

COMMENSURATORS OF NORMAL SUBGROUPS OF LATTICES

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ABSTRACT. We study a question of Shalom concerning arithmeticity of discrete subgroups of semisimple Lie groups with dense commensurators. We answer Shalom's question positively for normal subgroups of lattices. This generalizes a result of the second author and T. Koberda.

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1. INTRODUCTION

1.1. Main result. Let G be a real or p -adic semisimple Lie group with finite center and without compact factors, or a finite product of such groups. More precisely, we consider $G = \mathbb{G}(k)$ where k is a local field of characteristic zero and \mathbb{G} is a semisimple algebraic group defined over k and also products of groups of this type. Let $\Gamma \subseteq G$ be a discrete subgroup with commensurator Δ . For lattices, a landmark theorem of Margulis shows that Δ detects arithmeticity of Γ :

Theorem 1.1 (Margulis). *Let G be as above and let Γ be an irreducible lattice in G . Then Γ is arithmetic if and only if Δ is dense in G .*

It is natural to speculate that this theorem holds more generally, assuming only that Γ is discrete and Zariski dense. This question was first asked by

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Greenberg for $SO(n, 1)$ in 1974 and later asked more generally by Shalom [Gre74, Mj11].

Question 1.2 (Greenberg-Shalom). Let G be a semisimple Lie group with finite center and without compact factors. Suppose $\Gamma \subseteq G$ is a discrete, Zariski-dense subgroup of G whose commensurator $\Delta \subseteq G$ is dense. Is Γ an arithmetic lattice in G ?

Greenberg has given a positive answer for finitely generated subgroups of $G = \mathrm{SL}(2, \mathbb{R})$ [Gre74]. Building on work by Leininger-Long-Reid [LLR11], Mj gave a positive answer for finitely generated subgroups of $\mathrm{SL}(2, \mathbb{C})$ [Mj11]. Koberda-Mj have given a positive answer for certain normal subgroups Γ of lattices in rank 1, e.g. those with abelian quotients [KM21, KM19]. For all other cases of G , as well as general infinitely generated subgroups of the above, Question 1.2 is open. Shalom's question is closely related to a number of other problems which will be discussed in Section 1.3 below. More generally it is known that any group as in the question has full limit set. This was observed by Greenberg in $SO(n, 1)$ and Mj in general [Gre74, Mj11]. In that context it is natural to look at the case of normal subgroups of lattices, which are a robust source of groups with full limit set. Other constructions of groups with full limit set are possible, but normal subgroups of lattices seem a better candidate for having large commensurator. Our main result is a positive answer to Question 1.2 for normal subgroups of lattices:

Theorem 1.3. *Let G be a finite product of almost simple algebraic groups defined over local fields of characteristic zero, and let $\Lambda \subseteq G$ be an irreducible lattice. Suppose $\Gamma \subseteq \Lambda$ is an infinite normal subgroup with dense commensurator $\Delta \subseteq G$. Then Γ is an arithmetic lattice and has finite index in Λ .*

This implies a suitable version for reducible lattices, see Theorem 5.1. With a few modifications to the proof, we obtain the same result when G is defined over a local field with positive characteristic when Λ is either uniform or arithmetic, see Theorem 3.1. If G has higher rank, then Γ has finite index in Λ by Margulis' Normal Subgroups theorem. We note here that we do not give a new proof of that theorem here and we use it in our proof.

1.2. Outline of proofs. We start by proving that Δ also commensurates Λ (Section 2.2). The rest of the proof is based on the study of a relative profinite completion L of Δ with respect to its commensurated subgroup Γ , the similarly defined relative completion H of Δ with respect to Λ , and the natural map $L \rightarrow H$ between them.

We study the kernel N of this map in several steps. First we quotient by the normal closure C (in L) of $\bar{\Gamma} \cap N$, and we show that $L/C \rightarrow H$ is (up to finite index) a central extension (Subsection 2.3). Then we study the (continuous) cohomology class of this extension in $H^2(H, N/C)$, and

we show it must be torsion. This readily implies that N/C must be finite (Section 2.5).

Finally, in Section 2.6, we use a direct argument to show that C is compact and since N/C is finite, N is also compact. This immediately implies that Λ/Γ is finite.

In Section 3, we discuss the required modifications for the proof in positive characteristic. The proof is largely the same, except where we use finite generation of Λ . Instead, we use a structural result of Lubotzky that such a lattice is a free product of a free subgroup and lattices in unipotent groups. This description is precise enough that it allows us to circumvent the use of finite generation.

In Section 4 we establish a technical result on subgroups of S -arithmetic lattices, which generalizes a result of Venkataramana (with essentially the same proof).

1.3. Motivations and connections. In this subsection we explain some connections between Shalom's Question 1.2 and other well-known problems. In the first subsection we recall a conjecture of Margulis and Zimmer and explain Shalom's original motivation for his conjecture. In the second subsection, we point to a newer set of connections which to the best of our knowledge have not previously appeared in the literature. We only sketch these and leave further details to our forthcoming article with Nic Brody [BFMvL22].

1.3.1. Margulis-Zimmer Conjecture. A major motivation for Shalom's question is the following conjecture advertised by Margulis-Zimmer in the late '70s, seeking to classify commensurated subgroups of higher rank lattices Λ . Here we say $\Gamma \subseteq \Lambda$ is *commensurated* if $\Lambda \subseteq \text{Comm}_G(\Gamma)$.

Conjecture 1.4 (Margulis-Zimmer, see [SW13]). Let \mathbb{G} be a semisimple algebraic group defined over a global field k and S a finite set of valuations of k . Assume \mathbb{G} has higher S -rank. Then any commensurated subgroup of the S -arithmetic lattice Λ is either finite or S' -arithmetic for some $S' \subseteq S$.

Here, a global field is a number field or a finite extension of a function field $\mathbb{F}_p(t)$, and the S -rank of \mathbb{G} is the sum of the k_ν -ranks over all valuations $\nu \in S$, and \mathbb{G} is said to have *higher S -rank* if its S -rank is at least 2.

The connection to Shalom's question occurs as soon as there are at least two valuations, and one of them is non-Archimedean. To be most transparent, we assume $\Lambda = \text{SL}(n, \mathbb{Z}[1/p])$, but this explanation easily generalizes. Let $\Gamma < \Lambda$ be a commensurated subgroup contained in $\text{SL}(n, \mathbb{Z})$. Then Γ is a discrete subgroup of $\text{SL}(n, \mathbb{R})$ and in the latter group, its commensurator Λ is dense. The Margulis-Zimmer conjecture in this case predicts that Γ is either finite, or of finite index in $\text{SL}(n, \mathbb{Z})$.

