TIGHT TREES AND MODEL GEOMETRIES OF SURFACE BUNDLES OVER GRAPHS

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Abstract. We generalize the notion of tight geodesics in the curve complex to tight trees. We then use tight trees to construct model geometries for certain surface bundles over graphs. This extends some aspects of the combinatorial model for doubly degenerate hyperbolic 3-manifolds developed by Brock, Canary and Minsky during the course of their proof of the Ending Lamination Theorem. Thus we obtain uniformly Gromov-hyperbolic geometric model spaces equipped with geometric $G-$actions, where G admits an exact sequence of the form

$$1 \to \pi_1(S) \to G \to Q \to 1.$$

Here S is a closed surface of genus $g > 1$ and Q belongs to a special class of free convex cocompact subgroups of the mapping class group $MCG(S)$.

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1. Introduction

A combinatorial model for doubly degenerate hyperbolic 3-manifolds was developed by Brock, Canary and Minsky in [Min10, BCM12] during the course of their proof of the Ending Lamination Theorem. The combinatorial machinery guiding the construction of the combinatorial model in [Min10, BCM12] is based on the technology of hierarchy paths developed by Masur and Minsky in [MM99, MM00]. Let $\mathcal{C}(S)$ denote the curve complex of a closed surface S. Then the boundary $\partial\mathcal{C}(S)$ consists of the ending laminations $\mathcal{EL}(S)$ [Kla99]. For a pair of ending laminations $\mathcal{L}_\pm \in \partial\mathcal{C}(S) = \mathcal{EL}(S)$, let γ be a tight geodesic in the curve complex $\mathcal{C}(S)$ joining $\mathcal{L}_\pm$. A hierarchy of paths joining $\mathcal{L}_\pm$ is then constructed in [MM00, Min10] with γ as the base tight geodesic. The hierarchy forms the combinatorial backbone for the model (see also [Bow16, Ohs98] for some alternate treatments).

Convex cocompact subgroups of the mapping class group: We shall extend some aspects of the combinatorial model to treat a class of free convex cocompact subgroups of the mapping class group $MCG(S)$. A subgroup Q of $MCG(S)$ is said to be **convex cocompact** [FM02] if some orbit of Q in the Teichmüller space Teich(S) is quasiconvex. We shall say that Q is $K-$convex cocompact if the weak hull of the limit set of Q quotiented by Q has diameter at most K; equivalently some Q orbit is $K-$quasiconvex.

Associated to any $Q \subset MCG(S)$, there is an exact sequence [FM02, Section 1.2] of the form

$$1 \to \pi_1(S) \to G \to Q \to 1.$$

It follows from work of Farb-Mosher [FM02] Hamenstadt [Ham05] and Kent-Leininger [KL08] that the following are equivalent:

(1) Q is convex cocompact,
(2) the extension G occurring in the above exact sequence is hyperbolic (see also [MS12] for an extension to surfaces with punctures),
(3) Any orbit of Q in $\mathcal{C}(S)$ is qi-embedded.

Our principal aim in this paper is to construct uniformly Gromov-hyperbolic geometric model spaces equipped with geometric $G-$actions, where G is as above and Q belongs to a special class of free convex cocompact subgroups of the mapping class group $MCG(S)$.

Identifying Q with an orbit in $\mathcal{C}(S)$, the Gromov boundary ∂Q of Q can be canonically identified with a Cantor set in $\mathcal{EL}(S)$ as well as in the Thurston boundary

$\mathcal{PML}(S) = \partial\mathrm{Teich}(S)$ of Teichmüller space. In order to construct a model space for G, we shall first need to construct tight geodesics and a hierarchy of paths for every pair of points p, q in $\partial(Q)$.

Model geometries: A crucial issue that arises in the process is to check for consistency: When two such tight geodesics γ_1, (resp. γ_2) joining p_1, q_1 (resp. p_2, q_2) cross at a vertex v then the hierarchy paths joining p_1, q_1 (resp. p_2, q_2) subordinate to v need to be consistent. This is one of the new and somewhat subtle features that appears when ∂Q is a Cantor set as opposed to the case where $Q = \mathbb{Z}$ and ∂Q has exactly two points. There are two cases in which we can handle this problem corresponding to the following two model geometries of doubly degenerate 3-manifolds:

1. bounded geometry [Min01],
2. a special case of the split geometry model investigated in [Mj14, Mj16].

1.1. **Statement of results.** The first case that we address is that of **bounded geometry,** where we assume that there exists $\epsilon > 0$ such that for all $p, q \in \partial Q \subset \partial\mathrm{Teich}(S)$, the Teichmüller geodesic joining p, q lies in the $\epsilon-$thick part of Teichmüller space. Suppose now that Q is free. Let Γ_Q be a Cayley graph of Q with respect to a free generating set and $\Phi : \Gamma_Q \to Teich(S)$ be a piecewise geodesic equivariant map. The pull-back of the universal bundle to Γ_Q will be denoted as $M_{Q,\Phi}$. Then (see Proposition 5.9) we have:

Proposition 1.1. *Given $K, \epsilon \geq 0$, there exists $\delta > 0$ such that the following holds: Let Q be a free $K-$convex cocompact subgroup and let $o \in Teich(S)$ with $Q.o \subset Teich_\epsilon(S)$. There exists $\Phi : \Gamma_Q \to Teich(S)$ such that the universal cover $\widetilde{M_{Q,\Phi}}$ is $\delta-$hyperbolic.*

Generalizing the notion of a tight geodesic from [MM99, MM00], we say that a simplicial map $i : T \to \mathcal{C}(S)$ from a (not necessarily regular) simplicial tree T of bounded valence defines an $L-$**tight tree of non-separating curves** if for every vertex v of T, $i(v)$ is non-separating, and for every pair of distinct vertices $u \neq w$ adjacent to v in T,

$$d_{\mathcal{C}(S\setminus i(v))}(i(u), i(w)) \geq L.$$

The following (see Proposition 2.10, essentially due to Bromberg) shows that $L-$ tight trees are isometrically embedded.

Proposition 1.2. *There exists $L \geq 3$,such that the following holds. Let S be a closed surface of genus at least 3, and let $i : T \to \mathcal{C}(S)$ define an L–tight tree of non-separating curves. Then i is an isometric embedding.*

Let $i : T \to \mathcal{C}(S)$ be a tight tree of non-separating curves and let v be a vertex of T. The link of v in T is denoted as $lk(v)$. Let $W_v = S \setminus i(v)$. Then $i(lk(v))$ consists of a uniformly bounded number of vertices in $\mathcal{C}(W_v)$. Hence the weak convex hull $CH(i(lk(v))$ of $i(lk(v))$ in $\mathcal{C}(W_v)$ admits a uniform approximating tree T_v. We refer to T_v as the **tree-link** of v. The **blow-up** $\mathrm{BU}(T)$ of T is a **metric tree** obtained from T by replacing the $\frac{1}{2}-$neighborhood of each $v \in T$ by the tree-link T_v.

An $L-$tight tree is said to be $R-$**thick** if for any vertices u, v, w of T and any proper essential subsurface W of $S \setminus i(v)$ (including essential annuli),

$$d_W((i(u), i(w)) \leq R,$$

where $d_W(\cdot\,,\cdot)$ denotes distance in $\mathcal{C}(W)$ between subsurface projections onto W. For an $L-$tight, $R-$thick tree T we construct a bundle $P : M_T \to \mathrm{BU}(T)$ over the blow-up $\mathrm{BU}(T)$ of T. M_T will take the place of the model manifold of [Min10]. The pre-image $P^{-1}(T_v)$ will be called the **building block** corresponding to v and will be denoted as M_v. Inside every M_v, there is a natural copy of $S^1 \times T_v$ corresponding to the simple closed curve $i(v) \subset S$. We refer to it as the **Margulis riser** corresponding to v and denote it by $\mathcal{R}_v$. Margulis risers in M_T take the place of Margulis tubes in hyperbolic 3-manifolds. (The terminology "riser" is borrowed from [MMS19] where they form parts of tracks). One can think of the geometry of Margulis tubes in [Min10] as a consequence of performing hyperbolic Dehn surgery on a thickened neighborhood of Margulis risers. Equivalently, one thickens the Margulis risers, removes the interior and performs hyperbolic Dehn filling. For convex cocompact free subgroups of $MCG(S)$, there is no such canonical filling. Thus Margulis risers are the best replacement we could find for Margulis tubes.

For l a bi-infinite geodesic in T, let $l_\pm$ denote the ending laminations given by the ideal end-points of $i(l)$ in the boundary of $\mathcal{C}(S)$ and let N_l denote the doubly degenerate hyperbolic 3-manifold with ending laminations $l_\pm$. We denote the vertices of T occurring along l by $\mathcal{V}(l)$. If L is large enough, then each $i(v)$ gives a Margulis tube $\mathbf{T}_v$ in N_l. Let $N_l^0 = N_l \setminus \bigcup_{v\in\mathcal{V}(l)} \mathbf{T}_v$.

Let $\mathrm{BU}(l)$ denote the bi-infinite geodesic in $\mathrm{BU}(T)$ after blowing up l in T. Also let M_l denote the bundle over $\mathrm{BU}(l)$ induced from $\Pi : M_T \to \mathrm{BU}(T)$. Let $M_l^0 = M_l \setminus \bigcup_{v\in\mathcal{V}(l)} \mathcal{R}_v$.

Theorem 1.3. *(See Theorem 3.35) Given $R \geq 0$, there exist $K \geq 1, e > 0$ such that if $i : T \to \mathcal{C}(S)$ is an $L-$tight $R-$thick tree of non-separating curves, then the following holds:*
There exists a metric d_{weld} on M_T such that $P : M_T \to \mathrm{BU}(T)$ satisfies the following properties:

1. *The induced metric on a Margulis riser $\mathcal{R}_v$ is the metric product $S^1_e \times T_v$, where S^1_e is a round circle with radius e.*
2. *For any bi-infinite geodesic l in T, N_l^0 and M_l^0 are $K-$bi-Lipschitz homeomorphic.*
3. *Further, if there exists a subgroup Q of $MCG(S)$ acting cocompactly and geometrically on $i(T)$, then this action can be lifted to an isometric fiber-preserving isometric action of Q on (M_T, d_{weld}).*
4. *$P : (M_T, d_{weld}) \to \mathrm{BU}(T)$ is uniformly proper.*

The universal cover $(\widetilde{M_T}, d_{weld})$ contains flat strips $\mathbb{R} \times T_v$ coming from the universal covers of the Margulis risers $\mathcal{R}_v = S^1_e \times T_v$. We show that this is the only obstruction to effectively hyperbolizing $\widetilde{M_T}$. Equip each $\mathcal{R}_v$ with a product pseudometric that is zero on the first factor S^1 and agrees with the metric on T_v on the second. This replacement of a product metric by a pseudo-metric is called partial electrification in [MR08] and in the specific context of Margulis tubes, it is called tube-electrification in [Mj14]. The resulting pseudometric on M_T is denoted as d_{te}. The main Theorem of the paper is the following (see Theorem 3.36):

Theorem 1.4. *Given R, there exists δ such that if $i : T \to \mathcal{C}(S)$ is an $L-$tight $R-$thick tree of non-separating curves, then $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic. Further, $(\widetilde{M_T}, d_{weld})$ is strongly $\delta-$hyperbolic relative to the collection $\widetilde{\mathcal{R}}$ of lifts of Margulis risers.*

A coarse version of the model theorem of [Min10, BCM12] would say that there exists $\delta > 0$ such that the model geometry M corresponding to any doubly degenerate hyperbolic manifold satisfies the property that $\widetilde{M}$ is $\delta-$hyperbolic. Let $\widetilde{\mathcal{T}}$ denote the collection of lifts of models of Margulis tubes to $\widetilde{M}$. It follows that $\widetilde{M}$ is *strongly $\delta-$hyperbolic relative to* the collection $\widetilde{\mathcal{T}}$ (see Definition 4.11). The second statement of Theorem 1.4 above generalizes this statement to the coarse model (M_T, d_{weld}) for bundles over tight trees. In fact, the hypothesis on non-separating curves can be removed completely (Corollary 6.4) for the second statement. The first statement of Theorem 1.4 is finer and captures one of the parameters of model Margulis tubes, viz. the imaginary coefficient of model Margulis tubes in [Min10, BCM12]). For this statement, the hypothesis on non-separating curves can be relaxed somewhat (see Definition 2.18 and Theorem 3.36) but cannot be removed altogether (see the examples in Section 6.2).

Steps of the proof and technical issues: Theorem 1.4 is an effective hyperbolization theorem for surface bundles over trees. The broad strategy is as follows:

(1) First, a geometric model is constructed for the bundle M_T over T with fiber S (see the discussion before Theorem 1.3 for a summary).
(2) For any bi-infinite geodesic l in T, we would have liked to show that the restriction M_l of the bundle M_T to l is uniformly bi-Lipschitz to the combinatorial model of [Min10] for a doubly degenerate hyperbolic 3-manifold. This is not quite true and the construction needs to be modified (see Item (1) of Theorem 1.3 above for a precise statement). We think of M_l as a bundle over a line.
(3) Use the converse to the Bestvina-Feighn combination theorem to extract effective and uniform flaring constants for the bundles M_l over lines.
(4) Feed the uniform flaring constants back into the bundle over T to obtain effective hyperbolization.

A number of difficulties arise in making the above strategy work as stated. We have already mentioned the consistency check that needs to be done when tight geodesics γ_1, γ_2 cross at a vertex v. We briefly elaborate on the difficulty alluded to in Item (2) above. In the case we shall be most interested in this paper, the vertex v will give rise to Margulis tubes $\mathbf{T}_1, \mathbf{T}_2$ in the doubly degenerate manifolds M_1, M_2 corresponding to γ_1, γ_2. It turns out that gluing the Margulis tubes $\mathbf{T}_1, \mathbf{T}_2$, even partially, in M_1, M_2 to construct a hyperbolic model over $\gamma_1 \cup \gamma_2$ is simply not possible. We can nevertheless partially glue the boundaries $\partial\mathbf{T}_1, \partial\mathbf{T}_2$. The precise process involved is a certain welding construction introduced by the author in [Mj14]. This construction, however, gives rise to flat strips obstructing effective and uniform hyperbolization of the bundle as mentioned before Theorem 1.4. To circumvent this, we tube-electrify the Margulis risers to finally obtain a uniformly hyperbolic pseudometric.

Outline of the paper: In Section 2, we introduce the notion of tight trees in $\mathcal{C}(S)$ and show that such trees T are necessarily isometrically embedded. For links of vertices in T, we describe a blowup construction: we replace a small neighborhood of a vertex v by an associated finite tree called a tree-link T_v. The topological building blocks for the model we construct later in the paper are of the form $M_v = S \times T_v$. The blown up tree is denoted as $\mathrm{BU}(T)$.

A geometric structure for the building blocks M_v is introduced in Section 3. Motivated by the model geometries of doubly degenerate hyperbolic 3-manifolds constructed by Minsky [Min01, Min94, Min10] and adapted in [Mj16, Mj14] we describe the model geometry of M_v. Assembling these together give us a metric d_{weld} on the bundle M_T over the blowup $\mathrm{BU}(T)$ of T. An auxiliary partially electrified version d_{te} of d_{weld} is also defined here.

In Section 4 we recall and adapt some basic technical tools that we require for the proof of Theorems 3.35 and 1.4 (see Theorem 3.36). We describe an effective version of the Bestvina-Feighn combination theorem and its converse for hyperbolic spaces. We also describe relatively hyperbolic analogs.

Uniform hyperbolicity of $(\widetilde{M_T}, d_{te})$ is established in Section 5.

2. Trees in the Curve Complex

The aim of this section is twofold:

(1) To use subsurface projections [MM00] to give a sufficient condition for an isometric embedding of a tree T in the curve complex $\mathcal{C}(S)$ (Lemma 2.6, Propositions 2.10 and 2.12).
(2) To describe the topological structure of building blocks M_v corresponding to vertices v of T. The main point here is to construct a blow up of the vertex v to a finite tree T_v, called the **tree-link** of v, and hence a blown-up tree $\mathrm{BU}(T)$ from T.

A remark on notation. We shall use $MCG(S)$ to denote the mapping class group of a closed surface S and $Mod(S)$ to denote its moduli space.

2.1. Subsurface projections.

The **complexity** of a surface Y of genus g with b boundary components is given by

$$\xi(Y) = 3g + b - 3.$$

The curve complex of Y is denoted as $\mathcal{C}(Y)$ and the arc-and-curve complex of Y is denoted as $\mathcal{AC}(Y)$. There is a coarsely defined 2–Lipschitz retraction ψ_Y from $\mathcal{AC}(Y)$ to $\mathcal{C}(Y)$, given by performing surgery using boundary curves [MM00, Lemma 2.2]. In particular for any arc $a \in \mathcal{AC}(Y)$, $\mathrm{dia}_Y(\psi_Y(a)) \leq 2$.

Definition 2.1. Let $Y \subset S$ be an essential proper subsurface. If $\gamma \in \mathcal{C}(S)$ can be homotoped to be disjoint from Y, define $\pi_Y(\gamma) = \emptyset$. If γ is homotopic to an essential curve in Y, then $\pi_Y(\gamma) = \gamma$. Else homotope γ to intersect ∂Y minimally. Then $\gamma \cap Y$ is a set of vertices of a simplex in $\mathcal{AC}(Y)$. Define

$$\pi_Y(\gamma) = \bigcup_i \psi_Y(a_i).$$

Definition 2.1 can be easily extended to laminations. For $\mathcal{L}$ a geodesic lamination on S, and Y an essential subsurface of S, let $\mathcal{L}|_Y = \mathcal{L} \cap W$. Then $\mathcal{L}|_Y$ gives an element of the arc-and-curve complex $\mathcal{AC}(Y)$ after identifying the arcs and closed curves of $\mathcal{L}|_Y$ with their relative isotopy classes (see [Min01, Section 2.2] for instance). By performing surgery on the arcs along boundary components of Y we obtain elements of $\mathcal{C}(Y)$. Hence $\pi_Y(\mathcal{L})$ may be defined as in Definition 2.1.

Definition 2.2. Let $Y \subset S$ be an essential subsurface with $\xi(Y) > 1$. For any collection of vertices $\mathcal{V}$ of $\mathcal{C}(S)$, define

$$\text{dia}_Y(\mathcal{V}) = \text{dia}_Y\left(\bigcup\{\pi_Y(v)|v \in \mathcal{V}\}\right).$$

If F is a subgraph of $\mathcal{C}(S)$, define

$$\text{dia}_Y(F) = \text{dia}_Y(F^{(0)}).$$

Finally, for X, Z proper essential subsurfaces of S, define

$$\text{dia}_Y(X, Z) = \text{dia}_Y(\partial X, \partial Z).$$

The same definitions work for laminations.

Theorem 2.3. (Bounded geodesic image theorem) [MM00, Theorem 3.1], [Web15, Corollary 1.3] *There exists $M > 0$ satisfying the following. Let S be a surface of finite type, and let $Y \subset S$ be an essential subsurface of complexity at least 2. Let γ be a (finite or infinite) geodesic segment in $\mathcal{C}(S)$ so that $\pi_Y(v) \neq \emptyset$ for every vertex v of γ. Then $\text{dia}_Y(\gamma) \leq M$.*

Note that M in Theorem 2.3 above is a universal constant.

Theorem 2.4. (Behrstock Inequality) [Beh06]**:** *For a surface S of finite type, there exists $D \geq 0$ such that for any three essential subsurfaces X, Y, Z of S,*

$$\min\{\text{dia}_Y(X, Z), \text{dia}_Z(X, Y)\} \leq D.$$

Two essential subsurfaces Y, Y' of S are said to fill S if there exists no simple closed curve in S that can be homotoped off Y as well as off Y'. For multicurves v, w on S, the subsurface filled by v, w is denoted as $F(v, w)$. We adapt the notion of a tight sequence from [MM00] below (and caution the reader that what we call a tight geodesic here is referred to as a tight sequence in [MM00]).

Definition 2.5. A sequence of multicurves $\{v_i\}$ is said to be a **tight geodesic** if:

(1) for any simple closed curve $\alpha_i \in v_i$ and $\alpha_j \in v_j$, $d_{\mathcal{C}(S)}(\alpha_i, \alpha_j) = |i - j|$.
(2) $v_i = \partial F(v_{i-1}, v_{i+1})$.

We shall now furnish a sufficient condition for proving that a sequence of multicurves is a tight geodesic. We are grateful to Ken Bromberg for telling us a proof of the following:

Lemma 2.6. *There exists $L \geq 3$ such that the following holds.*
Let $v_0, \cdots, v_n$ be a sequence of multicurves in $\mathcal{C}(S)$ such that

(1) *For all i, there exists an essential subsurface Y_i of S such that $\partial Y_i = v_i$ and $v_{i-1}, v_{i+1} \subset Y_i$.*
(2) *$d_{\mathcal{C}(Y)}(v_{i-1}, v_{i+1}) \geq L$.*

Then $\{v_0, \cdots, v_n\}$ is a tight geodesic.

