NASH EQUILIBRIA VIA DUALITY AND HOMOLOGICAL SELECTION

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Abstract. Cost functions in problems concerning the existence of Nash Equilibria are traditionally multilinear in the mixed strategies. The main aim of this paper is to relax the hypothesis of multilinearity. We use basic intersection theory, Poincaré Duality and the Dold-Thom Theorem to establish existence of Nash Equilibria under fairly general topological hypotheses. The Dold-Thom Theorem provides us with a homological version of a selection Theorem, which may be of independent interest.

Contents

1. Introduction

We shall work in the framework of non-cooperative games with mixed strategies, where the domain for every player is a finite dimensional simplex. The cost functions (or alternately payoff functions) in traditional problems of Nash Equilibria are multilinear in the mixed strategies. The main aim of this paper is to relax the hypothesis that cost functions are multilinear. The original proof of existence of Nash Equilibria [12] uses fairly simple Algebraic Topology, namely the Brouwer or Kakutani Fixed Point Theorems. This approach has been refined in various ways [6, 3, 5] (see also [4] for an effective approach using a minimax technique). Our

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approach in this paper is quite different inasmuch as we use standard but more sophisticated tools from Algebraic Topology to establish existence of Nash Equilibria under considerably more general conditions on cost functions. The tools we use are basic intersection theory, Poincaré Duality and the Dold-Thom Theorem. We use the Dold-Thom Theorem to prove the existence of certain relative cycles contained in graphs of multifunctions. The chains thus constructed may be thought of as homological versions of selections [8, 9, 10]. This furnishes us with a Homological Selection Theorem 2.5, which might be of independent interest.

The conditions we use on the cost functions are detailed in Section 1.4. We give a brief sketch here for a 2-player non-cooperative game to stress the soft topological nature of the hypotheses used. Consider a game between players 1 and 2 with mixed strategy spaces $\mathbb{A}_1, \mathbb{A}_2$. We use the notation $\mathbb{A}_{-1} = \mathbb{A}_2$ and $\mathbb{A}_{-2} = A_1$.

A cost function $r_i(x, y)$ for player i $(i = 1, 2)$ is said to be polynomial-like if
1) the set of local minima of the best response multifunction $R_i(y)$ (for each $y \in \mathbb{A}_{-i}$) of the function $r_i(x, y)$ $(x \in \mathbb{A}_i)$ is finite; and
2) Counted with multiplicity, the set of local minima $R_i(y)$ is continuous on $\mathbb{A}_{-i}$.
A weakly polynomial-like cost function is one whose set of local minima can be arbitrarily well-approximated by the set of local minima of a polynomial-like function.

Two kinds of hypotheses will be relevant in this paper.
(A1) For each i, the map R_i is continuous.
(A2) For each i, the map R_i is weakly polynomial-like.

The main Theorem of this paper (Theorem 3.2) proves that assumption **(A2)** is sufficient to guarantee the existence of Nash equilibria. In Section 4 we shall give an example to show that assumption **(A1)** is not sufficient to guarantee the existence of Nash equilibria. A word about this counterexample ahead of time. It is easy to see that the multifunction $f(z) = \pm\sqrt{z}$ from the unit disk Δ in the complex plane to itself has no continuous selection. However the graph $gr(f)(\subset \Delta \times \Delta)$ does support a non-zero relative cycle in $H_2(\Delta \times \Delta, \partial\Delta \times \Delta)$ and hence admits a *homological selection* in our terminology (see Section 2). The counterexample in Section 4.1, which is a continuous map from $\mathbb{D}^2$ to $Sub_3(\mathbb{D}^2)$, shows that a continuous multifunction need not admit even a homological selection. (Here $Sub_3(\mathbb{D}^2)$ denotes the collection of subsets of $\mathbb{D}^2$ with at most 3 points equipped with the Hausdorff metric.) It follows (cf. Section 4.3) that any higher dimensional analog of weighted branched surfaces in codimension greater than one cannot be a straightforward generalization of 2-dimensional weighted branched surfaces in 3-manifolds (cf. [2]).
Convention: For this paper, we fix the metric on the $n-$dimensional ball $\mathbb{D}^n$ as $d(x, y) = |x - y| = sup_i(|x_i - y_i|)$.

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1.1. **Nash Equilibria.** We refer to [1] for the basics of Game Theory. An N-person non-cooperative game is determined by $2N$ objects $(\mathbb{A}_1, \ldots, \mathbb{A}_N, r_1, \ldots, r_N)$ where $\mathbb{A}_i$ denotes the *strategy space* of player i and $r_i : \mathbb{A} \overset{def}{=} \mathbb{A}_1 \times \cdots \times \mathbb{A}_N \to \mathbb{R}$ is the cost function for player i. We shall call $\mathbb{A}$ the *total strategy space*. Each $\mathbb{A}_i$ will, for the purposes of this paper, be the space of probability measures on a finite set of cardinality $(n_i + 1)$ and hence homeomorphic to $\mathbb{D}^{n_i}$. Thus, in Game Theory terminology each element of $\mathbb{A}_i$ is a *mixed strategy* or equivalently a probability measure on a finite set. Thus, $\mathbb{A}_i$ is the space of mixed strategies or player i. Vertices of the simplex $\mathbb{A}_i$ are also referred to as *pure strategies*. A mixed strategy may therefore be regarded as a *probability vector* with $(n_i + 1)$ components. Each player i independently chooses his strategy $a_i \in \mathbb{A}_i$. For a strategy-tuple $a = (a_1, \cdots, a_N)$, player i pays an immediate cost $r_i(a)$.

Goal: Each player wants to minimize his cost.

Notation: We denote $\mathbb{A}_{-i} \stackrel{def}{=} \mathbb{A}_1 \times \cdots \mathbb{A}_{i-1} \times \mathbb{A}_{i+1} \times \mathbb{A}_N$ for all i. For any strategy-tuple $(a_1, \cdots, a_N) \in \mathbb{A}$, $a_{-i} \stackrel{def}{=} (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N) \in \mathbb{A}_{-i}$. Also for convenience of notation we change the order of the variables and assume that $r_i : \mathbb{A}_i \times \mathbb{A}_{-i} \to \mathbb{R}$ is the i-th cost function.

