MODELS OF ENDS OF HYPERBOLIC 3-MANIFOLDS: A SURVEY

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ABSTRACT. We survey bi-Lipschitz models for degenerate ends of hyperbolic 3-manifolds used in the resolution of the Ending Lamination Conjecture. A quick introduction to Fuchsian and Kleinian surface groups is included. We end with the use of the bi-Lipschitz model to prove the existence of Cannon-Thurston maps and local connectivity of limit sets.

CONTENTS

1. INTRODUCTION: FUCHSIAN SURFACE GROUPS

The aim of this survey is to describe models for degenerate ends of hyperbolic 3-manifolds. The main problem in the area was Thurston's Ending Lamination Conjecture [Thu82], now a theorem due to Brock,

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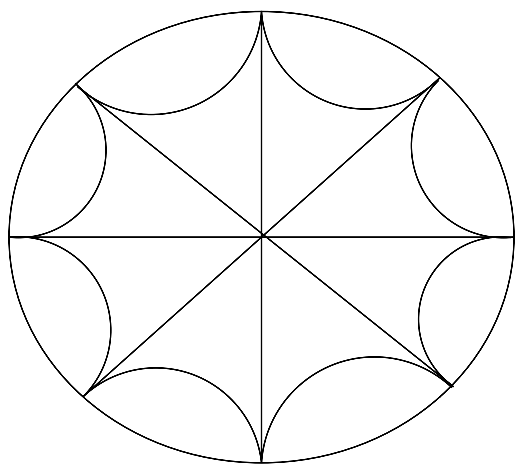
Canary and Minsky [Min10, BCM12]. The Ending Lamination Theorem says roughly that the asymptotic topology of an end, encoded by the *ending lamination* captures the geometry of the end completely. As a direct consequence of the techniques developed in [Min10, BCM12], a number of other major conjectures in the area were directly resolved. Arguably the most important of these was the Bers' density conjecture asserting that quasi-Fuchsian groups are dense in the space $\mathcal{AH}(S)$ of discrete faithful representations of the fundamental group $\pi_1(S)$ of a surface S into $PSL_2(\mathbb{C})$ equipped with the algebraic topology. The main tool developed in [Min10, BCM12] is a combinatorial model for a degenerate end. This model was used by the author in [Mj14] to prove that Cannon-Thurston maps exist and hence connected limit sets of finitely generated Kleinian groups are locally connected, thus settling another problem in [Thu82]. We should also mention here work of Bowditch [Bow15] giving a different proof of the Ending Lamination Theorem.

We shall survey some of these developments here. But to set the context, we shall first give a quick introduction to Fuchsian and Kleinian surface groups. A curious commentary on mathematical terminology and attribution is the fact that Fuchs had little to do with Fuchsian groups and Klein had little to do with Kleinian groups (though he did have something to do with Fuchsian groups); while Poincaré had everything to do with both. The theme of this survey is the study of discrete faithful representations of surface groups into $PSL_2(\mathbb{C})$ and so the story begins with Poincaré's study of discrete faithful representations of surface groups into $PSL_2(\mathbb{R})$. The Lie group $PSL_2(\mathbb{R})$ can be equivalently described as the group of Mobius transformations $Mob(\Delta)$ of the unit disk, and hence as the group of Mobius transformations $Mob(\mathbb{H})$ of the (conformally equivalent) upper half plane. The natural metric of constant negative curvature on the upper half plane is given by $ds^2 = \frac{dx^2+dy^2}{y^2}$—the *hyperbolic metric.* The resulting space is denoted as $\mathbb{H}^2$. The orientation preserving isometries of $\mathbb{H}^2$ are exactly the conformal automorphisms of $\mathbb{H}^2$. The boundary circle S^1 compactifies $\mathbb{H}^2$, or equivalently, Δ. This has a geometric interpretation. It codes the 'ideal' boundary of $\mathbb{H}^2$, consisting of asymptote classes of geodesics. The topology on S^1 is induced by a metric which is defined as the angle subtended at $0 \in \Delta$. The geodesics turn out to be semicircles meeting the boundary S^1 at right angles.

To give the reader an idea of where we are headed let us proceed to construct an explicit example of a discrete subgroup of $Isom(\mathbb{H}^2)$. The genus two orientable surface Σ_2 can be described as a quotient space of an octagon with edges labelled $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}$, where the boundary has the identification induced by this labeling. In order to construct a metric of constant negative curvature on it, we have to ensure that each point has a small neighborhood isometric to a small ball in $\mathbb{H}^2$. To ensure this it is enough to do the above identification on a regular hyperbolic octagon (all sides and all angles equal) such that the sum of the interior angles is 2π. To ensure this, we have to make each interior angle equal $\frac{2\pi}{8}$. The infinitesimal regular octagon at the tangent space to the origin has interior angles equal to $\frac{3\pi}{4}$. Also the ideal regular octagon in $\mathbb{H}^2$ has all interior angles zero. See figure below.



Hence by the Intermediate value Theorem, as we increase the size of the octagon from an infinitesimal one to an ideal one, we shall hit interior angles all equal to $\frac{\pi}{4}$ at some stage. The group G that results from side-pairing transformations corresponds to a Fuchsian group, or equivalently, a discrete faithful representation of the fundamental group of a genus 2 surface into $Isom(\mathbb{H}^2)$. We let $\rho : \Sigma_2 \to PSL_2(\mathbb{R})$ denote the associated representation. A quick look at the construction above shows that it can be generalized considerably. All we required are:

(1) Sides identified are isometric,
(2) Total internal angle for the octagon is 2π.

By modifying lengths and angles subject to these constraints, this allows us to obtain the entire space of hyperbolic structures on Σ_2, also called the Teichmúller space $Teich(\Sigma_2)$.

2. Kleinian Surface Groups

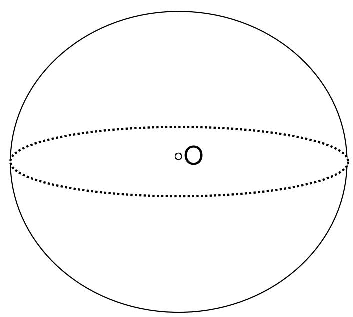
We now turn to $PSL_2(\mathbb{C})$. As above, this Lie group appears in three incarnations: $PSL_2(\mathbb{C}) = Mob(\widehat{\mathbb{C}}) = Isom^+(\mathbb{H}^3)$. A Kleinian group

G is a discrete subgroups of $PSL_2(\mathbb{C})$. This gives us three closely intertwined perspectives on the field:

(1) Studying discrete subgroups G of the group of Mobius transformations $Mob(\widehat{\mathbb{C}})$ emphasizes the *Complex Analytical/Dynamic* aspect.
(2) Studying discrete subgroups G of $PSL_2(\mathbb{C})$ emphasizes the *Lie group/matrix group* theoretic aspect.
(3) Studying discrete subgroups G of $Isom^+(\mathbb{H}^3)$ emphasizes the *Hyperbolic Geometry* aspect.

We shall largely emphasize the third perspective. Since G is discrete, we can pass to the quotient $M^3 = \mathbb{H}^3/G$. Thus we are studying hyperbolic structures on 3-manifolds. The hyperbolic metric is given by $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$ on upper half space. Note that the metric blows up as one approaches $z = 0$. Equivalently we could consider the ball model, where the boundary $S^2 = \widehat{\mathbb{C}}$ consists of ideal end-points of geodesic rays as before. The metric on $\widehat{\mathbb{C}}$ is given by the angle subtended at $0 \in \mathbb{H}^3$.

Since $Isom(\mathbb{H}^2) \subset Isom(\mathbb{H}^3)$, we can look upon the discrete group G constructed in Section 1 above also as a discrete subgroup of $Isom(\mathbb{H}^3)$.



In the above picture two things need to be observed.
1) the orbit $G.o$ accumulates on the equatorial circle. This is called the *limit set* Λ_G.
2) The complement of Λ_G consists of two round open discs. On each of these disks, G acts freely (i.e. without fixed points) properly discontinuously, by conformal automorphisms. Hence the quotient is two copies of the 'same' Riemann surface (i.e. a one dimensional complex analytic manifold). The complement $\widehat{\mathbb{C}} \setminus \Lambda_G = \Omega_G$ is called the *domain of discontinuity* of G.

We proceed with slightly more formal definitions identifying $\widehat{\mathbb{C}}$ with the sphere S^2.

Definition 2.1. *If $x \in \mathbf{H}^3$ is any point, and G is a discrete group of isometries, the limit set $\Lambda_G \subset S^2$ is defined to be the set of accumulation points of the orbit $G.x$ of x.*

The domain of discontinuity for a discrete group G is defined to be $\Omega_G = S^2 \setminus \Lambda_G$.

