

ONE-RELATOR SASAKIAN GROUPS

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ABSTRACT. We prove that any one-relator group G is the fundamental group of a compact Sasakian manifold if and only if G is either finite cyclic or isomorphic to the fundamental group of a compact Riemann surface of genus $g > 0$ with at most one orbifold point of order $n \geq 1$. We also classify all groups of deficiency at least two that are also the fundamental group of some compact Sasakian manifold.

1. INTRODUCTION

Fundamental groups of compact Sasakian manifolds are called *Sasakian groups*. While the Kähler groups have been studied for long (see [ABCKT] and references therein), the study of their odd-dimensional relatives, viz. Sasakian groups, has been started relatively recently [Ch, BM2]. In this paper, we extend the results of [BM1, Ko] to Sasakian groups by analyzing Sasakian one-relator groups and Sasakian groups of deficiency greater than one.

We prove the following generalizing the corresponding statement for Kähler groups [Ko] (see Theorem 3.3):

Theorem 1.1. *Let G be a Sasakian group with $\text{def}(G) > 1$. Then G must be an orbifold surface group with genus greater than 1.*

The following theorem classifies one-relator Sasakian groups (see Theorem 4.10):

Theorem 1.2. *Let G be an infinite one-relator group. Then G is Sasakian if and only if it is isomorphic to*

$$\langle a_1, b_1, \dots, a_i, b_i, \dots, a_g, b_g \mid \left(\prod_{i=1}^g [a_i, b_i]^n \right) \rangle,$$

where g and n are some positive integers.

We observe that each of the groups

$$\langle a_1, b_1, \dots, a_g, b_g \mid \left(\prod_{i=1}^g [a_i, b_i]^n \right) \rangle, \quad g, n > 0,$$

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can in fact be realized as the fundamental group of a closed Sasakian manifold as follows. It was shown in [BM1] that these groups are fundamental groups of smooth projective varieties. Since the fundamental group of a smooth projective variety is also a Sasakian group [Ch, Proposition 1.2], the above groups are also Sasakian groups.

It was shown in [Ch] that all finite groups are Sasakian. Since the only finite one-relator groups are finite cyclic groups, it follows that finite one-relator Sasakian groups are precisely the finite cyclic groups.

A useful tool we establish along the way is the following generalization of the corresponding fact for Kähler groups in [ABCKT, Ch. 4] (see Theorem 3.1):

Theorem 1.3. *If G is a Sasakian group that satisfies $\beta_1^2(G) > 0$, then G is virtually a surface group.*

In Theorem 1.3 $\beta_1^2(G)$ denotes the first l^2 betti number of G .

2. SASAKIAN GROUPS OF POSITIVE DEFICIENCY

We refer to [BoGa] for the definition and basic properties of Sasakian manifolds. The Sasakian structure on any Sasakian manifold M can be perturbed in order to make the Sasakian structure quasi-regular [Ru], [OV, p. 161, Theorem 1.2]. In particular, every Sasakian group is the fundamental group of some closed quasi-regular Sasakian manifold. In view of this, henceforth all Sasakian manifolds considered here will be assumed to be quasi-regular.

Let M be a quasi-regular closed Sasakian manifold. The group $U(1)$ acts on M ; let

$$\alpha : M \times U(1) \longrightarrow M \tag{2.1}$$

be the action map. The Kähler orbifold base $M/U(1)$ of M will be denoted by B [Ru], [BoGa, p. 208, Theorem 7.1.3]. Let

$$f : M \longrightarrow M/U(1) = B \tag{2.2}$$

be the quotient map. Let

$$G = \pi_1(M)$$

be the fundamental group. For notational convenience we will omit mentioning the base point for any fundamental group.