Definition 1.1. *A strategy-tuple $a^* = (a_1^*, \ldots, a_N^*) \in \mathbb{A}$ is called a Nash equilibrium if*

$$r_i^* \stackrel{def}{=} r_i(a^*) \leq r_i(a_i, a_{-i}^*), \ \forall a_i \in \mathbb{A}_i, \ i = 1, \ldots, N, \tag{1}$$

where,

Given a metric space (X, d), let $\mathcal{H}_c(X)$ denote the space of all compact subsets of X equipped with the Hausdorff metric d_H i.e, for all $A, B \in \mathcal{H}_c(X)$,

$$d_H(A, B) \stackrel{def}{=} \max\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y)\}. \tag{2}$$

Definition 1.2. *For any player i, given a strategy-tuple $a_{-i} \in \mathbb{A}_{-i}$ of the other players, we define his best-response $R_i(a_{-i})$ as follows*

$$R_i(a_{-i}) \stackrel{def}{=} \arg\min_{a_i \in \mathbb{A}_i} r_i(a_i, a_{-i}) \in \mathcal{H}_c(\mathbb{A}_i). \tag{3}$$

Here $\arg\min$ denotes the *argument* of the minima set, i.e. the set of values $x \in \mathbb{A}_i$ such that $r_i(x, a_{-i})$ is minimum as a function of x.

Note: It will not affect our arguments at all if we take $\arg\min$ to denotes the argument of the *local* minima set, i.e. the set of values $x \in \mathbb{A}_i$ such that $r_i(x, a_{-i})$ is a local minimum as a function of x provided we modify Definition 1.1 appropriately. In such a case, we are looking at *local Nash equilibria* which are defined by demanding that the r_i^* appearing in Definition 1.1 are *local minima* rather than global minima as defined there.

Definition 1.3. *For any i, the graph $gr(R_i)$ of R_i is defined as*

$$gr(R_i) \stackrel{def}{=} \{(a_{-i}, a_i) : a_i \in R_i(a_{-i}), \ a_{-i} \in \mathbb{A}_{-i}\}. \tag{4}$$

1.2. **Motivational Example: Nash Equilibria for Bi-matrix Games.** We motivate our main Theorem by giving a simplified version of our proof of the existence of Nash equilibria in the special case of standard 2-person bimatrix games.

Consider a two-player bimatrix game where each player i has strategy space $\mathbb{A}_i \stackrel{def}{=} \{(x_i, 1 - x_i) : 0 \leq x_i \leq 1\}$, for $i = 1, 2$. The cost function for player i is a bilinear function of the form $r_i(x_1, x_2) = x_1^T M_i x_2$, where M_1, M_2 are 2×2 matrices and x_1, x_2 are probability vectors (mixed strategies) for players $1, 2$.

Then the best response $R_i(x_j)$ $(i \neq j)$ of player i when player j chooses $x_j \in [0,1]$ is given by

$$R_i(x_j) = \arg\min_{x_i \in [0,1]} r_i(x_i, x_j), \tag{5}$$

which is a *singleton* set $\{x_i^*(x_j)\}$ due to the component-wise linearity of the payoff function. Moreover, the dependence of x_i^* on x_j is continuous. Hence $\mathrm{gr}(R_i)$ for $i = 1, 2$ are continuous paths in $[0,1] \times [0,1]$ where $\mathrm{gr}(R_1)$ is a path from the bottom $([0,1] \times \{0\})$ to the top $([0,1] \times \{1\})$ and $\mathrm{gr}(R_2)$ is a path from the left$(\{0\} \times [0,1])$ to the right $(\{1\} \times [0,1])$. Hence $\mathrm{gr}(R_1)$ and $\mathrm{gr}(R_2)$ must intersect (for a formal proof of this intuitively clear fact, see Lemma 2 of [7] for instance). Hence there exists (x_1^*, x_2^*) such that $(x_1^* = R_1(x_2^*), x_2^* = R_2[x_1^*])$ i.e. a Nash equilibrium.

1.3. **Ordered and Unordered Tuples of Points.** Let $\mathbb{D}^d \stackrel{def}{=} \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{j=1}^d x_j^2 \leq 1\}$ denote the d-dimensional disk. Given a set A, let $\#A$ denote its cardinality. For positive integers k and compact metric spaces X we define the topological space of subsets of cardinality $\leq k$,

$$Sub_k(X) \stackrel{def}{=} \{subsets\, of\, X\, of\, cardinality\, \leq k\}.$$

This has a topology induced as a subset of $C_c(X)$. An analogous construction is the configuration space,

$$C_k(X) \stackrel{def}{=} \{(x_1, \ldots, x_k) \in X^k\}/\Sigma_k \tag{6}$$

where Σ_k is the symmetric group on k symbols. The elements of $C_k(X)$ can be written multiplicatively $x_1^{i_1} \ldots x_j^{i_j}$ with $i_1 + \ldots + i_j = k$ or as equivalence classes $[x_1, \ldots, x_k]$. There is a metric on $C_k(X)$ given by

$$d_k([x_1, \ldots, x_k], [y_1, \ldots, y_k]) = min_{\sigma \in \Sigma_k}(sup_i(d(x_{\sigma(i)}, y_i))).$$

We have a map $q_k : C_k(X) \to Sub_k(X)$ that maps $[x_1, \ldots, x_k]$ to $\{x_1, \ldots, x_k\}$. It is a surjective continuous map (as $d_H(\{x_1, \ldots, x_k\}, \{y_1, \ldots y_k\}) \leq d_k([x_1, \ldots, x_k], [y_1, \ldots, y_k])$).

Proposition 1.4. *The quotient topology on $Sub_k(X)$ induced from the map q_k is the same as the topology induced by the metric d_H.*

Proof. The argument above already shows that if U is an open set in the Hausdorff metric the inverse image is open and so it is open in the quotient topology. Conversely suppose U is such that $q_k^{-1}(U)$ is open. Then we must show that U is open in the Hausdorff topology. Consider a point $p \in U$. If $\#(p) = k$ then $q_k^{-1}(p)$ has exactly one point which we also write p and because $q_k^{-1}(U)$ is open, an open ball of $B(p, \delta)$ is contained in it.

Let the minimum distance between points of p be λ. We note that if a point q is of distance less than $\lambda/2$ from p then it has the same cardinality as p and $d_k(p, q) = d_H(p, q)$. Therefore, if $r = min(\delta, \lambda/2)$, the ball $B(p, r)$ is contained in $q_k^{-1}(U)$. The ball in the Hausdorff metric of radius r is $= q_k(B(p, r))$ which is contained in U since $r \leq \delta$, showing that U contains an open neighborhood of p.