Using well-chosen unipotent generating sets, Venkataramana has proven Conjecture 1.4 for arithmetic lattices $\Gamma = \mathbb{G}(\mathbb{Z})$ in simple groups defined over \mathbb{Q} [Ven87]. Shalom-Willis have proved Conjecture 1.4 in more instances,

including the first that are not simple [SW13]. Their proof crucially relies on fine arithmetic properties for lattices in these groups, namely bounded generation by unipotents. This is the only prior work on Conjecture 1.4. Our results provide some new, partial evidence for the Margulis-Zimmer conjecture in some special cases.

1.3.2. Irreducible subgroups of products. In this subsection, we provide some indications of results with Nic Brody that will appear later in [BFMvL22]. The general motivation here is to study discrete subgroups of products that project indiscretely to the factors. The only published reference we know that mentions questions of this type is [FLSS18], though that article only mentions much weaker questions than the following:

Question 1.5. Let G_1 and G_2 be semisimple groups over local fields and $\Lambda < G_1 \times G_2$ a discrete subgroup with both projections dense. Is Λ in fact an irreducible lattice in the product?

At first glance, this question seems overly strong. We state an easy proposition that shows that it is a natural generalization of Shalom's question.

Proposition 1.6. *Let G_1 and G_2 be semisimple algebraic groups over local fields with G_2 totally disconnected. Let $\Lambda < G_1 \times G_2$ be discrete with both projections dense. Then there is a subgroup $\Gamma < \Lambda$ which projects discretely to G_1 and Γ is commensurated by Λ .*

The proposition is easy to prove by picking Γ to be the intersection of Λ with a compact open subgroup of G_2 . In fact the proposition really only requires that G_1 and G_2 be locally compact topological groups with G_2 Hausdorff and totally disconnected (then G_2 contains a compact open subgroup). We note here that in the proof of Proposition 1.6, the group Γ we construct is most likely infinitely generated and not normal in any subgroup of Λ . So no existing work on Shalom's question 1.2 bears on Question 1.5.

The difficulty in applying this approach is to guarantee the density of projections. Of course if either projection is discrete, Λ is best studied as a discrete subgroup of that factor rather than the product. So it is natural to assume the projections are indiscrete, in which case it is often also easy to prove they are dense. This will allow us to see that a number of natural questions would be answered by a positive answer to Shalom's question. These include

- (1) the non-existence of surface groups as discrete subgroups of products of rank one p -adic groups; this has applications to the arithmetic of three manifold groups [FLSS18];
- (2) the non-existence of discrete free groups with dense projections in many products, answering a question asked by Yves Benoist;
- (3) an old question of Lyndon and Ullman on groups generated by parabolics in $\mathrm{SL}(2, \mathbb{R})$, recently made a conjecture by Kim and Koberda [LU69, KK19].

Detailed explanations of all these connections and some others will appear in the forthcoming paper with Brody [BFMvL22].

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2. PROOFS

2.1. Schlichting completions. We start by introducing a main tool in our proofs, so-called Schlichting completions. See [SW13, Section 3] for more details on the facts reviewed here.

Definition 2.1. Let Υ be a countable group and $\Theta < \Upsilon$ a commensurated subgroup. The *Schlichting completion* $\Upsilon//\Theta$ is defined to be the closure of $\Upsilon < \text{Symm}(\Upsilon/\Theta)$ in the topology of pointwise convergence.

Then $\Upsilon//\Theta$ is a totally disconnected, locally compact, Hausdorff, second countable group. Υ maps onto a dense subgroup of $\Upsilon//\Theta$ with kernel the normal core of Θ in Υ , and the image of Θ in Υ has compact open closure. If in addition Υ is finitely generated, then $\Upsilon//\Theta$ is compactly generated.

Example 2.2. The Schlichting completion of $\Upsilon := \mathbb{G}(\mathbb{Z}[1/p])$ with respect to $\Theta := \mathbb{G}(\mathbb{Z})$ is $\Upsilon//\Theta = \mathbb{G}(\mathbb{Q}_p)$. For us, Δ is S -arithmetic, and $\Delta//\Lambda$ has finite index in $\mathbb{G}(k_S)$.

In the case that Υ is finitely generated, $\Upsilon//\Theta$ can be realized as a group of automorphisms of a locally finite graph, namely its Cayley-Abels graph (with respect to a compact generating set). Its construction is due to Abels [Abe74]. Our discussion follows [Mon01, 11.3]. Let H be a totally disconnected, compactly generated Hausdorff topological group. Then H has a compact open subgroup K . We fix a bi- K -invariant compact symmetric generating set C . Then the Cayley-Abels graph $\mathcal{G} = \mathcal{G}(H, C, K)$ of H with respect to C and K is defined to have vertex set H/K and edges between hK and hcK for all $h \in H$ and $c \in C$. By compactness of C , this graph is d -regular for $d = |C/K| < \infty$, and since C generates H , it is connected. Further, H acts on \mathcal{G} by graph automorphisms with vertex stabilizers given by conjugates of K . In particular, this action is isometric and hence proper.

If Υ is a finitely generated group with commensurated subgroup Θ , then $K := \overline{\Theta}^H$ is a compact open subgroup. With this choice, the Cayley-Abels graph has vertex set $H/K = \Upsilon/\Theta$.

2.2. Arithmeticity of Λ . As remarked in the introduction, since Γ is normal in Λ and Λ is an irreducible lattice in G , Margulis' Normal Subgroups Theorem allows us to assume that G is a rank one simple group. We make this assumption without further comment for the rest of the paper. We start by proving:

Proposition 2.3. *Under the assumptions of Theorem 1.3, Δ commensurates Λ . In particular, Λ is arithmetic.*

Proof. The second part follows from the first by Margulis' commensurator rigidity theorem. For the first claim, let $\delta \in \Delta$. We claim that Λ normalizes a finite index subgroup of $\Gamma \cap \Gamma^\delta$. This is clear if Γ is finitely generated (so that it only admits finitely many subgroups of a given index), but in general we argue as follows:

The group $\Lambda_\delta := \langle \Lambda, \delta \rangle$ is finitely generated and has commensurated subgroup Γ . Fix a Cayley-Abels graph of $\Lambda_\delta // \Gamma$ with vertex set Λ_δ / Γ . Then $\Gamma \cap \Gamma^\delta$ is the pointwise stabilizer in Λ_δ of $\{e\Lambda_\delta, \delta\Lambda_\delta\} \subseteq \Lambda_\delta / \Gamma$. Let R be the distance between these points, and let Γ' be the pointwise stabilizer of the ball of radius R centered at $e\Lambda_\delta \in \Lambda_\delta / \Gamma$ (with respect to the graph metric on the Cayley-Abels graph).