Take a regular point $x_0 \in B$. This means that the action of $U(1)$ on the fiber $f^{-1}(x_0)$ is free. Let $i : f^{-1}(x_0) \longrightarrow M$ be the inclusion map of the regular fiber. The map $U(1) \longrightarrow f^{-1}(x_0)$ defined by $\lambda \longrightarrow x_1\lambda$, where $x_1 \in f^{-1}(x_0)$ is any given point, is a diffeomorphism. The image $i_*\pi_1(f^{-1}(x_0)) = i_*\mathbb{Z}$ is a (finite or infinite) cyclic group which we shall denote by C ; to clarify, C may be the trivial group. Let

$$Q := \pi_1^{orb}(B)$$

be the orbifold fundamental group of B regarded as the orbifold quotient of M by the action of $U(1)$. Then there is a short exact sequence:

$$1 \longrightarrow C \longrightarrow G = \pi_1(M) \longrightarrow Q \longrightarrow 1. \tag{2.3}$$

Lemma 2.1 ([BlGo], [Fu], [Ta], [Ch]). *For the group G in (2.3), the first betti number $b_1(G)$ is even.*

Note that from Lemma 2.1 it follows that a nontrivial free group with finitely many generators can't be a Sasakian group.

The *deficiency* of a finite presented group Γ is the maximum of $n - r$ taken over all possible finite presentations of Γ , where n and r are the numbers of generators of relations respectively.

Corollary 2.2. *If G in (2.3) is of positive deficiency, then $b_1(G) \geq 2$. If G in (2.3) is an infinite one-relator group, then $b_1(G) \geq 2$.*

Proof. Since G has positive deficiency, we have $b_1(G) \geq 1$. From Lemma 2.1 it follows that $b_1(G) \geq 2$.

Since infinite one-relator groups have positive deficiency, the second statement follows from the first statement. \square

We shall denote the first l^2 betti number of a group H by $\beta_1^2(H)$. As Kotschick did in [Ko], we shall make essential use of the following theorem due to Gaboriau and Lück.

Theorem 2.3 ([Lu], [Ga]). *If $1 \rightarrow N \rightarrow H \rightarrow Q_1 \rightarrow 1$ is a short exact sequence of infinite finitely generated groups with H finitely presented, then $\beta_1^2(H) = 0$.*

Corollary 2.4. *Assume that G in (2.3) is of positive deficiency (for example, it is an infinite one-relator group). If C in (2.3) is infinite, then $\beta_1^2(G) = 0$.*

Proof. In view of Theorem 2.3 it suffices to show that Q in (2.3) is infinite. To prove by contradiction, suppose Q is finite. Then G has a subgroup $C = \mathbb{Z}$ of finite index in it. Let N be the finite covering of M such that $C = \pi_1(N) \subset \pi_1(M) = G$. So N is a quasiregular closed Sasakian manifold with fundamental group \mathbb{Z} . This contradicts Lemma 2.1. Hence we conclude that Q is infinite. \square

We shall also need the following fundamental inequality:

Lemma 2.5 ([Gr]). *Let $\text{def}(H)$ denote the deficiency of H . Then $\text{def}(H) - 1 \leq \beta_1^2(H)$.*

Combining Corollary 2.4 and Lemma 2.5 the following is obtained.

Lemma 2.6. *Assume that G in (2.3) is of deficiency at least two. Then the cyclic group C in (2.3) is finite.*

Proof. Since G is of deficiency at least two, Lemma 2.5 says that $\beta_1^2(G) \geq 1$. On the other hand, if C is infinite, then $\beta_1^2(G) = 0$ by Corollary 2.4. Hence the cyclic group C in (2.3) is finite. \square

The following lemma is proved in [Ko].

Lemma 2.7 ([Ko, Lemma 3]). *If C in (2.3) is finite, then Q in (2.3) satisfies the condition*

$$\beta_1^2(Q) = p \cdot \beta_1^2(G),$$

where p is the order of C .

Lemma 2.8. *Assume that G in (2.3) is of deficiency at least two. Then Q in (2.3) is the fundamental group of a smooth projective orbifold with $\beta_1^2(Q) > 0$.*

Proof. Since M in (2.2) is a closed quasi-regular Sasakian manifold, the quotient B in (2.2) is a projective orbifold.

Since G is of deficiency at least two, Lemma 2.6 implies that C in (2.3) is finite. In view of Lemma 2.5, the given condition, that the deficiency of G is at least two, also implies that $\beta_1^2(G) > 0$. Hence $\beta_1^2(Q) > 0$ by Lemma 2.7. \square

3. SASAKIAN GROUPS WITH DEFICIENCY GREATER THAN ONE

3.1. From Sasakian groups to virtual surface groups. A group H is said to be *virtually a surface group* if some finite index subgroup of H is the fundamental group of a closed surface of positive first Betti number. The following theorem was proved in [Gr], [ABR], [ABCKT, p. 47, Theorem 4.1] for Kähler groups.