Proof. First, observe that since $L \geq 3$, it follows that $\partial Y(v_{i-1}, v_{i+1}) = v_i$, i.e. v_{i-1}, v_{i+1} fill Y_i. Choosing $L \geq 5$, it follows that for any simple closed curves $\sigma_{i-1} \in v_{i-1}$ and $\sigma_{i+1} \in v_{i+1}$, $\sigma_{i-1}, \sigma_{i+1}$ fill Y_i. It suffices therefore to prove that for simple closed curves $\sigma_i \in v_i$, $\{\sigma_0, \cdots, \sigma_n\}$ is a geodesic. Choose $L \geq 4D + 1$ (where D is as in Theorem 2.4).

Recall the notation of Definition 2.2. We now use the Behrstock inequality Theorem 2.4 to show that if $i < j < k$ then $d_{Y_j}(v_i, v_k)$ is uniformly coarsely equal to $d_{Y_j}(v_{j-1}, v_{j+1})$. More precisely for D as in Theorem 2.4, for $i < j < k$,

$$|d_{Y_j}(v_i, v_k) - d_{Y_j}(v_{j-1}, v_{j+1})| \leq 2D.$$

We argue by induction. Assuming that the statement is true for $k \leq m$ we shall show that if $i < j < m+1$ then the statement holds. By induction $d_{Y_{j-1}}(v_i, v_j)$ is coarsely (up to an additive $2D$) equal to $d_{Y_{j-1}}(v_{j-2}, v_j) \geq L$ (by hypothesis). Hence $d_{Y_{j-1}}(v_i, v_j) \geq L - 2D \geq 2D + 1$. By Theorem 2.4 this means that $d_{Y_j}(v_i, v_{j-1})$ is uniformly small, bounded by D (for $i = j-1$ this is trivial.) Similarly we have $d_{Y_j}(v_{j+1}, v_{m+1})$ is uniformly small, bounded by D. Hence by the triangle inequality if $i < j < m+1$,

$$|d_{Y_j}(v_i, v_{m+1}) - d_{Y_j}(v_{j+1}, v_{j-1})| \leq 2D.$$

This proves the claim by induction.

Claim 2.7. *Y_i and Y_j fill if $i \neq j$.*

We complete the proof modulo this claim. Choose $L > 2M$, where M is as in the Bounded Geodesic Image Theorem. By the Bounded Geodesic Image Theorem, for every $0 < i < n$ any geodesic between v_0 and v_n must pass through a curve η_i that does not intersect Y_i. By Claim 2.7, η_i intersects Y_j for all $j \neq i$. Hence $\eta_i \neq \eta_j$ for all $i \neq j$ and hence any geodesic between v_0 and v_n has length $n-1$. This implies that the original sequence $\{v_0, \cdots, v_n\}$ is a tight geodesic. □

Proof of Claim 2.7: To show that any two Y_i and Y_j fill, we first observe that since ∂Y_i is contained in Y_{i+1} and ∂Y_{i+1} is contained in Y_i, Y_i and Y_{i+1} fill S.

Next assume $i + 1 < j$. If Y_i and Y_j do not fill there is a curve c disjoint from both Y_i and Y_j. In particular, c is contained in Y_{i+1} (since Y_i and Y_{i+1} fill S). Hence the (subsurface) projections of both v_i and v_j to Y_{i+1} will be disjoint from c (as proper arcs) or at a uniformly bounded distance from c (if we turn them into curves a la Masur-Minsky). This contradicts $d_{Y_{j+1}}(v_i, v_j) \geq L$. □

2.2. **Tight trees.** Let S be a surface of finite type and $\mathcal{C}(S)$ its curve-complex. The collection of simplices in $\mathcal{C}(S)$ will be denoted as $\mathcal{C}_\Delta(S)$. For a tree T, the set of vertices of T will be denoted as $V(T)$. We generalize the notion of a tight geodesic to an isometric embedding of a tree as follows:

Definition 2.8. For any geodesic (finite, semi-infinite, or bi-infinite) $\gamma = \{\cdots, v_{-1}, v_0, v_1, \cdots\}$ in T, and a map $i : V(T) \to \mathcal{C}_\Delta(S)$, a choice of simple closed curves $\sigma_i \in i(v_i)$ will be called a path in $\mathcal{C}(S)$ induced by γ.

A map $i : V(T) \to \mathcal{C}_\Delta(S)$ will be called an isometric embedding if any path induced in $\mathcal{C}(S)$ by a geodesic γ in T is a geodesic in $\mathcal{C}(S)$.

Much of the discussion in this subsection and Section 2.3 gets simplified if we assume that we are dealing with a sequence of simple non-separating curves. We therefore define this special case first.

Definition 2.9. An $L-$**tight tree of non-separating curves** in the curve complex $\mathcal{C}(S)$ consists of a (not necessarily regular) simplicial tree T of bounded valence and a simplicial map $i : T \to \mathcal{C}(S)$ such that for every vertex v of T and for every pair of distinct vertices $u \neq w$ adjacent to v in T,

$$d_{\mathcal{C}(S \setminus i(v))}(i(u), i(w) \geq L.$$

An $L-$tight tree of non-separating curves for some $L \geq 3$ will simply be called a tight tree of non-separating curves.

The proof of Lemma 2.6 immediately gives us the following (Chris Leininger first told us the proof of this special case of Lemma 2.6):

Proposition 2.10. *There exists $L \geq 3$ such that the following holds. Let S be a closed surface, and let $i : T \to \mathcal{C}(S)$ define an L–tight tree of non-separating curves. Then i is an isometric embedding.*

We now extend the above definition to allow the possibility of multicurves, as well as separating curves.

Definition 2.11. An $L-$**tight tree** in the curve complex $\mathcal{C}(S)$ consists of a (not necessarily regular) simplicial tree T of bounded valence and a map $i : V(T) \to \mathcal{C}_\Delta(S)$ such that

(1) for every vertex v of T, $S \setminus i(v)$ consists of exactly one or two components. Further, if $S \setminus i(v)$ consists of two components and $i(v)$ contains more than one simple closed curve, then each component of $i(v)$ is individually non-separating. In this situation, v is called a **separating vertex** of T.
(2) for every pair of adjacent vertices $u \neq v$ in T, and any vertices u_0, v_0 of the simplices $i(u), i(v)$ respectively,
$$d_{\mathcal{C}(S)}(u_0, v_0) = 1.$$
(3) There is a **distinguished component** Y_v of $S \setminus i(v)$ such that for any vertex u adjacent to v in T, $i(u) \subset Y_v$ (automatic if $i(v)$ is non-separating). For $i(v)$ separating, we shall refer to $Y_v' := S \setminus Y_v$ as the **secondary** component for v.
(4) for every pair of distinct vertices $u \neq w$ adjacent to v in T, and any vertices u_0, w_0 of the simplices $i(u), i(w)$ respectively,
$$d_{\mathcal{C}(Y_v)}(u_0, w_0) \geq L.$$

An $L-$tight tree for some $L \geq 3$ will simply be called a tight tree.

Lemma 2.6 gives us the following generalization of Proposition 2.10:

Proposition 2.12. *Let $L > \max(2M, 4D)$, where M is the constant from the Bounded Geodesic Image Theorem and D is the Behrstock constant from Theorem 2.4. Let S be a closed surface of genus at least 2, and let $i : V(T) \to \mathcal{C}_\Delta(S)$ define an L–tight tree (as in Definition 2.11). Then i is an isometric embedding.*

Proof. It suffices to show (cf. Definition 2.8) that any path induced in $\mathcal{C}(S)$ by a geodesic γ in T is a geodesic in $\mathcal{C}(S)$. But this last statement follows immediately from Lemma 2.6. □

Standing Assumption 2.13. *We shall henceforth assume throughout the paper that whenever we refer to an $L-$tight tree, $L > \max(2M, 4D)$ as in the hypothesis of Proposition 2.12.*

2.3. **Topological building blocks from links.** In this subsection, we shall first describe a construction of building blocks from a tight tree of non-separating curves motivated by Minsky's construction in [Min10]. We shall then proceed to indicate the modifications necessary for more general tight trees. In this section, we shall describe only the topological part of the construction, postponing the geometric aspect of it to Section 3.

For (X, d) a hyperbolic metric space, and $\mathcal{V} \subset X$, $CH(\mathcal{V})$ will denote the union of all geodesics joining $v_i, v_j \in \mathcal{V}$ and will be called the **weak convex hull** of $\mathcal{V}$.

Let $i : T \to \mathcal{C}(S)$ be a tight tree of non-separating curves and let v be a vertex of T. The link of v in T is denoted as $lk(v)$. Then $i(lk(v))$ consists of a uniformly bounded number of vertices in $\mathcal{C}(S)$ (since T has bounded valence). Let m_T denote this bound.

Since S is fixed, there exists $\delta_0 > 0$ such that for any essential connected subsurface W of S, $\mathcal{C}(W)$ is δ_0-hyperbolic. In fact, there is a universal $\delta_0 \leq 17$ independent even of S [HPW15], but we shall not need this. It follows that for any essential connected subsurface W of S and any collection $\mathcal{V} = \{v_1, \cdots, v_k\}$ of $k \leq m_T$ vertices of $\mathcal{C}(W)$, there exists a finite tree $T_{\mathcal{V}} \subset \mathcal{C}(W)$ uniformly approximating $CH(\mathcal{V})$, i.e. there exists a surjective map $\mathbb{P} : CH(\mathcal{V}) \to T_{\mathcal{V}}$ such that

1. the pre-image of any point in $T_{\mathcal{V}}$ under $\mathbb{P}$ has diameter uniformly bounded by $(2\delta_0 + 1)m_T$ (the exact constant is not important; it will suffice for our purposes to have a uniform bound in terms of δ_0 and m_T).
2. $d_{\mathcal{C}(W)}(v_i, v_j) = d_{T_{\mathcal{V}}}(\mathbb{P}(v_i), \mathbb{P}(v_j))$.
3. The vertices $\{\mathbb{P}(v_i)\}$ are precisely the extremal/leaf vertices of $T_{\mathcal{V}}$, i.e. $T_{\mathcal{V}}$ is precisely the convex hull of the collection of points $\{\mathbb{P}(v_i)\}$ in $T_{\mathcal{V}}$.

Note that the tree $T_{\mathcal{V}}$ constructed from $\mathcal{V}$ is not unique, but only coarsely so, in the sense that any two such trees are uniformly quasi-isometric to $CH(\mathcal{V})$ by maps taking $\Pi(v_i)$ to v_i.

In the light of Proposition 2.10 we define:

Definition 2.14. For a tight tree $i : T \to \mathcal{C}(S)$ of non-separating curves, there exists $k \geq 1$ such that for all $v \in T$ there exists a tree T_v (by the above discussion) satisfying the following:
For $W = S \setminus i(v)$, there exists a surjective $k-$quasi-isometry

$$\mathbb{P}_W : CH(i(lk(v)) \to T_v,$$

where $CH(i(lk(v))$ denotes the weak convex hull of $i(lk(v))$ in $\mathcal{C}(W)$.

We shall refer to T_v as the **tree-link** of v.

Definition 2.15. Let $i : T \to \mathcal{C}(S)$ be a tight tree of non-separating curves and let v be a vertex of T. The **topological building block corresponding to** v is

$$M_v = S \times T_v.$$

Thus the topological building block corresponding to v is the trivial $S-$bundle over its tree-link. Note that M_v contains a distinguished 'annulus' $i(v) \times T_v$, where, as before, $i(v)$ is identified with a non-separating simple closed curve on S. We shall refer to $i(v) \times T_v \subset M_v$ as the **Margulis riser** in M_v or simply as the Margulis riser corresponding to v. The reason for this terminology will become clearer when we describe the geometric structure on M_v.

In order to assemble the building blocks corresponding to vertices together, we shall need an auxiliary 'blow-up' construction of the tree T. We pass to the first barycentric subdivision $S_1(T)$ of T and label the mid-point of an edge in T joining v_i, v_j by v_iv_j. These vertices will be referred to simply as the mid-point vertices of $S_1(T)$. For each vertex v of T we define the **half-star** $\mathrm{hs}(v) \subset S_1(T)$ of v to be the (usual) star of v in $S_1(T)$.

Definition 2.16. Let $i : T \to \mathcal{C}(S)$ be a tight tree of non-separating curves. The **blow-up** $\mathrm{BU}(T)$ of T is a tree obtained from $S_1(T)$ by replacing each half-star $\mathrm{hs}(v)$ by the tree-link T_v.

More precisely, we proceed in two steps:
First, attach for each v, the metric tree-link T_v to $S_1(T)$ by gluing $\mathbb{P}(v_i)$ to the mid-point vertex v_iv as $\mathbb{P}(v_i)$ ranges over all the terminal vertices of T_v. In the second step remove the interiors of the half-stars $\mathrm{hs}(v)$ from $S_1(T)$ for all $v \in V$.

We retain the labels of the mid-point vertices of $S_1(T)$ in $\mathrm{BU}(T)$ and refer to them as the mid-point vertices of $\mathrm{BU}(T)$.

The topological model for the tight tree is obtained by gluing together the topological building blocks M_v corresponding to v according to the combinatorics of the blow-up $\mathrm{BU}(T)$. Topologically this is simply the product:

Definition 2.17. Let $i : T \to \mathcal{C}(S)$ be a tight tree of non-separating curves. The **topological model corresponding to** T is

$$M_T = S \times \mathrm{BU}(T).$$

Let $P : M_T \to \mathrm{BU}(T)$ denote the natural projection. Note that $\mathrm{BU}(T)$ has distinguished finite subtrees corresponding to the tree-links T_v. We also identify $P^{-1}(T_v)$ with M_v. Note that a mid-point vertex vw in $\mathrm{BU}(T)$ is the intersection of the tree-links of v, w:

$$\{vw\} = T_v \cap T_{lk(i(w))}.$$

We shall denote $P^{-1}(vw)$ by S_{vw} and refer to them as **mid-surfaces**.

2.4. **Balanced trees.** We now indicate the modifications necessary for a general tight tree. Let $i : V(T) \to \mathcal{C}_\Delta(S)$ be a tight-tree. Tree-links T_v are defined as in Definition 2.14 with the understanding that for $i(v)$ separating, the weak convex hull $CH(i(lk(v)))$ is constructed in the curve complex $\mathcal{C}(Y_v)$ of the distinguished component Y_v of $S \setminus i(v)$. It remains to construct tree-links for the secondary component Y'_v when $i(v)$ is separating. To construct tree-links for the secondary component Y'_v we need to restrict the class of tight trees we are considering.

For w adjacent to v let T'_w denote the connected component of $T \setminus \{v\}$ containing w. Let $\Pi'_v(T'_w)$ denote the subsurface projection of $i(V(T'_w))$ onto $\mathcal{C}(Y'_v)$.

Definition 2.18. A tight tree $i : V(T) \to \mathcal{C}_\Delta(S)$ is said to be a **balanced** tree with parameters D, k if

(1) For every separating vertex v of T,
$$\mathrm{dia}(\Pi'_v(T'_w)) \leq D.$$
(2) Let $i(lk(v))' \subset \mathcal{C}(Y'_v)$ denote the collection of curves $w_0 \in \Pi'_v(T'_w)(\subset \mathcal{C}(Y'_v))$ as w ranges over all vertices adjacent to v in T. Let $CH(i(lk(v))')$ denote the weak convex hull of $i(lk(v))'$ in $\mathcal{C}(Y'_v)$. We require that there exists a surjective $k-$quasi-isometry
$$\mathbb{P}' : CH(i(lk(v))') \to T_v$$

to the tree-link T_v such that for a vertex w of T adjacent to v,

$$\mathbb{P}'(\Pi'_v(T'_w)) = \mathbb{P}(w),$$

(where $\mathbb{P}$ is the projection defined in Definition 2.14).

Building blocks for balanced trees: The notions of topological building block (Definition 2.15) in the case of balanced trees, blow-up (Definition 2.16), topological model (Definition 2.17) now go through exactly as before. The notion of a balanced tree (Definition 2.18) ensures that the weak convex hulls $CH(i(lk(v))) \subset \mathcal{C}(Y_v)$ and $CH(i(lk(v))') \subset \mathcal{C}(Y'_v)$ are coarsely quasi-isometric to each other and to the tree-link T_v.

3. Geometry of building blocks

The purpose of this section is to construct a model geometry on the topological building blocks M_v (Definition 2.15) and the topological model $M_T = S \times \mathrm{BU}(T)$ (Definition 2.17) corresponding to a tight tree of non-separating vertices, and more generally for a balanced tree T (Definition 2.18).

3.1. Model geometries of doubly degenerate 3-manifolds.

It will be convenient to recall some model geometries on doubly degenerate 3-manifolds as these form the motivation and the background for the model geometry on M_v.

3.1.1. *A quick summary.*

Ingredients 3.1. *The model geometry on doubly degenerate 3-manifolds M that is relevant to that on the topological building block M_v is built from the following ingredients:*

1. *the general combinatorial model in* [Min10] *built from the hierarchy machinery of tight geodesics and hierarchy paths* [MM99, MM00]*;*
2. *the model for bounded geometry doubly degenerate 3-manifolds built from a thick Teichmüller geodesic in* [Min94].
3. *The main theorem of* [Min01] *establishing a combinatorial model for bounded geometry doubly degenerate 3-manifolds along with a dictionary between the combinatorics of such a model and the model geometry from a a thick Teichmüller geodesic in Item (2) above.*

We briefly describe these 3 ingredients in this section and use them in Definition 3.6 to define the geometry that will lead to the model geometry on M_v. As usual Teich(S) will denote the Teichmüller space of S.

Item(1): The combinatorial model of [Min10]**.** The general combinatorial model on a doubly degenerate 3-manifold M in [Min10] is built as follows. Let $\mathcal{L}_\pm$ denote the ending laminations of M. Identify $\mathcal{L}_\pm$ with a pair of points on the boundary $\partial\mathcal{C}(S)$ of the curve complex (using Klarreich's theorem [Kla99]). Let $\gamma \subset \mathcal{C}(S)$ be a tight geodesic joining $\mathcal{L}_\pm$. The hierarchy path joining $\mathcal{L}_\pm$ is built inductively by [Min10, MM00]. One starts with γ as the base geodesic. For every vertex (or simplex) v in γ, one (roughly speaking) constructs geodesics in $\mathcal{C}(S \setminus \{v\})$ joining the predecessor of v to its successor. This process is repeated inductively for the geodesics constructed at this second stage and so on. The general combinatorial model in [Min10] is built from standard building blocks ($S_{0,4} \times I$ or $S_{1,1} \times I$

equipped with some standard metrics) by assembling them according to the combinatorics dictated by the hierarchy path joining $\mathcal{L}_\pm$.

Item (2): the model for bounded geometry 3-manifolds M with ending laminations $\mathcal{L}_\pm$. Recall that M is homeomorphic to $S \times \mathbb{R}$. We first replace the laminations $\mathcal{L}_\pm$ on S by singular foliations $\mathcal{F}_\pm$ (differing from $\mathcal{L}_\pm$ by bounded homotopies) and equip S with a singular Euclidean metric where the $x-$(resp. $y-$)co-ordinate is given by $\mathcal{F}_+$ (resp. $\mathcal{F}_-$). Note that fixing co-ordinates implicitly converts $\mathcal{F}_\pm$ into *measured* singular foliations. Then the model geometry on M is locally given by a singular Sol-type metric (see [CT07] or [Min94, p. 567])

$$ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2,$$

where t parametrizes the $\mathbb{R}-$direction in $M = S \times \mathbb{R}$. So far we have not used the bounded geometry hypothesis. There exists a more canonical parametrization of the $\mathbb{R}-$direction when M has bounded geometry. In [Min92, Min94], Minsky showed that when M has bounded geometry the Teichmüller geodesic γ in Teich(S) joining $\mathcal{F}_\pm \in \partial$Teich$(S)$ is thick, i.e. it projects to a geodesic lying inside a compact region in moduli space $Mod(S)$:

Definition 3.2. A geodesic γ in Teich(S) is said to be ϵ_0-**thick** if the systole of any surface $S_x,\ x \in \gamma$ (thought of as a hyperbolic surface) is bounded below by ϵ_0.

A geodesic γ in Teich(S) is **thick** if it is ϵ_0-thick for some $\epsilon_0 > 0$.

For a thick Teichmüller geodesic γ, joining $\mathcal{F}_\pm \in \partial$Teich$(S)$ the parameter t may be identified with the arc-length of the Teichmüller geodesic γ.

Rafi [Raf14] characterized thick Teichmüller geodesics in terms of subsurface projections. To state this characterization we recall that in Definitions 2.1 and 2.2 the notion of subsurface projections was defined. As pointed out after Definition 2.1 these notions can be naturally extended to $d_Y(\lambda, \mu)$ for laminations λ, μ on S and arbitrary essential subsurfaces Y of S [Min01, p. 150-151].