Now we let p have cardinality $< k$ so that there are many points mapped to p by q_k. Suppose these points are $p_1, p_2, \ldots, p_n$. Since $q_k^{-1}(U)$ is open,

(1) It is contains a ϵ neighborhood of each of the p_i (we can choose this uniformly since n is finite), and

(2) 2ϵ is less than the minimum distance of points of p.

Now suppose there is a point q in the ϵ neighborhood of p. The latter condition ensures that a point in q is ϵ close to a unique point of p inducing a function $\Lambda : q \to p$. Hence, any weighted product of elements of q so that the total weight is k and each q point has a non zero weight, is ϵ close in the metric d_k to the product obtained by applying Λ pointwise. Since every point in p is ϵ close to at least one point of q, the element obtained maps to p. Thus, there is one $q_i \in q_k^{-1}(q)$ which is ϵ close to p_i, implying $q \in U$. Therefore U contains an ϵ neighborhood of p. This completes the proof. □

1.4. **Hypotheses for Existence of Equilibria.** The next definition establishes the generality in which we shall work in this paper. The terminology is motivated by a continuously parametrized family of complex polynomials of a fixed degree, where the set of roots counted with multiplicity varies continuously.

Definition 1.5. *A cost function $r_i : \mathbb{A}_i \times \mathbb{A}_{-i} \to \mathbb{R}$ is said to be* **polynomial-like** *if the response function $R_i : \mathbb{A}_{-i} \to \mathbb{A}_i$ can be lifted to a map $R_i^M : \mathbb{A}_{-i} \to C_M(\mathbb{A}_i)$ some configuration space $C_M(\mathbb{A}_i)$.*

A cost function $r_i : \mathbb{A}_i \times \mathbb{A}_{-i} \to \mathbb{R}$ is said to be **weakly polynomial-like** *if for all $\epsilon > 0$ there exists $R_{i,\epsilon} : \mathbb{A}_{-i} \to \mathbb{A}_i$ such that*

(1) *$R_{i,\epsilon}$ can be lifted to a map $R_i^M : \mathbb{A}_{-i} \to C_M(\mathbb{A}_i)$ some configuration space $C_M(\mathbb{A}_i)$,*
(2) *$d_H(R_i(a_{-i}), R_i^M(a_{-i})) < \epsilon$ for all $a_{-i} \in A_{-i}$.*

Equivalently, a cost function r_i is polynomial-like if
1) the set of local minima $R_i(a_{-i})$ of the function $r_i(a_i, a_{-i})$ is finite for each $a_{-i} \in \mathbb{A}_{-i}$; and
2) Counted with multiplicity, the set of local minima $R_i(a_{-i})$ is continuous on $\mathbb{A}_{-i}$. Also, a weakly polynomial-like cost function is one whose set of local minima can be arbitrarily well-approximated by the set of local minima of a polynomial-like function.

There are two assumptions that will be relevant in this paper.
(A1) For each i, the map R_i is continuous in a_{-i}.
(A2) For each i, the map R_i is weakly polynomial-like in a_{-i}.

Condition **(A1)** is clearly a consequence of **(A2)**. The main aim of this paper is to prove the existence of Nash equilibria under assumption **(A2)**. In Section 4 we shall give a counter-example to show that assumption **(A1)** is not sufficient to guarantee the existence of Nash equilibria.

2. Homological Selection

In this section we will prove a result about the homology of $gr(R_i)$ under the assumption (**A2**). The result will be used to prove the existence of Nash equilibria in the next section.

For a topological space X with a basepoint $*$, we define the space $SP^r(X)$ to be the quotient of $C_r(X)$ under the relation $* = 1$ (here we adopt the notational convention that elements of $C_r(X)$ are written multiplicatively). This space is called the symmetric product of X. It is n-connected if X is. Using the natural inclusions of $SP^r(X)$ inside $SP^{r+1}(X)$ we can form the union $SP^\infty(X)$ the infinite symmetric product of X. The homotopy groups $\pi_n(SP^\infty(X))$ are the homology groups $H_n(X)$ of X by the Dold-Thom theorem ([11] Section 4.K) and the Hurewicz homomorphism is induced by the inclusion of $X = SP^1(X) \hookrightarrow SP^\infty(X)$. We start with the following lemma, whose proof-idea is due to A. Dranishnikov.

Lemma 2.1. *Suppose that $f : \mathbb{D}^m \to C_r(\mathbb{D}^n)$ is a continuous function that has the same value $(0, \dots, 0)$ on the boundary $\partial\mathbb{D}^m = S^{m-1}$, and define $gr(f) = \{(x, y) \in \mathbb{D}^m \times \mathbb{D}^n \mid y \text{ is a factor in } f(x)\}$. Consider the map $i : S^{m-1} \to gr(f)$ with $i(x) = (x, (0, \dots, 0))$. Then*

$$i_* \otimes \mathbb{Q} : H_{m-1}(S^{m-1}) \otimes \mathbb{Q} \to H_{m-1}(gr(f)) \otimes \mathbb{Q}$$

is zero.

Proof. We will apply the Dold-Thom theorem. Consider $\beta(f) : \mathbb{D}^m \to SP^r(gr(f))$ given by the formula

$$\beta(f)(x) = (x, y_1)^{i_1} \dots (x, y_k)^{i_k}$$

if $f(x) = y_1^{i_1} \dots y_k^{i_k}$. This is a continuous embedding of the disk in the r-fold symmetric power of the graph. The boundary of this map is the diagonal embedding $e_r : x \mapsto (x, 0)^r$. It factors as the composite $SP^1(S^{m-1}) \overset{e_r}{\to} SP^r(S^{m-1}) \overset{SP^r(i)}{\to} SP^r(gr(f))$. We also have an inclusion $i_r : S^{m-1} \simeq SP^1(S^{m-1}) \to SP^r(S^{m-1})$ which is defined as $i_r(x) = x.*^{r-1}$.