Note that the stabilizer (in Δ) of a point in Δ / Γ only depends on its image in Δ / Λ : Indeed, if $\delta \in \Delta$ and $\lambda \in \Lambda$, then the stabilizer of $\delta\Gamma$ is just Γ^δ , whereas the stabilizer of $\delta\lambda\Gamma$ is $\Gamma^{\delta\lambda} = \Gamma^\delta$. Therefore Γ' is just the pointwise stabilizer of all fibers that are within distance R of Λ / Γ (which is the fiber over $e\Lambda \in \Lambda_\delta / \Lambda$), and since this set of fibers is Λ -invariant, Λ normalizes Γ' .

Since Λ normalizes Γ and Γ' , we can consider the conjugation action $\Lambda \rightarrow \text{Aut}(\Gamma / \Gamma')$. Since $\text{Aut}(\Gamma / \Gamma')$ is finite, a finite index subgroup $\Lambda' \subseteq \Lambda$ leaves $(\Gamma \cap \Gamma^\delta) / \Gamma'$ invariant, so that Λ' normalizes $\Gamma \cap \Gamma^\delta$. Likewise, there is a finite index subgroup $\Lambda'' \subseteq \Lambda^\delta$ that normalizes $\Gamma \cap \Gamma^\delta$. However, since $\Gamma \cap \Gamma^\delta$ is a discrete Zariski-dense subgroup of G , its normalizer $N_G(\Gamma \cap \Gamma^\delta)$ is also discrete, and hence so is the subgroup $\langle \Lambda', \Lambda'' \rangle$ of $N_G \Gamma'$. Since $\langle \Lambda', \Lambda'' \rangle$ contains a lattice and is discrete, it is itself a lattice. Therefore Λ' and Λ'' are both finite index in $\langle \Lambda', \Lambda^\delta \rangle$ and hence are commensurable. \square

Arithmeticity of Λ lets us reduce to the case where Δ is an S -arithmetic lattice by way of the following generalization of a result of Venkataramana (see [LZ01, Proposition 2.3]):

Lemma 2.4. *Let $\Theta \subseteq \mathbb{G}(k)$ be a subgroup containing Λ whose projection to $\mathbb{G}(k_s)$ is bounded for almost all places s , and let S be the (finite) set of places where the projection of Θ is unbounded. Then Θ is S -arithmetic.*

Venkataramana's result is stated in [LZ01] in characteristic zero and under the assumption that S contains an archimedean place. The proof of the above lemma is rather technical and we postpone it until Section 4.

Remark 2.5. By the above lemma, for any $\delta \in \Delta \setminus \Lambda$, the group $\langle \Lambda, \delta \rangle$ is S -arithmetic for some finite set of places S . We will now replace Δ by this subgroup so that henceforth Δ is S -arithmetic. This reduction will be used in Section 2.5.

2.3. Maps of descent and restriction. Let $L := \Delta // \Gamma$ be the completion of Δ with respect to Γ (i.e. the closure in the topology of pointwise convergence of Δ acting on Δ/Γ). Likewise let $H := \Delta // \Lambda$ be the completion of Δ with respect to Λ . Note that L and H are locally compact second countable groups. Further $\bar{\Gamma}^L$ (the closure of Γ in L) and $\bar{\Lambda}^H$ are compact open subgroups (since they are isotropy groups of points in the coset spaces).

The construction of Schlichting completions is functorial with respect to the inclusion $\Gamma \hookrightarrow \Lambda$:

Proposition 2.6. *There is a continuous surjection $q : L \rightarrow H$ such that the canonical projection $\pi : \Delta/\Gamma \rightarrow \Delta/\Lambda$ is q -equivariant.*

Proof. It is clear that π is Δ -equivariant, so Δ permutes the fibers of π . We claim that the same is true for L . Indeed, suppose $g \in L$ and choose $g_n \in \Delta$ such that $g_n \rightarrow g$ pointwise on Δ/Γ . Let $x, y \in \Delta/\Gamma$ be points that lie in the same fiber of π . Then for any $n \geq 1$, their translates $g_n x$ and $g_n y$ also lie in the same fiber of π . On the other hand, for $n \gg 1$, we have that $g x = g_n x$ and $g y = g_n y$. Hence $g x$ and $g y$ belong to the same fiber of π .

Therefore every $g \in L$ descends to a map $q(g) \in \text{Sym}(\Delta/\Lambda)$. It is obvious that if $g_n \rightarrow g$ pointwise on Δ/Γ , then $q(g_n) \rightarrow q(g)$ on Δ/Λ , and hence q is continuous. Since $\Delta \subseteq L$ is dense and $q(\Delta) \subseteq H$, we conclude that $q : L \rightarrow H$ is a continuous surjection. \square

Proposition 2.7. *Set $N := \ker(q)$. Then $N \subseteq \bar{\Lambda}^L$.*

Proof. Consider the N -orbit of the basepoint $e\Gamma \in \Delta/\Gamma$. Since N is the kernel of q and $\pi : \Delta/\Gamma \rightarrow \Delta/\Lambda$ is q -equivariant, we see that $N \cdot e\Gamma \subseteq q^{-1}(e\Lambda) = \Lambda/\Gamma$. Since $\bar{\Gamma}^L$ is the isotropy group of $e\Gamma \in \Delta/\Gamma$, we find that $N \subseteq \Lambda \bar{\Gamma}^L$ (the product on the right-hand side is taken as sets). The right-hand side is clearly inside $\bar{\Lambda}^L$. \square

Since N is normal in $\bar{\Lambda}^L$, the product $N \bar{\Gamma}^L$ (as sets) is a subgroup of $\bar{\Lambda}^L$. We have:

Proposition 2.8. *$N \bar{\Gamma}^L$ has finite index in $\bar{\Lambda}^L$.*

Proof. We start by observing the map $q : L \rightarrow H$ is open: Indeed, since Δ is countable and L is locally compact, L is also σ -compact, and a classical result of Freudenthal is that any continuous surjective homomorphism between Hausdorff locally compact groups with σ -compact domain, is open [CdIH16,

Cor. 2.D.6]. Therefore $\bar{\Gamma}^H = q(\bar{\Gamma}^L)$ is an open subgroup of the compact group $\bar{\Lambda}^H$, and hence has finite index. Therefore $N\bar{\Gamma}^L = q^{-1}(\bar{\Gamma}^H)$ has finite index in $q^{-1}(\bar{\Lambda}^H) = \bar{\Lambda}^L$. \square

By replacing Λ with its finite index subgroup $\Lambda \cap \bar{\Gamma}^H$ and Δ by the normal closure of Λ , we can arrange that:

Standing assumptions 2.9. $\bar{\Lambda}^H = \bar{\Gamma}^H$ and hence we have equality in the above proposition, i.e. $\bar{\Lambda}^L = N\bar{\Gamma}^L$.

Note that $\Lambda/\Gamma = q^{-1}(e\Lambda)$ is $\bar{\Lambda}^L$ -invariant (inside Δ/Γ). Let $\text{res} : \bar{\Lambda}^L \rightarrow \text{Sym}(\Lambda/\Gamma)$ denote the restriction map.