Theorem 3.3. [Raf14] *Let γ be a bi-infinite geodesic in* Teich(S) *with end-points $\mathcal{L}_\pm \in \mathcal{PML}(S) = \partial$Teich$(S)$. Then γ is of bounded geometry if and only if there exists $D > 0$ such that for every essential subsurface W of S (including annular domains), $d_W(\mathcal{L}_+, \mathcal{L}_-) \leq D$.*

Definition 3.4. For a bounded geometry doubly degenerate 3-manifold M without parabolics (homeomorphic to $S \times \mathbb{R}$) with ending laminations $\mathcal{L}_\pm$, the **thick Minsky model** on M is given by the singular Sol-type metric

$$ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2,$$

where t parametrizes (according to arc length) the Teichmüller geodesic γ joining $\mathcal{L}_\pm$ and x, y are co-ordinates for singular foliations boundedly homotopic to $\mathcal{L}_\pm$.

For $S = S_{g,n}$ with marked points, let M^h (homeomorphic to $S \times \mathbb{R}$) be a bounded geometry doubly degenerate 3-manifold with ending laminations $\mathcal{L}_\pm$ and let M denote M^h minus a small neighborhood of cusps. The singular Sol-type metric $ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2$ on $S \times \mathbb{R}$ is given as before; but the latter contains a distinguished set of geodesics through each of the marked points $p_1, \cdots, p_n$ given by (p_i, t). we refer to these as **cusp geodesics**.

This will be elaborated upon in Section 3.1.2 below.

Item (3): the relationship between the thick Minsky model of a bounded geometry manifold M as per Definition 3.4 in Item (2) and its combinatorial model given in Item (1). The main theorem of [Min01] establishes the necessary dictionary (note the similarity with Theorem 3.3).

Theorem 3.5. [Min01, p. 144] *Let M be a doubly degenerate hyperbolic 3-manifold M with ending laminations $\mathcal{L}_\pm$. Then M is of bounded geometry if and only if there exists $D > 0$ such that for every essential subsurface W of S (including annular domains), $d_W(\mathcal{L}_+, \mathcal{L}_-) \leq D$.*

We turn now to a special geometry that will be relevant to this paper. We describe in terms of subsurface projections the conditions that define the model relevant to the geometry on the topological building block M_v (Definition 2.15).

Definition 3.6. Let M (homeomorphic to $S\times\mathbb{R}$) be a doubly degenerate hyperbolic 3-manifold with ending laminations $\mathcal{L}_\pm$. Then M will be said to be of **special split geometry** with parameters L, R if it satisfies the following conditions:
1) Let γ be a tight geodesic joining $\mathcal{L}_\pm$ in $\mathcal{C}(S)$. Then for every simplex v of γ and every component Y of $S \setminus v$,

$$d_Y(\mathcal{L}_+, \mathcal{L}_-) \geq L.$$

We refer to the components Y of $S \setminus v$, $v \in \gamma$ as **principal component domains**.
2) For every proper essential subsurface W of S that is *not* a principal component domain,

$$d_W(\mathcal{L}_+, \mathcal{L}_-) \leq R.$$

Further, if each simplex of the base tight geodesic γ is a single vertex v corresponding to a non-separating simple closed curve on S then M is said to be of special split geometry **with non-separating curves**.

It follows from [Min10, Theorem 8.1] that for L large enough, each vertex v gives a Margulis tube:

Lemma 3.7. *For every $\epsilon_0 > 0$, there exists L_0 such that for $L \geq L_0$ the following holds. Let M be of special split geometry with parameters L, R as in Definition 3.6. Then every vertex of the tight geodesic γ in Definition 3.6 gives an ϵ_0-Margulis tube in M.*

3.1.2. *The bounded geometry model.* We turn now to the second item of Ingredients 3.1 and adapt it to bundles over quasiconvex subsets of $\mathcal{C}(S)$ or Teich(S). Let N^h be a doubly degenerate hyperbolic 3-manifold corresponding to a surface S with or without punctures. Let N denote N^h minus a small neighborhood of the cusps. We normalize so that the boundary components of N are isometric to products $S^1_e \times \mathbb{R}$, where S^1_e are round circles of radius e. We define the **systole** of a manifold or more generally a length space to be the infimum of the length of closed geodesics (thus ignoring cusps). If there exists $\epsilon > 0$ such that the systole of N^h (and hence N) is bounded below by ϵ, then N^h is said to be of **bounded geometry**.

The following theorem is essentially due to Minsky [Min94] (see, however, the paragraphs following Theorem 3.8 for references to the literature, from where the refinement we need can be culled). It establishes a bi-Lipschitz equivalence between the hyperbolic structure on a bounded geometry doubly degenerate hyperbolic 3-manifold and its thick Minsky model:

Theorem 3.8. [Min94, Cor. 5.10] *For $S = S_{g,n}$ a surface of genus g and n punctures, and $\epsilon > 0$, let N^h be a doubly degenerate hyperbolic 3-manifold corresponding to S with injectivity radius bounded by $\epsilon > 0$, and N denote N^h minus a small neighborhood of the cusps as above. Then there exists $L \geq 1$ such that the following holds:*
Let $\mathcal{L}_\pm$ as above be the ending laminations of N^h, let l be the bi-infinite geodesic in Teich(S) *joining $\mathcal{L}_\pm$. Let Q^h denote the thick Minsky model as in Definition 3.4. Let Q denote Q^h minus a small neighborhood of the cusp geodesics (Definition 3.4) with boundary components normalized to be isometric to products $S^1_e \times \mathbb{R}$. Then Q and N are $L-$bi-Lipschitz homeomorphic.*

We point out that Minsky established a quasi-isometric map at the level of universal covers that is a lift of a possibly non-injective map between N, Q in the case that $N^h = N$, i.e. in the absence of cusps. We refer the reader to [Lot97, Proposition 8] for the relevant refinement in the absence of cusps. The conclusion also follows from the full strength of the ending lamination theorem [Min10, BCM12] applied to this special case.

A word about cusps. A coarse model in the case of surfaces with punctures may be found in [Bow02], [Mj09, Section 1.1] or [Mj10, Section 5]. One removes a small neighborhood of the cusps from N^h to obtain N as in the statement of Theorem 3.8. Then one proves the existence of a sequence of pleated surfaces in N^h, such that the intersections of these pleated surfaces with N give a sequence of equispaced pleated surfaces with boundary. The main theorem of [Min92] applies equally to the punctured surface case to show that the Teichmüller distance between successive pleated surfaces with boundary is uniformly bounded both above and below. However, as in [Min94], these pleated surfaces may be immersed, and not embedded. To upgrade immersed surfaces to embedded surfaces, Lott's argument in [Lot97, Proposition 8] applies equally to the manifold with boundary N, concluding the proof.

Theorems 3.5 and 3.8 thus establish two different descriptions of bounded geometry doubly degenerate hyperbolic 3-manifolds.

Universal bundles: The thick Minsky model in Definition 3.4 will need to be generalized to a situation where the base space is a quasiconvex subset of Teich(S) rather than a geodesic. To do this it will be more convenient to obtain a description in terms of hyperbolic metrics on S rather than the singular Euclidean metric in Definition 3.4. The natural structure is given in terms of **universal bundles or universal curves** over Teich(S). The following remark recalls the necessary notion from [Wol90].

Remark 3.9. The moduli space $Mod(S)$ is a quasiprojective variety [Mum77]. A finite-sheeted cover of $Mod(S)$ is actually a manifold (and also a quasiprojective variety) and can be naturally equipped with a Kähler metric: the Weil-Petersson metric [Wol90, p. 420]. We specialize to the case where $g \geq 2, n = 1$ and denote $S = S_{g,0}$. Then $Mod(S_{g,1})$ admits a natural bundle structure fibering over $Mod(S) = Mod(S_{g,0})$ with fiber over $x \in Mod(S_{g,0})$ the curve x. This is called the universal curve [Wol90, p. 419]. The cover of $Mod(S_{g,1})$ corresponding to the fundamental group $\pi_1(S_{g,0})$ of the fiber is then called the **universal bundle** UTeich(S) over Teich(S). The fiber S_x over $x \in$ Teich(S) is then the marked hyperbolic structure given by x. The metric induced on S_x is the restriction of the Weil-Petersson metric on $U\text{Teich}(S)$ and equals the hyperbolic metric (up to a global scale factor).

Finally, for any $X \subset \text{Teich}(S)$ the restriction of $U\text{Teich}(S)$ to X gives topologically a product $UX = S \times X$. The metric on UX is the path-metric induced from $U\text{Teich}(S)$ on UX.

There is a natural fiberwise uniformization map Φ from the thick Minsky model to the universal curve over a thick Teichmüller geodesic l. Since the systole of every fiber $S_x,\ x \in l$ is uniformly bounded below, there exists $K \geq 1$, depending only on the lower bound on systole, such that Φ^{-1} is $K-$bi-Lipschitz on S_x for every x. It follows that the universal bundle over l with its metric is bi-Lipschitz homeomorphic to the thick Minsky model under a fiber-preserving homeomorphism.

Remark 3.10. An alternate coarse description of universal bundles for S closed may be given as follows. Let $X \subset \text{Teich}(S)$ be contained in the $\epsilon-$thick part $\text{Teich}_\epsilon(S)$ of $\text{Teich}(S)$, i.e. for every $x \in X$ the hyperbolic surface S_x has systole at least ϵ. Further, suppose that X is quasiconvex (with respect to the Teichmüller metric). Note that the quotient $\text{Teich}_\epsilon(S)/MCG(S)$ by the mapping class group is compact and hence the inclusion $MCG(S).o \subset \text{Teich}_\epsilon(S)$ is a quasi-isometry where $o \in \text{Teich}_\epsilon(S)$ is some base-point. Hence there is a subset $K \subset MCG(S)$ such that $K.o$ is quasi-isometric to X with the same constants. Further, if

$$1 \to \pi_1(S) \xrightarrow{i} MCG(S,*) \xrightarrow{q} MCG(S) \to 1$$

denote the Birman exact sequence, then $q^{-1}(K)$ projects to K under q and there is a coarsely fiber-preserving quasi-isometry between $q^{-1}(K)$ and the universal cover of the universal curve over X (see the notion of metric bundles in [MS12, Definition 1.2] for more details).

3.1.3. *Relations between model geometries.* In what follows, we shall need to go between three different geometries of doubly degenerate hyperbolic manifolds:

(1) The hyperbolic metric.
(2) The combinatorial model [Min10].
(3) A model obtained by interbreeding the thick model of Theorem 3.8 above with the combinatorial model *in a certain special case* that we shall amplify below. This last model will be called a **special split geometry** model following [Mj14].

It follows essentially from the ending lamination theorem [Min94, Min10, BCM12] that these three different geometries will give us metrics that are uniformly bi-Lipschitz to each other. We shall say more about Item (3) above below (see especially Theorem 3.23).

3.1.4. *The special split geometry model.* We shall now proceed to elaborate on the special split geometry model given in Definition 3.6 and provide an alternate description of the model that we shall need later. The alternate description is culled out of [Min10, Mj14] (see especially [Mj14, Sections 1.1.2, 1.1.3, 4.1], and the description of models of 'graph amalgamation geometry' in [Mj16]).

A piecewise smooth embedded incompressible surface S in a hyperbolic 3-manifold is said to have $(\epsilon, D)-$bounded geometry if, with respect to the induced path metric

(1) the systole of S is bounded below by ϵ
(2) the diameter of S is bounded above D.

We summarize the part of the discussion in Section 4.1 of [Mj14] that will be necessary for us. Let N be a doubly degenerate hyperbolic 3-manifold of special split geometry (Definition 3.6) with ending laminations $l_\pm$. Let $E_\pm$ be the two ends of N and $\gamma = \{\cdots, v_{i-1}, v_i, v_{i+1}, \cdots\}$ be the tight geodesic of simplices joining $l_\pm$ occurring in Definition 3.6. Then Proposition 4.2 of [Mj14] gives a sequence of bounded geometry surfaces $\{S_i\},\ i \in \mathbb{Z}$ exiting the ends $E_\pm$. Proposition 4.3 of [Mj14] now shows that the region between S_i, S_{i+1} has a finite number of Margulis tubes corresponding to the simple closed curves occurring as vertices of the simplex v_i. Further, away from these Margulis tubes, the systole of N is uniformly bounded away from zero. We summarize the conclusions of this construction (see p. 36 of [Mj14]) as follows.

Proposition 3.11. *For all R there exist $\epsilon, C, D > 0$ such that the following holds. Let N be a doubly degenerate hyperbolic 3-manifold of special split geometry with parameters $L \geq 3,\ R$ and ends $E_\pm$. Then (see figure below):*

(1) *There exists a sequence $\{S_i\}, i \in \mathbb{Z}$ of disjoint, embedded, incompressible, $\epsilon, D-$bounded geometry surfaces exiting the ends $E_\pm$ as $i \to \pm\infty$ respectively. The surfaces are ordered so that $i < j$ implies that S_j is contained in the unbounded component of $E_+ \setminus S_i$. The topological product region between S_i and S_{i+1} is denoted B_i and is termed a* **split block**.
(2) *corresponding to each such product region B_i, there exists a finite number of Margulis tubes corresponding to disjoint simple closed curves on S_i. The disjoint union of these Margulis tubes is called a multi-Margulis tube and denoted as $\mathbb{T}_i$. Then $\mathbb{T}_i \subset B_i$. Further, $\mathbb{T}_i \cap S_i$ and $\mathbb{T}_i \cap S_{i+1}$ are (multi-)annuli on S_i and S_{i+1} respectively, with core curves homotopic to the core curve of $\mathbb{T}_i$. We think of $\mathbb{T}_i$ as splitting the ith* **split block** *B_i and call it a* **splitting tube**. *The complementary components K_{ij} of $B_i \setminus \mathbb{T}_i$ and their lifts $\widetilde{K_{ij}}$ to $\widetilde{N}$ are called* **split components**. *The top and bottom boundary surfaces S_{i+1}, S_i of B_i are called* **split surfaces**.
(3) *The core curves of $\mathbb{T}_i$ correspond to a simple closed multicurve τ_i on S.*
(4) *Further $B_i \setminus \mathbb{T}_i$ has systole uniformly bounded below by ϵ for all i.*
(5) *The geometry of the Margulis tubes $\mathbb{T}_i$ is as follows. For a component of a splitting tube $\mathbb{T}_i$ in a split block B_i, the vertical boundaries $A_i^\pm$, corresponding to the left and right vertical annuli in the figure below, are $C-$bi-Lipschitz homeomorphic to products, $A_i^\pm = S^1 \times [0, l_i^\pm]$, of the unit circle(a normalization condition) and an interval of length $l_i^\pm$. The horizontal boundaries of $\mathbb{T}_i$ are $C-$bi-Lipschitz homeomorphic to (each other and to) products, $S^1 \times [-e, e]$ for a fixed (small) e independent of i.*

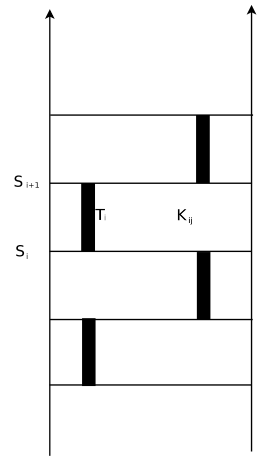

Figure: *A schematic representation of the model geometry*

Remark 3.12. When the tight geodesic γ of Definition 2.9 joining the ending laminations $\mathcal{L}_\pm$ of N in Proposition 3.11 consists of simple closed curves τ_i, then each multi-Margulis tube $\mathbb{T}_i$ is in fact a Margulis tube with core curve isotopic to τ_i.

Remark 3.13. The geometry of the Margulis tubes in Proposition 3.11 really originates in the geometry of such tubes in the combinatorial model of [Min10]. Using the bi-Lipschitz homeomorphism of [BCM12] between the combinatorial model and the hyperbolic metric we obtain the structure of Margulis tubes given in Proposition 3.11.

Remark 3.14. We remark that the general case of *weak split geometry* described in [Mj14, Remark 4.9] allows for each multi-Margulis tube $\mathbb{T}_i$ to split a uniformly bounded number of blocks. For special split geometry, this number is precisely one.

Let $l_i = \min(l_i^+, l_i^-)$ Let $\Phi_i^\pm : S^1 \times [0, l_i^\pm] \to S^1 \times [0, l_i]$ be maps that are identity in the first factor and affine surjective maps in the second factor.

Definition 3.15. [Mj14, p. 38] A **welded split block** $B_{i,weld}$ (homeomorphic to $S \times [0,1]$) is a split block equipped with the following quotient path metric on each splitting tube:

(1) horizontal boundaries $S^1 \times [-e, e]$ quotiented down to $S^1 \times \{0\}$ by projecting the second co-ordinate to 0,
(2) the vertical boundaries of splitting tubes are identified with each other via the maps $\Phi_i^\pm$.

The resulting annuli in $B_{i,weld}$ after the identification shall simply be called **standard annuli in** $B_{i,weld}$. The resulting metric on $B_{i,weld}$ will be denoted by $d_{i,weld}$. We shall also refer to l_i as the **height** of the standard annulus in B_i, or simply the **height** of B_i.

The composition of the two maps above give a quotienting map $f_i : \partial T_i \to S^1 \times [0, l_i]$.

The definition of a welded manifold we have used here is slightly different from the one in [Mj14], where all the l_i's were equal to one.

We shall equip $B_{i,weld}$ with a new pseudometric. Equip the standard annulus $S^1 \times [0, l_i]$ with the product of the zero metric on the S^1-factor and the Euclidean metric on the $[0, l_i]$ factor. Let $(S^1 \times [0, l_i], d_0)$ denote the resulting pseudometric.

Definition 3.16. [Mj14, p. 39] The **tube-electrified metric** d_{te} is defined to be the pseudometric metric that agrees with d_{weld} away from the standard annuli in B_{weld} and with d_0 on the standard annuli in $B_{i,weld}$.

To distinguish it from $(B_{i,weld}, d_{i,weld})$ the new space and pseudometric will be denoted as $(B_{i,te}, d_{i,te})$. Note that all the top and bottom split surfaces of split blocks B_i (before or after tube-electrification) are homeomorphic to a fixed hyperbolic S via uniformly bi-Lipschitz homeomorphisms.

Gluing successive welded blocks along common split surfaces we obtain the **welded model manifold** (N_{weld}, d_{weld}) homeomorphic to $S \times \mathbb{R}$ corresponding to the original doubly degenerate manifold N.

3.2. **Model geometry of topological building blocks M_v.** The purpose of this section is twofold. First, it furnishes an alternate explicit model geometry (the special split model geometry) for the split blocks of Proposition 3.11 by interbreeding the thick Minsky model (Theorem 3.8) with the combinatorial model of [Min10]. Secondly, armed with the model geometries of bounded geometry doubly degenerate 3-manifolds (Theorem 3.8) and the special split geometry model (Proposition 3.11), we describe a model geometry (i.e. a metric) on the topological building blocks M_v (Definition 2.15). The metric on M_v shall be denoted as d_v and the metrized building block (M_v, d_v) shall be called the **geometric building block**.

Remark 3.17. Suppose that the tree T of Definition 2.11 is a simplicial tree l with underlying space $\mathbb{R}$ and with vertices at $\mathbb{Z}$. Let $l_\pm$ denote the ending laminations corresponding to the end-points of $i(l) \subset \mathcal{C}_\Delta(S)$ in $\partial\mathcal{C}(S)$. Let M_l be the (unique up to isometry [Min10, BCM12]) doubly degenerate hyperbolic 3-manifold with ending laminations $l_\pm$. Then M_l is of special split geometry as in Proposition 3.11 Let v be a vertex in the vertex set $\mathbb{Z}$. Then the model geometries using d_{weld}, d_{te} that we describe below on M_v will respectively be uniformly bi-Lipschitz to the metrics on the welded split block (Definition 3.15) and the tube-electrified metric d_{te} of Definition 3.16.

Recall that the topological building block corresponding to v is given by $M_v = S \times T_v$.

Definition 3.18. A **special split geometry** on M_v with parameters k, ϵ is built from the following:

(1) A $k-$bi-Lipschitz section $\sigma_W : T_v \to \mathrm{Teich}_\epsilon(W)$ for each component W of $S \setminus i(v)$ such that $\sigma_W(T_v)$ is $k-$quasiconvex in $\mathrm{Teich}(W)$. Note that by Definition 2.18, T_v is coarsely independent of the component W.
(2) The Margulis riser $i(v) \times T_v$ corresponding to v is metrized by equipping it with the product metric so that each circle of $i(v)$ is a round circle S^1_e of radius $e > 0$.
(3) Let W_v denote the universal metric bundle (see Remark 3.9 and the discussion following it) over $\sigma_W(T_v)$ with a neighborhood of the cusps removed. We further demand that each annular boundary component of W_v

(corresponding to circular boundary components of W) is a metric product $S^1_e \times \sigma_W(T_v)$ (equivalently, we excise the cusps of a fiber over any $x \in \sigma_W(T_v)$ in such a way that the boundary curves are isometric to S^1_e).