Proposition 2.2. [Thu80]*[Proposition 8.1.2] If G is not elementary, then every non-empty closed subset of S^2 invariant by G contains the limit set Λ_G.*

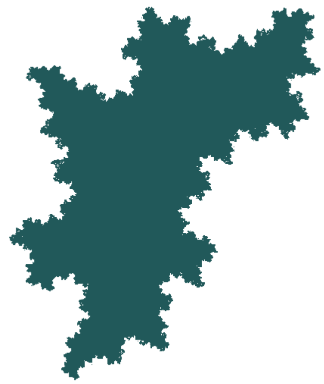
2.1. **Quasi-Fuchsian groups.** Suppose that G is abstractly isomorphic to the fundamental group of a finite area hyperbolic surface S, and $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ is a representation with image G. Suppose further that ρ is *strictly type-preserving*, i.e. $g \in \pi_1(S)$ represents an element in a peripheral (cusp) subgroup if and only $\rho(g)$ is parabolic. In this situation we shall refer to G as a *Kleinian surface group*. A recurring theme in the context of finitely generated, infinite covolume Kleinian groups is that the general theory can be reduced to the study of surface Kleinian groups. Thus we study the discrete faithful elements of the representation space $Rep(\pi_1(S), PSL_2(\mathbb{C}))$.

Regarding G as a subgroup of $Mob(\widehat{\mathbb{C}})$, the *dynamics* of the action of G on $\widehat{\mathbb{C}}$) emerges. Recall that the limit set Λ_G of G is the set of accumulation points of the orbit $G.o$ in $\widehat{\mathbb{C}}$ for some (any) $o \in \mathbb{H}^3$. The limit set may be thought of as the locus of chaotic dynamics of the action of G on $\mathbb{C}$. The complement $\widehat{\mathbb{C}} \setminus \Lambda_G = \Omega_G$ is called the *domain of discontinuity* of G.

On the other hand regarding G as a subgroup of $Isom(\mathbb{H}^3)$, we obtain a quotient hyperbolic 3-manifold $M = \mathbb{H}^3/G$ with fundamental group G.

A substantial part of the theory of Kleinian groups tries to understand the relationship between the dynamic and the hyperbolic geometric descriptions of G.

The **Bers simultaneous Uniformization Theorem** states that given any two conformal structures τ_1, τ_2 on a surface, there is a discrete subgroup G of $Mob(\widehat{\mathbb{C}})$ whose limit set is *topologically* a circle, and whose domain of discontinuity quotients to the two Riemann surfaces τ_1, τ_2. See figure below.



The limit set is a quasiconformal map of the round circle.

Definition 2.3. *A Kleinian surface group G is* **quasi-Fuchsian** *if its limit set Λ_G is homeomorphic to a circle.*

These *(quasi Fuchsian)* groups can be thought of as *deformations* of Fuchsian groups (Lie group theoretically) or quasiconformal deformations (analytically). Ahlfors and Bers proved that these are precisely all quasiconvex surface Kleinian groups. Let $\mathfrak{QF}(S)$ consist of quasi Fuchsian representations $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$, where two such representations are regarded as equivalent if they are conjugate in $PSL_2(\mathbb{C})$. Thus the Ahlfors-Bers theorem can be summarized as:

Theorem 2.4. $\mathfrak{QF}(S) = Teich(S) \times Teich(S)$.

The **convex hull** CH_G of Λ_G is the smallest non-empty closed convex subset of $\mathbb{H}^3$ invariant under G. It can be constructed by joining all pairs of points on limit set by bi-infinite geodesics and iterating this construction. The quotient of CH_G by G, which is homeomorphic to $S \times [0, 1]$, is called the **convex core** $CC(M)$ of $M = \mathbb{H}^3/G$.

A slight generalization of quasi-Fuchsian groups is given by the following:

Definition 2.5. *A Kleinian group is* **geometrically finite** *if for some $\epsilon > 0$, the volume of the $\epsilon-$neighborhood $N_\epsilon(CC(M))$ is finite.*

A Kleinian group is **geometrically infinite** *if it is not geometrically finite.*

The 'thickness' of $CC(M)$ for a quasi Fuchsian Kleinian surface group, measured by the distance between $S \times \{0\}$ and $S \times \{1\}$ is a geometric measure of the complexity of the quasi Fuchsian group G. If we assume that S has no parabolics, then $\mathfrak{QF}(S)$ coincides precisely with geometrically finite representations without parabolics. For any such G, $M \setminus CC(M)$ consists of two *hyperbolically flaring* pieces, called **geometrically finite ends**, each homeomorphic to $S \times [0, \infty)$. The metric on such a geometrically finite end E is of the form $dt^2 + sinh(t)ds^2$.

On the other hand, geometrically infinite Kleinian surface groups G are those for which the convex core $CC(M)$ is homeomorphic to $S \times [0, \infty)$ or $S \times (-\infty, \infty)$. The geometry of such ends is far more complicated and we shall return to this topic later.

2.2. **Laminations and Pleated Surfaces.** An essential technical tool in the study of Kleinian surface groups is the theory of laminations and pleated surfaces [Thu80].

Definition 2.6. *A geodesic lamination on a hyperbolic surface is a foliation of a closed subset with geodesics.*

A geodesic lamination equipped with a transverse measure is a measured lamination.

The space of measured laminations on S is denoted as $\mathcal{ML}(S)$. Projectivizing (by removing the zero element and identifying elements that differ by a scaling) we obtain the projectivized measured lamination space $\mathcal{PML}(S)$.

A fundamental theorem of Thurston shows that $\mathcal{PML}(S)$ is homeomorphic to S^{6g-7} (assuming that S is closed of genus g). Further, the Thurston compactification of $Teich(S)$ adjoins $\mathcal{PML}(S)$ to the Teichmuller space (homeomorphic to $\mathbb{R}^{6g-6}$) in a natural way, so that the action of the mapping class group $Mod(S)$ on $Teich(S)$ extends naturally and continuously to the compactification.

Geodesic laminations arise naturally in a number of contexts in the study of hyperbolic 2- and 3- manifolds.

(1) as stable and unstable laminations corresponding to a pseudo-anosov diffeomorphism of a hyperbolic surface.
(2) as the pleating locus of a component of the convex core boundary $\partial CC(M)$ of a hyperbolic 3-manifold M.
(3) as the ending lamination corresponding to a geometrically infinite end of a hyperbolic 3-manifold.

We shall in this section discuss briefly how each of these examples arise.

2.2.1. *Stable and Unstable laminations.* We consider the torus T^2 equipped with a diffeomorphism ϕ, whose action on homology is given by a 2×2 matrix with irrational eigenvalues, e.g. $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$. Then the eigendirections give rise to two sets of foliations by dense copies of $\mathbb{R}$: the stable and unstable foliation. Such a diffeomorphism is called Anosov. Anosov diffeomorphisms of the torus may be characterized in terms of their action on $\pi_1(T)$ as not having periodic conjugacy classes.

Now consider the stable (or unstable) foliation (minus a point $*$) on $S = (T^2 \setminus \{*\})$. Equip S with a complete hyperbolic structure of finite volume and straighten every leaf of the foliation to a complete geodesic. The resulting union of leaves is called the stable (or unstable) lamination of the diffeomorphism ϕ on the hyperbolic surface S.

One of the fundamental pieces of Thurston's work [FLP79] shows that the existence of such a stable and unstable lamination generalizes to all hyperbolic surfaces. A diffeomorphism ϕ of a hyperbolic surface S preserving punctures (or boundary components according to taste) is called pseudo Anosov if the action of ϕ_* on $\pi_1(S)$ has no periodic conjugacy classes. Thurston proved the existence of a unique stable and unstable lamination without any closed leaves for any pseudo Anosov homeomorphism ϕ acting on a hyperbolic surface S.

2.2.2. *Pleating locus.* We quote a picturesque passage from [Thu80]:

> Consider a closed curve σ in Euclidean space, and its convex hull $H(\sigma)$. The boundary of a convex body always has non-negative Gaussian curvature. On the other hand, each point p in $\partial H(\sigma) \setminus \sigma$ lies in the interior of some line segment or triangle with vertices on σ. Thus, there is some line segment on $\partial H(\sigma)$ through p, so that $\partial H(\sigma)$ has non-positive curvature at p. It follows that $\partial H(\sigma) \setminus \sigma$ has zero curvature, i.e., it is developable. If you are not familiar with this idea, you can see it by bending a curve out of a piece of stiff wire (like a coat-hanger). Now roll the wire around on a big piece of paper, tracing out a curve where the wire touches. Sometimes, the wire may touch at three or more points; this gives alternate ways to roll, and you should carefully follow all of them. Cut out the region in the plane bounded by this curve (piecing if necessary). By taping the paper together, you can envelope the wire in a nice paper model of its convex hull. The physical process of unrolling a developable surface onto the plane is the origin of the notion of the developing map.
>
> The same physical notion applies in hyperbolic three-space. If K is any closed set on S^2 (the sphere at infinity), then $H(K)$ is convex, yet each point on $\partial H(K)$ lies

> on a line segment in $\partial H(K)$. Thus, $\partial H(K)$ can be developed to a hyperbolic plane. (In terms of Riemannian geometry, $\partial H(K)$ has extrinsic curvature 0, so its intrinsic curvature is the ambient sectional curvature, -1. Note however that $\partial H(K)$ is not usually differentiable). Thus $\partial H(K)$ has the natural structure of a complete hyperbolic surface.