Note that $H_{m-1}(SP^r(S^{m-1})) \cong \mathbb{Z}$. At the level of homology groups, it is clear that $e_{r*}(1) = r.i_{r*}(1)$. The space $SP^r(S^{m-1})$ is $(m-2)$-connected so by Hurewicz's theorem we obtain $e_r \simeq r.i_r$. Therefore we have,

$$\begin{array}{ccccccc}
S^{m-1} & \xrightarrow{\simeq} & SP^1(S^{m-1}) & \xrightarrow{i} & SP^1(gr(f)) & \xrightarrow{\simeq} & gr(f) \\
 & & \downarrow{\scriptstyle e_r \simeq r i_r} & \overset{e_r}{\searrow} & \downarrow & & \\
 & & SP^r(S^{m-1}) & \longrightarrow & SP^r(gr(f)) & & \\
 & & \downarrow & & \downarrow & & \\
 & & SP^\infty(S^{m-1}) & \longrightarrow & SP^\infty(gr(f)) & &
\end{array}$$

We know that $\pi_*(SP^\infty(X) \cong H_*(X; \mathbb{Z})$ by the Dold-Thom Theorem. So, $i_* : H_{m-1}(S^{m-1}) \to H_{m-1}(gr(f))$ is the same function as $\pi_{m-1}(SP^\infty(i)) : \pi_{m-1}(SP^\infty(S^{m-1})) \to \pi_{m-1}(SP^\infty(gr(f)))$. Since $\beta(f)$ gives a nullhomotopy of $e_r \simeq r.i_r$, we get that

$$r.SP^{m-1}(i) \simeq 0$$
$$\implies r.SP^\infty(i) \simeq 0$$
$$\implies r.i_* = 0$$
$$\implies i_* \otimes \mathbb{Q} = 0.$$

□

Remark 2.2. *In the case $n = 1$ of the above lemma, we get an easier proof which works even for a map $\mathbb{D}^m \to Sub_r(I)$. Here sending a subset to its minimum is a continuous function from $Sub_r(I)$ to I. Therefore we have a disk $(x \mapsto (x, min(f(x)))$ in $gr(f)$ whose boundary is $S^{m-1} = \partial gr(f)$. This shows that $i_* = 0$ and hence, $i_* \otimes \mathbb{Q} = 0$.*

Remark 2.3. *In the case $m = 1$ also the above conclusion is correct. The graph $gr(f)$ is path connected so one can choose a path from the point over -1 to the point over 1. Removing loops one obtains a 1-cell with boundary $\partial gr(f)$.*

Lemma 2.1 has an interesting interpretation in the context of Selection Theorems (cf. [8, 9, 10]. We make the following definition.

Definition 2.4. *Suppose that $f : \mathbb{D}^m \to \mathcal{H}_c(\mathbb{D}^n)$ is a continuous function from $\mathbb{D}^m$ into the space of compact subsets of $\mathbb{D}^n$. Let $gr(f) = \{(x, y) \in \mathbb{D}^m \times \mathbb{D}^n \,|\, y \in f(x)\}$ and let $gr(\partial f) = \{(x, y) \in \partial\mathbb{D}^m \times \mathbb{D}^n \,|\, y \in f|_{\partial\mathbb{D}^m}(x)\}$, where $f|_{\partial\mathbb{D}^m}$ denotes the restriction of f to $\partial\mathbb{D}^m$. A non-zero $m-$dimensional chain c_m supported in $gr(f)$ is said to be a* **homological selection** *of f if its boundary $\partial^m c_m$ is supported in $gr(\partial f)$ and the projection $H_m(gr(f), gr(\partial f)) \to H_m(\mathbb{D}^m, \partial\mathbb{D}^m)$ maps c_m to a non zero class. In such a situation we say that f admits a homological selection.*

Theorem 2.5 below is now a consequence of Lemma 2.1.

Theorem 2.5. Homological Selection Theorem: *Suppose that $f : \mathbb{D}^m \to C_r(\mathbb{D}^n)$ is a continuous function that has the same value $(0, \ldots, 0)$ on the boundary $\partial\mathbb{D}^m = S^{m-1}$. Then f admits a homological selection.*

Proof. We know from Lemma 2.1 that the map

$$H_{m-1}(S^{m-1}) \otimes \mathbb{Q} \cong H_{m-1}(gr(\partial f)) \otimes \mathbb{Q} \to H_{m-1}(gr(f)) \otimes \mathbb{Q}$$

is zero. We consider the commutative square

$$\begin{array}{ccc} H_m(gr(f), \partial gr(f)) & \xrightarrow{\partial} & H_{m-1}(gr(\partial f)) \\ {\scriptstyle \Pi_{i*}}\downarrow & & \downarrow{\scriptstyle \cong} \\ H_m(\mathbb{D}^m, \partial\mathbb{D}^m) & \xrightarrow{\cong} & H_{m-1}(\partial\mathbb{D}^m) \end{array}$$

It suffices to show that the boundary map from $H_m(gr(f), \partial gr(f))$ to $H_{m-1}(gr(\partial f))$ is non zero. The latter group is the top homology of the sphere, hence $\cong \mathbb{Z}$. The next term in the long exact sequence is $H_{m-1}(gr(f))$. Equivalently we must have the map $H_{m-1}(\partial gr(f)) \to H_{m-1}(gr(f))$ has non zero kernel, which because the left group is $\mathbb{Z}$ is the same as saying the above map is 0 after tensoring with $\mathbb{Q}$. Hence by Lemma 2.1 we are done. □

Remark 2.6. *The hypothesis that $f : \mathbb{D}^m \to C_r(\mathbb{D}^n)$ has the same value $(0, \ldots, 0)$ on the boundary $\partial\mathbb{D}^m = S^{m-1}$ can be dropped. Any $f : \mathbb{D}^m \to C_r(\mathbb{D}^n)$ can be extended by a linear homotopy to such a function f_1 by enlarging $\mathbb{D}^m$ slightly by adding an annulus A_m. We note that $gr(A_m) = \{(x, y) \in A_m \times \mathbb{D}^n \,|\, y \in f|_{A_m}(x)\}$ deformation retracts to the outer boundary S^{m-1}. Therefore,*

$$H_m(gr(f_1), gr(\partial f_1)) \cong H_m(gr(f_1), gr(A_m)) \cong H_m(gr(f), gr(\partial f))$$

the second isomorphism coming from excision. Since f_1 admits a homological selection by Theorem 2.5 the same class under the above isomorphisms gives a homological selection for f.

Given any set A in a metric space (X, d) and any $\epsilon > 0$, we denote the ϵ-neighborhood of A by A^ϵ. Given any $\epsilon > 0$ and for all i, $\mathrm{gr}(R_i)^\epsilon$ is intrinsically a manifold with boundary and an open subset of $\mathbb{A}$. Let Π_i denote the (continuous) projection $\Pi_i : \mathrm{gr}(R_i)^\epsilon \to \mathbb{A}_{-i}$. Let $\dim(\mathbb{A}_{-i}) = \sum_{j=1, j\neq i}^{N} n_j \equiv d_i$. Also , let

$d_0 = \sum_{j=1}^{N} n_j$ so that $d_i = (d_0 - n_i)$. As usual, we denote the boundary of a manifold M by ∂M.