(4) Let A_W be an annular boundary component of some W_v (W ranges over components of $S \setminus i(v)$). Then there exists a **simple** closed curve $v_A \subset v$ such that A_W corresponds to the Margulis riser $v_A \times T_v$ and is isometric to the metric product $S^1_e \times \sigma_W(T_v)$. We glue the annular boundary component A_W to the Margulis riser $v_A \times T_v$ via the map (Id, σ_W^{-1}).

(5) We do this for every component W of $S \setminus i(v)$.

The resulting quotient metric on M_v is denoted d_v. M_v equipped with d_v will be called the **building block of special split geometry** corresponding to v. The natural projection from (M_v, d_v) to T_v will be denoted by P_v.

Definition 3.19. We shall say that a map $f : (A, d_A) \to (B, d_B)$ of metric spaces is $c-$**proper** if for any $B_1 \subset B$ of diameter at most one, $f^{-1}(B_1)$ has diameter at most c. If f is $c-$proper for some c we shall simply say that it is uniformly proper.

We observe an immediate consequence of Definition 3.18.

Lemma 3.20. *Given k, ϵ, there exists c such that if M_v, as in Definition 3.18, is of special split geometry with parameters k, ϵ, then $P_v : (M_v, d_v) \to T_v$ is $c-$proper.*

Remark 3.21. Special case of a single non-separating curve:
We describe a quick informal way of thinking about the geometric building block (M_v, d_v) when v consists of a single non-separating curve, so that $W = S \setminus v$ is connected. Here, $\sigma_W(T_v)$ is a $k-$quasiconvex tree in the $\epsilon-$thick part Teich(W). The metric on W_v away from the cusps is the universal bundle metric over $\sigma_W(T_v)$. Thus, away from the cusps, the metric on W_v is like the bounded geometry metric given by Theorem 3.8. After excising the cusps this bundle is glued to the metric product Margulis riser $S^1_e \times T_v$ by a map that is identity in the first co-ordinate and σ_W^{-1} in the second.

We proceed to define a tube-electrified (pseudo-)metric on M_v following Definition 3.16. Equip each Margulis riser $S^1_e \times T_v$ with the product of the zero metric on the S^1_e-factor and the usual (tree) metric on the T_v factor. Let $(S^1 \times T_v, d_0)$ denote the resulting pseudometric.

Definition 3.22. The **tube-electrified metric** d_{te} on M_v is defined to be the pseudometric that agrees with d_v away from the Margulis risers in M_v and with d_0 on the Margulis risers in M_v.

M_v equipped with the tube-electrified metric d_{te} will be denoted as (M_v, d_{te}) (as in Definition 3.16).

An alternate description of the model geometry on M_v (Definition 3.18) can be given in terms of hierarchy paths along the lines of the dictionary established by Theorem 3.5. We give a quick informal recapitulation following [Min01]. Let $\mathcal{M}(S)$ and $\mathcal{P}(S)$ denote respectively, the marking complex and the pants complex of S. Fix a base-point $o \in$ Teich$_e(S)$ and let $MCG(S)$ denote the mapping class group of S acting on Teich$_e(S)$. Note that $MCG(S)$ (with respect to a word metric for a finite generating set) and $\mathcal{M}(S)$ are quasi-isometric. Let $\mathbb{P}_M :$ Teich$_e(S) \to \mathcal{M}(S)$ denote a projection (coarsely well-defined, see [MM99, MM00]) taking a point x of Teich$_e(S)$ to a nearest point $g.o$ in the mapping class group orbit $MCG(S).o$

and hence via a quasi-isometry to $\mathcal{M}(S)$. Also, let $\mathbb{P}_C : \mathrm{Teich}_e(S) \to \mathcal{C}(S)$ denote a projection (again coarsely well-defined, see [MM99, Min10]) taking a point x of $\mathrm{Teich}_e(S)$ to the collection of short curves (where shortness is defined by a Bers' constant). We may and will assume that $\mathbb{P}_C$ factors through $\mathbb{P}_M$.

We shall need a slight generalization of Theorems 3.3 and 3.5 due to Rafi [Raf14] and Minsky [Min01]. Using the projection $\mathbb{P}_C$, subsurface projections $\pi_W(x)$ of points $x \in \mathrm{Teich}(S)$ onto the curve complex $\mathcal{C}(W)$ of an essential subsurface W and distances $d_W(x, y)$ between $x, y \in \mathrm{Teich}(S)$ can be defined in a straightforward fashion [Min01, Raf14]. The hierarchy machinery of Masur-Minsky in the papers [MM00, Min01] is needed to state the Theorem below. Theorems 3.3 and 3.5 have been stated for bi-infinite geodesics. However, in [Min01, Raf14] these are proven for geodesic segments and rays as well using the projection $\mathbb{P}_C$ above. We restate these in the form we need them (see Section 2.6 and the Bounded Geometry Theorem on p. 144 of [Min01]):

Theorem 3.23. [Min01, Raf14] *For $K \geq 0$ and $\epsilon > 0$, there exists $R > 0$ such that if H is a bounded $K-$quasiconvex subset of $\mathrm{Teich}_\epsilon(S)$ then for any $x, y \in H$ and any proper essential subsurface W of S, the hierarchy path in W subordinate to any tight geodesic joining $\mathbb{P}_C(x), \mathbb{P}_C(y)$ is either empty or has length at most R.*

Conversely, for any $R > 0$ there exists $\epsilon, K > 0$, such that the following holds. Suppose that

1. *$u, v \in \mathcal{C}_\Delta(S)$ are maximal simplices equipped with transversals $t(u), t(v)$,*
2. *for any proper essential subsurface W of S (including annular domains), the hierarchy path in W subordinate to any tight geodesic joining $\mathbb{P}_C(x), \mathbb{P}_C(y)$ is either empty or has length at most R.*

Then

1. *the set of points x (resp. y) in $\mathrm{Teich}(S)$ where $u, t(u)$ (resp. $v, t(v)$) are short (bounded by the Bers' constant, say) lies in a ball of radius K in $\mathrm{Teich}_\epsilon(S)$.*
2. *The Teichmüller geodesic joining such pairs x, y lies in $\mathrm{Teich}_\epsilon(S)$.*

Definition 3.24. A subset X of $\mathcal{C}(S)$ is $R-$thick, if for any $v \in X$ and $v_1, v_2 \in X$ adjacent to v, and any component W of $S \setminus v$,

1. any geodesic γ joining v_1, v_2 in $\mathcal{C}(W)$ is of length at most R,
2. any geodesic in a hierarchy path joining v_1, v_2 and subordinate to a geodesic γ as in the previous condition is of length at most R.

As an immediate consequence of Theorem 3.23 we have the following:

Corollary 3.25. *For $S = S_{g,n}$, let ϕ be a pseudo-Anosov homeomorphism. Then there exists $R > 0$ such that any tight geodesic γ in $\mathcal{C}(S)$ preserved by ϕ is $R-$thick.*

More generally, let $\phi_1, \cdots, \phi_k$ freely generate a free convex cocompact subgroup $Q = F_k$. There exists R such that if Q preserves a quasi-isometrically embedded tree $T_Q \subset \mathcal{C}(S)$, then T_Q is also $R-$thick.

Definition 3.26. Let $i : V(T) \to \mathcal{C}_\Delta(S)$ be a balanced tree (see Definition 2.18) and $v \in T$. Let W be a component of $S \setminus i(v)$ and let $T_{v,W}$ denote a bi-Lipschitz embedded image of the tree-link T_v of v in $\mathcal{C}(W)$ (with parameters as in Definition 2.18).

For any two terminal vertices u, w of $T_{v,W}$, any tight geodesic γ_W joining them in $\mathcal{C}(W)$, and any proper essential subsurface W' of W, a tight geodesic supported on W' and occurring in a hierarchy of geodesics subordinate to γ_W will be called a **geodesic subordinate to the tree-link** $T_{v,W}$.

If there exists a component W of $S \setminus i(v)$ such that γ is a geodesic subordinate to the tree-link $T_{v,W}$, then γ is called a **geodesic subordinate to the tree-link** T_v.

If there exists a vertex v of T such that γ is a geodesic subordinate to the tree-link T_v, then γ is simply called a **geodesic subordinate to the tree** T.

As a consequence of Theorem 3.23, we have the following alternate description of a building block M_v of special split geometry corresponding to v. The Corollary follows by applying Theorem 3.23 to the tree-link of v.

Corollary 3.27. *For all $k, \epsilon > 0$, there exists $R > 0$ such that the following holds: If a model building block of special split geometry has parameters $k, \epsilon > 0$ then every geodesic subordinate to the tree-link T_v has length at most R.*

Conversely, given $R > 0$, there exists $k, \epsilon > 0$ such that the following holds. For a topological building block M_v with tree-link T_v if every geodesic subordinate to the tree-link T_v has length at most R then M_v admits a special split geometry structure with parameters $k, \epsilon > 0$.

The advantage of Corollary 3.27 over Definition 3.18 is that the problem is reduced to looking only at the curve complex rather than varying Teichmüller spaces.

Remark 3.28. We observe that the welded split block in Definition 3.15 is a special case of a model building block of special split geometry when the tree link T_v is an interval of the form $[0, n]$ with vertices at the integer points.

A word of caution: The split block of Proposition 3.11 may be quite different from the welded split block in Definition 3.15 as far as the geometry of the tubes $\mathbb{T}_i$ are concerned. In the split block, the Margulis tubes have the geometry of solid hyperbolic tori. In the welded split block, these are replaced by flat annuli.

We expand on Remark 3.17 and explicitly state here the relationship between the geometry of split blocks in totally degenerate 3-manifolds (Proposition 3.11) and the special split geometry of M_v as in Definition 3.18. Let $i : V(T) \to \mathcal{C}_\Delta(S)$ be a balanced tree and $v \in T$. Let l be a bi-infinite geodesic in T through v. We further equip l with the simplicial tree structure induced by T. Let $\mathrm{BU}(T)$ denote the blown-up tree and let $\mathrm{BU}(l)$ denote the blow up of l. Let $T_v(l)$ denote the tree-link of v in $\mathrm{BU}(l)$ and let $M_v(l)$ denote the associated geometric building block. Let $P_v : M_v \to T_v$ and $P_v(l) : M_v(l) \to T_v(l)$ denote the natural projections.

Lemma 3.29. *Given $R, D, k \geq 1$, $n \geq 2$ there exists $C \geq 1$ such that the following holds:*
Let $i : V(T) \to \mathcal{C}_\Delta(S)$ be an $L-$tight $R-$thick balanced tree with parameters D, k (see Definition 2.18) such that each vertex of T has valence at most n. Let $T_v, M_v, l, T_v(l), M_v(l), P_v, P_v(l)$ be as above. Then there exist

1. *a $C-$bi-Lipschitz embedding $\psi_v : T_v(l) \to T_v$ taking the end-points of $T_v(l)$ to the corresponding end-points of T_v.*
2. *a $C-$bi-Lipschitz embedding $\phi_v : M_v(l) \to M_v$*

such that $\psi_v \circ P_v(l) = P_v \circ \phi_v$, i.e. ϕ_v preserves fibers.

Proof. The construction of the tree-link in Definition 2.14 guarantees the existence of a $C-$bi-Lipschitz embedding $\psi_v : T_v(l) \to T_v$ taking the end-points of $T_v(l)$ to the corresponding end-points of T_v, where C depends only on n.

The construction of the model geometry on M_v in Definition 3.18 now guarantees the bi-Lipschitz embedding ϕ_v with constant C depending only on the parameters k, ϵ of the model geometries of $M_v(l), M_v$. Since k, ϵ depend only on R by Corollary 3.27, the Lemma follows. □

For doubly degenerate manifolds of special split geometry, the height l_i of the block B_i has a nice interpretation that we now recall. From the construction of the Minsky model for such manifolds, [Min10, Theorem 8.1] (see the summary in [Mj14, Sections 1.1.2 and 3]) l_i may be taken to be approximately equal to $d_{\mathcal{C}(S\setminus v_i)}(v_{i-1}, v_{i+1})$:

Proposition 3.30. *Given $R > 0$, there exists c_0 such that the following holds. Let l be an $L-$tight $R-$thick tree whose underlying topological space is homeomorphic to $\mathbb{R}$ and whose vertices v_i are simple non-separating curves. Let M_l be the corresponding model manifold of special split geometry. Then for every vertex v_i of T, the height l_i of the ith split block B_i may be chosen to equal $l_i^+ = l_i^-$ (thus $C = 1$ in Proposition 3.11) and*

$$|d_{\mathcal{C}(S\setminus v_i)}(v_{i-1}, v_{i+1}) - l_i| \leq c_0.$$

3.3. **Model geometry on the topological model $M_T = S \times \mathrm{BU}(T)$.** We now describe how to glue the geometric building blocks together to obtain a model geometry on $M_T = S \times \mathrm{BU}(T)$. Since the model geometry will be quite similar to the metric in Definition 3.15, the resulting metric on M_T will also be denoted as d_{weld}. There are two points of view one can adopt in describing the model geometry: hierarchy paths or geodesics in Teichmüller space. It will be more convenient to define the model using hierarchy paths as observed after Corollary 3.27.

Definition 3.31. A balanced tree $i : V(T) \to \mathcal{C}_\Delta(S)$ is said to be $L-$tight and $R-$thick if

1. (1) it is $L-$tight in the sense of Definition 2.9, and
2. (2) all geodesics subordinate to the tree T have length at most R.

To recover the model geometry on $M_T = S \times \mathrm{BU}(T)$ from Definition 3.31 we shall need the model geometry used in the Ending Lamination Theorem [Min10, BCM12] of Brock-Canary-Minsky. Note that for any $v \in T$, Corollary 3.27 furnishes a model building block M_v of special split geometry as a bundle over the tree-link T_v. To construct the model geometry on M_T, it remains to assemble the pieces given by M_v. Note also that:

1. (1) Every terminal vertex of T_v corresponds to a mid-point vertex vw of the blown-up tree $\mathrm{BU}(T)$ (Definition 2.16), where w is adjacent to v in T.
2. (2) For every terminal vertex vw of T_v, the mid-surface S_{vw} (Definition 2.17) is of (uniformly, independent of v, w) bounded geometry, i.e. it has injectivity radius uniformly bounded below and diameter uniformly bounded above.

In order to assemble the pieces given by M_v therefore, it suffices to determine (at least coarsely) the gluing maps between M_v and M_w at S_{vw} as v, w range over adjacent vertices in T. Since S_{vw} is of uniformly bounded geometry, it will suffice to show that, up to a choice of a base-point in $\mathrm{Teich}_\epsilon(S)$ (where ϵ is as in Corollary

3.27), S_{vw} lies in a uniformly (independent of v, w) bounded ball in $\mathrm{Teich}_\epsilon(S)$. It is precisely this fact that is furnished by the Minsky model as summarized and explained in Sections 1.1.2 and 1.1.3 of [Mj14].

We briefly recall the necessary facts and the argument for completeness. We shall find it convenient to think of T as rooted, with root vertex $*$. Let l be any bi-infinite geodesic in T through $*$. Then $i(l)$ is a tight geodesic in $\mathcal{C}(S)$ by our hypothesis on $i : T \to \mathcal{C}_\Delta(S)$ and gives a bi-infinite tight geodesic in $\mathcal{C}(S)$ converging to ending laminations $l_\pm \in \mathcal{EL}(S) = \partial\mathcal{C}(S)$ [Kla99]. Given such a tight geodesic, Minsky [Min10] constructs a combinatorial model M_l for a hyperbolic 3-manifold N_l with ending laminations $l_\pm$. Finally, Brock-Canary-Minsky [BCM12] prove that M_l is uniformly bi-Lipschitz homeomorphic to N_l. The construction of M_l in [Min10, Theorem 8.1] shows in particular that the bounded geometry surfaces in M_l correspond to markings and hence give coarsely well-defined points of $\mathrm{Teich}(S)$ (once a base surface is chosen and identified with a base-point of $\mathrm{Teich}(S)$).

Proposition 3.11 now shows that if moreover l is $L-$tight (for some $L \geq 3$) and $R-$thick, then

(1) M_l admits a bi-Lipschitz homeomorphism to a model of special split geometry (Definition 3.6). Further, the bi-Lipschitz constant and the parameters $\epsilon, D > 0$ occurring in Proposition 3.11 depend only on R.
(2) The split surface (Item (2) of Proposition 3.11) between split blocks corresponding to adjacent vertices v, w in l gives a coarsely well-defined element $S(v, w)$ of $\mathrm{Teich}(S)$.

We restate the last conclusion more precisely. Given $R > 0$ there exists $r, \epsilon > 0$ such that the following holds:
Let $i : V(T) \to \mathcal{C}_\Delta(S)$ be $L-$tight and $R-$thick. Then for any pair of adjacent vertices $v, w \in T$, and any bi-infinite geodesic $i(l)$, passing through $i(v), i(w)$ and $*$, the split surface between split blocks corresponding to $v, w \in l$ lies in $N_r(S(v, w)) \subset \mathrm{Teich}_\epsilon(S)$. Note that $r, \epsilon > 0$ depend on R but not L.

Thus we have a coarsely well-defined element $S(v, w)$ of $\mathrm{Teich}(S)$ corresponding to the mid-surface S_{vw} independent of the bi-infinite geodesic l passing through v, w. We summarize the above discussion as follows:

Theorem 3.32. *There exists $C_0 \geq 1$ depending only on the topology of S and given $R > 0$, $D_0, k_0 \geq 1$ there exist $r, \epsilon, > 0, C, D, k \geq 1$ such that the following holds: Suppose that $i : V(T) \to \mathcal{C}_\Delta(S)$ is an $L-$tight $R-$thick balanced tree with parameters D_0, k_0 as in Definition 2.18. Let $*$ be a root of T. Let l be any bi-infinite tight geodesic in $i(T)$ through $*$ with end-points $l_\pm \in \mathcal{EL}(S) = \partial\mathcal{C}(S)$. Then*

(1) The doubly degenerate hyperbolic 3-manifolds N_l with end-invariants $l_\pm$ are of special split geometry with constants $\epsilon, D > 0, C \geq 1$ as in Proposition 3.11.
(2) The model manifold M_l is C_0-bi-Lipschitz homeomorphic to N_l.

Further, for any pair of adjacent vertices $v, w \in T$, there exists $S(v, w) \in \mathrm{Teich}_\epsilon(S)$ such that for any geodesic l in T, passing through $i(v), i(w), *$, the split surface between split blocks in M_l corresponding to $v, w \in l$ lies in $N_r(S(v, w)) \in \mathrm{Teich}_\epsilon(S)$.

Definition 3.33. Theorem 3.32 implies in particular that the mid-surfaces S_{vw} of $\mathrm{BU}(T)$ are (coarsely) well-defined points of $\mathrm{Teich}(S)$. Thus the image of $lk(v)(\subset \mathrm{BU}(T))$ in $\mathrm{Teich}(S)$ is (coarsely) well-defined under a qi-section as a finite set of points (of uniformly bounded cardinality). Interpolating the model building blocks

(M_v, d_v) of special split geometry finally gives us the **model metric** d_{weld} on M_T. The pair (M_T, d_{weld}) will be called the **model of special split geometry** on the topological model M_T.

Replacing each (M_v, d_v) in (M_T, d_{weld}) with the tube-electrified (pseudo-)metric (M_v, d_{te}) (Definition 3.22) gives us the tube-electrified metric d_{te} on M_T. The pair (M_T, d_{te}) will be called the **tube electrified model of special split geometry** on the topological model M_T. $P : (M_T, d_{weld}) \to \mathrm{BU}(T)$ and $P : (M_T, d_{te}) \to \mathrm{BU}(T)$ will denote the natural projections.

The lift of the metric d_{weld} (resp. d_{te}) to the universal cover $\widetilde{M_T}$ is also denoted by d_{weld} (resp. d_{te}). Also, $P : (\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$ and $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ will denote the natural projections.

We should remind the reader of the caveat in Remark 3.28: the model metrics on (M_v, d_v) differ from the model metrics on the split blocks of Proposition 3.11 at the Margulis tubes.

Lemma 3.20 and Theorem 3.32 give us the following:

Lemma 3.34. *Given a surface S, $D, k \geq 1$ and $R > 0$, there exist $c \geq 1$ such that the following holds:*
Suppose that $i : V(T) \to \mathcal{C}_\Delta(S)$ is an $L-$tight $R-$thick balanced tree with parameters D, k as in Definition 2.18. Then $P : (M_T, d_{weld}) \to \mathrm{BU}(T)$ and $P : (M_T, d_{te}) \to \mathrm{BU}(T)$ are $c-$proper.