This forces $\partial H(K)$ equipped with its intrinsic metric to be a hyperbolic surface. However, there are complete geodesics along which it is bent (but not crumpled). Thus each boundary component S, and hence its universal cover $\widetilde{S}$, carries a metric that is intrinsically hyperbolic. However, in $\mathbf{H}^3$, the universal cover $\widetilde{S}$ is bent along a geodesic lamination. S is an example of a **pleated surface**:

Definition 2.7. [Thu80]*[Definition 8.8.1] A* **pleated surface** *in a hyperbolic three-manifold N is a complete hyperbolic surface S of finite area, together with an isometric map $f : S \to N$ such that every $x \in S$ is in the interior of some straight line segment which is mapped by f to a straight line segment. Also, f must take every cusp of S to a cusp of N*

The pleating locus of the pleated surface $f : S \to M$ is the set $\gamma \subset S$ consisting of those points in the pleated surface which are in the interior of unique line segments mapped to line segments.

Proposition 2.8. [Thu80]*[Proposition 8.8.2] The pleating locus γ is a geodesic lamination on S. The map f is totally geodesic in the complement of γ.*

2.2.3. *Ending Laminations.* The notion of an ending lamination comes up in the context of a geometrically infinite group. We shall deal with these groups in greater detail in the next section. Thurston defined the notion of a geometrically tame end E of a manifold M as follows.

Definition 2.9. *An end E of a hyperbolic manifold M is* **geometrically tame** *if there exists a sequence of pleated surfaces exiting E.*

For such an end E, choose a sequence of simple closed curves $\{\sigma_n\}$ exiting E. Let $S = \partial E$ be the bounding surface of E. Then the limit of such a sequence in the projectivized measured lamination space $\mathcal{PML}(S)$ (the reader will not be much mistaken if (s)he thinks of the Hausdorff limit on the bounding surface S of E) is a lamination λ. It turns out that λ is independent of the sequence $\{\sigma_n\}$ and is called the **ending lamination** of the end E.

2.3. **Degenerate Groups.** The most intractable examples of surface Kleinian groups are obtained as limits of quasi Fuchsian groups. In fact, it has been recently established by Minsky et al. [Min10] [BCM12] that the set of all surface Kleinian groups (or equivalently all discrete faithful representations of a surface group in $PSL_2(\mathbb{C})$) are given by quasiFuchsian groups and their limits. This is known as the Bers density conjecture.

To construct limits of quasi Fuchsian groups, one allows the thickness of the convex core $CC(M)$ to tend to infinity. There are two possibilities:
a) Let only τ_1 degenerate. i.e. $I \to [0, \infty)$ **(simply degenerate case)**
b) Let both τ_1, τ_2 degenerate, i.e. $I \to (-\infty, \infty)$ **(doubly degenerate case)**

Thurston's **Double Limit Theorem** [Thu86] says that these limits exist. In the doubly degenerate case the limit set is all of $\widehat{\mathbb{C}}$.

3. Building Blocks and Model Geometries

Let N be the convex core of a hyperbolic 3-manifold without parabolics. The proof of the tameness conjecture [Ago04, CG06] shows that any end E of N is homeomorphic to $S \times [0, \infty)$, where S is a closed (or more generally finite volume hyperbolic) surface; in other words ends of hyperbolic 3-manifolds are **topologically tame**. Further, Thurston-Bonahon [Thu80, Bon86] and Canary [Can93] establish that geometrically infinite ends are geometrically tame i.e. there exists a sequence of pleated surfaces exiting them. However, the geometry of such ends can be quite complicated. We shall now proceed to describe model geometries of ends of hyperbolic 3-manifolds following [Min01, Min10, BCM12, Mj10, Mj11, Mj16, Mj14] and leading to the most general case.

In what follows in this section we shall describe different kinds of models for building blocks of E: thick, thin, amalgamated. Each building block is homeomorphic to $S \times [0, 1]$, where S is a closed surface of genus greater than one. What is common to all these three model building blocks is that the top and bottom boundary components are uniformly bi-Lipschitz to a fixed hyperbolic S. In the next section, a more general model geometry will be described.

Definition 3.1. *A model E_m is said to be built up of blocks of some prescribed geometries* **glued end to end***, if*

(1) *E_m is homeomorphic to $S \times [0, \infty)$*

(2) *There exists* $L \geq 1$ *such that* $S \times [i, i+1]$ *is* $L-$*bilipschitz to a block of one of the prescribed geometries*

$S \times [i, i+1]$ *will be called the* $(i+1)$**th block** *of the model* E_m.

The **thickness** *of the* $(i+1)$**th block is the length of the shortest path between** $S \times \{i\}$ **and** $S \times \{i+1\}$ **in** $S \times [i, i+1] (\subset E_m)$**.**

3.1. Bounded Geometry.

Definition 3.2. [Min01, Min94] *An end* E *of a hyperbolic* M *has* **bounded geometry** *if there is a (uniform) lower bound for lengths of closed geodesics in* E.

We briefly sketch the steps of the proof of the Ending Lamination Theorem by Minsky [Min94] in the special case of bounded geometry ends.

Let E be a simply degenerate end of N. Then E is homeomorphic to $S \times [0, \infty)$ for some closed surface S of genus greater than one. Thurston [Thu80] established the density of pleated surfaces:

Lemma 3.3. *There exists* $D_1 > 0$ *such that for all* $x \in N$*, there exists a pleated surface* $g : (S, \sigma) \rightarrow N$ *with* $g(S) \cap B_{D_1}(x) \neq \emptyset$.

The following Lemma now follows easily from the fact that $inj_N(x) > \epsilon_0$:

Lemma 3.4. [Bon86],[Thu80] *There exists* $D_2 > 0$ *such that if* $g : (S, \sigma) \rightarrow N$ *is a pleated surface, then* $dia(g(S)) < D_2$.

The following Theorem due to Minsky [Min92] follows from compactness of pleated surfaces and a geometric limit argument.

Theorem 3.5. [Min92] *Fix* S *and* $\epsilon > 0$. *Given* $a > 0$ *there exists* $b > 0$ *such that if* $g : (S, \sigma) \rightarrow N$ *and* $h : (S, \rho) \rightarrow N$ *are homotopic pleated surfaces which are isomorphisms on* π_1 *and* $inj_N(x) > \epsilon$ *for all* $x \in N$, *then*

$$d_N(g(S), h(S)) \leq a \Rightarrow d_{Teich}(\sigma, \rho) \leq b,$$

where d_{Teich} *denotes Teichmuller distance.*

The **universal curve** over $X \subset Teich(S)$ is a bundle whose fiber over $x \in X$ is x itself. One can now show:

Lemma 3.6. *There exist* K, ϵ *and a homeomorphism* h *from* E *to the universal curve* S_γ *over a Lipschitz path* γ *in Teichmuller space, such that* $\tilde{h}$ *from* $\tilde{E}$ *to the universal cover of* S_γ *is a* (K, ϵ)*-quasi-isometry.*

Proof. We can assume that $S\times\{0\}$ is mapped to a pleated surface $S_0 \subset N$ under the homeomorphism from $S\times[0,\infty)$ to E. We shall construct inductively a sequence of 'equispaced' pleated surfaces $S_i \subset E$ exiting the end. Assume that $S_0, \cdots, S_n$ have been constructed such that:

(1) If E_i be the non-compact component of $E\backslash S_i$, then $S_{i+1} \subset E_i$.
(2) Hausdorff distance between S_i and S_{i+1} is bounded above by $3(D_1 + D_2)$.
(3) $d_N(S_i, S_{i+1}) \geq D_1 + D_2$.
(4) From Lemma 3.5 and condition (2) above there exists D_3 depending on D_1, D_2 and S such that $d_{Teich}(S_i, S_{i+1}) \leq D_3$

Next choose $x \in E_n$, such that $d_N(x, S_n) = 2(D_1 + D_2)$. Then by Lemma 3.3, there exists a pleated surface $g : (S, \tau) \to N$ such that $d_N(x, g(S)) \leq D_1$. Let $S_{n+1} = g(S)$. Then by the triangle inequality and Lemma 3.4, if $p \in S_n$ and $q \in S_{n+1}$,

$$D_1 + D_2 \leq d_N(p, q) \leq 3(D_1 + D_2).$$

This allows us to continue inductively. The Lemma follows. □

Finally, Minsky [Min92, Min94] establishes that the Lipschitz path in Lemma 3.6 tracks a unique Teichmüller geodesic ray. As a consequence he shows that any two ends of bounded geometry with the same ending lamination are bi-Lipschitz homeomorphic to each other. The Ending Lamination Theorem in this case now follows in a straightforward fashion by using a Theorem due to Sullivan [Sul81].

Lemma 3.6 also furnishes a bi-Lipschitz model E_m for E by gluing a sequence of **thick blocks** end-to-end.

We describe below the construction of a thick block, as this will be used in all the model geometries that follow.

Let S be a closed hyperbolic surface with a fixed but arbitrary hyperbolic structure.

Thick Block

Fix a constant L and a hyperbolic surface S. Let $B_0 = S \times [0, 1]$ be given the product metric. If B is $L-$bilipschitz homeomorphic to B_0, it is called an $L-$**thick block**.