Theorem 3.1. *If G in (2.3) satisfies the condition that $\beta_1^2(G) > 0$, then G is virtually a surface group.*

Proof. Let C, G, Q be as in (2.3). As we saw in the proof of Corollary 2.4, if Q is finite then C is finite. On the other hand, if Q is infinite, then Theorem 2.3 implies that C is finite. Therefore, we conclude that C is finite.

Hence, by Lemma 2.7 we have $\beta_1^2(Q) > 0$. Note that Q equals the orbifold fundamental group of the orbifold B in (2.2).

Consider the Sasakian metric g_M on the quasi-regular Sasakian manifold M in (2.2). In the proof of [ABCKT, p. 47, Theorem 4.1] substitute (M, g_M) in place of the Kähler metric. Then it is straight-forward to check that all results in Section 4.2 and Section 4.3 of [ABCKT] remain valid.

Let

$$\psi : \widetilde{M} \longrightarrow M$$

be the universal covering of the quasiregular Sasakian manifold M in (2.2). Consider the action of the fundamental group $\pi_1(M)$ on \widetilde{M} . Set

$$\widehat{M} := \widetilde{M}/C,$$

where $C \subset \pi_1(M) = G$ is the subgroup in (2.3). Let

$$p_0 : \widehat{M} \longrightarrow M \tag{3.1}$$

be the natural projection.

It can be shown that the action of $U(1)$ on M in (2.1) canonically lifts to an action of $U(1)$ on \widehat{M} . To prove this, let

$$\chi = \alpha \circ (p_0 \times \text{Id}_{U(1)}) : \widehat{M} \times U(1) \longrightarrow M$$

be the composition, where α is the map in (2.1). The image of the homomorphism

$$\chi_* : \pi_1(\widehat{M} \times U(1)) \longrightarrow \pi_1(M)$$

is clearly the subgroup $C \subset G = \pi_1(M)$. From this it follows that the action of $U(1)$ on M canonically lifts to an action of $U(1)$ on \widehat{M} . Let

$$\varphi : \widehat{M} \longrightarrow \widehat{B} := \widehat{M}/U(1) \tag{3.2}$$

be the orbifold quotient for this action of $U(1)$ on \widehat{M} .

So we have a commutative diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{p_0} & M \\ \downarrow \varphi & & \downarrow f \\ \widehat{B} & \xrightarrow{q} & B \end{array} \quad (3.3)$$

where p_0 , φ and f are the maps in (3.1), (3.2) and (2.2) respectively. The map q in (3.3) is an étale Galois covering, in the orbifold category, with Galois group $Q = G/C$.

Following the proof of Theorem 4.14 in [ABCKT, p. 53, Section 4.4] we see that

- there is a proper holomorphic map to the unit disk

$$h : \widehat{B} \longrightarrow \mathbb{D}^2 := \{z \in \mathbb{C} \mid |z| < 1\}, \quad (3.4)$$

- and a homomorphism $\rho : Q \longrightarrow \text{Aut}(\mathbb{D}^2) = \text{PSL}(2, \mathbb{R})$,

such that

- (1) the fibers of h are connected,
- (2) h is Q -equivariant for the action of Q on \mathbb{D}^2 given by ρ and the Galois action of Q on \widehat{B} , and
- (3) the homomorphism $h^* : \mathcal{H}_{(2)}^1(\mathbb{D}^2) \longrightarrow \mathcal{H}_{(2)}^1(\widehat{B})$ corresponding to h is an isomorphism.

Consider the complex one-dimensional orbifold $\mathcal{O} = \mathbb{D}^2/\rho(Q)$ with orbifold fundamental group $Q_1 = \rho(Q)$. Since h in (3.4) is Q -equivariant, we conclude that

- (1) h induces a holomorphic (orbifold) map $h_1 : B \longrightarrow \mathcal{O}$ inducing a surjective homomorphism $h_{1*} : Q \longrightarrow Q_1$ of orbifold fundamental groups, and
- (2) the fibers of h_1 are compact, connected, because h is proper with connected fibers.