We now state and prove the following crucial proposition.

Proposition 2.7. *Suppose that a cost function $r_i : \mathbb{A}_i \times \mathbb{A}_{-i} \to \mathbb{R}$ is weakly polynomial-like. Let R_i be the response function of r_i. Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and for all i, the induced homomorphism of Π_i in the relative (co)homology*

$$\Pi_i^* : H^{d_i}(\mathbb{A}_{-i}, \partial\mathbb{A}_{-i}) \longrightarrow H^{d_i}(gr(R_i)^\epsilon, \partial gr(R_i)^\epsilon)$$

or equivalently

$$\Pi_{i*} : H_{d_i}(gr(R_i)^\epsilon, \partial gr(R_i)^\epsilon) \longrightarrow H_{d_i}(\mathbb{A}_{-i}, \partial\mathbb{A}_{-i})$$

is nonzero.

Proof. Suppose that in an ϵ neighborhood of R_i there is a continuous function f from $\mathbb{A}_{-i}$ for which $H_{d_i}(gr(f), \partial gr(f)) \to H_{d_i}(\mathbb{A}_{-i}, \partial\mathbb{A}_{-i})$ is non zero. This non zero map can be written as the composite

$$H_{d_i}(gr(f), \partial gr(f)) \to H_{d_i}(\mathrm{gr}(R_i)^\epsilon, \partial\mathrm{gr}(R_i)^\epsilon) \longrightarrow H_{d_i}(\mathbb{A}_{-i}, \partial\mathbb{A}_{-i})$$

which forces the latter map to be non zero. This is the statement we need.

Since r_i is weakly polynomial-like, for every $\epsilon > 0$ there exists f such that
a) $gr(f)$ is contained in an ϵ-neighborhood of $gr(R_i)$, and
b)f can be lifted to the space $C_N(\mathbb{D}^n)$.
Lemma 2.1 now completes the proof. □

3. Existence of Nash Equilibria

In this section, we prove the existence of Nash equilibria under the assumption (**A2**).

For any $k \in \mathbb{N}$ we denote $[k] = \{1, 2, \dots, k\}$. We begin with the useful lemma.

Lemma 3.1. *Given closed subset sequences $\{A_{ji}\}_{i\in\mathbb{N}}, j \in [k]$ of a compact subset S of X such that $\bigcap_j A_{ji} \neq \emptyset, \forall i$ and $d_H(A_{ji}, A_j) \overset{i\to\infty}{\longrightarrow} 0$ for given non-empty subsets $A_j, j \in [k]$ of S, then $\bigcap_j A_j \neq \emptyset$.*

Proof. Choose any $\epsilon > 0$. Since, for any $j \in [k]$, $d_H(A_{ji}, A_j) \overset{i\to\infty}{\longrightarrow} 0$, there exists $i_0(j)$ such that for all $i \geq i_0(j)$, $A_{ji} \subseteq A_j^\epsilon$. Let $i_0 = \max_{j\in[k]}\{i_0(j)\}$. Now, since by assumption $\bigcap_j A_{ji} \neq \emptyset, \forall i \geq i_0$, for all j, we have $\bigcap_j A_j^\epsilon \neq \emptyset$. Now suppose $\bigcap_j A_j = \emptyset$. Then we can construct a non-empty set

$$\Xi \equiv \{(\Lambda_1, \Lambda_2) \in 2^{[k]} \times 2^{[k]} :\ \Lambda_1 \cap \Lambda_2 = \emptyset,\ \cap_{i\in\Lambda_1}\mathbb{A}_i \neq \emptyset,\ \cap_{j\in\Lambda_2}A_j \neq \emptyset\}.$$

For each tuple $(\Lambda_1, \Lambda_2) \in \Xi$, there exists

$$\epsilon_{\Lambda_1,\Lambda_2} \equiv \min_{i\in\Lambda_1} \inf_{x\in\mathbb{A}_i} \inf_{y\in\cap_{j\in\Lambda_2}A_j} d(x, y) > 0.$$

Choose $\epsilon_0 \equiv \frac{1}{3}\min_{(\Lambda_1,\Lambda_2)\in\Xi}\epsilon_{\Lambda_1,\Lambda_2}$. Then, for any $0 < \epsilon < \epsilon_0$ and for all $(\Lambda_1,\Lambda_2)\in\Xi$,

$$\begin{aligned}\cap_{i\in\Lambda_1}\mathbb{A}_i^\epsilon \bigcap \cap_{j\in\Lambda_2}A_j^\epsilon &\subseteq \cup_{i\in\Lambda_1}\mathbb{A}_i^\epsilon \bigcap \cap_{j\in\Lambda_2}A_j^\epsilon\\ &\subseteq (\cup_{i\in\Lambda_1}\mathbb{A}_i)^\epsilon \bigcap \cap_{j\in\Lambda_2}A_j^\epsilon\\ &\subseteq (\cup_{i\in\Lambda_1}\mathbb{A}_i)^\epsilon \bigcap (\cap_{j\in\Lambda_2}A_j)^{2\epsilon}\\ &= \emptyset.\end{aligned}$$

Hence $\bigcap_j A_j^\epsilon = \emptyset$ for $0 < \epsilon < \epsilon_0$ which is a contradiction. □

Before we state our main theorem we recall the notion of **external cup product, or cross product** of relative cohomology classes ([11] p. 220). Since $\mathbb{A} = \mathbb{A}_i \times \mathbb{A}_{-i}$, we have an isomorphism $H^{n_i}(\mathbb{A}_i,\partial\mathbb{A}_i)\otimes H^{d_i}(\mathbb{A}_{-i},\partial\mathbb{A}_{-i}) \to H^n(\mathbb{A},\partial\mathbb{A})$ ([11] Theorem 3.20). This isomorphism is implemented by the cross product $a\times b = p_1^*(a)\cup p_2^*(b)$, where $p_1 : (\mathbb{A}_i\times\mathbb{A}_{-i},\partial\mathbb{A}_i\times\mathbb{A}_{-i}) \to (\mathbb{A}_i,\partial\mathbb{A}_i)$ and $p_2 : (\mathbb{A}_{-i}\times\mathbb{A}_i,\partial\mathbb{A}_{-i}\times\mathbb{A}_i) \to (\mathbb{A}_{-i},\partial\mathbb{A}_{-i})$ are projections of pairs. Also, since $H^{n_i}(\mathbb{A}_i,\partial\mathbb{A}_i) = H^{d_i}(\mathbb{A}_{-i},\partial\mathbb{A}_{-i}) = H^n(\mathbb{A},\partial\mathbb{A}) = \mathbb{Z}$, (relative) Poincaré Duality holds in this set up.