The following statement is now a consequence of Lemma 3.6 [Min94] (see also [Mit98, Mj10]).

Remark 3.7. *For any bounded geometry end, there exists L such that E is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing $L-$thick blocks end-to-end.*

3.1.1. *Cannon-Thurston Maps for Bounded Geometry.* The bounded geometry model of Remark 3.7 above was used to prove the existence of Cannon-Thurston maps in this case (see [Min94] for the case of surface groups and [Mit98, Kla99, Bow07, Miy02, Mj09] for other Kleinian groups). The main tool in [Mit98] is the construction of a **hyperbolic ladder** $\mathcal{L}_\lambda \subset \widetilde{E}$ for *any* geodesic in $\widetilde{S}$. Then $\widetilde{E}$ can be thought of as (is quasi-isometric to) a tree T of spaces, where

(1) T is the simplicial tree underlying $\mathbb{R}_+$ with vertices at $\{0\} \cup \mathbb{N}$.
(2) All the vertex and edge spaces are (intrinsically) isometric to $\widetilde{S}$.
(3) The edge space to vertex space inclusions are quasi-isometries.

Given a geodesic $\lambda = \lambda_0 \subset \widetilde{S}$, we briefly outline the construction of a **ladder** $\mathcal{L}_\lambda \subset \widetilde{E}$ containing λ. Index the vertices by $n \in \{0\} \cup \mathbb{N}$. Since the edge-to-vertex vertex inclusions are quasi-isometries, this induces a quasi-isometry ϕ_n from $\widetilde{S} \times \{n\}$ to $\widetilde{S} \times \{n+1\}$ for $n \geq 0$. These quasi-isometries ϕ_n induce maps Φ_n from geodesic segments in $\widetilde{S} \times \{n\}$ to geodesic segments in $\widetilde{S} \times \{n+1\}$ for $n \geq 0$ by sending a geodesic in $\widetilde{S} \times \{n\}$ joining a, b to a geodesic in $\widetilde{S} \times \{n+1\}$ joining $\phi_n(a), \phi_n(b)$. Inductively define:

- $\lambda_{j+1} = \Phi_j(\lambda_j)$ for $j \geq 0$,
- $\mathcal{L}_\lambda = \bigcup_j \lambda_j$.

$\mathcal{L}_\lambda$ turns out to be quasiconvex in X [Mit98]. This suffices to prove the existence of Cannon-Thurston maps.

3.2. i-bounded Geometry.

The next model geometry is satisfied by degenerate Kleinian punctured-torus groups as shown by Minsky in [Min99].

Definition 3.8. *An end E of a hyperbolic M has* **i-bounded geometry** [Mj11] *if the boundary torus of every Margulis tube in E has bounded diameter.*

An alternate description of i-bounded geometry can be given as follows. We start with a closed hyperbolic surface S. Fix a finite collection $\mathcal{C}$ of disjoint simple closed geodesics on S and let $N_\epsilon(\sigma_i)$ denote an ϵ neighborhood of σ_i, ($\sigma_i \in \mathcal{C}$). Here ϵ is chosen small enough so that no two lifts of $N_\epsilon(\sigma_i)$ to the universal cover $\widetilde{S}$ intersect.

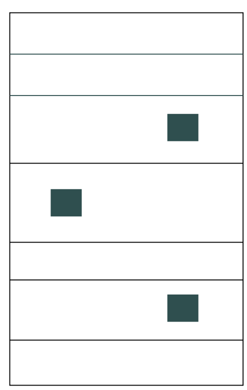
Thin Block
Let $I = [0, 3]$. Equip $S \times I$ with the product metric. Let $B^c = (S \times I - \cup_i N_\epsilon(\sigma_i) \times [1, 2]$. Equip B^c with the induced path-metric.

For each resultant torus component of the boundary of B^c, perform Dehn filling on some $(1, n_i)$ curve (the n_i's may vary from block to block but we do not add on the suffix for B to avoid cluttering notation), which goes n_i times around the meridian and once round the longitude. n_i will be called the **twist coefficient**. Foliate the torus boundary of B^c by translates of $(1, n_i)$ curves and arrange so that the solid torus Θ_i thus glued in is hyperbolic and foliated by totally geodesic disks bounding the $(1, n_i)$ curves. Θ_i equipped with this metric will be called a **Margulis tube**.

The resulting copy of $S \times I$ obtained, equipped with the metric just described, is called a **thin block**. The following statement is a consequence of [Mj11].

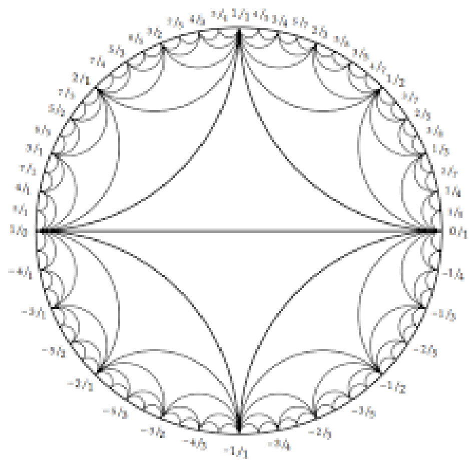
Proposition 3.9. *An end E of a hyperbolic 3-manifold M has* **i-bounded geometry** *if and only if it is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing thick and thin blocks end-to-end.*

The figure below illustrates a model E_m, where the black squares denote Margulis tubes and the (long) rectangles without black squares represent thick blocks.



Model of i-bounded geometry (schematic)

Minsky proved the Ending Lamination Theorem for punctured torus groups in [Min99] by constructing a bi-Lipschitz model for ends. This is done by looking at the sequence of short curves exiting the end and tracking the sequence on the curve complex $\mathcal{C}(S_{1,1})$ of a punctured torus $S_{1,1}$. The curve complex in this case is the Farey graph (see below):



An ending lamination corresponds to an irrational number z on the boundary circle. A geodesic ray r ending at z then crosses a well-defined sequence of ideal triangles. Let $v_1, v_2, \cdots$ be the sequence of ideal vertices of these triangles. The vertices are ordered so that

(1) the edge (v_i, v_{i+1}) belongs to two successive ideal triangles that r crosses, the preceding triangles do not contain v_{i+1} and the succeeding triangles do not contain v_i.
(2) There are exactly n_i consecutive triangles with vertex v_i crossed by r.

The Minsky model in this case has the ith block B_i with short curves (v_i, v_{i+1}) on its two boundary components. Further, n_i agrees precisely with the Dehn filling coefficient described above. The main theorem of [Min99] then proves that an end E with ending lamination coded by z is bi-Lipschitz homeomorphic to the model thus constructed. The Ending Lamination Theorem for Kleinian punctured torus groups immediately follows using Sullivan's theorem [Sul81].

3.2.1. *Cannon-Thurston maps for i-bounded geometry.* The existence of Cannon-Thurston maps for punctured torus groups was first proven by McMullen [McM01]. Following [Mj11], we give a quick sketch of the construction of the ladder $\mathcal{L}_\lambda$ and the proof of the existence of Cannon-Thurston maps in this case. First electrify all the Margulis tubes. This ensures that in the resulting electric geometry, each block is of bounded geometry. More precisely, there is a (metric) product structure on $S \times [0,3]$ such that each $\{x\} \times [0,3]$ has uniformly bounded length in the electric metric.

Further, since the curves in $\mathcal{C}$ are electrified in a block, Dehn twists are isometries from $S \times \{1\}$ to $S \times \{2\}$ in a thin block. This allows the construction of the ladder $\mathcal{L}_\lambda$ to go through as before and ensures that it is quasiconvex in the resulting electric metric.

Finally given an electric geodesic lying outside large balls modulo Margulis tubes one can recover a genuine hyperbolic geodesic tracking it outside Margulis tubes. The existence of Cannon-Thurston maps follows in this case.

3.3. **Amalgamation Geometry.** The next model geometry was used for proving the existence of Cannon-Thurston maps in a special case [Mj16]. As before, we start with a closed hyperbolic surface S. An amalgamated geometry block is similar to a thin block, except that we impose very mild control on the geometry of $S \times [1,2] \setminus (\bigcup_j N_\epsilon(\sigma_{i_j}) \times [1,2])$.