Proposition 2.10. *res is continuous (with respect to the topology of pointwise convergence), has image Λ/Γ (acting by translations) and kernel $\bar{\Gamma}^L$.*

Proof. Clearly if a sequence of maps converge pointwise on Δ/Γ , they do so on the subset Λ/Γ . Therefore res is continuous. Since Λ is dense in $\bar{\Lambda}^L$ and $\text{res}(\Lambda) = \Lambda/\Gamma$ is closed in $\text{Sym}(\Lambda/\Gamma)$, the image of res is exactly Λ/Γ . It remains to determine $\ker(\text{res})$: Clearly $\bar{\Gamma}^L \subseteq \ker(\text{res})$. Conversely, suppose $g \in \bar{\Lambda}^L$ satisfies $\text{res}(g) = e$. Write $g = \lim_n \lambda_n$ for some $\lambda_n \in \Lambda$. Then λ_n fixes the basepoint $e\Gamma \in \Delta/\Gamma$ for $n \gg 1$, and hence $\lambda_n \in \Gamma$ for $n \gg 1$. \square

Since Γ is normal in Λ , we also have that $\bar{\Gamma}^L$ is normal in $\bar{\Lambda}^L$. In particular, $\bar{\Gamma}^L \cap N$ is a compact open normal subgroup of N .

Claim 2.11. res restricts to an isomorphism $N/(\bar{\Gamma}^L \cap N) \xrightarrow{\cong} \Lambda/\Gamma$.

Proof. Recall from Proposition 2.10 that

$$\text{res} : \bar{\Lambda}^L \rightarrow \Lambda/\Gamma$$

is surjective and $\ker(\text{res}) = \bar{\Gamma}^L$. So res descends to an isomorphism

$$\bar{\Lambda}^L/\bar{\Gamma}^L \xrightarrow{\cong} \Lambda/\Gamma.$$

On the other hand, by the modification of Λ in Standing assumption 2.9, we have $\bar{\Lambda}^L = N\bar{\Gamma}^L$, so that

$$\bar{\Lambda}^L/\bar{\Gamma}^L = (N\bar{\Gamma}^L)/\bar{\Gamma}^L \cong N/(\bar{\Gamma}^L \cap N).$$

\square

2.4. Down towards a central extension. Write $C := \ll \bar{\Gamma}^L \cap N \gg_L$ for the normal closure in L of $\bar{\Gamma}^L \cap N$. Note that C is a normal subgroup of L contained in N , and is open in N (because $\bar{\Gamma}^L \cap N$ is open in N), hence also closed.

Now we divide by C and study the extension

$$1 \rightarrow N/C \rightarrow L/C \rightarrow H \rightarrow 1.$$

Restriction of this short exact sequence to $\overline{\Lambda}^L/C$ (as middle term) gives

$$(2.1) \quad 1 \rightarrow N/C \rightarrow \overline{\Lambda}^L/C \rightarrow \overline{\Lambda}^H \rightarrow 1.$$

Claim 2.12. The short exact sequence (2.1) splits trivially as an extension of topological groups.

Proof. We provide a left splitting. First note that by the above claim, res/C descends to a map

$$\text{res}/C : \overline{\Lambda}^L/C \rightarrow (\Lambda/\Gamma)/\text{res}(C)$$

that restricts to an isomorphism $N/C \rightarrow (\Lambda/\Gamma)/\text{res}(C)$. Then the desired left splitting is simply the composition of res/C with the inverse of its restriction to N/C :

$$(\text{res}/C)|_{N/C}^{-1} \circ (\text{res}/C) : \overline{\Lambda}^L/C \longrightarrow N/C.$$

It is clear this map restricts to identity on N/C . This is a splitting of topological groups because both res/C and its inverse are continuous. \square

Hence we have a closed subgroup $K \subseteq \overline{\Lambda}^L/C$ (namely the kernel of the above left splitting) such that

$$\overline{\Lambda}^L/C \cong (N/C) \times K$$

(and this is compatible with the sequence, i.e. the copy of N/C is the one given by the inclusion in the sequence, and K projects isomorphically to $\overline{\Lambda}^H$.)

We note for use below:

Claim 2.13. $\Gamma \subseteq K$.

Proof. K is precisely the kernel of the left-splitting produced in the proof of Claim 2.12, i.e. the kernel of

$$(\text{res}/C)|_{N/C}^{-1} \circ (\text{res}/C) : \overline{\Lambda}^L/C \longrightarrow N/C.$$

The kernel of this map coincides with $\ker(\text{res}/C) = (C\overline{\Gamma}^L)/C$ and therefore contains (the image of) Γ . \square

Now consider the normal closure $\ll K \gg_{L/C}$.

Claim 2.14. Under the map $L/C \rightarrow H$, the image of $\ll K \gg_{L/C}$ has finite index.

Proof. Under $q : L \rightarrow H$, the image of K is $\overline{\Lambda}^H$ and therefore the image of $\ll K \gg_{L/C}$ is $\ll \overline{\Lambda}^H \gg_H$. Taking instead the normal closure $\ll \Lambda \gg_\Delta$ in Δ and noting that by Lemma 2.4, Δ is S -arithmetic and in particular satisfies the Margulis normal subgroups theorem, we see that $\ll \Lambda \gg_\Delta$ has

finite index in Δ . Since Δ is dense in H and $\ll \bar{\Lambda}^H \gg_{H \subseteq H}$ is open (and hence closed), it follows that $\ll \bar{\Lambda}^H \gg_H$ has finite index in H . \square

Write H' for the image of $\ll K \gg_{L/C}$ in H , and let L' be the preimage of H' in L/C , so that we can write

$$(2.2) \quad 1 \rightarrow N/C \rightarrow L' \rightarrow H' \rightarrow 1.$$

Since K commutes with N/C , so does its normal closure. In particular $\ll K \gg_{L/C}$ intersects N/C in a central subgroup Z that is normal in L/C . Now we quotient by Z and find the sequence

$$(2.3) \quad 1 \rightarrow (N/C)/Z \rightarrow L'/Z \rightarrow H' \rightarrow 1$$

which splits trivially as

$$L'/Z \cong (N/C)/Z \times (\ll K \gg_{L/C} / Z).$$

Claim 2.15. Z has finite index in N/C .

Proof. We have a natural map $\Delta \rightarrow (L/C)/Z$. Let $\Delta' \subseteq \Delta$ consist of those elements whose projection to L/C lies in L' . So Δ' is a finite index subgroup of Δ . Composition of $\Delta' \rightarrow L' \rightarrow L'/Z$ with the projection from

$$L'/Z \cong (N/C)/Z \times (\ll K \gg_L / Z)$$

to its first factor restricts to a map $\rho : \Delta' \rightarrow (N/C)/Z$. Then ρ is surjective: Indeed, the image of Δ in L' is dense and N/C is discrete. Furthermore, ρ has infinite kernel: Indeed $\ker(\rho) = \Delta \cap (\ll K \gg_L / Z)$. Since $\Gamma \subseteq K$ by Claim 2.13, ρ is trivial on Γ .