Using Definition 1.3 the existence of Nash equilibria can be stated as follows.

Theorem 3.2. *Consider an $n-$person non-cooperative game with mixed strategy spaces satisfying assumption **(A2)**. Let $r_1,\cdots,r_n$ be the cost functions and $R_1,\cdots,R_n$ be the response functions. Then, $\bigcap_{i=1}^N gr(R_i)\neq\emptyset$. Equivalently, the game has at least one Nash equilibrium.*

Proof. We continue with the notation of Proposition 2.7, which gives the following:

$$\Pi_{i*} : H_{d_i}\left(\mathrm{gr}(R_i)^\epsilon,\partial\mathrm{gr}(R_i)^\epsilon\right) \longrightarrow H_{d_i}\left(\mathbb{A}_{-i},\partial\mathbb{A}_{-i}\right)$$

is nonzero. For each i, choose a relative d_i-cycle $z_i \in H_{d_i}\left(\mathrm{gr}(R_i)^\epsilon,\partial\mathrm{gr}(R_i)^\epsilon\right)(\subset H_{d_i}(\mathbb{A}_{-i},\partial\mathbb{A}_{-i}))$ such that $w_i = \Pi_{i*}(z_i)\neq 0$.

Let $w_i^*(\neq 0)\in H^{n_i}(\mathbb{A}_i,\partial\mathbb{A}_i)$ be its (relative) Poincaré Dual. Then $\bigcup_{i=1\cdots N} w_i^* \in H^n(\mathbb{A},\partial\mathbb{A})$ is a non-zero cohomology class. Hence, by (relative) Poincaré Duality again the intersection of the supports of the chains z_i is non-empty, i.e. $\bigcap_{i=1\cdots N}\mathrm{supp}(z_i)\neq\emptyset$. One way to see this is to take simplicial approximations of the chains z_i homologous to z_i contained in $\mathrm{gr}(R_i)^\epsilon$ and ensure that their supports are in general position. Then $\bigcap_{i=1\cdots N}\mathrm{supp}(z_i)$ is the support of a zero-cycle (Poincaré) dual to $\bigcup_{i=1\cdots N} w_i^*$. Hence the intersection of the closures $Cl(\mathrm{gr}(R_i)^\epsilon)$ is non-empty, i.e. $\bigcap_{i=1\cdots N} Cl(\mathrm{gr}(R_i)^\epsilon)\neq\emptyset$. Since this is true for all $\epsilon > 0$, it follows from Lemma 3.1 that $\bigcap_{i=1\cdots N}\mathrm{gr}(R_i)\neq\emptyset$. □

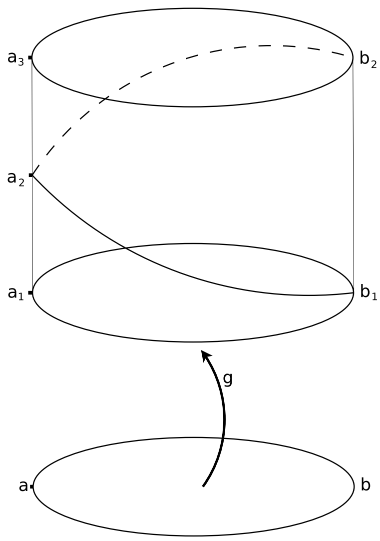
4. A Counterexample

In this section we give an example where (**A2**) is not satisfied though (**A1**) is satisfied and the conclusion in Proposition 2.7 is not true, thereby affirming the necessity of the assumption. Proposition 1.4 indicates that the topology on $Sub_n(\mathbb{D}^m)$ is the quotient topology inherited from $C_n(\mathbb{D}^m)$ by forgetting weights/multiplicities. This shows that there is no trivial counterexample.

The example below will involve a map $\mathbb{D}^2 \to Sub_3(\mathbb{D}^2)$ (that is, $m = n = 2$) . Note that this is the smallest case where such an example can exist: by Remark 2.2 the conclusion in Proposition 2.7 is satisfied if $n = 1$. By Remark 2.3 we have the case $m = 1$, and since $Sub_2(X) = C_2(X)$ we obtain the result for the case $\mathbb{D}^2 \to Sub_2(\mathbb{D}^2)$. We also use the counterexample to demonstrate that Nash equilibria might not exist or equivalently that Theorem 3.2 does not hold if we assume only (**A1**). However, before we get into the counterexample, a remark is in order.

Remark 4.1. Let $\mathcal{P}(\mathbb{D}^j)$ denote the set of probability measures on $\mathbb{D}^j$ equipped with the weak topology. Also let $\mathcal{P}_n(\mathbb{D}^j)$ be the subspace of probability measures supported on (at most) n points. Any continuous function $f : \mathbb{D}^l \to Sub_n(\mathbb{D}^j)$ can be lifted to $\mathcal{P}_n(\mathbb{D}^j)$ by Michael's Selection Theorem [10]. Equivalently any continuous function $f : \mathbb{D}^l \to Sub_n(\mathbb{D}^j)$ can be lifted to a continuous **weighted real valued multifunction** with total weight constant at each point, where all weights are non-negative. The following counterexample shows that we cannot upgrade this to a continuous weighted **rational-valued** (or equivalently integer-valued, after normalization) multifunction with constant total weight at each point.

4.1. **Counterexample to Proposition 2.7.** The idea of the counterexample is simple in outline. Consider a map $g : S^1 \to Sub_3(\mathbb{D}^1)$ given by $g(e^{2\pi i\theta}) = \{0, 1, \theta\}$, where $\theta \in [0, 1]$ and $\mathbb{D}^1$ is identified with $[0, 1]$. Any lift $\widetilde{g} : S^1 \to SP^k(\mathbb{D}^1)$ for some $k \geq 3$ assigns non-negative integer weights to each point in $g(x)$ for all $x \in S^1$. Choose a regular point $a \in S^1$ (for g), i.e. a point such that $\#g(a) = 3$. Let $g(a) = \{a_1, a_2, a_3\}$ (see figure below). Let w_i be the weight assigned to a_i so that $(w_1 + w_2 + w_3) = k$.