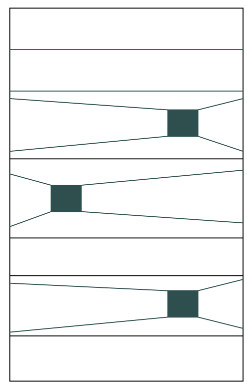
Definition 3.10. Amalgamated Block As before $I = [0,3]$. We will describe a geometry on $S \times I$. There exist ϵ, L (these constants will be uniform over blocks of the model E_m) such that

1. $B = S \times I$. Let $K = S \times [1,2]$ under the identification of B with $S \times I$.
2. We call K the geometric core. In its intrinsic path metric, it is $L-$bilipschitz to a convex hyperbolic manifold with boundary consisting of surfaces L-bilipschitz to a fixed hyperbolic surface. It follows that there exists $D > 0$ such that the diameter of $S \times \{i\}$ is bounded above by D (for $i = 1,2$).
3. There exists a simple closed multicurve on S each component of which has a geodesic realization on $S \times \{i\}$ for $i = \{1,2\}$ with (total) length at most ϵ_0. Let γ denote its geodesic realization in K.
4. There exists a regular neighborhood $N_k(\gamma) \subset K$ of γ which is homeomorphic to a union of disjoint solid tori, such that $N_k(\gamma) \cap S \times \{i\}$ is homeomorphic to a union of disjoint open annuli for $i = 1,2$. Denote $N_k(\gamma)$ by T_γ and call it the Margulis tube(s) corresponding to γ.
5. $S \times [0,1]$ and $S \times [2,3]$ are given the product structures corresponding to the bounded geometry structures on $S \times \{i\}$, for $i = 1,2$ respectively.
6. The lift to the universal cover of each component of $K \setminus N_k(\gamma)$ is quasiconvex.

Definition 3.11. *An end E of a hyperbolic 3-manifold M has* **amalgamated geometry** *if it is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing thick and amalgamated blocks end-to-end.*

Remark 3.12. *It turns out that i-bounded geometry is a special case of amalgamated geometry. The difference is that amalgamated geometry imposes relatively mild conditions on the geometry of the complement* $K - T_\gamma$. *The components of* $K - T_\gamma$ *shall be called* **amalgamation components** *of* K.

The figure below illustrates schematically what the model looks like. Filled squares correspond to solid tori along which amalgamation occurs. The adjoining piece(s) denote amalgamation blocks of K. The blocks which have no filled squares are the *thick blocks* and those with filled squares are the *amalgamated blocks*



Model of amalgamated geometry (schematic)

To construct the ladder $\mathcal{L}_\lambda$ we electrify amalgamation components as well as Margulis tubes. This ensures that in the electric metric,

(1) Each amalgamation block has bounded geometry
(2) The mapping class element taking $S \times \{1\}$ to $S \times \{2\}$ induces an isometry of the electrified metrics.

Quasiconvexity of $\mathcal{L}_\lambda$ in the electric metric now follows as before. To recover the data of hyperbolic geodesics from quasigeodesics lying close to $\mathcal{L}_\lambda$, we use (uniform) quasiconvexity of amalgamation components and existence of Cannon-Thurston maps follows.

4. Hierarchies and the Ending Lamination Theorem

We recapitulate the essential aspects of hierarchies and split geometry from [MM99, MM00, Min10]. The exposition here follows [Mj14] in part.

4.1. **Hierarchies.** We fix some notation first:

- $\xi(S_{g,b}) = 3g + b$ is the *complexity* of a compact surface $S = S_{g,b}$ of genus g and b boundary components

- For an essential subsurface Y of S, $\mathcal{C}(Y)$ will be its curve complex and $\mathcal{P}(Y)$ its pants complex.
- γ_α will be a collection of disjoint simple closed curves on S corresponding to a simplex $\alpha \in \mathcal{C}(Y)$
- α, β in $\mathcal{C}(Y)$ fill an essential subsurface Y of S if all non-trivial non-peripheral curves in Y have essential intersection with at least one of γ_α or γ_β, where γ_α and γ_β are chosen to intersect minimally.
- Given α, β in $\mathcal{C}(S)$, form a regular neighborhood of $\gamma_\alpha \cup \gamma_\beta$, and fill in all disks and one-holed disks to obtain Y which is said to be *filled* by α, β.
- For an essential subsurface $X \subset Z$ let $\partial_Z(X)$ denote the *relative boundary* of X in Z, i.e. those boundary components of X that are non-peripheral in Z.
- A **pants decomposition** of a compact surface S, possibly with boundary, is a disjoint collection of 3-holed spheres $P_1, \cdots, P_n$ embedded in S such that $S \setminus \bigcup_i P_i$ is a disjoint collection of non-peripheral annuli in S, no two of which are homotopic.
- A **tube** in an end $E \subset N$ is a regular $R-$neighborhood $N(\gamma, R)$ of an unknotted geodesic γ in E.

Definition 4.1. Tight Geodesics and Component Domains Let Y be an essential subsurface in S. If $\xi(Y) > 4$, a **tight** sequence of simplices $\{v_i\}_{i \in \mathcal{I}} \subset \mathcal{C}(Y)$ (where $\mathcal{I}$ is a finite or infinite interval in $\mathbb{Z}$) satisfies the following:
1) For any vertices w_i of v_i and w_j of v_j where $i \neq j$, $d_{\mathcal{C}(Y)}(w_i, w_j) = |i - j|$,
2) For $\{i-1, i, i+1\} \subset \mathcal{I}$, v_i equals $\partial_Y F(v_{i-1}, v_{i+1})$.
If $\xi(Y) = 4$ then a tight sequence is the vertex sequence of a geodesic in $\mathcal{C}(Y)$.
A *tight geodesic* g in $\mathcal{C}(Y)$ consists of a tight sequence $v_0, \cdots, v_n$, and two simplices in $\mathcal{C}(Y)$, $\mathbf{I} = \mathbf{I}(g)$ and $\mathbf{T} = \mathbf{T}(g)$, called its initial and terminal markings such that v_0 (resp. v_n) is a sub-simplex of $\mathbf{I}$ (resp. $\mathbf{T}$). The length of g is n. v_i is called a simplex of g. Y is called the *domain or support* of g and is denoted as $Y = D(g)$. g is said to be supported in $D(g)$.

For a surface W with $\xi(W) \geq 4$ and v a simplex of $\mathcal{C}(W)$ we say that Y is a *component domain of* (W, v) if Y is a component of $W \setminus \mathbf{collar}(v)$, where $\mathbf{collar}(v)$ is a small tubular neighborhood of the simple closed curves.

If g is a tight geodesic with domain $D(g)$, we call $Y \subset S$ a *component domain of* g if for some simplex v_j of g, Y is a component domain of $(D(g), v_j)$.

Definition 4.2. Hierarchies
A **hierarchy path** in $\mathcal{P}(S)$ joining pants decompositions P_1 and P_2 is a path $\rho : [0, n] \to P(S)$ joining $\rho(0) = P_1$ to $\rho(n) = P_2$ such that
1) There is a collection $\{Y\}$ of essential, non-annular subsurfaces of S, called component domains for ρ, such that for each component domain Y there is a connected interval $J_Y \subset [0, n]$ with $\partial Y \subset \rho(j)$ for each $j \in J_Y$.
2) For a component domain Y, there exists a tight geodesic g_Y supported in Y such that for each $j \in J_Y$, there is an $\alpha \in g_Y$ with $\alpha \in \rho(j)$.
A **hierarchy path** in $\mathcal{P}(S)$ is a sequence $\{P_n\}_n$ of pants decompositions of S such that for any $P_i, P_j \in \{P_n\}_n$, $i \leq j$, the finite sequence $P_i, P_{i+1}, \cdots, P_{j-1}, P_j$ is a hierarchy path joining pants decompositions P_i and P_j.
The collection H of tight geodesics g_Y supported in component domains Y of ρ will be called the **hierarchy** of tight geodesics associated to ρ.

Definition 4.3. A **slice** of a hierarchy H associated to a hierarchy path ρ is a set τ of pairs (h, v), where $h \in H$ and v is a simplex of h, satisfying the following properties:

1. A geodesic h appears in at most one pair in τ.
2. There is a distinguished pair (h_τ, v_τ) in τ, called the bottom pair of τ. We call h_τ the bottom geodesic.
3. For every $(k, w) \in \tau$ other than the bottom pair, $D(k)$ is a component domain of $(D(h), v)$ for some $(h, v) \in \tau$.

A **resolution** of a hierarchy H associated to a hierarchy path $\rho : I \to \mathcal{P}(S)$ is a sequence of slices $\tau_i = \{(h_{i1}, v_{i1}), (h_{i2}, v_{i2}), \cdots, (h_{in_i}, v_{in_i})\}$ (for $i \in I$, the same indexing set) such that the set of vertices of the simplices $\{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ is the same as the set of the non-peripheral boundary curves of the pairs of pants in $\rho(i) \in \mathcal{P}(S)$.

In [Min10], Minsky constructs a model manifold M_ν associated to end-invariants ν. $M_\nu[0]$ denotes M_ν minus the collection of Margulis tubes and horoball neighborhoods of cusps. $M_\nu[0]$ is built up as a union of standard 'blocks' of a finite number of topological types.

Definition 4.4. Minsky Blocks *(Section 8.1 of* [Min10]*)*
A tight geodesic in H supported in a component domain of complexity 4 is called a 4-geodesic and an edge of a 4-geodesic in H is called a 4-edge.

Given a 4-edge e in H, let g be the 4-geodesic containing it, and let $D(e)$ be the domain $D(g)$. Let e^- and e^+ denote the initial and terminal vertices of e. Also **collar** v *denotes a small tubular neighborhood of v in $D(e)$.*

To each e a Minsky block $B(e)$ is assigned as as follows:

$$B(e) = (D(e)\times[-1,1])\setminus(\mathbf{collar}(e^-)\times[-1,-1/2)\cup\mathbf{collar}(e^+)\times(1/2,1]).$$

That is, $B(e)$ is the product $D(e)\times[-1,1]$, with solid-torus trenches dug out of its top and bottom boundaries, corresponding to the two vertices e^- and e^+ of e.