Proof. By Corollary 3.27, there exist k, ϵ depending on R such that each M_v is of split geometry with parameters k, ϵ. Theorem 3.32 now shows that the mid-surfaces S_{vw} of $\mathrm{BU}(T)$ are coarsely well-defined points of $\mathrm{Teich}(S)$: the constant r occurring in the conclusion of Theorem 3.32 depends only on R. Hence $P : (M_T, d_{weld}) \to \mathrm{BU}(T)$ is $c-$proper. It follows that $P : (M_T, d_{te}) \to \mathrm{BU}(T)$ is $c-$proper. □

3.4. **The Main Theorems.** We are now in a position to present the main theorems of this paper. We carry forward the notation from the discussion preceding Lemma 3.29: l is a bi-infinite geodesic in T and $\mathrm{BU}(l)$ denotes the bi-infinite geodesic in $\mathrm{BU}(T)$ after blowing up l in T. Further, let $\mathcal{V}(l)$ denote the collection of vertices of T on l, N_l denote the doubly degenerate hyperbolic 3-manifold with ending laminations given by $l_\pm$, the ideal end-points of $i(l)$. Let $\mathbf{T}_v$ denote the Margulis tube in N_l corresponding to v. Let $N_l^0 = N_l \setminus \bigcup_{v \in \mathcal{V}(l)} \mathbf{T}_v$. Also let M_l denote the bundle over $\mathrm{BU}(l)$ induced from $\Pi : M_T \to \mathrm{BU}(T)$. Let $M_l^0 = M_l \setminus \bigcup_{v \in \mathcal{V}(l)} \mathcal{R}_v$.

Theorem 3.35. *Given $R > 0$, $D, k \geq 1$, there exist $K, c \geq 1, e > 0$ such that the following holds. Let $i : V(T) \to \mathcal{C}_\Delta(S)$ be an $L-$tight $R-$thick balanced tree with parameters D, k as in Definition 2.18. There exists a metric d_{weld} on M_T such that $P : M_T \to \mathrm{BU}(T)$ satisfies the following:*

1. *The induced metric on a Margulis riser $\mathcal{R}_v$ is the metric product $S^1_e \times T_v$, where S^1_e is a round circle with radius e.*
2. *For any bi-infinite geodesic l in T, N_l^0 and M_l^0 are $K-$bi-Lipschitz homeomorphic.*
3. *Further, if there exists a subgroup Q of $MCG(S)$ acting cocompactly and geometrically on $i(T)$, then this action can be lifted to an isometric fiber-preserving isometric action of Q on (M_T, d_{weld}).*
4. *$P : (M_T, d_{te}) \to \mathrm{BU}(T)$ is $c-$proper.*

Proof. Item (1) follows immediately from the construction in Definition 3.18 and Lemma 3.29.

Item (2) follows from Proposition 3.11 and Lemma 3.29.

Item (3) follows from the observation that the constructions of the tree-link in Definition 2.14, the blow-up in Definition 2.16, and the model geometry in Definition 3.18 can all be done equivariantly with respect to the action of Q.

Item (4) follows from Lemma 3.34. $\square$

The lift of the pseudometric d_{te} on (M_T, d_{te}) to $\widetilde{M_T}$ is also denoted by d_{te}.

Theorem 3.36. *Given $R > 0$, $D, k \geq 1$, there exists $\delta_0, L_0 \geq 0$ such that the following holds. Let $i : V(T) \to \mathcal{C}_\Delta(S)$ be an $L-$tight $R-$thick balanced tree with $L \geq L_0$ and parameters D, k as in Definition 2.18. Then $(\widetilde{M_T}, d_{te})$ is δ_0-hyperbolic.*

In the statement of Theorem 3.36 we have explicitly mentioned the constant L_0 from Standing Assumption 2.13. The proof of Theorem 3.36 will occupy the rest of the paper.

4. Effective combination theorems and relative hyperbolicity

Before proving Theorem 3.36 we shall recall, organize and adapt some known material on combination theorems and relative hyperbolicity. The fundamental combination theorem in the context of trees of spaces is due to Bestvina and Feighn [BF92]. Its converse is due, in various forms, to Gersten [Ger98], Bowditch [Bow02] and others. This was generalized to the context of relative hyperbolicity in [MR08, MS12]. An effective (i.e. with constants) generalization is due to Gautero [GH09, Theorem 2] [Gau16, Theorem 2.20] (see especially Sections 7, 8 of the last paper), [GW11]. In the context that we are interested in, the base-tree will be a metric tree where some of the edges (corresponding to edges of the tree-links T_v, see Definition 2.18) might have non-integral length. Strictly speaking, therefore we are in the context of a metric bundle in the sense of [MS12] where the fibers are uniformly hyperbolic (see Remark 4.7 below for going back and forth between trees of spaces and metric bundles). The combination theorem and its converse for metric bundles are proven in [MS12, Theorem 4.3, Proposition 5.8]. It is also shown in [MS12] that the metric bundle is (with effective uniform constants) quasi-isometric to a metric graph bundle.

We shall be specifically interested in the following bundles:

(1) The universal cover $(\widetilde{M_T}, d_{weld})$ of the bundle (M_T, d_{weld}),
(2) The universal cover $(\widetilde{M_T}, d_{te})$ of the bundle (M_T, d_{te}).

Both have as base the blown-up tree $\mathrm{BU}(T)$ (see Definition 2.16). We shall denote the projection map to the base as $P : (\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$ or $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$. Metric bundles over trees are examples of trees of spaces (Section 4.1 below) as well as metric bundles in the sense of [MS12] and both points of view will be important. We shall in Section 4.4 use the terminology of metric bundles and adapt the statements of [Gau16, GH09, MS12] to the context of $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$.

4.1. **Trees of hyperbolic spaces and effective combination theorem.** We recall the notion of a tree of spaces.

Definition 4.1. [BF92] Let (X,d) be a geodesic metric space and T a simplicial tree with vertex set $\mathcal{V}(T)$ and edge set $\mathcal{E}(T)$. $P : X \to T$ is said to be a tree of geodesic metric spaces satisfying the *quasi-isometrically embedded condition* (or *qi condition*) if there exists a map $P : X \to T$, and constants $K \geq 1, \epsilon \geq 0$ satisfying the following:

(1) For all vertices $v \in \mathcal{V}(T)$, $X_v = P^{-1}(v) \subset X$ with the induced path metric d_v is a geodesic metric space X_v. Further, the inclusions $i_v : X_v \to X$ are uniformly proper, i.e. for all $M > 0$, $v \in T$ and $x, y \in X_v$, there exists $N > 0$ such that $d(i_v(x), i_v(y)) \leq M$ implies $d_{X_v}(x,y) \leq N$.
(2) Let $e \in \mathcal{E}(T)$ with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point of e. There exist continuous maps $f_e : X_e \times [0,1] \to X$, such that $f_e|_{X_e \times (0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric. Further, f_e is fiber-preserving, i.e. projection to the second co-ordinate in $X_e \times [0,1]$ corresponds via f_e to projection to the tree $P : X \to T$.
(3) Identifying e with $[0,1]$, $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ are (K,ϵ)-quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ will occasionally be referred to as f_{e,v_1} and f_{e,v_2} respectively.

K, ϵ will be called the constants or parameters of the qi-embedding condition.

A tree of spaces $P : X \to T$ as in Definition 4.1 above is said to be a tree of hyperbolic metric spaces, if there exists $\delta > 0$ such that the vertex and edge spaces X_v, X_e are all δ-hyperbolic for all vertices v and edges e of T.

Definition 4.2. [BF92] A disk $f : [-m,m] \times I \to X$ is a **hallway** of length $2m$ if it satisfies:

(1) $f^{-1}(\cup X_v : v \in T) = \{-m, \cdots, m\} \times I$
(2) f maps $i \times I$ to a geodesic in X_v for some vertex space X_v.
(3) f is transverse, relative to condition (1) to $\cup_e X_e$.

Definition 4.3. [BF92] A hallway $f : [-m,m] \times I \to X$ is **ρ-thin** if $d(f(i,t), f(i+1,t)) \leq \rho$ for all i, t.

A hallway $f : [-m,m] \times I \to X$ is said to be **λ-hyperbolic** if

$$\lambda l(f(\{0\} \times I)) \leq \max \{l(f(\{-m\} \times I)), l(f(\{m\} \times I)).$$

The quantity $\min_i \{l(f(\{i\} \times I))\}$ is called the **girth** of the hallway.

A hallway is **essential** if the edge path in T resulting from projecting the hallway under $P \circ f$ onto T does not backtrack (and is therefore a geodesic segment in the tree T).

Definition 4.4. Hallways flare condition [BF92]**:** The tree of spaces, X, is said to satisfy the **hallways flare** condition if there are numbers $\lambda > 1$ and $m \geq 1$ such that for all ρ there is a constant $H := H(\rho)$ such that any ρ-thin essential hallway of length $2m$ and girth at least H is λ-hyperbolic. In general, λ, m will be called the constants of the hallways flare condition. If, in addition ρ is fixed, H will also be called a constant of the hallways flare condition.

We recall the notion of a metric bundle from [MS12]:

Definition 4.5. Let (X, d_X) and (B, d_B) be geodesic metric spaces. Let $c, K \geq 1$ be constants and $h : \mathbb{R}^+ \to \mathbb{R}^+$ a function. $P : X \to B$ is called an $(h, c, K)-$ **metric bundle** if

(1) P is 1-Lipschitz.
(2) For each $z \in B$, $X_z = P^{-1}(z)$ is a geodesic metric space with respect to the path metric d_z induced from (X, d_X). Further, we require that the inclusion maps $i_z : (X_z, d_z) \to X$ are uniformly metrically proper as measured with respect to h, i.e. for all $z \in B$ and $u, v \in X_z$, $d_X(i_z(u), i_z(v)) \leq N$ implies that $d_z(u, v) \leq f(N)$.
(3) For $z_1, z_2 \in B$ with $d_B(z_1, z_2) \leq 1$, let γ be a geodesic in B joining them. Then for any $z \in \gamma$ and $x \in X_z$, there is a path in $p^{-1}(\gamma)$ of length at most c joining x to both X_{z_1} and X_{z_2}.
(4) For $z_1, z_2 \in B$ with $d_B(z_1, z_2) \leq 1$ and $\gamma \subset B$ a geodesic joining them, let $\phi : X_{z_1} \to X_{z_2}$, be a(ny) map such that for all $x_1 \in X_{z_1}$ there is a path of length at most c in $P^{-1}(\gamma)$ joining x_1 to $\phi(x_1)$. Then ϕ is a $K-$quasi-isometry.

If in addition, there exists δ' such that each X_z is $\delta'-$hyperbolic, then $P : X \to B$ is called an $(h, c, K)-$ metric bundle of $\delta'-$hyperbolic spaces.

It is pointed out in [MS12] that condition (4) follows from the previous three (with some K); but it is more convenient to have it as part of our definition. For any hyperbolic metric space F with more than two points in its Gromov boundary ∂F, there is a coarse **barycenter map** $\phi : \partial^3 F \to F$ mapping any unordered triple (a, b, c) of distinct points in ∂F to a centroid of the ideal triangle spanned by (a, b, c). We shall say that the barycenter map $\phi : \partial^3 F \to F$ is $N-$coarsely surjective if F is contained in the N-neighborhood of the image of ϕ. A $K-$**qi-section** $\sigma : B \to X$ is a $K-$qi-embedding from B to X such that $P \circ \sigma$ is the identity map. The following Proposition guarantees the existence of qi-sections for metric bundles.

Proposition 4.6. [MS12, Section 2.1] *For all $\delta^{'}, N, c, K \geq 0$ and proper $f : \mathbb{N} \to \mathbb{N}$ there exists K_0 such that the following holds.*
Suppose $p : X \to B$ is an (f, c, K)-metric bundle of $\delta'-$hyperbolic spaces such that the barycenter maps $\phi_b : \partial^3 F_b \to F_b$ are uniformly $N-$coarsely surjective, Then there is a K_0-qi section through each point of X.

Remark 4.7. A word of clarification is necessary regarding the relationship between

(1) Metric bundles over trees in the sense of Definition 4.5, and
(2) A tree of spaces satisfying the qi-embedded condition in the sense of Definition 4.1 with the additional restriction that the edge-space to vertex-space maps in Item (3) of Definition 4.1 are (K, ϵ)-quasi-isometries rather than just (K, ϵ)-quasi-isometric embeddings. We refer to such a tree of spaces as a **homogeneous** tree of spaces.

It is clear that a homogeneous tree of spaces is an example of a metric bundle over a tree. The converse is not, strictly speaking, true as the metric on fibers F_b in Definition 4.5 is allowed to change continuously.

However, all the underlying trees BU(T) of metric bundles (Definition 2.16) occurring in this paper can be assumed to be simplicial trees (with edges of length one) as they approximate geodesic polygons in curve complexes. Further, as shown in [MS12, Lemma 1.21], any metric bundle over a tree can be approximated by a homogeneous tree of spaces. (In [MS12] a more general result was proven approximating general metric bundles by metric graph bundles.) The constants (K, ϵ) occurring in Definition 4.5 are then determined by the parameters (h, c, K) occurring in Definition 4.5.

We shall thus assume henceforth, without mentioning it explicitly, that whenever we are talking of a metric bundle over a tree as a homogeneous tree of spaces, we have approximated the former by the latter as in [MS12, Lemma 1.21].

We shall now state the main theorem of [BF92] in an effective form, using [Gau16, Theorem 2.20] where the proof does not require uniform properness of the space. A converse may be found in [GH09, Theorem 2] (see also [Ger98, Bow02]). We shall however, state the theorem and its converse [MS12, Section 5.3] in the restrictive setting of a metric bundle over a tree, where it is easier to state.

Theorem 4.8. *Suppose that there exist $\delta_0 \geq 0$ and $\rho \geq 1$ such that $P : X \to T$ is a metric bundle over a tree satisfying the following conditions:*

(1) *X_z is δ_0-hyperbolic, for every $z \in T$.*
(2) *through every $x \in X$ there is a $\rho-$qi-section $\sigma_x : T \to X$.*

Then given $K_0, \epsilon_0, \lambda_0, m_0, H_0$ there exists $\delta > 0$ such that the following holds:
If X satisfies the qi-embedded condition with constants $K \leq K_0, \epsilon \leq \epsilon_0$ and the hallways flare condition with constants $\lambda \geq \lambda_0, m \leq m_0, H \leq H_0$ for hallways bounded by $\rho-$qi-sections, then X is $\delta-$hyperbolic.

Conversely, given $\delta > 0$, there exist $K_0 \geq 1, \epsilon_0 \geq 0$ and $\lambda_0 > 1, m_0 \in \mathbb{N}, H_0 \geq 0$ such that if X is $\delta-$hyperbolic, then as a tree of hyperbolic metric spaces X satisfies

(1) *the qi-embedded condition with constants $K \leq K_0, \epsilon \leq \epsilon_0$.*
(2) *hallways bounded by $\rho-$qi-sections satisfy the flare condition with constants $\lambda \geq \lambda_0, m \leq m_0, H \leq H_0$.*

4.2. **Effective relative hyperbolicity.** We shall also need to quantify relative hyperbolicity. If X is strongly hyperbolic relative to a collection $\mathcal{H}$ of parabolic subsets (see [Far98, Bow12] for definitions) we can attach a hyperbolic cone H_h to each $H \in \mathcal{H}$ as follows.

Definition 4.9. For any geodesic metric space (H, d), the *hyperbolic cone* (analog of a horoball) H^h is the metric space $H \times [0, \infty) = H^h$ equipped with the path metric d_h obtained from two pieces of data
1) $d_{h,t}((x,t),(y,t)) = 2^{-t} d_H(x,y)$, where $d_{h,t}$ is the induced path metric on $H \times \{t\}$. Paths joining $(x,t),(y,t)$ and lying on $H \times \{t\}$ are called *horizontal paths.*
2) $d_h((x,t),(x,s)) = |t-s|$ for all $x \in H$ and for all $t, s \in [0, \infty)$, and the corresponding paths are called *vertical paths.*
3) for all $x, y \in H^h$, $d_h(x,y)$ is the path metric induced by the collection of horizontal and vertical paths.

Definition 4.10. Let X be a geodesic metric space and $\mathcal{H}$ be a collection of mutually disjoint uniformly separated subsets of X. X is said to be strongly hyperbolic relative to $\mathcal{H}$, if the quotient space $\mathcal{G}(X, \mathcal{H})$, obtained by attaching the hyperbolic cones H^h to $H \in \mathcal{H}$ by identifying $(z, 0)$ with z for all $H \in \mathcal{H}$ and $z \in H$, is a complete hyperbolic metric space. The collection $\{H^h : H \in \mathcal{H}\}$ is denoted as $\mathcal{H}^h$. The induced path metric is denoted as d_h.

As per Bowditch's definition of relative hyperbolicity [Bow12] following Gromov [Gro85], X is strongly hyperbolic relative to $\mathcal{H}$ if $\mathcal{G}(X, \mathcal{H})$ is hyperbolic. We make this effective as follows:

Definition 4.11. We say that X is strongly $\delta-$hyperbolic relative to a collection $\mathcal{H}$ of parabolic subsets if $\mathcal{G}(X, \mathcal{H})$ is $\delta-$hyperbolic.

4.2.1. *Partial Electrification.* In this subsection, we give a quantitative version of the notion of partial electrification following [MR08, MP11, MS12].

Definition 4.12. Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple such that the following holds for some $K, \epsilon, \delta > 0$:

(1) X is a geodesic metric space. $\mathcal{H}$ is a collection of subsets H_α of X. X is strongly $\delta-$hyperbolic relative to $\mathcal{H}$.
(2) $\mathcal{L}$ is a collection of δ-hyperbolic metric spaces L_α and $\mathcal{G}$ is a collection of coarse $(K, \epsilon)-$Lipschitz maps $g_\alpha : H_\alpha \to L_\alpha$. Note that the indexing set for $H_\alpha, L_\alpha, g_\alpha$ is common.

The **partially electrified space** or *partially coned off space* $\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ corresponding to $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ is obtained from X by gluing in the (metric) mapping cylinders for the maps $g_\alpha : H_\alpha \to L_\alpha$. The metric on $\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ is denoted by d_{pel}.

In the particular case that each L_α is a point and g_α is a constant map, this gives back the electrified, or **coned-off** space $\mathcal{E}(X, \mathcal{H})$ in the sense of Farb [Far98]. For the next two statements, see [MP11, Lemmas 1.20. 1.21], (also [MR08],[MS12, Lemma 1.50]).

Lemma 4.13. *For $K, \epsilon, \delta > 0$ there exists δ', C such that the following holds: Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple as in Definition 4.12 above with constants $K, \epsilon, \delta > 0$. Then $(\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L}), d_{pel})$ is a $\delta'-$hyperbolic metric space and the sets L_α are $C-$quasiconvex.*

Lemma 4.14. *Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple with constants as in Definition 4.12 above. Given $K_0, \epsilon_0 \geq 0$, there exists $C_0 > 0$ such that the following holds:*
Let γ_{pel} and γ denote respectively a (K_0, ϵ_0) partially electrified quasigeodesic in $(\mathcal{PE}(X, \mathcal{H}, \mathcal{G}, \mathcal{L}), d_{pel})$ and a (K_0, ϵ_0)quasigeodesic in $(\mathcal{G}(X, \mathcal{H}), d_h)$ joining a, b. Then $\gamma \setminus \bigcup_{H_\alpha \in \mathcal{H}} H_\alpha$ lies in a C-neighborhood of (any representative of) γ_{pel} in (X, d). Further, outside of a C-neighborhood of the horoballs that γ meets, γ and γ_{pel} track each other, i.e. lie in a C-neighborhood of each other.

4.3. **Effective relatively hyperbolic combination theorem.** We follow [Gau16] and [MR08] here and subsequently indicate the modifications needed for us.

Definition 4.15. A tree $P : X \to T$ of geodesic metric spaces is said to be a tree of relatively hyperbolic metric spaces if in addition to the conditions of Definition 4.1

(4) each vertex space X_v is strongly hyperbolic relative to a collection of subsets $\mathcal{H}_v$ and each edge space X_e is strongly hyperbolic relative to a collection of subsets $\mathcal{H}_e$. The individual sets $H_{v,\alpha} \in \mathcal{H}_v$ or $H_{e,\alpha} \in \mathcal{H}_e$ will be called **horosphere-like sets**.

(5) the maps f_{e,v_i} above $(i = 1, 2)$ are **strictly type-preserving**, i.e. $f_{e,v_i}^{-1}(H_{v_i,\alpha})$, $i = 1, 2$ (for any $H_{v_i,\alpha} \in \mathcal{H}_{v_i}$) is either empty or some $H_{e,\beta} \in \mathcal{H}_e$. Also, for all $H_{e,\beta} \in \mathcal{H}_e$, there exists v and $H_{v,\alpha}$, such that $f_{e,v}(H_{e,\beta}) \subset H_{v,\alpha}$.
(6) There exists $\delta > 0$ such that each $\mathcal{E}(X_v, \mathcal{H}_v)$ is δ-hyperbolic.
(7) The induced maps (see below) of the coned-off edge spaces into the coned-off vertex spaces $\widehat{f_{e,v_i}} : \mathcal{E}(X_e, \mathcal{H}_e) \to \mathcal{E}(X_{v_i}, \mathcal{H}_{v_i})$ $(i = 1, 2)$ are uniform quasi-isometries. This is called the **qi-preserving electrification condition**

Given the tree of spaces with vertex spaces X_v and edge spaces X_e there exists a naturally associated tree whose vertex spaces are $\mathcal{E}(X_v, \mathcal{H}_v)$ and edge spaces are $\mathcal{E}(X_e, \mathcal{H}_e)$ obtained by simply coning off the respective horosphere like sets. Condition (4) of the above definition ensures that we have natural inclusion maps of edge spaces $\mathcal{E}(X_e, \mathcal{H}_e)$ into adjacent vertex spaces $\mathcal{E}(X_v, \mathcal{H}_v)$.