By continuity, the weights on the strands containing each of the a_i's is constant over all regular points. Let $b = e^{2\pi i}$ be the only non-regular point and let $g(b) = \{b_1, b_2\}$. Let u_i be the weight assigned to b_i so that $(u_1+u_2) = k$. Then $u_1 = w_1+w_2$ and $u_2 = w_2 + w_3$. It follows that $w_2 = 0$, i.e. the weight assigned to a_2 is zero.

The main idea behind the counterexample below is to *wedge three copies of the above picture together appropriately.* More precisely, wedge three copies S^1_1, S^1_2, S^1_3 of the circle S^1 along a regular point a and permute the multifunction values $g(a) = \{a_1, a_2, a_3\}$ over the three copies during the gluing operation such that by the computation above, each w_i is forced to be zero. Equivalently, a_i is forced to lie in the middle strand on $g(S^1_i)$. Next this multifunction g is extended to a wedge

W of 3 disks D_1, D_2, D_3, where D_i is bounded by S^1_i, $i = 1, 2, 3$. Finally embed W into a disk D and extend the multifunction to all of D.

We now start with a formal description. Fix a path $P(t)$ in $\mathbb{D}^2$. Define $g_P : \mathbb{D}^2 \to Sub_3(\mathbb{D}^2)$ as follows:
On the ray from (0,0) to (1/2,0) define $g_P(t, 0) = \{P(0), P(2t)\}$ so that $g_P(0, 0) = \{P(0)\}$. Now define g on the circle of radius t by

$$g_P(tcos2\pi a, tsin2\pi a) = \{P(0), P(2t), P(2at)\}$$

We extend g_P to $\mathbb{D}^2$ by using a linear homotopy from the map on the circle of radius half to the constant function $(0, 0)$, and putting this homotopy as the function on the outward rays. Then g_P is continuous, and there is no lift to $SP^3(\mathbb{D}^2)$ as any assignment of weights to the points forces the weight of the points $P(at)$ for $a \neq 0$ over $(tcos2\pi a, tsin2\pi a)$ to be zero (see argument above). Note however that one can pass to a continuous submultifunction (leaving out the subset of the above form − the middle strand in the picture) on which weights can be defined and hence the conclusion of Proposition 2.7 remain true.

The topological space $gr(g_P)$ is homeomorphic to the sphere S^2 with the north pole and south pole identified along with a disk whose boundary is the wedge of the longitude and the meridian. We now write a CW complex structure of $gr(g_P)$: There are two 0-cells v, w with v as the pinched point (north pole = south pole = the point lying over $(0, 0)$) and w as the point on the boundary lying over the point $(1, 0)$; There are three 1-cells a_1, a_2, α, where a_i are the two lines lying over the line joining $(0, 0)$ to $(1, 0)$, and α is the boundary. There are three 2-cells e_1, e_2, e_3, the three inverse images of the disk minus the line joining $(0, 0)$ to $(1, 0)$. The boundaries are given by:

$$\partial e_1 = \alpha \cup a_1 \cup (-1)a_1,$$
$$\partial e_2 = \alpha \cup a_2 \cup (-1)a_2,$$
$$\partial e_3 = \alpha \cup a_1 \cup a_2$$

and

$$\partial a_i = v \cup w \, (i = 1, 2),$$
$$\partial \alpha = v \cup v.$$

We use the above g_P to get a counterexample by wedging three similar maps at a regular point. Formally, we use three arcs of a circle as our paths. Fix a circle C, define $h_C : \mathbb{D}^2 \to Sub_3(\mathbb{D}^2)$ as follows. Divide C into three arcs I_1, I_2, I_3 by choosing three points A_1, A_2, A_3 in order on it and defining $I_1 = A_2A_3$, $I_2 = A_3A_1$, $I_3 = A_1A_2$. Define $P_i = C - I_i$ for $i = 1, 2, 3$. The orientation of P_i is defined by starting at the end of I_i and ending at the beginning (for example, P_1 starts at A_3 passes through A_1 and ends at A_2). In $\mathbb{D}^2$ consider three smaller disks D_i $(1 \leq i \leq 3)$ in the interior wedged at the point $(0, 0)$. On the disk D_i define h_C as the function g_{P_i} in the half disk (by fixing an orientation preserving homeomorphism between the half disk and D_i in such a way that the point corresponding to $(0, 0)$ is mapped by g_{P_i} to $\{A_1, A_2, A_3\}$). On the complement of $\cup D_i$ we use a linear homotopy to get the value $(0, 0)$ on the boundary. We prove that the boundary circle is a free generator for $H_1(gr(h_C); \mathbb{Z})$ thus violating the conclusion of Proposition 2.7.

The topological space $gr(h_C)$ can be obtained from the spaces $G_i = gr(g_{P_i})$ (all the three G_i are homeomorphic) by slicing each over a regular point of the disk (with three inverse images) and gluing the three together via the three cyclic permutations. Then we can write down a CW complex structure on $gr(h_C)$ as follows: There are six 0-cells v_i, w_i $(i = 1, 2, 3)$ corresponding to the notation in the case of G_i; There are nine 1-cells a_1^i, a_2^i $(i = 1, 2, 3)$ (the same definition as before in the G_i) and $\alpha_{1,2}$, $\alpha_{2,3}$, $\alpha_{3,1}$ where $\alpha_{i,j}$ joins v_i to v_j on the boundary; There are still three 2-cells (since connecting the disks only results in a bigger disk) e_1, e_2, e_3. The boundary maps are given by the following(where the boundary $\alpha = \alpha_{1,2}, \cup\alpha_{2,3}, \cup\alpha_{3,1}$):

$$\partial e_1 = \alpha \cup a_1^1 \cup (-1)a_1^1 \cup a_2^2 \cup (-1)a_2^2 \cup a_1^3 \cup a_2^3,$$
$$\partial e_2 = \alpha \cup a_1^1 \cup a_2^1 \cup a_1^2 \cup (-1)a_1^2 \cup a_2^3 \cup (-1)a_2^3,$$
$$\partial e_3 = \alpha \cup a_2^1 \cup (-1)a_2^1 \cup a_1^2 \cup a_2^2 \cup a_1^3 \cup (-1)a_1^3$$

and

$$\partial a_j^i = v^i \cup w^i \,(j = 1, 2,\ i = 1, 2, 3),$$
$$\partial \alpha_{k,l} = v_k \cup v_l \,(k = 1, 2, 3,\ l = 1, 2, 3).$$

Now we can compute the CW homology of $gr(h_C)$. We obtain that $H_1(gr(h_C)) \cong \mathbb{Z}$ with generator α which is homologous to each $a_1^i \cup a_2^i$. This violates the conclusion of Proposition 2.7, proving the counterexample. □

4.2. **Counterexample to Theorem 3.2.** Theorem 3.2 asserts the existence of Nash equilibria under the condition (**A2**). Using the above counter-example, we construct an example where Nash equilibria do not exist. To do this we realize the graph of any continuous function $f : \mathbb{A}_{-i} \to Sub_N(\mathbb{A}_i)$ as the set of minima of a cost function.

