p_1] \cup [p_1, p_2] \cup [p_2, P_2]$ so that $d_{\mathbb{H}}(O, [P_1, P_2]) \overset{+}{\asymp} R$ but at the same time we could have $N \gg R$. This would

cause problems in Section 7 when we need to find uniform estimates for a sequence of groups with fixed R_0 but $N \to \infty$.

References

[1] M. Bridson and A. Haefliger. Metric spaces of non-positive curvature. *Springer Grundlehren* 319, Springer, Berlin, 1999.

[2] J. Brock. Iteration of mapping classes and limits of hyperbolic 3-manifolds. *Inventiones Math.* 1043, 523 − 570, 2001.

[3] R. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* 6, 1 − 35, 1993.

[4] J. Cannon and W. P. Thurston. Group Invariant Peano Curves. *Geometry and Topology* 11, 1315 − 1355, 2007.

[5] R. Evans. Deformation spaces of hyperbolic 3-manifolds: strong convergence and tameness. *Ph.D. Thesis, University of Michigan*, 2000.

[6] R. Evans. Weakly type-preserving sequences and strong convergence. *Geometriae Dedicata* 108, 71 − 92, 2004.

[7] W. Fenchel and J. Nielsen. Discontinuous groups of isometries in the hyperbolic plane. *de Gruyter Studies in Math.* 319, Berlin 2003.

[8] W. Floyd. Group completions and limit sets of Kleinian groups. *Inventiones Math.* 57, 205 − 218, 1980.

[9] S. Francaviglia. Constructing equivariant maps for representations. *Ann. Inst. Fourier* 59, 393 − 428, 2009.

[10] E. Ghys and P. de la Harpe (eds.). Sur les groupes hyperboliques d'après Mikhael Gromov. *Progress in Math. Vol 83, Birkhauser, Boston*, 1990.

[11] S. Kerckhoff and W. Thurston. Non-continuity of the action of the modular group at the Bers' boundary of Teichmüller Space. *Inventiones Math.* 100, 25 − 48, 1990.

[12] T. Kuusalo. Boundary mappings of geometric isomorphisms of Fuchsian groups. *Ann. Acad. Sci. Fennicae Ser. A Math.* 545, 1 − 7, 1973.

[13] T. Jørgensen and A. Marden. Algebraic and geometric convergence of Kleinian groups. *Math. Scand.* 66, 47 − 72, 1990.

[14] R. Mañé, P. Sad and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. École. Norm. Sup.*, 16, 193 − 217, 1983.

[15] A. Marden. *Outer Circles: An introduction to hyperbolic 3-manifolds.* Cambridge University Press, 2007.

[16] C. T. McMullen. Local connectivity, Kleinian groups and geodesics on the blow-up of the torus. *Inventiones Math.* 97, 95 − 127, 2001.

[17] Y. N. Minsky. The classification of punctured-torus groups. *Ann. of Math.* 149, 559 − 626, 1999.

[18] Y. N. Minsky. The classification of Kleinian surface groups I: Models and Bounds. *Ann. of Math.* 171, 1 − 107, 2010.

[19] M. Mitra. Cannon-Thurston maps for hyperbolic group extensions. *Topology* 37, 527 − 538, 1998.

[20] H. Miyachi. Moduli of continuity of Cannon-Thurston maps. *In Spaces of Kleinian groups; London Math. Soc. Lecture Notes 329, Cambridge University Press*, 121 −150, 2006.

[21] M. Mj. Cannon-Thurston Maps for Kleinian Groups. *preprint, arXiv:1002.0996*, 2010.

[22] M. Mj and C. Series. Limits of limit sets II. *In preparation.*

[23] J. Nielsen. Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen. *Acta Math.* 50, 189 − 358, 1927.

[24] P. Tukia. On isomorphisms of geometrically finite Möbius groups. *IHES Publ.* 61, $127 - 140$, 1985.

[25] P. Tukia. A remark on a paper by Floyd. *In Holomorphic functions and moduli Vol II; MSRI Publ. 11, Springer, New York*, 165 –172, 1988.

mahan@rkmvu.ac.in; mahan.mj@gmail.com
http://people.rkmvu.ac.in/∼mahan/
School of Mathematical Sciences, RKM Vivekananda University
P.O. Belur Math, Dt. Howrah, WB 711202, India

C.M.Series@warwick.ac.uk
http://www.maths.warwick.ac.uk/∼masbb/
Mathematics Institute, University of Warwick
Coventry CV4 7AL, UK