The resulting tree of coned-off spaces $P : \mathcal{TC}(X) \to T$ will be called the **induced tree of coned-off spaces**. The resulting space will thus be denoted as $\mathcal{TC}(X)$ when thought of as a tree of spaces. The **cone locus** of $\mathcal{TC}(X)$ is the graph (actually a forest) whose vertex set $\mathcal{V}$ consists of the cone-points c_v in the vertex set and whose edge-set $\mathcal{E}$ consists of the cone-points c_e in the edge set.

Each such connected component of the cone-locus will be called a **maximal cone-subtree**. The collection of *maximal cone-subtrees* will be denoted by $\mathcal{T}$ and elements of $\mathcal{T}$ will be denoted as T_α. Further, each maximal cone-subtree T_α naturally gives rise to a tree T_α of horosphere-like subsets depending on which cone-points arise as vertices and edges of T_α. The metric space that T_α gives rise to will be denoted as C_α and will be referred to as a **maximal cone-subtree of horosphere-like spaces**. The induced tree of horosphere-like sets will be denoted as $g_\alpha : C_\alpha \to T_\alpha$. The collection of these maps will be denoted as $\mathcal{G}$. The collection of C_α's will be denoted as $\mathcal{C}$. Note thus that each T_α thus appears in two guises:
1) as a subset of $\mathcal{TC}(X)$
2) as the underlying tree of C_α

An essential hallway of length $2m$ is **cone-bounded** if $f(i \times \partial I)$ lies in the cone-locus for $i = \{-m, \cdots, m\}$.

Definition 4.16. Cone-bounded hallways strictly flare condition: The tree of spaces, X, is said to satisfy the *cone-bounded hallways flare* condition if there are numbers $\lambda > 1$ and $m \geq 1$ such that any cone-bounded hallway of length $2m$ is λ-hyperbolic. λ, m will be called the constants of the strict flare condition.

Theorem 4.17. [MR08, Gau16] *Given $K_0 \geq 1, \epsilon_0 \geq 0, \delta_0 \geq 0, \lambda_0 > 1, m_0 \geq 1, \rho_0 > 1, H_0 \geq 0$ there exists $\delta > 0$ such that the following holds: Let $P : X \to T$ be a metric bundle over a tree such that*

(1) *X_z is δ_0-relatively hyperbolic, for every $z \in T$.*
(2) *through every $x \in X$ there is a ρ_0-qi-section $\sigma_x : T \to X$.*

If X satisfies the qi-embedded condition with constants $K \leq K_0, \epsilon \leq \epsilon_0$, the hallways flare condition with constants $\lambda \geq \lambda_0, m \leq m_0, H \leq H_0$ with respect to hallways bounded by ρ_0-qi-sections, and the cone-bounded hallways strictly flare condition with parameters $\lambda \geq \lambda_0, m \leq m_0$, then X is $\delta-$relatively hyperbolic.

4.4. M_T **as a bundle over** $\mathrm{BU}(T)$**.** We shall now specialize and adapt the above results to the case that will be of relevance to us:

(1) $P:(\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$, and
(2) $P:(\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$.

Remark 4.18. A word of caution is necessary here. It is easy to see that $P:(\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$ is a metric bundle as per Definition 4.5. However, $P:(\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ violates the properness condition (Item 2 in Definition 4.5). It also violates condition 5 (the strictly type-preserving condition) and hence condition 7 (the qi-preserving electrification condition) of Definition 4.15. We therefore need a way around these conditions. Instead of doing this in the fullest possible generality we shall simply focus on the relevant example, namely

$$P:(\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T),$$

and proceed to check the properties of metric bundles and trees of relatively hyperbolic spaces that go through. Much of the discussion in the remainder of this subsection is aimed at addressing the issue just discussed and pointing out adaptations of existing arguments in the literature (particularly [Gau16, GH09, MS12]) that help us circumvent it.

We first observe that through every point of $(\widetilde{M_T}, d_{weld}), (\widetilde{M_T}, d_{te})$ there exist uniform qi-sections.

Lemma 4.19. *Given $g \geq 2$, there exists ρ_0 such that the following holds. Let $P:(\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$ and $P:(\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ be as in Definition 3.33 with fiber S of genus g. Then through every $x \in (\widetilde{M_T}, d_{weld})$ and $x \in (\widetilde{M_T}, d_{te})$ there exists a ρ_0-qi-section.*

Proof. Since the partial electrification map from $(\widetilde{M_T}, d_{weld})$ to $(\widetilde{M_T}, d_{te})$ is 1-Lipschitz, it suffices to prove the Lemma for $P:(\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$. Further, since sections can be lifted from (M_T, d_{weld}) to $(\widetilde{M_T}, d_{weld})$, it suffices to prove the Lemma for $P:(M_T, d_{weld}) \to \mathrm{BU}(T)$.

For a building block M_v of special split geometry and $P: M_v \to T_v$ the natural projection onto the associated tree-link, there is an isometric section $\sigma_v: T_v \to M_v$ lying inside the Margulis riser $\mathcal{R}_v = S^1 \times T_v$ since the latter is a metric product. The fibers $P^{-1}(z)$ of $P:(M_T, d_{weld}) \to \mathrm{BU}(T)$ have diameter bounded by some $D = D(g)$, by the Gauss-Bonnet Theorem. Choosing $\rho_0 = 2D+1$, we can construct a ρ_0-qi-section from $\mathrm{BU}(T)$ to (M_T, d_{weld}) by connecting the sections σ_v using paths lying in the mid-surfaces. □

Ladders in trees of spaces: We shall need the technology of ladders from [Mit98b, Mit98a] below. We extract the necessary features from the ladder construction of [Mit98b, Mit98a] and adapt it here to the language of hallways. The following is a restatement of [Mit98b, Theorem 3.6] in our context (see also the construction of the ladder in [Mit98b, Section 3]). The corresponding statement for $(\widetilde{M_T}, d_{te})$ follows from Lemma 4.19.

Theorem 4.20. *Given $\delta \geq 0, K \geq 1, \epsilon \geq 0$ there exists D such that the following holds.*
Let

(1) *(X, d) be either a tree of $\delta-$hyperbolic spaces as in Definition 4.1 with parameters K, ϵ and let X_v be a vertex space,*

(2) *or* $X = (\widetilde{M_T}, d_{te})$ *with* $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$, *and* $X_v = P^{-1}(v)$ *for some* $v \in \mathrm{BU}(T)$.

Then for every geodesic segment $\mu \subset (X_v, d_v)$ *there exists a* $D-$*qi-embedded subset* $\mathcal{L}_\mu$ *of* X *such that the following holds.*

(1) $X_v \cap \mathcal{L}_\mu = \mu$,
(2) *For* $X = (\widetilde{M_T}, d_{te})$ *and every* $w \in \mathrm{BU}(T)$, $X_w \cap \mathcal{L}_\mu$ *is a geodesic* μ_w *in* (X_w, d_w).
(3) *For* X *a tree of hyperbolic metric spaces and every* $w \in T$, $X_w \cap \mathcal{L}_\mu$ *is either empty or a geodesic* μ_w *in* (X_w, d_w). *Further, there exists a subtree* $T_1 \subset T$ *such that the collection of vertices* $w \in T$ *satisfying* $X_w \cap \mathcal{L}_\mu \neq \emptyset$ *equals the vertex set of* T_1.
(4) *There exists* $\rho_0 \geq 1$ *such that through every* $z \in \mathcal{L}_\mu$, *there exists a* ρ_0-*qi-section* σ_z *of* $[v, P(z)]$ *contained in* $\mathcal{L}_\mu$ *satisfying*

$$\sigma_z(P(z)) = z, \quad \sigma_z(v) \in \mu.$$

(5) *There exist constants* λ_0, m_0, H_0 *such that for every* $\mu_w = X_w \cap \mathcal{L}_\mu$ *the following holds:*
There is a hallway $\mathcal{H}_w$ *bounded by* ρ_0-*qi-sections as in (4) above containing* μ_w *satisfying the hallways flare condition with* $\lambda \geq \lambda_0, m \leq m_0, H \leq H_0$. *Further,* $\mathcal{H}_w \cap X_v$ *is a geodesic subsegment of* μ.

Further, there exists a $D-$*coarse Lipschitz retraction* $\Pi_\mu : X \to \mathcal{L}_\mu$, *i.e.*

(1) $d(\Pi_\mu(x), \Pi_\mu(y)) \leq Dd(x, y) + D$, $\forall\, x, y \in X$,
(2) $\Pi_\mu(x) = x$, $\forall\, x \in \mathcal{L}_\mu$.

The qi-embedded set $\mathcal{L}_\mu$ is called a **ladder** in [Mit98a, Mit98b]. Theorem 4.20 shows in particular that there is a $(2D, 2D)-$ quasigeodesic of (X, d_X) joining the end-points of μ and lying on $\mathcal{L}_\mu$.

Remark 4.21. Note that in Theorem 4.20, we have **not** assumed that X is hyperbolic: no assumptions on the global geometry of X are necessary here.

Ladders in $(\widetilde{M_T}, d_{te})$ **or** $(\widetilde{M_T}, d_{weld})$**:** Given two ρ_0-qi-sections $X_1, X_2 \subset (\widetilde{M_T}, d_{te})$ or $(\widetilde{M_T}, d_{weld})$ as in Lemma 4.19, we construct a **ladder** $C(X_1, X_2)$ by joining the points $X_1 \cap F_b$ and $X_1 \cap F_b$ by a geodesic in F_b (see [MS12, Section 2.2]). The coarse Lipschitz retraction property of Theorem 4.20 goes through in this context also. Further, in Theorem 4.20, the constant D depends only on δ, K, ϵ. By Remark 4.7 we can pass from a metric bundle to a homogeneous tree of spaces. Unraveling definitions, K, ϵ depend on R and parameters D, k in Definition 2.18 of an $L-$tight $R-$thick balanced tree. Now, for $(\widetilde{M_T}, d_{te})$ or $(\widetilde{M_T}, d_{weld})$, δ depends only on the genus g. Thus we have the following:

Lemma 4.22. *Given* $g \geq 2$ *and* $R, D, k \geq 1$, *there exists* ρ_0 *such that the following holds.*
Let $i : V(T) \to \mathcal{C}_\Delta(S)$ *be an* $L-$*tight,* $R-$*thick balanced tree with parameters* D, k *as in Definition 2.18. Let* $C(X_1, X_2)$ *be a ladder in* $(\widetilde{M_T}, d_{weld})$ *or* $(\widetilde{M_T}, d_{te})$ *as above. Then through every* $x \in C(X_1, X_2)$, *there exists a* ρ_0-*qi-section contained in* $C(X_1, X_2)$.

The definition of hallways (Definition 4.2) now continues to make sense for $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ and $P : (\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$ with the following modification:

the maps $f : [-m, m] \times \{0\} \to (\widetilde{M_T}, d_{te})$, $f : [-m, m] \times \{1\} \to (\widetilde{M_T}, d_{te})$ (or $(\widetilde{M_T}, d_{weld})$) in Definition 4.2 are restrictions of ρ_0-qi-sections from $\mathrm{BU}(T)$ to $(\widetilde{M_T}, d_{te})$, where ρ_0 is as in Lemma 4.22. Note that $P \circ f$ is an isometry onto its image. To distinguish from the hallways of Definition 4.2, we shall call them **qi-section bounded hallways**. With this clarification, the flaring condition of Definition 4.4 continues to make sense for $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ and qi-section bounded hallways. We now state the following consequence of Theorem 4.8 in the form that we shall need it:

Corollary 4.23. *Given $\lambda_0, m_0, H_0, \delta_0$ there exists $\delta > 0$ such that the following holds:*
For $b \in \mathrm{BU}(T)$, let $F_b = P^{-1}(b)$ equipped with the induced path metric and suppose that F_b is δ_0-hyperbolic for all $b \in \mathrm{BU}(T)$. If $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ satisfies the flare condition with constants $\lambda \geq \lambda_0, 1 \leq m \leq m_0, 1 \leq H \leq H_0$ for qi-section bounded hallways, then $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic.

Proof. The proof is a transcription of the relevant steps from [MS12] and [Gau16, Theorem 5.2] and we only give a sketch.

Step 1: Lemma 4.19 guarantees the existence of ρ_0-qi-sections through every point and ρ_0-qi-sections in ladders. This replaces [MS12, Proposition 2.12].

Step 2: Now [MS12, Theorem 3.2] shows that $C(X_1, X_2)$ is $C-$qi-embedded in $(\widetilde{M_T}, d_{te})$ where C depends only on δ_0 and the parameters R, D, k of the $L-$tight $R-$thick balanced tree (see Definition 2.18 and also Lemma 4.22 for the dependence on constants).

Step 3: Then $C(X_1, X_2)$ is a bundle over $\mathrm{BU}(T)$ with fibers closed intervals. Further, it satisfies the flare condition with respect to qi-section bounded hallways. We now invoke Theorem 4.8 to conclude that there exists δ_1 such that each $C(X_1, X_2)$ is δ_1-hyperbolic. Note that it is at this step that we are circumventing the use of properness of the metric bundle as in [MS12, Section 3] by using [Gau16] instead, cf. Remark 4.18. We recall that the proof given by Gautero of Theorem 4.8 in [Gau16, Theorems 2.20, 5.2] does not use properness of the total space and proceeds by directly deducing effective hyperbolicity from exponential divergence of geodesics. The last condition (exponential divergence of geodesics) in turn is an immediate consequence of flaring. In particular, the proof in [Gau16] does not go via the original linear isoperimetric inequality proof of [BF92].

Step 4: The rest of the proof follows [MS12, Section 4]. Given any 3 points, $x, y, z \in (\widetilde{M_T}, d_{te})$, let X_x, X_y, X_z be ρ_0-qi-sections through x, y, z respectively. The union of the ladders $C(X_x, X_y)$, $C(X_y, X_z)$, $C(X_x, X_z)$ is denoted as $C(X_x, X_y, X_z)$ is called a **tripod-bundle** in [MS12, Definition 4.1]. Let $\phi_b(x, y, z)$ denote a barycenter in F_b of $(X_x \cap F_b), (X_y \cap F_b), (X_z \cap F_b)$. Then the set

$$X_b := \{\phi_b(x, y, z) | b \in \mathrm{BU}(T)\}$$

gives a qi section [MS12, Proposition 4.2]. The tripod-bundle $C(X_x, X_y, X_z)$ can be δ_0-approximated by the union of three ladders $C(X_b, X_x), C(X_b, X_y), C(X_b, X_z)$

and any two of them intersect along X_b.

Step 5: By Step (3) above, each of $C(X_b, X_x), C(X_b, X_y), C(X_b, X_z)$ is δ_1-hyperbolic and they all intersect along the qi-embedded subset X_b. Hence by Theorem 4.8, there exists δ_2 depending only on δ_1 and the qi-embeddedness constant ρ_0 of X_b (see Lemma 4.22) such that

$$C(X_b, X_x) \cup C(X_b, X_y) \cup C(X_b, X_z)$$

is δ_2-hyperbolic.

Step 6: Finally, by a standard path-family argument (see [MS12, Theorem 4.3]) $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic, where δ depends only on δ_0 (the hyperbolicity constant of fiber spaces) R, D, k (the parameters of the $L-$tight $R-$thick balanced tree). □

For the converse direction, we refer the reader to Section 5.3 of [MS12], which proves the necessity of flaring. We briefly indicate how to adapt the argument here. First, $\delta-$hyperbolicity of $(\widetilde{M_T}, d_{te})$ guarantees that there exists H (depending on δ) such that qi-section-bounded hallways of girth (cf. Definition 4.3) lying between H and $H+1$ flare (see Lemma 5.9 of [MS12]) so long as ρ_0 is chosen (again depending on δ) to ensure that ρ_0-thin hallways exist connecting a point of $P^{-1}(z_1)$ to some point of $P^{-1}(z_2)$ for any $z_1, z_2 \in \mathrm{BU}(T)$ with $d_{\mathrm{BU}(T)}(z_1, z_2) \leq 1$. Next, [MS12] (see the paragraph in [MS12, Section 5.3] called 'Flaring of general ladders') shows how to decompose a general hallway into flaring hallways of girth between H and H. Thus we conclude the converse direction of Theorem 4.8 for $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$:

Corollary 4.24. *Given $\delta > 0, \rho_0$, there exist λ_0, m_0, H_0 such that the following holds:*
If $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic and ρ_0 is as in Lemma 4.19, then $(\widetilde{M_T}, d_{te})$ satisfies the hallways flare condition with respect to ρ_0-qi-section bounded hallways, with constants $\lambda \geq \lambda_0, m \leq m_0, H \leq H_0$.

Finally, we shall combine Corollary 4.24 with Lemma 4.13. To do this, observe that for $P : (M_T, d_{weld}) \to \mathrm{BU}(T)$, any tree-link $T_v \subset \mathrm{BU}(T)$, $z \in T_v$, the pre-image $P^{-1}(z) = S_z$ is of uniformly bounded geometry. Hence,

(1) The fibers $(\widetilde{S_z}, d_{weld})$ of $P : (\widetilde{M_T}, d_{weld}) \to \mathrm{BU}(T)$ are uniformly hyperbolic.
(2) The fibers $(\widetilde{S_z}, d_{te})$ of $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ are uniformly hyperbolic as these are obtained by electrifying uniformly separated (independent of z) uniform quasigeodesics (again with constant independent of z) in $(\widetilde{S_z}, d_{weld})$.

We denote the collection of Margulis risers as

$$\mathcal{R}_{\mathcal{M}} \ := \ \{v \times T_v | T_v \subset \mathrm{BU}(T) \text{ is a tree} - \text{link}\},$$

and the set of all lifts of $\mathcal{R}_{\mathcal{M}}$ to $\widetilde{M_T}$ as $\widetilde{\mathcal{R}_{\mathcal{M}}}$.

Proposition 4.25. *Given $\delta, \rho_0 > 0$, there exist $\lambda_0 > 1, m_0 \geq 1, H_0 \geq 0$ and $C \geq 0$ such that the following holds:*
If $(\widetilde{M_T}, d_{weld})$ is strongly $\delta-$hyperbolic relative to $\widetilde{\mathcal{R}_{\mathcal{M}}}$ and ρ_0 is as in Lemma 4.19, then $(\widetilde{M_T}, d_{te})$ satisfies the hallways flare condition with respect to ρ_0-qi-section

bounded hallways, with constants $\lambda \geq \lambda_0, m \leq m_0, H \leq H_0$. Further, each element of $\widetilde{\mathcal{R}_\mathcal{M}}$ is $C-$quasiconvex in $(\widetilde{M_T}, d_{te})$.

Proof. We first observe that $(\widetilde{M_T}, d_{te})$ is obtained from $(\widetilde{M_T}, d_{weld})$ by partially electrifying the $\mathbb{R}-$directions in $\mathbb{R} \times T_v$ for every lift $\mathbb{R} \times T_v$ of a Margulis riser to $\widetilde{M_T}$. We now consider the quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ with

(1) $(\widetilde{M_T}, d_{weld})$ in place of X,
(2) $\widetilde{\mathcal{R}_\mathcal{M}}$ in place of $\mathcal{H}$,
(3) Indexing the elements of $\widetilde{\mathcal{R}_\mathcal{M}}$ by $\widetilde{\mathcal{R}_{\mathcal{M}\alpha}}$, define

$$g_\alpha : (\widetilde{\mathcal{R}_{\mathcal{M}\alpha}}, d_{weld}) \to (\widetilde{\mathcal{R}_{\mathcal{M}\alpha}}, d_{te})$$

to be the map that partially electrifies the $\mathbb{R}-$directions in $\mathbb{R} \times T_v$ for every lift $\mathbb{R} \times T_v$ of a Margulis riser. Then $\mathcal{G}$ is the collection of maps g_α and $\mathcal{L}$ is the collection of spaces $(\widetilde{\mathcal{R}_{\mathcal{M}\alpha}}, d_{te})$.

Lemma 4.13 applied to this quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ then shows that there exist $\delta_0, C \geq 0$ such that

(1) $(\widetilde{M_T}, d_{te})$ is δ_0-hyperbolic.
(2) $(\widetilde{\mathcal{R}_{\mathcal{M}\alpha}}, d_{te})$ is $C-$quasiconvex in $(\widetilde{M_T}, d_{te})$ for every α.