The horizontal boundary *of $B(e)$ is*

$$\partial_\pm B(e) \equiv (D(e)\setminus \mathbf{collar}(e^\pm))\times\{\pm 1\}.$$

The horizontal boundary is a union of three-holed spheres. The rest of the boundary is a union of annuli and is called the vertical boundary. The top (resp. bottom) horizontal boundaries of $B(e)$ are $(D(e)\setminus \mathbf{collar}(e^+))\times\{1\}$ (resp. $(D(e)\setminus \mathbf{collar}(e^-))\times\{-1\}$.

The blocks constructed this way are the *internal blocks* of [Min10]. A related set of blocks are also constructed to form *boundary blocks* which figure in the construction of manifolds corresponding to simply degenerate surface groups.

One of the main aims of [Min10] is to construct a model manifold using these building blocks. A model $M_\nu[0]$ is constructed in [Min10] by taking the disjoint union of all the Minsky blocks and identifying them along three-holed spheres in their gluing boundaries. The rule is that whenever two blocks B and B' have the same three-holed sphere Y appearing in both ∂^+B and ∂^-B', these boundaries are identified using the identity on Y. The hierarchy serves to organize these gluings and insure that they are consistent.

The following theorem is then shown in [Min10].

Theorem 4.5. *(Theorem 8.1 of Minsky* [Min10]*) $M_\nu[0]$ admits a proper flat orientation-preserving embedding $\Psi : M_\nu[0]\to S\times\mathbb{R}$.*

4.1.1. *Tori and Meridinal Coefficients.* Let T be the boundary of a Margulis tube in $M_\nu[0]$. The boundary of a Margulis tube has the structure of a Euclidean torus and gives a unique point ω_T in the upper half plane, the Teichmuller space of the torus. The real and imaginary components of ω_T have a geometric interpretation.

Suppose that the Margulis tube T corresponds to a vertex $v\in\mathcal{C}(S)$. Let tw_T be the signed length of the annulus geodesic corresponding to v, i.e. it counts with sign the number of Dehn twists about the curve represented by v. Next, note that by the construction of the Minsky model, the **vertical** boundary of T consists of two sides - the **left vertical boundary** and **right vertical boundary**. Each is attached to vertical boundaries of Minsky blocks. Let the total number of blocks whose vertical boundaries, are glued to the vertical boundary of T be n_T. Similarly, let the total number of blocks whose vertical boundaries, are glued to the left (resp. right) vertical boundary of T be n_{Tl} (resp. n_{Tr}) so that $n_T = n_{Tl} + n_{Tr}$.

In Section 9 of [Min10], Minsky shows:

Theorem 4.6. [Min10] *There exists $C_0 \geq 0$, such that the following holds.*
$\omega - (tw_T + in_T) \leq C_0$

Definition 4.7. *The union of $M_\nu[0]$ as in Theorem 4.5 and model Margulis tubes as above is the* **Minsky model**.

In [BCM12], Brock-Canary-Minsky show that the Minsky model manifold built in [Min10] is in fact bi-Lipschitz homeomorphic to the hyperbolic manifold with the same end-invariants.

Theorem 4.8. Bi-Lipschitz Model Theorem:
Given a surface S of genus g, there exists L (depending only on g) such that for any doubly degenerate hyperbolic 3-manifold M without accidental parabolics homotopy equivalent to S, there exists an L-biLipschitz map from M to the Minsky Model for M.

Once the bi-Lipschitz model theorem 4.8 is in place, the ending lamination theorem is an immediate consequence of Sullivan's theorem [Sul81] as before:

Theorem 4.9. Ending Lamination Theorem [BCM12]**:**
Let M_i $(i = 1, 2)$ be two doubly hyperbolic 3-manifolds homeomorphic to $S \times \mathbb{R}$ with the same ending laminations. Then M_1 and M_2 are isometric.

5. Split Geometry

We come now to the last model geometry of this paper and indicate how it is used to apply the bi-Lipschitz model Theorem 4.8 to prove the existence of Cannon-Thurston maps in general.

5.1. **Split level Surfaces.** Let E be a degenerate end of a hyperbolic 3-manifold N. Let $\mathcal{T}$ denote a collection of disjoint, uniformly separated tubes in ends of N such that

(1) All Margulis tubes in E belong to $\mathcal{T}$.

(2) there exists $\epsilon_0 > 0$ such that the injectivity radius $injrad_x(E) > \epsilon_0$ for all $x \in E \setminus \bigcup_{T\in\mathfrak{T}} Int(T)$.

Let $(Q, \partial Q)$ be the unique hyperbolic pair of pants such that each component of ∂Q has length one. Q will be called the *standard* pair of pants. An isometrically embedded copy of $(Q, \partial Q)$ in $(M(0), \partial M(0))$ will be said to be *flat*.

Definition 5.1. A **split level surface** associated to a pants decomposition $\{Q_1, \cdots, Q_n\}$ of a compact surface S (possibly with boundary) in $M(0) \subset M$ is an embedding $f : \cup_i(Q_i, \partial Q_i) \to (M(0), \partial M(0))$ such that
1) Each $f(Q_i, \partial Q_i)$ is flat
2) f extends to an embedding (also denoted f) of S into M such that the interior of each annulus component of $f(S \setminus \bigcup_i Q_i)$ lies entirely in $F(\bigcup_{T\in\mathfrak{T}} Int(T))$.

Let S_i^s denote the union of the collection of flat pairs of pants in the image of the embedding S_i. Note that $S_i \setminus S_i^s$ consists of annuli properly embedded in Margulis tubes.

The class of *all* topological embeddings from S to M that agree with a split level surface f associated to a pants decomposition $\{Q_1, \cdots, Q_n\}$ on $Q_1 \cup \cdots \cup Q_n$ will be denoted by $[f]$.

We define a partial order $\leq_E$ on the collection of split level surfaces in an end E of M as follows:
$f_1 \leq_E f_2$ if there exist $g_i \in [f_i]$, $i = 1, 2$, such that $g_2(S)$ lies in the unbounded component of $E \setminus g_1(S)$.

A sequence S_i of split level surfaces is said to exit an end E if $i < j$ implies $S_i \leq_E S_j$ and further for all compact subsets $B \subset E$, there exists $L > 0$ such that $S_i \cap B = \emptyset$ for all $i \geq L$.

Definition 5.2. *A curve v in $S \subset E$ is* l**-thin** *if the core curve of the Margulis tube $T_v(\subset E \subset N)$ has length less than or equal to l. A tube $T \in \mathfrak{T}$ is l-thin if its core curve is l-thin. A tube $T \in \mathfrak{T}$ is l-thick if it is not l-thin.*
A curve v is said to split a pair of split level surfaces S_i and S_j $(i < j)$ if v occurs as a boundary curve of both S_i and S_j. A pair of split level surfaces S_i and S_j $(i < j)$ is said to be an l**-thin pair** *if there exists an l-thin curve v splitting both S_i and S_j.*

The collection of all l-thin tubes is denoted as $\mathfrak{T}_l$. The union of all l-thick tubes along with $M(0)$ is denoted as $M(l)$.

Definition 5.3. *A pair of split level surfaces S_i and S_j $(i < j)$ is said to be* k-**separated** *if*
a) for all $x \in S_i^s$, $d_M(x, S_j^s) \geq k$
b) Similarly, for all $x \in S_j^s$, $d_M(x, S_i^s) \geq k$.

As a consequence of the bi-Lipschitz Model Theorem 4.8 we have the following:

Theorem 5.4. [Min10] [BCM12] *Let N be the convex core of a simply or doubly degenerate hyperbolic 3-manifold minus an open neighborhood of the cusp(s). Let S be a compact surface, possibly with boundary, such that N is homeomorphic to $S \times [0, \infty)$ or $S \times \mathbb{R}$ according as N is simply or doubly degenerate. There exist $L \geq 1$, $\theta, \omega, \epsilon, \epsilon_1 > 0$, a collection $\mathcal{T}$ of (θ, ω)-thin tubes containing all Margulis tubes in N, a 3-manifold M, and an L-bilipschitz homeomorphism $F : N \to M$ such that the following holds.*
Let $M(0) = F(N \setminus \bigcup_{T \in \mathcal{T}} Int(T))$ and let $F(\mathcal{T})$ denote the image of the collection $\mathcal{T}$ under F. Let $\leq_E$ denote the partial order on the collection of split level surfaces in an end E of M. Then there exists a sequence S_i of split level surfaces associated to pants decompositions P_i exiting E such that