Since Δ' is S -arithmetic and ρ has infinite kernel, by Margulis' normal subgroups theorem, its image is finite. Since the image of ρ is isomorphic to $(N/C)/Z$, we conclude that Z has finite index in N/C . \square

2.5. Splitting of the central extension. The goal of this section is to prove:

Lemma 2.16. *Assume the notation of the Main Theorem 1.3, and define C as above. Then $C \subseteq N$ has finite index.*

Proof. Since $Z \subseteq N/C$ is finite index, it suffices to prove that Z is finite. Recall that L' is a suitable finite index subgroup of L . Consider the short exact sequence

$$(2.4) \quad 1 \rightarrow Z \rightarrow L' \rightarrow L'/Z \rightarrow 1.$$

Write $H'' := L'/Z$ and recall that $H'' \cong (N/C)/Z \times H'$. The above short exact sequence is a central extension, and hence is determined by a cohomology class in $H^2(H'', Z)$. We claim this cohomology class is actually contained in the continuous cohomology $H_c^2(H'', Z)$. Here L' is equipped with the topology as a subgroup of the Schlichting completion $L = \Delta // \Gamma$, and Z is equipped with the subspace topology (and hence is discrete) of L' and H'' is equipped with the quotient topology. To establish continuity,

recall that a central extension corresponds to a continuous cohomology class if the extension is a product as topological spaces (Hu [Hu52, 5.3]), i.e. if L' is homeomorphic to $Z \times H''$.

Claim 2.17. L' is homeomorphic to $Z \times H''$.

Proof. Since $H'' \cong (N/C)/Z \times H'$, and N/C is a discrete group and hence homeomorphic to $Z \times (N/C)/Z$, it suffices to show that L' is homeomorphic to $(N/C) \times H'$.

By Claim 2.12 the short exact sequence

$$1 \rightarrow N/C \rightarrow L' \rightarrow H' \rightarrow 1$$

splits as an extension of topological groups when restricted to the compact open subgroup $\bar{\Lambda}^L/C$ of L' , i.e.

$$1 \rightarrow N/C \rightarrow \bar{\Lambda}^L/C \rightarrow \bar{\Lambda}^H \rightarrow 1$$

is split and this splitting is topological. Since $\bar{\Lambda}^L/C$ is open in L' , we can choose a (countable, discrete) set $\{l_i\}_{i \in I}$ of its coset representatives, and write

$$L' = \bigsqcup_{i \in I} l_i \bar{\Lambda}^L/C.$$

Using $\bar{\Lambda}^L/C \cong (N/C) \times \bar{\Lambda}^H$ and that N/C is normal in L' , we have a homeomorphism

$$\begin{aligned} L' &\cong \bigsqcup_{i \in I} l_i((N/C) \times \bar{\Lambda}^H) \\ &= N/C \times \bigsqcup_i l_i \bar{\Lambda}^H \\ &= N/C \times H', \end{aligned}$$

as desired. \square

This cohomological rephrasing is only useful if we can compute the relevant cohomology group $H_c^2(H'', Z)$:

Claim 2.18. $H_c^2(H'', Z)$ is torsion.

Proof. First note that Z is finitely generated: Indeed, Z has finite index in N/C (Claim 2.15) so it suffices to show N/C is finitely generated. But N/C is a quotient of Λ/Γ (Claim 2.11) and hence is finitely generated because Λ is. Therefore Z is finitely generated and abelian, so that we can write $Z \cong \mathbb{Z}^r \oplus T$ for some finite abelian group T , and hence

$$H_c^2(H'', Z) \cong H_c^2(H'', \mathbb{Z})^r \oplus H_c^2(H'', T).$$

Since T is finite, the second term is clearly torsion.

Remark 2.19. For later use in the positive characteristic case (see Section 3), we note that we will not use that T is finite, but merely that it has bounded exponent.

Hence it remains to show that $H_c^2(H'', \mathbb{Z})$ is torsion. To do so, consider the short exact sequence of coefficients

$$(2.5) \quad 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z} \rightarrow 0.$$

Since this sequence is not split (as topological groups), this does not automatically yield a long exact sequence on continuous cohomology of (locally compact, second countable) topological groups. However, for totally disconnected groups, it is a result of Michael that there is a long exact sequence (see [Wig73, Thm M]). The part that is relevant for us is

$$(2.6) \quad \dots \rightarrow H_c^1(H'', \mathbb{C}/\mathbb{Z}) \rightarrow H_c^2(H'', \mathbb{Z}) \rightarrow H_c^2(H'', \mathbb{C}) \dots$$

To describe $H_c^2(H'', \mathbb{C})$, recall the following vanishing result for the continuous cohomology of semisimple groups:

Theorem 2.20 (Casselman-Wigner [CW74, Cor. 2]casselman-wigner). *Let F be a non-archimedean local field. The F -rational points of a connected, semisimple, algebraic group over F have vanishing continuous cohomology (with \mathbb{C} -coefficients) in positive degrees.*

Remark 2.21. The above citation is for the result with $F = \mathbb{Q}_p$. For the general case, see [CW74, Rem. (2), p. 210].

We will use this to show that $H_c^2(H'', \mathbb{C}) = 0$: Indeed, we have $H'' \cong (N/C)/Z \times H'$ and H' is a finite index subgroup of $\mathbb{G}(k_S)$ (see Remark 2.5). The cohomology of H'' is then computed using the Hochschild-Serre spectral sequence (see [CW74, Prop. 5]), whose second page is $E_2^{pq} = H_c^p(H''/H', H_c^q(H', \mathbb{C}))$. Since $H''/H' \cong (N/C)/Z$ is finite, its cohomology vanishes in positive degrees so that the second page is concentrated on $p = 0$, where its values are $H_c^\bullet(H', \mathbb{C})$. Further, H' is a finite index, open, normal subgroup of a finite product of groups to which Casselman-Wigner's result applies, and the same spectral sequence argument shows that $H_c^2(H', \mathbb{C}) = 0$, as desired.

Finally, to be able to use the long exact sequence (2.6), we consider $H_c^1(H'', \mathbb{C}/\mathbb{Z})$: From the splitting $H'' \cong (N/C)/Z \times H'$, it is clear that H'' has compact totally disconnected topological abelianization (i.e. the quotient by $\overline{[H'', H'']}$ is finite), and hence $H_c^1(H'', \mathbb{C}/\mathbb{Z})$ is torsion.