Consequently, there exists a short exact sequence

$$1 \longrightarrow K \longrightarrow Q \longrightarrow Q_1 \longrightarrow 1 \quad (3.5)$$

given by $\rho = h_{1*}$. It was observed at the beginning of the proof that $\beta_1^2(Q) > 0$. Also, note that Q_1 is an infinite group, because B being compact does not have any nonconstant map to a finite quotient of \mathbb{D}^2 . In view of these, by Theorem 2.3, the group K in (3.5) must be finite.

Since \mathcal{O} is a compact complex one-dimensional orbifold (it is compact because B has a nonconstant map to it), we conclude that Q_1 is virtually a surface group. Passing to a further finite index subgroup Q_2 of Q_1 if necessary, we may assume that Q_2 is a surface group acting trivially on the finite normal subgroup K in (3.5). Hence Q is virtually a surface group.

Finally, since C is finite, as pointed at the beginning of the proof, the same argument applied to G now shows that G is virtually a surface group. \square

By Theorem 3.1 and Lemma 2.5, we immediately have:

Proposition 3.2. *If G in (2.3) satisfies the condition that $\text{def}(G) > 1$, then G is virtually a surface group.*

3.2. From virtual surface groups to orbifold groups. In this subsection, we classify virtual surface groups of positive deficiency. Let \mathcal{G} be a virtual surface group, equivalently, there exists an exact sequence:

$$1 \longrightarrow \pi_1(S) \longrightarrow \mathcal{G} \longrightarrow F \longrightarrow 1, \quad (3.6)$$

where S is a compact surface of positive genus, and F is a finite group acting by automorphisms on $\pi_1(S)$.

Theorem 3.3. *Assume that G in (2.3) satisfies the condition $\text{def}(G) > 1$. Then G must be an orbifold surface group with genus greater than 1.*

Proof. By Proposition 3.2, G is a virtual surface group.

Euclidean case: Suppose that the genus of the surface S in (3.6) for $\mathcal{G} = G$ is one. Then there exists a 2-dimensional crystallographic group H such that

$$G = H \times F_1,$$

where F_1 is the subgroup of F in (3.6) acting trivially on $\pi_1(S)$. The deficiency of any crystallographic group other than $\mathbb{Z} \times \mathbb{Z}$, and the Klein bottle group, is non-negative. The deficiency of any finite group is also non-negative. Since $\mathbb{Z} \times \mathbb{Z}$ and the Klein bottle group have deficiency exactly one, and need at least two generators, it follows that if F_1 is non-trivial, then $\text{def}(G) \geq 0$. Thus, F_1 is trivial, and we have one of the two possibilities for G :

- (1) $\mathbb{Z} \times \mathbb{Z}$, and
- (2) the Klein bottle group.

Since the Klein bottle group has first betti number one, it cannot be Sasakian (see Lemma 2.1). Finally, $\mathbb{Z} \times \mathbb{Z}$ has deficiency equal to one and is ruled out by the hypothesis.

Hyperbolic case: Next, suppose that the genus of S in (3.6) is greater than one. By the Nielsen realization theorem [Ke], there exists a hyperbolic orbifold \mathcal{O} with $\pi_1(\mathcal{O}) = H$ such that

$$G = H \times B_1$$

with B_1 finite. As in the genus one case, it follows that B_1 is trivial. If \mathcal{O} is non-orientable, then $b_1(\mathcal{O})$ is odd, forcing \mathcal{O} to be an orientable hyperbolic orbifold (using Lemma 2.1 again). Further, from the given condition that the deficiency is greater than one it follows that the genus of \mathcal{O} is greater than one. \square

4. ONE RELATOR GROUPS

4.1. One relator Sasakian groups are projective orbifold groups. Murasugi has described in detail the centers of one-relator groups.

Theorem 4.1 ([Mu, Theorems 1, 2]). *Let $\mathcal{G} = \langle x_1, \dots, x_k \mid w^n \rangle$ be a one-relator group with $n \geq 1$. If $k \geq 3$, then the center $Z(\mathcal{G})$ of \mathcal{G} is trivial. If \mathcal{G} is not abelian with $k = 2$, and $Z(\mathcal{G})$ is not trivial, then $Z(\mathcal{G})$ is infinite cyclic.*

Theorem 4.2 ([KS, p. 219]). *Let \mathcal{G} be a one-relator group having a (nontrivial) finitely presented normal subgroup H of infinite index. Then \mathcal{G} is torsion-free and has two generators. Further, \mathcal{G} is an infinite cyclic or infinite dihedral extension of a finitely generated free group N (meaning $N \subset \mathcal{G}$) satisfying the following:*

- $H \subset N$ if H is not cyclic, and
- $H \cap N$ is trivial if H is cyclic.