1. *$S_i \leq_E S_j$ if $i \leq j$.*
2. *The sequence $\{P_i\}$ is a hierarchy path in $\mathcal{P}(S)$.*
3. *If $P_i \cap P_j = \{Q_1, \cdots Q_l\}$ then $f_i(Q_k) = f_j(Q_k)$ for $k = 1 \cdots l$, where f_i, f_j are the embeddings defining the split level surfaces S_i, S_j respectively.*
4. *For all i, $P_i \cap P_{i+1} = \{Q_{i,1}, \cdots Q_{i,l}\}$ consists of a collection of l pairs of pants, such that $S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l})$ has a single non-annular component of complexity 4. Further, there exists a Minsky block W_i and an isometric map G_i of W_i into $M(0)$ such that $f_i(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l})$ (resp. $f_{i+1}(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l}))$ is contained in the bottom (resp. top) gluing boundary of W_i.*
5. *For each flat pair of pants Q in a split level surface S_i there exists an isometric embedding of $Q \times [-\epsilon, \epsilon]$ into $M(0)$ such that the embedding restricted to $Q \times \{0\}$ agrees with f_i restricted to Q.*
6. *For each $T \in \mathcal{T}$, there exists a split level surface S_i associated to pants decompositions P_i such that the core curve of T is isotopic to a non-peripheral boundary curve of P_i. The boundary $F(\partial T)$ of $F(T)$ with the induced metric d_T from $M(0)$ is a Euclidean torus equipped with a product structure $S^1 \times S^1_v$, where any circle of the form $S^1 \times \{t\} \subset S^1 \times S^1_v$ is a round circle of unit length and*

is called a horizontal circle; and any circle of the form $\{t\} \times S^1_v$ *is a round circle of length* l_v *and is called a vertical circle.*

(7) *Let* g *be a tight geodesic other than the bottom geodesic in the hierarchy* H *associated to the hierarchy path* $\{P_i\}$*, let* $D(g)$ *be the support of* g *and let* v *be a boundary curve of* $D(g)$*. Let* T_v *be the tube in* $\mathcal{T}$ *such that the core curve of* T_v *is isotopic to* v*. If a vertical circle of* $(F(\partial T_v), d_{T_v})$ *has length* l_v *less than* $n\epsilon_1$*, then the length of* g *is less than* n*.*

Two consequences of Theorem 5.4 that are needed in [Mj14] are given below.

Lemma 5.5. [Mj14, Lemma 3.6] *Given* $l > 0$ *there exists* $n \in \mathbb{N}$ *such that the following holds.*
Let v *be a vertex in the hierarchy* H *such that the length of the core curve of the Margulis tube* T_v *corresponding to* v *is greater than* l*. Next suppose* $(h, v) \in \tau_i$ *for some slice* τ_i *of the hierarchy* H *such that* h *is supported on* Y*, and* D *is a component of* $Y \setminus \mathbf{collar}\, v$*. Also suppose that* $h_1 \in H$ *such that* D *is the support of* h_1*. Then the length of* h_1 *is at most* n*.*

Lemma 5.6. [Mj14, Lemma 3.7] *Given* $l > 0$ *and* $n \in \mathbb{N}$*, there exists* $L_2 \geq 1$ *such that the following holds:*
Let S_i, S_j $(i < j)$ *be split level surfaces associated to pants decompositions* P_i, P_j *such that*
a) $(j - i) \leq n$
b) $P_i \cap P_j$ *is a (possibly empty) pants decomposition of* $S \setminus W$*, where* W *is an essential (possibly disconnected) subsurface of* S *such that each component* W_k *of* W *has complexity* $\xi(W_k) \geq 4$*.*
c)For any k *with* $i < k < j$*, and* $(g_D, v) \in \tau_k$ *for* $D \subset W_i$ *for some* i*, no curve in* v *has a geodesic realization in* N *of length less than* l*.*
Then there exists an L_2*-bilipschitz embedding* $G : W \times [-1, 1] \to M$*, such that*
1) W *admits a hyperbolic metric given by* $W = Q_1 \cup \cdots \cup Q_m$ *where each* Q_i *is a flat pair of pants.*
2) $W \times [-1, 1]$ *is given the product metric.*
3) $f_i(P_i \setminus P_i \cap P_j) \subset W \times \{-1\}$ *and* $f_j(P_j \setminus P_i \cap P_i) \subset W \times \{1\}$*.*

5.2. **Split surfaces and weak split geometry.**

Definition 5.7. An L-bi-Lipschitz **split surface** in $M(l)$ associated to a pants decomposition $\{Q_1, \cdots, Q_n\}$ of S and a collection $\{A_1, \cdots, A_m\}$ of complementary annuli (not necessarily all of them) in S is an embedding $f : \cup_i Q_i \bigcup \cup_i A_i \to M(l)$ such that

1) the restriction $f : \cup_i(Q_i, \partial Q_i) \to (M(0), \partial M(0))$ is a split level surface
2) the restriction $f : A_i \to M(l)$ is an L-bi-Lipschitz embedding.
3) f extends to an embedding (also denoted f) of S into M such that the interior of each annulus component of $f(S \setminus (\cup_i Q_i \bigcup \cup_i A_i))$ lies entirely in $F(\bigcup_{T\in\mathcal{T}_l} Int(T))$.

A split level surface differs from a split surface in that the latter may contain bi-Lipschitz annuli in addition to flat pairs of pants. We denote split surfaces by Σ_i. Let Σ_i^s denote the union of the collection of flat pairs of pants and bi-Lipschitz annuli in the image of the split surface (embedding) Σ_i.

The next Theorem is one of the technical tools from [Mj14] and follows from combining the bi-Lipschitz model Theorem 5.4 with Lemmas 5.5 and 5.6.

Theorem 5.8. [Mj14, Theorem 4.8] *Let $N, M, M(0), S, F$ be as in Theorem 5.4 and E an end of M. For any l less than the Margulis constant, let $M(l) = \{F(x) : \mathrm{injrad_x}(N) \geq l\}$. Fix a hyperbolic metric on S such that each component of ∂S is totally geodesic of length one (this is a normalization condition). There exist $L_1 \geq 1$, $\epsilon_1 > 0$, $n \in \mathbb{N}$, and a sequence Σ_i of L_1-bilipschitz, ϵ_1-separated split surfaces exiting the end E of M such that for all i, one of the following occurs:*

1. *An l-thin curve splits the pair (Σ_i, Σ_{i+1}), i.e. the associated split level surfaces form an l-thin pair.*
2. *there exists an L_1-bilipschitz embedding*
$$G_i : (S \times [0,1], (\partial S) \times [0,1]) \to (M, \partial M)$$
such that $\Sigma_i^s = G_i(S \times \{0\})$ and $\Sigma_{i+1}^s = G_i(S \times \{1\})$

Finally, each l-thin curve in S splits at most n split level surfaces in the sequence $\{\Sigma_i\}$.

Pairs of split surfaces satisfying Alternative (1) of Theorem 5.8 will be called an **l-thin pair** of split surfaces (or simply a thin pair if l is understood). Similarly, pairs of split surfaces satisfying Alternative (2) of Theorem 5.8 will be called an **l-thick pair** (or simply a thick pair) of split surfaces.

Definition 5.9. *A model manifold satisfying the following conditions is said to have* **weak split geometry***:*

1. *A sequence of split surfaces S_i^s exiting the end(s) of M, where M is marked with a homeomorphism to $S \times J$ (J is $\mathbb{R}$ or $[0, \infty)$*

according as M is totally or simply degenerate). $S_i^s \subset S \times \{i\}$.

(2) *A collection of Margulis tubes $\mathcal{T}$ in N with image $F(\mathcal{T})$ in M (under the bilipschitz homeomorphism between N and M). We refer to the elements of $F(\mathcal{T})$ also as Margulis tubes.*

(3) *For each complementary annulus of S_i^s with core σ, there is a Margulis tube $T \in \mathcal{T}$ whose core is freely homotopic to σ such that $F(T)$ intersects S_i^s at the boundary. (What this roughly means is that there is an $F(T)$ that contains the complementary annulus.) We say that $F(T)$ splits S_i^s.*

(4) *There exist constants $\epsilon_0 > 0, K_0 > 1$ such that for all i, either there exists a Margulis tube splitting both S_i^s and S_{i+1}^s, or else $S_i (= S_i^s)$ and $S_{i+1} (= S_{i+1}^s)$ have injectivity radius bounded below by ϵ_0 and bound a* **thick block** *B_i, where a thick block is defined to be a K_0-bilipschitz homeomorphic image of $S \times I$.*

(5) *$F(T) \cap S_i^s$ is either empty or consists of a pair of boundary components of S_i^s that are parallel in S_i.*

(6) *There is a uniform upper bound $n = n(M)$ on the number of surfaces that $F(T)$ splits.*

Theorem 5.8 then gives:

Theorem 5.10. [Mj14] *Any degenerate end of a hyperbolic 3-manifold is bi-Lipschitz homeomorphic to a Minsky model and hence to a model of weak split geometry.*

5.2.1. *Split Blocks and Hanging Tubes.* We shall now define split geometry.