From the long exact sequence (2.6), it follows that $H_c^2(H'', \mathbb{Z})$ is also torsion, as desired. \square

Remark 2.22. Again for use in the positive characteristic, we note we only used that the topological abelianization of H'' is compact and totally disconnected. In characteristic zero, the proof in fact shows that $H_c^2(H'', \mathbb{Z})$ is finite.

So far we have found that $H_c^2(H'', \mathbb{Z})$ is torsion. We will now use this to show that Z is finite. Let $\omega \in H_c^2(H'', \mathbb{Z})$ be the class corresponding to the central extension

$$1 \rightarrow Z \rightarrow L' \rightarrow H'' \rightarrow 1,$$

and choose $m \geq 1$ such that $m\omega = 0$. Then the extension mL' corresponding to $m\omega$ is split. The natural map $L' \rightarrow mL'$ composed with the projection $mL' \rightarrow Z$ gives a continuous map $L' \rightarrow Z$. Further it restricts to multiplication by m on Z . Hence if Z is infinite, then the image of $L' \rightarrow Z$ is infinite. Since Δ' is dense in L' , the image of Δ in Z is also infinite. This contradicts the normal subgroups theorem. We conclude that Z is finite. \square

2.6. End of the proof of Theorem 1.3. In this section we prove:

Theorem 2.23. *Let Δ be a finitely generated group with commensurated, finitely generated subgroup $\Lambda \subseteq \Delta$. Suppose Γ is another commensurated subgroup of Δ that is normal in Λ . Further assume:*

- (1) $H_c^2(\Delta//\Lambda, \mathbb{C}) = 0$,
- (2) Any normal subgroup of Δ is finite or finite index.

Then $\Gamma \subseteq \Lambda$ has finite index.

Remarks 2.24.

- (1) Clearly, this implies Main Theorem 1.3.
- (2) Creutz-Shalom [CS14] have given conditions on $\Delta//\Lambda$ and $\Lambda//\Gamma$ that guarantee Assumption (2).

By Lemma 2.16, C has finite index in N and hence is cocompact in $\overline{\Lambda}^L = N\overline{\Gamma}^L$. In the remainder of this section we will prove that C is also compact, so that $\overline{\Lambda}^L$ is compact, and hence Γ has finite index in Λ .

To prove compactness of $C := \ll \overline{\Gamma}^L \cap N \gg$ (where the normal closure is taken in L), we start by investigating the conjugation action of L on N . The key property is that these automorphisms are inner modulo a compact subgroup:

Lemma 2.25. *For every $\ell \in L$, let $c_\ell : N \rightarrow N$ denote conjugation by ℓ . Then for every $\ell \in L$, there exists a compact open normal subgroup $C_\ell \subseteq N$ such that ℓ restricts to an automorphism of C_ℓ and descends to an inner automorphism of N/C_ℓ .*

Proof. For $\ell \in \overline{\Lambda}^L$, set $C_\ell := \overline{\Gamma}^L \cap N$. Then it is clear that $c_\ell(C_\ell) = C_\ell$. We have $N/C_\ell \cong \Lambda/\Gamma \cong \overline{\Lambda}^L/\overline{\Gamma}^L$, and c_ℓ descends to conjugation of the image of ℓ in $\overline{\Lambda}^L/\overline{\Gamma}^L$. This proves the claim for $\ell \in \overline{\Lambda}^L$.

Hence the claim is also true for ℓ conjugate into $\overline{\Lambda}^L$. Finally, we note that $\overline{\Lambda}^L$ normally generates L (since it contains N and $\overline{\Lambda}^H$ normally generates $H = L/N$). So it remains to prove that the composition of automorphisms that are inner modulo compact is inner modulo compact. Let φ, ψ be two automorphisms that are inner modulo C_φ and C_ψ . Consider $C_{\varphi\psi} := C_\varphi C_\psi = C_\psi C_\varphi$. Note that $C_{\varphi\psi}$ is a compact normal subgroup that is invariant under both φ and ψ : For example, to see invariance under φ , note that $C_{\varphi\psi}/C_\varphi$ is a compact normal subgroup of N/C_φ and hence invariant under any inner automorphism.

Finally, note that $\varphi \circ \psi$ descends to an automorphism of $N/(C_{\varphi\psi})$ that is the composition of inner automorphisms, and hence is also inner. \square

Next, we show that in the above claim, there is a single compact subgroup C_L of N that works for all $\ell \in L$ simultaneously. This will immediately imply compactness of C :

Lemma 2.26. *There exists a compact subgroup $C_L \subseteq N$ containing $\bar{\Gamma}^L \cap N$ such that C_L is normal in L . In particular, $C = \ll \bar{\Gamma}^L \cap N \gg$ is compact.*

Proof. Let S_Δ be a finite generating set of Δ . Since $\Delta \subseteq L$ is dense, it suffices to construct a compact subgroup $C_L \subseteq N$ containing $\bar{\Gamma}^L \cap N$ that is normalized by S_Δ . For $s \in S_\Delta$, choose a compact normal subgroup $C_s \subseteq N$ such that s is inner modulo C_s . Set

$$C_L := (\bar{\Gamma}^L \cap N) \prod_{s \in S_\Delta} C_s.$$

Then C_L is compact and just as in the proof of the previous lemma, it is normalized by any $s \in S_\Delta$ because C_L/C_s is invariant under any inner automorphism of N/C_s . \square

3. POSITIVE CHARACTERISTIC

In this section, we will prove the analogous theorem in positive characteristic. The proof is largely identical, except where finite generation of lattices is used in Section 2.5. The only issue that we cannot resolve in this generality is that if Λ is non-uniform (and hence infinitely generated), we were not able to prove its arithmeticity. Therefore we have the following modified statement in positive characteristic:

Theorem 3.1. *Let k be a local field of positive characteristic $p > 0$ and \mathbb{G} a connected semisimple algebraic group defined over k . Let Λ be a lattice in $G := \mathbb{G}(k)$ and suppose Γ is an infinite normal subgroup of Λ with commensurator Δ that is dense in G . Suppose that either Λ is uniform, or that Λ is arithmetic. Then Γ is an arithmetic lattice and hence has finite index in Λ .*

Proof. The proof is verbatim identical to the above proof in characteristic zero, except in the following places:

3.1. Schlichting completions. This parallels Section 2.1. No changes are necessary.

3.2. Arithmeticity of Λ . This section used finite generation of Λ . However, in positive characteristic, nonuniform lattices in rank 1 groups are infinitely generated (see Lubotzky's Theorem 3.2 below for a more precise statement), and therefore this part of the proof does not go through if Λ is nonuniform. However, if Λ is nonuniform, we assumed it is arithmetic so that we can simply skip this part of the proof.

As in Remark 2.5, we henceforth replace Δ by $\langle \Lambda, g \rangle$, where $g \in \Delta \setminus \Lambda$ is arbitrary. By Lemma 2.4, Δ is an S -arithmetic lattice for some finite set of places S and in particular satisfies the Normal Subgroups Theorem.