Proposition 4.2. *Let $f : \mathbb{A}_{-i} \to Sub_N(\mathbb{A}_i)$ be a continuous function. Then there is a function $F : \mathbb{A}_{-i} \times \mathbb{A}_i \to \mathcal{R}$ such that,*

$$f(x) = \{y \,|\, y\, is\, a\, local\, minimum\, of\, y \mapsto F(x, y)\}.$$

Proof. Consider the graph of f, $gr(f)$. This is a closed subset of the compact set $\mathbb{A}_{-i} \times \mathbb{A}_i$ and is therefore, compact. We define F to be the distance function to $gr(f)$ which is greater than zero if $(x, y) \notin gr(f)$ and 0 if $(x, y) \in gr(f)$. Thus, the set of minima of F are exactly the elements of the set $gr(f)$. □

The above proof also works if we replace local minima by global minima. We now write down the counterexample to the statement analogous to Theorem 3.2 obtained by replacing assumption (**A2**) by (**A1**). For the example below, the number of players $N = 2$, $\mathbb{A}_1$ and $\mathbb{A}_2$ are two dimensional disks. We have to construct functions $f_1 : \mathbb{A}_1 \to Sub_k(\mathbb{A}_2)$ and $f_2 : \mathbb{A}_2 \to Sub_l(\mathbb{A}_1)$ such that $gr(f_1)$ and $gr(f_2)$ do not intersect.

Define f_1 as the example h_C above by using the same formula on the disks $\mathbb{D}_i$ $(i = 1, 2, 3)$. Draw a continuous family of linear trajectories joining the boundary circle to the closest point on $\cup\partial\mathbb{D}_i$. Extend f_1 to $\mathbb{A}_1$ such that it is constant on each of the trajectories. Note that this defines uniquely a continuous function $f_1 : \mathbb{A}_1 \to Sub_3(\mathbb{A}_2)$.

Next, we shall choose f_2 judiciously so that $gr(f_2)$ does not intersect $gr(f_1)$. Note that $Im(f_1)$ is contained in $Sub_3(C)$, so that any intersection value must have second coordinate in the circle C. It is therefore enough to make the judicious choice over C and take any extension (such extensions are always possible since the inclusion of the circle C is a neighborhood deformation retract and $Sub_3(\mathbb{A}_1)$ is contractible).

Divide the circle C into three arcs α_1, α_2, α_3 where α_1 is the arc joining the mid-point of A_3A_1 to the mid-point of A_1A_2 and passing through A_1 (and similarly α_2, α_3). Define f_2 on the circle C to be the *single valued* function mapping α_i to $\partial\mathbb{D}_i$ such that
a) the map is an orientation preserving homeomorphism,
b) the boundary points of α_i are mapped to $(0,0)$, and
c) the point A_i on α_i is mapped to the point diametrically opposite to $(0,0)$ in $\partial\mathbb{D}_i$.
Recall that this point on $\partial\mathbb{D}_1$ (resp. $\partial\mathbb{D}_2$, $\partial\mathbb{D}_3$) is mapped by f_1 to $\{A_2, A_3\}$(resp. $\{A_1, A_3\}$, $\{A_1, A_2\}$). We extend f_2 to $\mathbb{A}_2$ by the recipe given in the previous paragraph.

To see that $gr(f_1) \cap gr(f_2) = \emptyset$, note that the contrary would imply that there is an $x \in \mathbb{A}_2$ with $x \in f_1(f_2(x))$ because f_2 is single valued on C. The point x cannot belong to the boundary of the α_i because then $f_1(f_2(x)) = \{A_1, A_2, A_3\}$ and none of the A_i lie on the boundary.

If x is in the interior of α_1 then $f_1(f_2(x)) = \{A_2, A_3, y\}$. Note that as the point x traverses from the mid-point of A_3A_1 to the mid-point of A_1A_2 the point y starts at A_1 and moves along the arc A_1A_2. When x is close to A_1 the point y is close to A_2. Therefore if $x \in A_3A_1 \cap \alpha_1$, $y \in A_1A_2$. Similarly, we observe that if $x \in A_1A_2 \cap \alpha_1$, then $y \in A_3A_1$. Therefore, $x \notin f_1(f_2(x))$.

The same proof works for x in the interior of α_2 and α_3. Therefore, f_1, f_2 as above give cost functions F_1, F_2 by Proposition 4.2 such that the game defined by the cost functions F_i for player i $(i = 1, 2)$ does not have a Nash equilibrium. *Thus Theorem 3.2 does not hold if assumption (**A2**) is replaced by (**A1**).* This completes the counterexample. □

4.3. **Generalizations of Branched Surfaces.** The theory of train-tracks in dimension two [13] and that of branched surfaces in dimension three [2] has been very fruitful. It is therefore tempting to try to generalize these to higher dimensions. A train-track (resp. branched surface) is a 1-complex (resp. 2-complex) in a 2-manifold (resp. 3-manifold) with a welldefined C^1 tangent space everywhere. The set of points where a train-track is not a 1-manifold is a discrete collection of isolated points, i.e. a 0-manifold. However, one dimension higher, the set of points where a branched surface is not a 2-manifold is only a 1-complex, and is called the branch locus. The branch locus is not a 1-manifold only at isolated points, where two lines of the branch locus cross each other transversely. It is important to note that both train-tracks and branched surfaces are codimension one objects. In applications, branches of train-tracks and branched surfaces are assigned weights consistently. The counterexample in Section 4.1 shows that the graph of a multifunction $f : \mathbb{D}^n \to Sub_k(\mathbb{D}^m)$ *cannot* serve as an analog of a non-trivially weighted branched surface in higher codimensions. On the other hand, Proposition 2.7 indicates that the graph of a multifunction $f : \mathbb{D}^n \to C_k(\mathbb{D}^m)$ does support a non-trivial relative cycle and could serve as the starting point for a theory of weighted branched surface in higher codimensions.

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