Definition 5.11. *Let $(\Sigma_i^s, \Sigma_{i+1}^s)$ be a thick pair of split surfaces in M. The closure of the bounded component of $M \setminus (\Sigma_i^s \cup \Sigma_{i+1}^s)$ between $\Sigma_i^s, \Sigma_{i+1}^s$ will be called a thick block.*

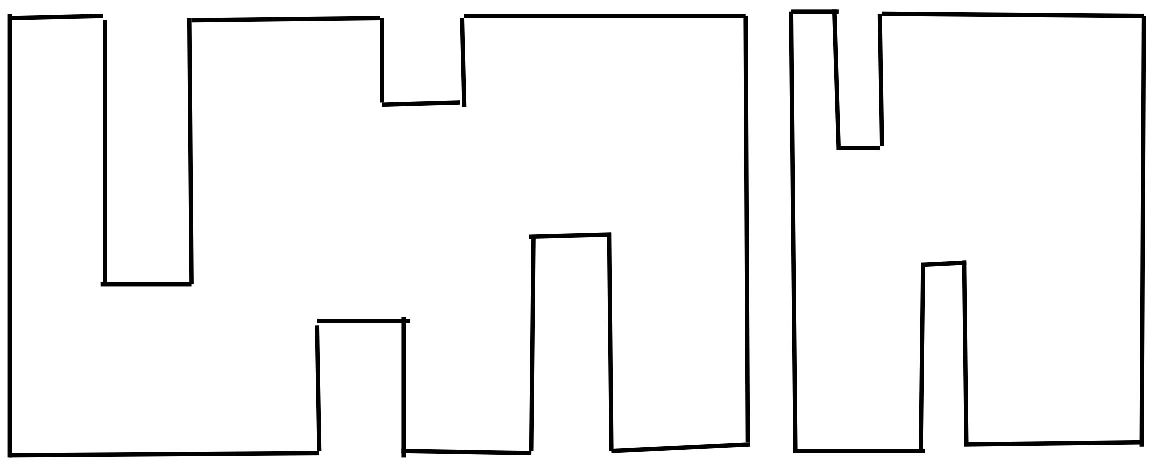
Note that a thick block is uniformly bi-Lipschitz to the product $S \times [0,1]$ and that its boundary components are $\Sigma_i^s, \Sigma_{i+1}^s$.

Definition 5.12. *Let $(\Sigma_i^s, \Sigma_{i+1}^s)$ be an l-thin pair of split surfaces in M and $F(\mathcal{T}_i)$ be the collection of l-thin Margulis tubes that split both $\Sigma_i^s, \Sigma_{i+1}^s$. The closure of the union of the bounded components of $M \setminus ((\Sigma_i^s \cup \Sigma_{i+1}^s) \bigcup_{F(T) \in F(\mathcal{T}_i)} F(T))$ between $\Sigma_i^s, \Sigma_{i+1}^s$ will be called a* **split block***.*

The closure of any bounded component is called a **split component**.

Each split component may contain Margulis tubes, which we shall call **hanging tubes** (see below) that *do not* split both $\Sigma_i^s, \Sigma_{i+1}^s$.

Topologically, a **split block** $B^s \subset B = S \times I$ is a topological product $S^s \times I$ for some *connected* S^s. However, the upper and lower boundaries of B^s need only be be split subsurfaces of S^s. This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**. See figure below:



Split Block with hanging tubes

The vertical lengths of *hanging tubes* are further required to be *uniformly bounded below by some* $\eta_0 > 0$. Further, each such annulus has cross section a round circle of length ϵ_0.

Definition 5.13. *Hanging tubes intersecting the upper (resp. lower) boundaries of a split block are called upper (resp. lower) hanging tubes.*

Electrifying split components, we obtain a new electric metric called the **graph metric** d_G on E.

Definition 5.14. *A model of weak split geometry is said to be of* **split geometry** *if the convex hull of each split component has uniformly bounded d_G−diameter. Equivalently we say that split components are* **uniformly graph quasiconvex**.

A key technical Proposition of [Mj14] asserts:

Proposition 5.15. *For E a degenerate end, split components are uniformly graph quasiconvex.*

As an immediate consequence of Theorem 5.10 and Proposition 5.15 we have the following.

Theorem 5.16. [Mj14] *Any degenerate end of a hyperbolic 3-manifold is bi-Lipschitz homeomorphic to a Minsky model and hence to a model of weak split geometry.*

Once it is established that M has split geometry, the proof of the existence of Cannon-Thurston maps proceeds as for amalgamation geometry by electrifying split components, constructing a hyperbolic ladder $\mathcal{L}_\lambda$ and finally recovering a hyperbolic geodesic from an electric one. We summarize this as follows.

Theorem 5.17. [Mj14] *Let $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ be a simply or doubly degenerate surface Kleinian group. Then a Cannon-Thurston map exists.*

References

[Ago04] I. Agol. Tameness of hyperbolic 3-manifolds. *preprint, arXiv:math.GT/0405568*, 2004.

[BCM12] J. F. Brock, R. D. Canary, and Y. N. Minsky. The Classification of Kleinian surface groups II: The Ending Lamination Conjecture. *Ann. of Math. 176 (1), arXiv:math/0412006*, pages 1–149, 2012.

[Bon86] F. Bonahon. Bouts de varietes hyperboliques de dimension 3. *Ann. Math. vol.124*, pages 71–158, 1986.

[Bow07] B. H. Bowditch. The Cannon-Thurston map for punctured surface groups. *Math. Z. 255*, pages 35–76, (2007).

[Bow15] B. H. Bowditch. The ending lamination theorem. *preprint, Warwick*, 2015.

[Can93] R. D. Canary. Ends of hyperbolic 3 manifolds. *J. Amer. Math. Soc.*, pages 1–35, 1993.

[CG06] D. Calegari and D. Gabai. Shrink-wrapping and the Taming of Hyperbolic 3-manifolds. *J. Amer. Math. Soc. 19, no. 2*, pages 385–446, 2006.

[FLP79] A. Fathi, M. Laudenbach, and V. Poenaru. Travaux de Thurston sur les surfaces. *Asterisque 66-67*, pages 1–284, 1979.

[Kla99] E. Klarreich. Semiconjugacies between Kleinian group actions on the Riemann sphere. *Amer. J. Math 121*, pages 1031–1078, 1999.

[McM01] C. T. McMullen. Local connectivity, Kleinian groups and geodesics on the blow-up of the torus. *Invent. math.*, 97:95–127, 2001.

[Min92] Y. N. Minsky. Teichmuller Geodesics and Ends of 3-Manifolds. *Topology*, pages 1–25, 1992.

[Min94] Y. N. Minsky. On Rigidity, Limit Sets, and End Invariants of Hyperbolic 3-Manifolds. *J. Amer.Math.Soc., vol.7*, pages 539–588, 1994.

[Min99] Y. N. Minsky. The Classification of Punctured Torus Groups. *Ann. of Math.149*, pages 559–626, 1999.

[Min01] Y. N. Minsky. Bounded geometry in Kleinian groups. *Invent. Math. 146*, pages 143–192, 2001.

[Min10] Y. N. Minsky. The Classification of Kleinian surface groups I: Models and Bounds. *Ann. of Math. 171(1), math.GT/0302208*, pages 1–107, 2010.

[Mit98] M. Mitra. Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces. *Jour. Diff. Geom.48*, pages 135–164, 1998.

[Miy02] H. Miyachi. Semiconjugacies between actions of topologically tame Kleinian groups. *preprint*, 2002.

[Mj09] M. Mj. Cannon-Thurston Maps for Pared Manifolds of Bounded Geometry. *Geom. Topol. 13; arXiv:math.GT/0503581*, pages 189–245, 2009.

[Mj10] M. Mj. Cannon-Thurston Maps and Bounded Geometry. *Teichmuller theory and moduli problem, Ramanujan Math. Soc. Lect. Notes Ser., 10, Ramanujan Math. Soc., Mysore, arXiv:math.GT/0603729*, pages 489–511, 2010.

[Mj11] M. Mj. Cannon-Thurston Maps, i-bounded Geometry and a Theorem of McMullen. *Actes du séminaire Théorie spectrale et géométrie, Grenoble, vol 28, 2009-10, arXiv:math.GT/0511104*, pages 63–108, 2011.

[Mj14] M. Mj. Cannon-Thurston Maps for Surface Groups. *Ann. of Math., 179(1)*, pages 1–80, 2014.

[Mj16] M. Mj. Cannon-Thurston Maps for Surface Groups: An Exposition of Amalgamation Geometry and Split Geometry. *Geometry, Topology, and Dynamics in Negative Curvature, London Mathematical Society Lecture Note Series volume 425, arXiv:math.GT/0512539*, pages 221–271, 2016.

[MM99] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Invent. Math.138*, pages 103–139, 1999.

[MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hierarchical structure. *Geom. Funct. Anal. 10*, pages 902–974, 2000.

[Sul81] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stonybrook Conference, Ann. of Math. Studies 97, Princeton*, 1981.

[Thu80] W. P. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes*, 1980.

[Thu82] W. P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.*, pages 357–382, 1982.

[Thu86] W. P. Thurston. Hyperbolic Structures on 3-Manifolds,II: Surface Groups and 3-Manifolds which Fiber over the Circle. *preprint, arXiv:math.GT/9801045*, 1986.

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