This proves the last statement of the proposition. The first statement now follows from Corollary 4.24. □

4.5. **Effective quasiconvexity and flaring.** The main purpose of this subsection is to prove Proposition 4.27. We shall apply it in its full strength in the companion paper [MMS19]. For the purposes of this paper, it is used mildly in the proofs of Propositions 5.15 and 5.17. Proposition 4.27 may be regarded as a fact supplementing the effective hyperbolicity and relative hyperbolicity Theorems 4.8 and 4.17.

For the purposes of this subsection, X will be

(1) Either a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition with constants K, ϵ and the hallways flare condition with constants λ_0, m_0. Further, if ρ_0 is given we shall assume an additional constant H_0 as a lower bound for girths of ρ_0-thin hallways. X is equipped with the usual projection map $P : X \to T$.
(2) OR $(\widetilde{M_T}, d_{te})$ corresponding to an $L-$tight $R-$thick tree T. $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ will denote the usual projection map. The constant ρ_0 will be as in Lemma 4.19 and the constants λ_0, m_0, H_0 will be as in Corollary 4.24.

Also (X_v, d_v) will, respectively, be a vertex space of X (in the tree of spaces case) or $P^{-1}(v)$ (in the $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$ case) and $Y \subset (X_v, d_v)$ will be a $C-$quasiconvex subset of (X_v, d_v).

Definition 4.26. We shall say that Y **flares in all directions with parameter** K if for any geodesic segment $[a, b] \subset (X_v, d_v)$ with $a, b \in Y$ and any $\rho-$thin hallway $f : [0, k] \times I \to X$ satisfying

(1) $\rho \leq \rho_0$,
(2) $f(\{0\} \times I) = [a, b]$,
(3) $l([a, b]) \geq K$,
(4) $k \geq K$,

the length of $f(\{k\} \times I)$ satisfies

$$l(f(\{k\} \times I)) \geq \lambda l([a,b]).$$

Proposition 4.27 below is probably well-known to experts (at least for trees of spaces) but we could not find an explicit statement in the literature.

Proposition 4.27. *Given K, C, there exists C_0 such that the following holds. Let $P : X \to T$ (or $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$) and X_v be as in Theorem 4.20 above. If Y is a $C-$quasiconvex subset of (X_v, d_v) and flares in all directions with parameter K, then Y is C_0-quasiconvex in (X, d_X).*

Conversely, given C_0, there exist K, C such that the following holds. For $P : X \to T$ (or $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$) and X_v as above, if $Y \subset X_v$ is C_0-quasiconvex in (X, d_X), then it is $C-$quasiconvex subset in (X_v, d_v) and flares in all directions with parameter K.

Proof. We first prove the forward direction. If the conclusion fails, then though Y flares in all directions, it is not quasiconvex in (X, d_X). In particular, for every $n \in \mathbb{N}$, there exists $\mu \subset X_v$ with end-points in Y such that there exists a $(2D, 2D)-$quasigeodesic μ^R (of (X, d_X)) joining the end-points of μ, lying on $\mathcal{L}_\mu$ and leaving the $n-$neighborhood of μ. Hence there exists a vertex w of T such that

(1) $d_T(v, w) = O(n)$,
(2) $\mu^R \cap X_w$ contains a pair of points a', b' such that $d_w(a', b')$ is minimal amongst lengths that exceed the minimal girth ($H(\rho_0)$ in Definition 4.4) required for flaring (see figure below). Since the flaring constant λ is fixed, it follows that $d_w(a', b') \leq \lambda H(\rho_0)$; in particular, $d_w(a', b')$ is uniformly bounded.

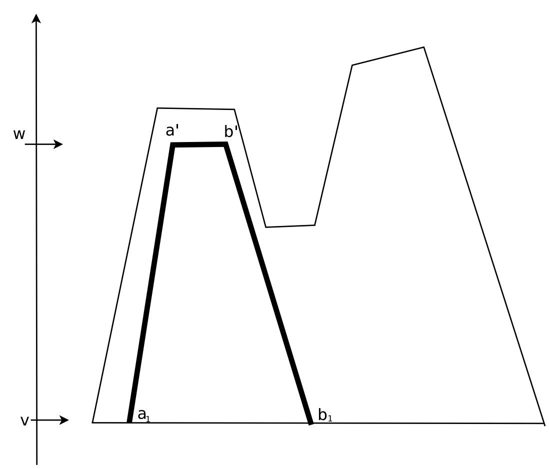

Figure: *Finding a flaring trapezium*

Let μ_w be a geodesic in (X_w, d_w) joining a', b'. By Theorem 4.20 it is contained in a hallway $\mathcal{H}_w$ such that $\mathcal{H}_w \cap X_v$ is a geodesic subsegment μ_0 of μ. Since Y is $C-$quasiconvex in (X_v, d_v), there exist $a_1, b_1 \in Y$ close to the end-points of μ_0.

Hence there exists a hallway $\mathcal{H}'_w$ (with slightly worse constants than $\mathcal{H}_w$, see Lemma 4.22)) such that

(1) $\mathcal{H}'_w \cap X_w = [a', b']$,
(2) $\mathcal{H}'_w \cap X_v = [a_1, b_1]$.

In particular (since $d_w(a', b') = O(1)$ is uniformly bounded), the geodesic $[a', b']$ *does not* flare in the direction $[v, w]$ (choosing n large enough). This contradiction proves the forward direction.

We now prove the converse direction. Since Y is C_0-quasiconvex in (X, d_X), it is C_0-quasiconvex in (X_v, d_v) the latter being a subspace of the former. Next, since Y is C_0-quasiconvex in (X, d_X), the following holds.
Let

(1) $a, b \in Y$ be vertices with $d_v(a, b)$ large enough,
(2) $[v, w] \subset T$ (or $\mathrm{BU}(T)$) be a geodesic segment starting at v. Let σ_a, σ_b be two qi-sections (with uniform constant K_0) of $[v, w]$ for $P : X \to T$ (or $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$).

Then σ_a, σ_b must flare with flaring constants depending on K_0 as soon as $d_T(v, w) \geq K$ (or $d_{\mathrm{BU}(T)}(v, w) \geq K$) for some K depending only on C. This is a simple quasification of the standard fact that geodesics diverge exponentially in a hyperbolic metric space (see [Mit97, Proposition 2.4] for instance). Since $w \in T$ (or $\mathrm{BU}(T)$) was arbitrary, it follows that Y flares in all directions with parameter K. □

5. Uniform hyperbolicity of M

In this section, we establish **uniform** estimates for the Gromov hyperbolicity of $(\widetilde{M_T}, d_{te})$. We restate Theorem 3.36 in the form that we shall prove it.

Theorem 5.1. *Given. $R \geq 1$, and $D, k \geq 1$ there exists $\delta > 0$ such that the following holds:*
For an $L-$tight $R-$thick balanced tree T with parameters $D, k \geq 1$,

(1) *$(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic.*
(2) *$(\widetilde{M_T}, d_{weld})$ is strongly $\delta-$hyperbolic relative to the collection $\widetilde{\mathcal{R}_{\mathcal{M}}}$ of lifts of Margulis risers,*

Note that by Definition 3.18 and Corollary 3.27, the hypothesis on existence of R in Theorem 5.1 is equivalent to the existence of $k_0 \geq 1, \epsilon_0 \geq 0$ such that M_T is a special split geometry model with parameters k_0, ϵ_0 corresponding to T.

The proof of Theorem 5.1 will be given in Section 5.3 and will use

(1) The fact that the Minsky model for doubly degenerate Kleinian surface groups with injectivity radius uniformly bounded below is uniformly bi-Lipschitz to the hyperbolic metric [Min94, Min10, BCM12].
(2) The Bestvina-Feighn combination theorem [BF92] and its converse in the effective form given by Corollaries 4.23, 4.24 and Proposition 4.25.
(3) The special split geometry of building blocks.

For the purposes of this section, N will denote a doubly degenerate hyperbolic 3-manifold corresponding to a surface S and a a doubly degenerate surface Kleinian group $\rho(\pi_1(S)) = \pi_1(N) \subset PSL(2, \mathbb{C})$. The ending laminations of N are denoted as $l_\pm$. Note that by work of Thurston [Thu80, Chapter 9] and Bonahon [Bon86], N is homeomorphic to $S \times \mathbb{R}$. Before dealing with the model $(\widetilde{M_T}, d_{weld})$ of special split geometry and proving Theorem 5.1, it will be convenient to focus on the simpler case of bounded geometry. This furnishes the same result under stronger hypotheses (Proposition 5.9) and will serve to delineate the ingredients of the proof.

We first recall from the Introduction some of the basics of convex cocompact subgroups of the mapping class group and refer the reader to [FM02] for details.

As before, S is a closed surface of genus g and $MCG(S)$ is its mapping class group. A subgroup H of $MCG(S)$ is said to be **convex cocompact** if some (every) orbit of H in the Teichmüller space $Teich(S)$ is quasiconvex. Associated to any $H \subset MCG(S)$, there is a natural associated exact sequence [FM02, Section 1.2] of the form

$$1 \to \pi_1(S) \to L_H \to H \to 1.$$

The following characterizes convex cocompactness:

Theorem 5.2. [FM02, Ham05] *A subgroup H of $MCG(S)$ is convex cocompact if and only if the extension L_H occurring in the associated exact sequence $1 \to \pi_1(S) \to L_H \to H \to 1$ is hyperbolic.*

Theorem 5.2 was proved for free groups by Farb and Mosher [FM02] as was the 'if' direction in general. Hamenstadt [Ham05] proved the only if direction. In [MS12, Proposition 5.17] this was extended to surfaces with punctures. The proof there was in fact effective (see also [Gau16, MR08]). We shall recall this in Section 5.2.

The next statement observes the absence of $\mathbb{Z}\oplus\mathbb{Z}$ in extensions of purely pseudo-Anosov subgroups of $MCG(S)$.

Proposition 5.3. [KL07, Theorem 8.1] *Let $H \subset MCG(S)$. If H is purely pseudo-Anosov, then L_G contains no Baumslag-Solitar subgroups and hence no copy of $\mathbb{Z} \oplus \mathbb{Z}$.*

For convenience of the reader, we outline the strategy that will go into the proof of Proposition 5.9. We shall modify this strategy to prove Theorem 5.1 in Section 5.3.

Scheme 5.4. *The steps of the proof of Proposition 5.9 are:*

1. *The Minsky model (Theorem 3.8) shows that the universal bundles over bi-infinite geodesics are uniformly hyperbolic.*
2. *The converse direction of the combination Theorem 4.8 furnishes effective flaring constants.*
3. *Feeding these effective flaring constants into the bundle $\widetilde{M_H}$ over Γ_H furnishes (effective) hyperbolicity of $\widetilde{M_H}$.*

5.1. **Thick Minsky model: No cusps.** We now turn to proving the analog of Theorem 5.1 for bounded geometry. By Theorem 3.8 the bounded geometry hypothesis is equivalent to the (union of the) assumptions that

1. The parameter R in the underlying $L-$tight and $R-$thick tree (cf. Definition 3.31) is uniformly bounded above.
2. There exists $L' \geq L$ such all the subsurface projections onto $S \setminus i(v)$ are bounded by L' for all $v \in V(T)$. (Note that this is stronger than in the statement of Theorem 5.1, where only R is bounded above.)

For the time being, we focus on the case of closed S. Let l be an $\epsilon-$thick bi-infinite Teichmüller geodesic (i.e. a bi-infinite Teichmüller geodesic contained in $\mathrm{Teich}_\epsilon(S)$) with end-points $l_\pm \in \partial\mathrm{Teich}(S) = \mathcal{PML}(S)$. By forgetting the underlying measure, we identify $l_\pm$ with the underlying elements of the ending lamination space $\mathcal{EL}(S)$. Let M_l be the universal curve over l equipped with the universal curve metric as in Remark 3.9. Then, by [Min01, Min94] M_l is uniformly bi-Lipschitz to the unique hyperbolic manifold $N(l\pm)$ with ending laminations $l_\pm$. As a consequence of Theorem 3.8, we thus have

Corollary 5.5. *For $S = S_{g,0}$ a closed surface of genus g, and $\epsilon > 0$, there exists $\delta > 0$ such that the following holds:*
For l an $\epsilon-$thick bi-infinite Teichmüller geodesic, the universal cover $\widetilde{M_l}$ of the Minsky model M_l, equipped with the universal curve metric, is $\delta-$hyperbolic.

Proof. This follows from the fact that M_l is $K-$bi-Lipschitz homeomorphic to a hyperbolic manifold $M(l\pm)$, with K depending only on g, ϵ; and hence $\widetilde{M_l}$ is $K-$bi-Lipschitz homeomorphic to $\mathbf{H}^3$. □

Uniform hyperbolicity of $\widetilde{M_l}$ in Corollary 5.5 ensures uniform flaring constants by the converse part of Theorem 4.8:

Corollary 5.6. *For $S = S_{g,0}$ a closed surface of genus g, and $\epsilon > 0$, there exists $\lambda_0, m_0, \rho_0, H_0 \geq 1$ such that the following holds:*
Let l be an $\epsilon-$thick bi-infinite Teichmüller geodesic and $P : M_l \to l$ denote the universal bundle over l. Let $P : \widetilde{M_l} \to l$ denote the lift to the universal cover. Then through every point of $\widetilde{M_l}$ there exists a ρ_0-qi-section of $P : \widetilde{M_l} \to l$. Further, hallways bounded by ρ_0-qi-sections in $\widetilde{M_l}$ satisfy the flaring condition with constants $\lambda \geq \lambda_0$, $n \leq n_0$ and $H \leq H_0$.

We shall say that a subgroup H of $MCG(S)$ is $K-$**convex cocompact** if some orbit of H in $Teich(S)$ is $K-$quasiconvex. Hence there exists $o \in Teich(S)$, such that for every $l_\pm \subset \partial H \subset \partial Teich(S)$, the Teichmüller geodesic l joining $l_\pm$ lies at bounded Hausdorff distance $D(= D(K))$ from $H.o$. For $a, b \in \partial H \subset \partial Teich(S)$ the Teichmüller geodesic l joining a, b is denoted as l_{ab}. Next, assume that H is free.

Construction 5.7. *Let H be a free, convex cocompact, purely pseudo-Anosov subgroup of $MCG(S)$. We can choose a free generating set for H, construct a Cayley graph Γ_H of H and also a map $\Phi : \Gamma_H \to Teich(S)$, such that*

1. *$\Phi(1) = o$,*
2. *Φ maps edges of Γ_H to geodesic segments,*
3. *For $a, b \in \partial\Gamma_H$, let (a, b) denote the bi-infinite geodesic joining a, b in Γ_H. Then $\Phi((a, b))$ and l_{ab} lie within bounded Hausdorff distance $D(= D(K))$ from each other. Further, we can (after choosing D depending only on S and K appropriately) parametrize (a, b) and l_{ab} proportional to their respective arc lengths, such that $d_{Teich}(\Phi(t), l_{ab}(t)) \leq D$.*
4. *The universal curve over $\Phi((a, b))$ is denoted as M_{ab}.*

The following is now a consequence of Corollary 5.5 (see also [Min01, Raf14]):

Corollary 5.8. *For $K \geq 0$ and $e > 0$, there exists $\delta > 0$ such that if*

1. *H is a $K-$convex cocompact subgroup, and*
2. *there exists $o \in Teich(S)$ with $H.o \subset Teich_e(S)$,*

then for all $a, b \subset \partial H \subset \partial Teich(S)$, the universal curve M_{ab} over $\Phi((a, b))$ with ending laminations a, b is $\delta-$hyperbolic.

Proposition 5.9. *Given $K, \epsilon \geq 0$, there exists $\delta > 0$ such that the following holds: Let H be a free $K-$convex cocompact subgroup and let $o \in Teich(S)$ with $H.o \subset Teich_\epsilon(S)$. Let Γ_H be a Cayley graph of H with respect to a free generating set* *and $\Phi : \Gamma_H \to Teich(S)$ be as in Construction 5.7 above. Let M_H be the universal bundle over $\Phi(\Gamma_H)$ (equipped with the universal bundle metric as before). Then the universal cover $\widetilde{M_H}$ is $\delta-$hyperbolic.*

Proof. For $a, b \in \partial\Gamma_H \subset \partial\text{Teich}(S)$, let l_{ab} denote the Teichmüller geodesics joining a, b and let (a, b) denote the bi-infinite geodesic in Γ_H joining a, b. By $K-$convex cocompactness and $\epsilon-$thickness, there exists e' such that l_{ab} lies in the $e'-$thick part of Teichmüller space for all $a, b \in \partial H$. Let M_{ab} denote the universal curve over $\Phi((a, b))$. Then by Corollary 5.8, there exists δ' such that $\widetilde{M_{ab}}$ is $\delta'-$hyperbolic.

Let $P : \widetilde{M_{ab}} \to (a, b)$ denote the natural projection. By Lemma 4.19, there exists ρ_0 such that through every point of $\widetilde{M_{ab}}$ there exists a ρ_0-qi-section of P. By (a straightforward quasification of) Corollary 5.6, there exist λ_0, m_0, H_0 (depending only on $K, \epsilon > 0$), such that the universal cover $\widetilde{M_{ab}}$ of the universal curve M over $\Phi((a, b))$ satisfies the flaring condition with $\lambda \geq \lambda_0$, $m \leq m_0$ and $H \leq H_0$ with respect to ρ_0-qi-section bounded hallways.

Hence, by (the forward part of) Theorem 4.8, there exists $\delta > 0$ depending only on λ_0, m_0, ρ_0 (and hence only on $K, \epsilon > 0$) such that $\widetilde{M_H}$ is $\delta-$hyperbolic. □

Remark 5.10. We note here that Proposition 5.9 and its proof go through if Γ_H is replaced by any convex subset of Γ_H, i.e. by a connected sub-tree of Γ_H. All we need to do is assume that the image of the convex subset (instead of the image of the whole Cayley graph) is $K-$quasiconvex and that it lies in the $e-$thick part of $Teich(S)$.

5.2. **Thick Minsky model: Cusped case.** We describe now a relative version of Proposition 5.9 when S has cusps. Though we shall not need it directly, we provide a statement and a sketch, as the proof is a fairly straightforward combination of [MS12, Proposition 5.17] and the proof of Proposition 5.9 above. We now state the quantitative version of [MS12, Proposition 5.17]:

Proposition 5.11. *Given K, e, there exists δ such that the following holds:*
Let $N = \pi_1(S)$ be the fundamental group of a surface $S(= S_{g,n})$ with n punctures. Let $N_1, \cdots, N_n$ be the cyclic peripheral subgroups. Let H be a $K-$convex cocompact subgroup of the pure *mapping class group of S having an orbit $H.o \subset Teich_e(S)$. Let*

$$1 \to N \to G \xrightarrow{p} H \to 1$$

be the induced exact sequence. The action of H centralizes each N_i. Let

$$1 \to N_i \to Z_G(N_i) \xrightarrow{p} H \to 1,$$

be the induced short exact sequences of peripheral groups, where $Z_G(N_i) = N_i \times H$ denotes the normalizer (equal to the centralizer) of N_i in G. Then G is strongly $\delta-$hyperbolic relative to the collection $\{N_G(K_i)\}, i = 1, \cdots, n$.

Conversely, if G is (strongly) hyperbolic relative to the collection $\{N_G(N_i)\}, i = 1, \cdots, n$, then H is convex-cocompact.

We now specialize to our case of interest, where H is free:

Corollary 5.12. *Let $S(= S_{g,n})$ be as in Proposition 5.11. Given $K, e \geq 0$, there exists $\delta > 0$ such that the following holds:*
Let H be a free $K-$convex cocompact subgroup and let $o \in Teich(S)$ with $H.o \subset Teich_e(S)$. Let Γ_H be a Cayley graph of H with respect to a free generating set and

$\Phi : \Gamma_H \to Teich(S)$ be as in Construction 5.7. Let M_H be the universal bundle over $\Phi(\Gamma_H)$ (equipped with the universal bundle metric as before) with a neighborhood of the cusps removed. Let S_0 denote S with the corresponding neighborhoods of the n punctures removed. Let $\mathcal{P}_0$ denote the connected components of $\partial S_0 \times \Phi(\Gamma_H)$. Let $\mathcal{P}$ denote the collection of lifts of $P_0 \in \mathcal{P}_0$ to the universal cover $\widetilde{M_H}$. Then $\widetilde{M_H}$ is strongly $\delta-$hyperbolic relative to the collection $\mathcal{P}$.

Proof. We sketch a proof of the Corollary carrying forward the notation from Proposition 5.9. First, by $K-$convex cocompactness, the universal curves M_{ab} have systole bounded below by some $\epsilon'(= \epsilon'(K, \epsilon))$. Next, electrify the cusps of S. This gives us

1. A tree of spaces where all the vertex and edge spaces are quasi-isometric to $(\widetilde{S}, d_e)$ with electrified horocycle boundary.
2. The universal cover $(\widetilde{M_{ab}}, d_{pel})$ of the universal curve M_{ab} over $\Phi((a, b))$ is consequently equipped with the *partially electrified* metric d_{pel}.