3.3. Maps of descent and restriction. No changes are necessary in 2.3.

3.4. Down to the central extension. No changes are necessary in 2.4.

3.5. Splitting of the central extension. Some modifications are required to establish that $H_c^2(H'', Z)$ is torsion (Claim 2.18). Namely, finite generation of Λ is used to obtain finite generation of Z . However, as we mentioned above, in positive characteristic, nonuniform lattices in rank one groups are not finitely generated. We note that by using Margulis Normal Subgroups Theorem, we can reduce to the case where G is rank one, non-compact and simple. In that case, we have the following result of Lubotzky that gives a ‘thick-thin decomposition’ of lattices in such G :

Theorem 3.2 (Lubotzky [Lub89, Thm. 2]). *Assume the notation of Theorem 3.1. Then Λ contains a finite index subgroup Λ' that can be written*

$$\Lambda' \cong F * \underset{1 \leq j \leq d}{*} U^j,$$

where F is a finitely generated free group and U^j are lattices in unipotent radicals of minimal parabolic subgroups of G .

Henceforth we fix Λ', F, U^j as in the above theorem. Note that the groups U^j are not finitely generated, and hence, neither is Λ . However, if $\mathbb{G}(k) \subseteq \mathrm{SL}(n, k)$, then any unipotent element of $\mathbb{G}(k)$ has order dividing $q := p^n$: Indeed, $A^{p^n} - \mathrm{Id} = (A - \mathrm{Id})^{p^n} = 0$. Then we argue as follows to prove that $H_c^2(H'', Z)$ is torsion: Write $N/C = \Lambda/\Theta$ and let S be the image in N/C of finitely many coset representatives of Λ/Λ' and the image of the generators of the free group F . Then N/C is virtually abelian and generated by the finite set S and the set $\sqrt[q]{\{e\}}$ of elements of exponent p^n .

Our first goal is to show that Z is also generated by a set of this form. To see this, note that $Z \cap \sqrt[q]{\{e\}}$ is a (characteristic) subgroup of Z (because Z is abelian), and $\sqrt[q]{\{e\}}/(Z \cap \sqrt[q]{\{e\}})$ is finite (because Z is central). Therefore $\Theta/(Z \cap \sqrt[q]{\{e\}})$ is finitely generated, and hence so is its finite index subgroup $Z/(Z \cap \sqrt[q]{\{e\}})$. Let \bar{S}_Z be a finite set of generators of $Z/(Z \cap \sqrt[q]{\{e\}})$, and let S_Z be a chosen set of pre-images of the elements of \bar{S}_Z . Then $S_Z \cup (Z \cap \sqrt[q]{\{e\}})$ generates Z , as desired.

Next, we will show that $Z \cong \mathbb{Z}^r \oplus T$ for some $r < \infty$ and bounded exponent group T . Indeed, since $Z/(\sqrt[q]{\{e\}} \cap Z)$ is finitely generated abelian, we see that $\sqrt[q]{\{e\}} \cap Z$ has finite index in the torsion subgroup $T := \mathrm{Tor}(Z)$, and hence T has bounded exponent. Further Z/T is finitely generated and torsion-free, so it is isomorphic to \mathbb{Z}^r for some $0 \leq r < \infty$, as desired.

The above argument replaces the use of finite generation of Λ in the proof of Claim 2.18. The rest of the proof goes through: The vanishing

of $H_c^2(H', \mathbb{C}) = 0$ is established as before. To show that $H_c^1(H'', \mathbb{C}/\mathbb{Z}) \cong \text{Hom}_c(H'', \mathbb{C}/\mathbb{Z})$ is torsion, it suffices to show that the topological abelianization of H'' is compact (because it is also totally disconnected). By a result of Tits (see [Tit64] or [Mar91, Thm I.1.5.6(ii)]), the commutator subgroup of H is given by H^+ , and compactness of H/H^+ is due to Borel-Tits ([BT73, 6.14] or [Mar91, I.2.3.1(b)]).

3.6. End of the proof of Theorem 3.1. No changes are necessary because in this part of the proof, we do not use finite generation of Λ , only that of Δ . And while in positive characteristic, lattices need not be finitely generated, higher rank lattices (in particular S -arithmetic ones) always are (due to Venkataramana when one of the simple factors is higher rank [Ven88] and Raghunathan if all simple factors have rank 1 [Rag89]). \square

4. SUPERGROUPS OF ARITHMETIC LATTICES

In this section we prove the technical Lemma 2.4 on subgroups of S -arithmetic lattices containing an arithmetic lattice. With minor additional assumptions, this is due to Venkataramana (see [LZ01, Prop. 2.3]) with essentially the same proof. This fact is probably well-known to experts but we have not been able to locate a version in full generality in the literature. For convenience we restate the result:

Lemma 4.1. *Let $\Theta \subseteq \mathbb{G}(k)$ be a subgroup containing Λ whose projection to $\mathbb{G}(k_s)$ is bounded for almost all places s , and let S be the (finite) set of places where the projection of Θ is unbounded. Then Θ is S -arithmetic.*

Proof. Set $\mathbb{G}(k_S) := \prod_{s \in S} \mathbb{G}(k_s)$ and let $K_S := \overline{\Lambda}^{\mathbb{G}(k_S)}$ be the closure of Λ . Then K_S is compact and open in a cocompact normal subgroup G_S of $\mathbb{G}(k_S)$. Here, openness follows by applying weak approximation (due to Kneser for the classical groups and Platonov in general (see e.g. [PR94])) to the simply-connected covering $\tilde{\mathbb{G}} \rightarrow \mathbb{G}$ and using that its image is a cocompact normal subgroup of $\mathbb{G}(k_S)$.

We claim that $\Theta \cap G_S$ also has unbounded projection at every $s \in S$, so that we can replace Θ by $\Theta \cap G_S$. We argue by contradiction so suppose that $\Theta \cap G_S$ has bounded projection at some $s \in S$. Then its fixed set in the Bruhat-Tits building is nonempty and, since $G_S \subseteq \mathbb{G}(k_S)$ is normal, also Θ -invariant. On the other hand the $(\Theta \cap G_S)$ -fixed set is finite because $\Theta \cap G_S$ contains $\Lambda \cap G_S$. It follows that a finite index subgroup of Θ fixes a point in the Bruhat-Tits building and hence Θ itself is bounded, but this is a contradiction.