As in the proof of [MS12, Proposition 5.17] (cf. Proposition 5.11 above) and Corollary 5.6, the resulting tree of spaces satisfies a uniform flaring condition, i.e. there exist $\lambda_0, m_0, \rho_0, H_0$ (depending only on $K, \epsilon > 0$), such that $(\widetilde{M_{ab}}, d_{pel})$ satisfies $(\lambda, m, \rho)-$flaring with $\lambda \geq \lambda_0$, $m \leq m_0$ and $\rho \leq \rho_0, H \leq H_0$.

Hence, by (the forward part of) Theorem 4.17, there exists $\delta > 0$ depending only on λ_0, n_0, ρ_0 such that $\widetilde{M_H}$ is strongly $\delta-$hyperbolic relative to the collection $\mathcal{P}$. □

5.3. **Uniform hyperbolicity of $(\widetilde{M_T}, d_{te})$.** We are now in a position to prove Theorem 5.1. Starting with a balanced tree $i : V(T) \to \mathcal{C}_\Delta(S)$, let BU$(T)$ denote the blown up tree. For $l \subset T$ a bi-infinite geodesic, $BU(l)$ will denote its blow-up in BU(T). The end-points of $BU(l)$ in $\mathcal{EL}(S) = \partial\mathcal{C}(S)$ will be denoted by $l_\pm$. We remind the reader of Standing Assumption 2.13 about $L-$tight trees.

Scheme 5.13. We now outline the steps of the proof of Theorem 5.1 and the modifications to Scheme 5.4 that we require.

Step 1: Let (M_l, d_{weld}) (resp. (M_l, d_{te})) denote the bundle $P : (M_T, d_{weld}) \to$ BU(T) (resp. $P : (M_T, d_{te}) \to$ BU(T)) restricted to $P : P^{-1}(BU(l)) \to BU(l)$. Let $\mathcal{M}_l$ denote the collection of intersections of Margulis risers with M_l. By Remark 3.28 and Theorem 3.35, (M_l, d_{weld}) (resp. (M_l, d_{te})) is precisely the model metric obtained from the welded split blocks of Definition 3.15 (resp. the tube-electrified split blocks of Definition 3.16). Note also that $\mathcal{M}_l$ consists precisely of the welded annuli in $P^{-1}(BU(l))$. Let $\widetilde{\mathcal{M}_l}$ denote the collection of lifts of $\mathcal{M}_l$ to the universal cover $(\widetilde{M_l}, d_{weld})$. Theorem 5.14 below will show that
a) $(\widetilde{M_l}, d_{weld})$ is (uniformly) strongly hyperbolic relative to the collection $\widetilde{\mathcal{M}_l}$, and
b) By Lemma 4.13, $(\widetilde{M_l}, d_{te})$ is (uniformly) hyperbolic and the elements of $(\widetilde{\mathcal{M}_l}, d_{te})$ are uniformly quasiconvex in it.

Step 2: The converse direction of the combination theorem in this context, Corollary 4.24 then furnishes effective flaring constants for $(\widetilde{M_l}, d_{te})$. Feeding these effective flaring constants into the bundle $P : (\widetilde{M_T}, d_{te}) \to$ BU(T)

furnishes (effective) hyperbolicity of $(\widetilde{M_T}, d_{te})$ by Corollary 4.23 and proves the first conclusion of Theorem 5.1.

Step 3: Finally we extract effective hyperbolicity of $(\widetilde{M_T}, d_{weld})$ *relative* to the collection $\widetilde{\mathcal{R}_{\mathcal{M}}}$ of lifts of Margulis risers and prove the second conclusion of Theorem 5.1.

Step 1:
For $BU(l)$ a blown-up bi-infinite geodesic in BU(T) with ending laminations $l_\pm$, let N_l be the doubly degenerate hyperbolic 3-manifold with end-invariants $l_\pm$. Theorem 3.32 and Theorem 3.35 yield the following as a consequence.

Theorem 5.14. [Min10, BCM12] *Given $R, k, D_0 \geq 0$, there exists ϵ, D, C_0 such that the following holds:*
Let T be an $L-$tight $R-$thick balanced tree with parameters D_0, k and let $(M_l, d_{weld}), (M_l, d_{te})$ be as above. Then

(1) *There exist model manifolds M_l^m of special split geometry with constants $\epsilon, D > 0$ as in Definition 3.11. such that M_l^m is C_0-bi-Lipschitz homeomorphic to N_l.*
(2) *The welded metrics and tube-electrified metrics of Definitions 3.15, 3.16, 3.33 associated with M_l^m are C_0-bi-Lipschitz homeomorphic to $(M_l, d_{weld}), (M_l, d_{te})$ respectively.*

Since Margulis tubes are convex in any N_l and uniformly separated from each other, it follows (see [Bow12] for instance) that there exists δ_0 such that

(1) $\widetilde{N_l}$ is uniformly hyperbolic for all l (since $\widetilde{N_l} = \mathbf{H}^3$),
(2) $\widetilde{N_l}$ is strongly δ_0-hyperbolic relative to the collection $\mathcal{M}_l$ of lifts $\widetilde{T}$ of Margulis tubes to $\widetilde{N_l}$.

Let $\partial\mathcal{M}_l$ denote the collection of boundaries $\{\partial\widetilde{T} | \widetilde{T} \in \mathcal{M}_l\}$, and let $Int(\mathcal{M}_l) = \{Int(\widetilde{T}) | \widetilde{T} \in \mathcal{M}_l\}$. Let $\widetilde{N_l^0} = \widetilde{N_l} \setminus \bigcup_{Int(\widetilde{T}) \in Int(\mathcal{M}_l)} Int(\widetilde{T})$. By strong δ_0-hyperbolicity of $\widetilde{N_l}$ relative to the collection $\mathcal{M}_l$, it follows that $\widetilde{N_l^0}$ is strongly δ_0-hyperbolic relative to the collection $\partial\mathcal{M}_l$.

Next, consider a standard annulus isometric to $S^1 \times [0, l_i]$ in a welded block $B_{i,wel}$ (see Definition 3.15) and let $f_i : \partial T_i \to S^1 \times [0, l_i]$ be the quotienting map defined in Definition 3.15. Let $\widetilde{f_i} : \partial\widetilde{T_i} \to \widetilde{S^1} \times [0, l_i]$ be lifts of f_i to $\widetilde{N_l^0}$ Further, assume that $\widetilde{S^1} \times [0, l_i]$ has been tube-electrified, by assigning the zero metric to the $\widetilde{S^1}-$direction (note that $\widetilde{S_i}$ is the real line $\mathbb{R}$), so that after this tube-electrification operation, we obtain the universal cover $\widetilde{N_{l,te}}$ of the tube-electrified model manifold $N_{l,te}$. Since the maps $\widetilde{f_i} : \partial\widetilde{T_i} \to \widetilde{S^1} \times [0, l_i]$ are clearly 1-Lipschitz, we have the following by Lemma 4.13, Theorem 5.14 and Proposition 4.27:

Proposition 5.15. Tube-electrified models are uniformly hyperbolic: *Given $R, k, D_0 \geq 0$, there exist $\delta', C \geq 0$ such that the following holds:*
Let T be an $L-$tight $R-$thick balanced tree with parameters D_0, k. For $BU(l)$ as before and $(M_l, d_{weld}), (M_l, d_{te})$ as in Theorem 5.14, the universal cover $(\widetilde{M_l}, d_{te})$ of the tube-electrified model manifold (M_l, d_{te}) is a $\delta'-$hyperbolic metric space. Further, each tube-electrified standard annulus (or equivalently, each tube-electrified Margulis tube) in $\mathcal{M}_l$ equipped with d_{te} is $C-$quasiconvex.

Alternate Proof: We furnish here an alternate proof of Proposition 5.15. We use the notation of Proposition 3.11.

Recall that each special split block B_i has injectivity radius bounded below by $\epsilon > 0$ away from the Margulis tube $\mathbf{T}_i$. Recall also that the core curve of $\mathbf{T}_i$ is denoted as τ_i. Hence, there exists $K \geq 1$ (independent of i), such that $B_i \setminus \mathbf{T}_i$ is $K-$bi-Lipschitz to the thick part of the universal curve (i.e. the universal bundle minus a neighborhood of the cusps) over a thick Teichmüller geodesic segment γ_i in $Teich_\epsilon(S \setminus \tau_i)$ for some uniform $\epsilon > 0$. Then, due to uniform thickness of Teichmüller geodesic segments γ_i, the bundle $(\widetilde{M}, d_{te})$ satisfies flaring conditions with uniformly bounded constants. Strong relative hyperbolicity of $(\widetilde{M}, d_{te})$ relative to $\mathcal{M}_l$ now follows from Theorem 4.17. □

Remark 5.16. In applications we have in mind, especially [MMS19], the full strength of the model from Theorem 5.14 used in the first proof of Proposition 5.15 becomes relevant. We have thus included two proofs, even though the alternate proof above does not use the full ending laminations machinery of [Min10, BCM12].

This completes Step 1 of Scheme 5.13.

Step 2:
Effective hyperbolicity of $(\widetilde{M_T}, d_{te})$ now follows the same route as the proof of Proposition 5.9 (see also Remark 5.10 and Proposition 4.27).

Proposition 5.17. *Given $R > 0$, there exists $\delta, C > 0$ such that the following holds:*
Let T be an $L-$tight $R-$thick tree. Then $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic.

Further, each element of the set of Margulis risers $\widetilde{\mathcal{R}_\mathcal{M}}$ is $C-$quasiconvex in $(\widetilde{M_T}, d_{te})$.

Proof. We follow the proof of Proposition 5.9. Uniform hyperbolicity of $(\widetilde{M_l}, d_{te})$ (Proposition 5.15) ensures uniform flaring constants by Corollary 4.24 for $(\widetilde{M_l}, d_{te})$ independent of $l \subset T$. This gives effective flaring constants for $(\widetilde{M_T}, d_{te})$ as a bundle over BU(T). Hence by Corollary 4.23 there exists $\delta > 0$ depending only on R such that $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic.

Next, since each Margulis riser in $(\widetilde{M_T}, d_{te})$ arises as a uniform quasi-isometric section of a tree-link T_v, there exists $C > 0$ such that each element of $\widetilde{\mathcal{R}_\mathcal{M}}$ is $C-$quasiconvex in $(\widetilde{M_T}, d_{te})$. □

This completes Step 2 of Scheme 5.13 and proves the first conclusion of Theorem 5.1.

Step 3:
We finally turn our attention to $(\widetilde{M_T}, d_{weld})$ and establish that $(\widetilde{M_T}, d_{weld})$ is hyperbolic *relative to* $\widetilde{\mathcal{R}_\mathcal{M}}$ with effective constants. The argument will be an adaptation of very similar arguments in [Gau16, MR08] and we will provide a road-map through it instead of reproducing all the details. The proof proceeds by first observing the analogous statement for $(\widetilde{M_T}, d_{te})$.

Proposition 5.18. *Given $R > 0$, there exists $\delta > 0$ such that the following holds: Let T be an $L-$tight $R-$thick tree. Then $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic relative to $\widetilde{\mathcal{R}_\mathcal{M}}$.*

Proof. Proposition 5.17 shows that $(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic and the Margulis risers in $\widetilde{\mathcal{R}_\mathcal{M}}$ are uniformly quasiconvex in $(\widetilde{M_T}, d_{te})$. Uniform separatedness of the elements of $\widetilde{\mathcal{R}_\mathcal{M}}$ is a consequence of the construction of $P : (\widetilde{M_T}, d_{te}) \to \mathrm{BU}(T)$.

The proof of uniform hyperbolicity of $(\widetilde{M_T}, d_{te})$ *relative to* the collection $\widetilde{\mathcal{R}_\mathcal{M}}$ is now a replica of the proof of Theorem 4.17 (the statement was culled from [MR08, Gau16]). We omit the details and mention only that the elements of $\widetilde{\mathcal{R}_\mathcal{M}}$ take the place of cone-loci in [MR08]; the rest of the proof is an exact copy. □

Proposition 5.19. *Given $R > 0$, there exists $\delta > 0$ such that the following holds: Let T be an $L-$tight $R-$thick tree. Then $(\widetilde{M_T}, d_{weld})$ is strongly $\delta-$hyperbolic relative to the collection $\widetilde{\mathcal{R}_\mathcal{M}}$ of lifts of Margulis risers.*

Further, if there exists L_1 such that the diameter of any tree-link T_v is bounded above by L_1 for every v, then $(\widetilde{M_T}, d_{weld})$ is hyperbolic.

Proof. **First statement of Proposition 5.19:** The proof of the first statement of Proposition 5.19, i.e. that there exists δ such that $(\widetilde{M_T}, d_{weld})$ is strongly $\delta-$hyperbolic relative to $\widetilde{\mathcal{R}_\mathcal{M}}$ is a replica of the proof of Theorem 2.20 of [Gau16]. Instead of reproducing the argument here, we shall now give specific references to the main steps of the proof from [Gau16] and translate its terminology and conclusion to our context.

First, we note that the main technical condition Gautero uses [Gau16, Definition 2.14] is what he calls the exponential separation property. In our context, this is equivalent to the effective flaring condition. and is provided by Corollary 4.24 applied to the conclusion of Proposition 5.18 above.

Next, the proof of [Gau16, Theorem 2.20] have, as its main steps, [Gau16, Theorem 5.2] and [Gau16, Proposition 7.4] (proved in [Gau16, Section 9.7]). The proofs of [Gau16, Theorem 5.2] and [Gau16, Proposition 7.4], in turn, depend precisely on the exponential separation property hypothesis, which, as we have observed is a consequence of Proposition 5.18 and Corollary 4.24. The first statement of the Proposition is now a translation, in the context of this paper, of [Gau16, Theorem 2.20].

Second statement of Proposition 5.19: The last statement of the Proposition now follows from Proposition 5.9 since the upper bound L_1 forces each bi-infinite geodesic l in T to lift to a geodesic in Teich_ϵ with ϵ uniformly bounded away from 0. □

This completes Step 3 of Scheme 5.13 and the proof of the second conclusion of Theorem 5.1. □

6. Generalizations and Examples

The purpose of this section is to generalize Theorem 5.1 to general $L-$tight, $R-$thick trees (Definition 2.9) rather than just balanced ones. This comes at a cost. Uniform properness (Conclusion (4) of Theorem 3.35) is no longer valid.

The tube electrification operation (Definitions 3.22 and 3.33 is devised to electrify *as little as possible.* In the more general cases below, we are forced to electrify more.

6.1. **Lipschitz trees.** An application of the technology developed in this paper is to prove cubulability of some surface-by-free hyperbolic groups [MMS19]. The main theorem of [MMS19] requires the construction of quasiconvex tracks in $(\widetilde{M_T}, d_{te})$. This in turn requires that all the distances between end-points (leaves) of any tree-link T_v is large. We thus define:

Definition 6.1. A finite metric tree $\mathcal{T}$ is said to be $\lambda-$long if the distance between any two end-points (leaves) of $\mathcal{T}$ is at least λ.

A geodesic from a leaf of a finite tree to another leaf will be called a **long edge**. A continuous map ϕ from a finite tree T_1 to a finite tree T_2 will be called **monotonic** if

(1) ϕ is a bijection on leaves,
(2) ϕ maps long edges monotonically (but not necessarily strictly monotonically) to long edges.

We shall now generalize Definition 2.18. We adapt the notation of Definition 2.18: T_v^+ denotes the tree-link obtained as an approximating tree of $CH(i(lk(v)))$. Let T_v^- denote an approximating tree of $CH(i(lk(v)))'$. Note that the constants of approximation depend only on the number of vertices in $i(lk(v))$ and hence only on the valence of v.

Definition 6.2. An $L-$tight $R-$thick tight tree $i : V(T) \to \mathcal{C}_\Delta(S)$ is said to be a **Lipschitz** tree with parameters D, k, λ if

(1) For every separating vertex v of T,
$$\mathrm{dia}(\Pi_v'(T_w')) \leq D.$$
(2) Let T_v^+, T_v^- be as above. There exists a $\lambda-$long tree T_v with the same cardinality of leaves as T_v^+, T_v^- and surjective $k-$Lipschitz monotonic maps $\mathbb{P}^+$ and $\mathbb{P}^-$ from T_v^+ and T_v^- respectively to T_v.

We have thus weakened the "coarse bi-Lipschitz" condition (equivalent to the surjective quasi-isometry condition) of Item (2) of Definition 2.18 to a coarse Lipschitz condition in Definition 6.2 above. The tube-electrification process goes through via Lipschitz maps with the following modifications:

(1) The tree links T_v are now the $\lambda-$long trees in Definition 6.2 above.
(2) The Margulis risers are isometric to $S_e^1 \times T_v$.

With these modifications, the proof of Theorem 5.1 goes through as before to yield:

Theorem 6.3. *Given $R \geq 1$, and $D, k, \lambda \geq 1$ there exists $\delta > 0$ such that the following holds:*
For an $L-$tight $R-$thick Lipschitz tree T with parameters D, k, λ,

(1) *$(\widetilde{M_T}, d_{weld})$ is strongly $\delta-$hyperbolic relative to the collection $\widetilde{\mathcal{R}_\mathcal{M}}$ of lifts of Margulis risers,*
(2) *$(\widetilde{M_T}, d_{te})$ is $\delta-$hyperbolic.*

Note again that hyperbolicity is not an issue in Theorem 6.3, but the tube electrification process electrifies more by

(1) Electrifying the $\mathbb{R}-$direction as before in Definition 3.22,
(2) Contracting the finite directions of Margulis risers as well via the Lipschitz maps $\mathbb{P}^\pm$.

6.2. **General tight Trees.** We finally turn to the case when no large λ is possible. To illustrate what can go wrong, define a tripod $\tau_x(a,b,c,A,B,C)$ to be a tree with a single trivalent vertex x and leaves a,b,c with $|xa| = A, |xb| = B, |xc| = C$. Now, glue $\tau_x(a,b,c,1,L,L/2)$ to $\tau_y(d,e,c,1,L,L/2)$ by identifying only the vertices labeled c to obtain a tree $T(a,b,d,e)$ with 4 leaves a,b,d,e so that $d(a,x) = 1, d(b,x) = L, d(x,y) = L, d(y,d) = 1, d(y,e) = L$; in particular $T(a,b,d,e)$ is $L-$long. Similarly, glue tripods $\tau_{x'}(a',e',c',L,1,L/2)$ to $\tau_{y'}(b',d',c',1,L,L/2)$ by identifying only the vertices labeled c' to obtain a tree $T(a',b',d',e')$. It follows that $d(a',x') = L, d(e',x') = 1, d(x',y') = L, d(y',d') = L, d(y',b') = 1$; in particular $T(a',b',d',e')$ is also $L-$long.

However, any tree $T(a'',b'',d'',e'')$ that receives monotonic continuous maps ϕ,ϕ' from both $T(a,b,d,e)$ and $T(a',b',d',e')$ such that $\phi(a) = \phi'(a') = a''$, $\phi(b) = \phi'(b') = b''$, and so on, has to necessarily be a star, i.e. the conditions $\phi(x) = \phi(y)$ and $\phi'(x') = \phi'(y')$ are forced. Let $\phi(x) = \phi(y) = \phi'(x') = \phi'(y') = z$. If further, ϕ,ϕ' are required to be $1-$Lipschitz, then $d(z,a''), d(z,b''), d(z,d''), d(z,e'')$ are all of length at most 1. Thus the only option for T_v is a star where all limbs have length one.

One can arrange so that $T(a,b,d,e)$ and $T(a',b',d',e')$ are approximating trees of $CH(ilkv)$ and $CH(ilkv)'$ in the notation of Definition 2.18. Thus, in the general case (when the restrictive hypotheses of Definition 2.18 is absent or the existence of a large λ in Definition 6.2 is not guaranteed), the best we can hope is for the tree T_v to be a star where each edge has length one. In this case, $\lambda = 2$ in Definition 6.2.

Let $(M_T, d_{te})^*$ denote the bundle with tube-electrified metric in the special case that each T_v in Definition 6.2 is a star with all edges of length one. Let $(\widetilde{M_T}, d_{te})^*$ denote the universal cover. Theorem 6.3 then gives:

Corollary 6.4. *Given. $R \geq 1$ there exists $\delta > 0$ such that the following holds: For an $L-$tight $R-$thick tight tree, $(\widetilde{M_T}, d_{te})^*$ is $\delta-$hyperbolic.*

To conclude we note that each riser $\mathcal{R}_v$ has diameter two in $(\widetilde{M_T}, d_{te})^*$. Thus $(\widetilde{M_T}, d_{te})^*$ is $(2,2)$-quasi-isometric to the space $\mathcal{E}((\widetilde{M_T}, d_{weld}), \tilde{\mathcal{R}_{\mathcal{M}}})$ obtained by electrifying the lifts of Margulis risers in $(\widetilde{M_T}, d_{weld})$. Thus, in the special case of balanced trees, Corollary 6.4 also follows immediately from the first statement of Theorem 5.1.

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