As mentioned above, we will now replace Θ by $\Theta \cap G_S$ so that we have $\Theta \subseteq G_S$. We claim that the image of Θ under the diagonal embedding $\text{diag} : \Theta \rightarrow G \times G_S$ is a lattice in $G \times G_S$. To see this, let $F \subseteq G$ be a Λ -fundamental domain of finite measure, and we will show that

$$F \times K_S \subseteq G \times G_S$$

is a $\text{diag}(\Theta)$ -fundamental domain for $\text{diag}(\Theta) \subseteq G \times G_S$. First, we argue the $\text{diag}(\Theta)$ -translates of $F \times K_S$ are disjoint. Suppose that $\theta \in \text{diag}(\Theta)$ is such that $\theta(F \cap K_S) \cap (F \times K_S) \neq \emptyset$. By inspecting the second factor and using K_S is a subgroup, we obtain that the projection of θ to the second factor lies in K_S . Since $\Theta \cap K_S = \Lambda$, we have $\theta \in \text{diag}(\Lambda)$. Then since F is a fundamental domain for $\Lambda \subseteq G$ and $\theta F \cap F \neq \emptyset$, we find that θ is trivial.

Now we show $\text{diag}(\Theta)(F \times K_S) = G \times G_S$. Since K_S is Λ -invariant and F is a fundamental domain for $\Lambda \subseteq G$, it suffices to show that $\Theta K_S = G_S$. Since in addition $K_S \subseteq G_S$ is open, it suffices to show Θ is dense in G_S .

But for every $s \in S$, the closure $H_s := \overline{\Theta}$ in G_s is unbounded and open. Therefore the pre-image \tilde{H}_s of H_s in the universal cover $\tilde{\mathbb{G}}(k_s)$ is also open and unbounded. However, any open, noncompact subgroup of $\tilde{\mathbb{G}}(k_s)$ contains the subgroup $\tilde{\mathbb{G}}(k_s)^+$ generated by all unipotents (this is an unpublished result of Tits with a published proof by Prasad [Pra82]) and $\tilde{\mathbb{G}}(k_s)^+ = \tilde{\mathbb{G}}(k_s)$ by Platonov's proof of the Kneser-Tits conjecture (for locally compact non-archimedean fields) [Pla69]. Finally, the image of $\tilde{\mathbb{G}}(k_s)$ in $\mathbb{G}(k_s)$ is cocompact (Borel-Tits [BT73, 3.19 and 20] or see [Mar91, I.2.3.4(i)]). So we conclude that $\overline{\Theta} \subseteq G_S$ is a closed subgroup whose projection to each factor $\mathbb{G}(k_s)$ is open and cocompact.

We now argue by induction on $|S|$ that $\overline{\Theta}$ is open and cocompact in G_S : Indeed, for $|S| = 1$ the above argument proves exactly that. Now suppose that $|S| > 1$ and fix some element $s \in S$. Let K_s be a maximal compact subgroup of $\mathbb{G}(k_s)$ containing Λ , and set $\Theta_s := \Theta \cap K_s$. By the inductive hypothesis, $\overline{\Theta}_s$ is open and cocompact in $G_{S \setminus s}$. Finally, when we add the last place s , the argument from the base case shows that $\overline{\Theta}$ is open and cocompact in G_S . \square

5. THE REDUCIBLE CASE

In this section, we prove a version of the main theorem for reducible lattices. For simplicity, we state only the result in characteristic zero. The interested reader can combine the proof here with the partial result in Theorem 3.1 to obtain the most general result.

Theorem 5.1. *Let G_i , $i \in I$, be finitely many semisimple algebraic groups defined over local fields of characteristic zero. Set $G := \prod_i G_i$ and let $\Lambda \subseteq G$ be a lattice such that the projection of Λ to every factor G_i is an irreducible lattice Λ_i .*

Suppose $\Gamma \subseteq \Lambda$ is a normal subgroup with dense commensurator. Then there is a subset $J \subseteq I$ of factors such that Γ has finite index in $\prod_{j \in J} \Lambda_j$.

Proof. Note that $\Lambda \subseteq \prod_{i \in I} \Lambda_i$ has finite index (since Λ is a lattice and $\prod_i \Lambda_i$ is discrete). Then there is a finite index subgroup of Γ that is normalized by $\prod_{i \in I} \Lambda_i$, and hence we can replace Λ by $\prod_{i \in I} \Lambda_i$.

Let $p_i : G \rightarrow G_i$ denote the projection onto a factor. Then $p_i(\Gamma)$ is a normal subgroup of Λ_i that is commensurated by $p_i(\Delta)$, and hence by the Main Theorem 1.3, $p_i(\Gamma)$ is either finite or has finite index in Λ_i .

Let J denote the subset of factors such that $p_i(\Gamma) \subseteq \Lambda_i$ has finite index, and set $G_J = \prod_{j \in J} G_j$. Then $\Gamma \cap G_J$ has finite index in Γ , and hence a further finite index subgroup Γ' is contained in G_J and also normal in Λ . By replacing G by G_J and Γ by Γ' , we may assume that $p_i(\Gamma) \subseteq \Lambda_i$ has finite index for all i . By replacing Λ by its finite index subgroup $\prod_i p_i(\Gamma)$, we may assume $p_i(\Gamma) = \Lambda_i$.

We will now show that $\Gamma \subseteq \prod_i \Lambda_i$ has finite index. Indeed, for any $i \in I$, the intersection with a factor $N_i := \Gamma \cap G_i$ is a normal subgroup of Λ_i , and it is easy to see $p_i(\Delta)$ commensurates N_i : Indeed for $\delta \in \Delta$, the natural map

$$N_i / (N_i \cap N_i^\delta) \rightarrow \Gamma / (\Gamma \cap \Gamma^\delta)$$

is injective because

$$\begin{aligned} N_i \cap (\Gamma \cap \Gamma^\delta) &= G_i \cap \Gamma \cap \Gamma^\delta \\ &= G_i \cap \Gamma \cap (G_i \cap \Gamma)^\delta \\ &= N_i \cap N_i^\delta. \end{aligned}$$

Therefore by the Main Theorem 1.3 applied to N_i , we obtain that $N_i \subseteq \Lambda_i$ is either finite or finite index. We will argue by contradiction that N_i must be infinite for all i , which will complete the proof.

Suppose then that there is $i \in I$ such that N_i is finite. Since $p_i(\Gamma) = \Lambda_i$ is residually finite, there is a finite index subgroup $\Gamma' \subseteq \Gamma$ such that $\Gamma' \cap N_i = 1$. By passing to a further finite index subgroup, we can assume that $\Gamma' \subseteq \Lambda$ is normal, and henceforth we replace Γ by Γ' , so that $N_i = 1$. It follows that the projection $q_i : G \rightarrow \prod_{j \neq i} G_j$ away from G_i restricts to an isomorphism on Γ , and hence Γ is the graph of a surjective homomorphism $q_i(\Gamma) \rightarrow \Lambda_i$. But the graph of a homomorphism $\varphi : A \rightarrow B$ is not normal in $A \times B$ unless its image is contained in the center of B . Since Λ_i is not abelian, this is a contradiction. \square

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