Proposition 4.3. *Assume that G in (2.3) is an infinite one relator group. Then*

- C in (2.3) is trivial, and
- $G = Q$ is a projective orbifold group.

Proof. By Theorem 4.1, C is trivial if $\text{def}(G) > 1$. In that case, $G = Q$ is a projective orbifold group.

On the other hand, if G is abelian, then $G = \mathbb{Z} \times \mathbb{Z}$ and G is a projective orbifold group, as it is the fundamental group of an elliptic curve.

Therefore, assume that $\text{def}(G) = 1$ and G is non-abelian.

Since $\text{def}(G) = 1$, the minimum number of generators of the one relator group G , which we will denote by k , is two.

Since C is contained in the center $Z(G)$ of G , if $Z(G)$ is trivial, then the proposition follows. So we assume that $Z(G)$ is non-trivial.

By Theorem 4.1, $C = \mathbb{Z}$. Hence, by Theorem 4.2, $G = N \times \mathbb{Z}$, where $N = F_r$ is free of rank r greater than one; we note that if N is free of rank one, then $G = \mathbb{Z} \times \mathbb{Z}$ and $C \neq \mathbb{Z}$.

Next, since G is one relator, we have $b_2(G) \leq 1$. On the other hand,

$$b_2(N \times \mathbb{Z}) = r > 1,$$

so we get a contradiction. Thus, if $k = 2$, then $Z(G)$ is trivial, forcing C to be trivial. Consequently, $G = Q$ is a projective orbifold group. \square

We shall need the following result.

Proposition 4.4 ([FKS, Theorem 1]). *Let $\mathcal{G} = \langle x_1, \dots, x_k \mid w^n \rangle$ be a one-relator group with $n \geq 1$. If $k = 1$, then G is torsion-free. Else, every torsion element in \mathcal{G} is conjugate to a power of w , and the subgroup generated by torsion elements in \mathcal{G} is the free product of the conjugates of w .*

4.2. From orbifold projective groups to projective groups. In this subsection we assume that G in (2.3) is a one-relator Sasakian group

$$G = \langle x_1, \dots, x_k \mid w^n \rangle.$$

If $\text{def}(G) > 1$, Theorem 3.3 shows that $G = \pi_1(\mathcal{O})$, where \mathcal{O} is a hyperbolic orbifold of genus greater than one.

Therefore, we assume that $\text{def}(G) = 1$, so $k = 2$.

Since $k = 2$, Proposition 4.3 further shows that

$$G = \pi_1(B), \tag{4.1}$$

where B is the projective orbifold in (2.2).

It can be shown that $\dim_{\mathbb{C}} B > 1$. Indeed, if $\dim_{\mathbb{C}} B = 1$, then M in (2.2) is a 3-dimensional Seifert-fibered manifold. Since $G = \pi_1(M)$ is infinite, it follows that the

center of G is infinite cyclic [He, Ch. 12]. But this contradicts Proposition 4.3. So we conclude that $\dim_{\mathbb{C}} B > 1$.

Structure of singularities: We analyze the locus of the points of the orbifold B in (2.2) with nontrivial inertia group. Such examples exist even when, at the global level, the Sasakian manifold is simply connected [CMST, MST, MT]. Let \mathcal{S} be a connected component of this locus of B . Let $U_{\mathcal{S}}$ denote a regular neighborhood of \mathcal{S} in B .

Lemma 4.5. *Let $2m$ denote the real dimension of B . There exists a lens space L of dimension $2l - 1$, with $l \geq 2$, such that the topological space for the orbifold $U_{\mathcal{S}}$ is homeomorphic to $cL \times \mathbb{D}_{2m-2l}$, where cL denotes the cone on L and \mathbb{D}_{2m-2l} denotes a ball of real dimension $2m - 2l$.*

Proof. We recall $B = M/U(1)$ with M being a smooth manifold. All the isotropy subgroups for the action of $U(1)$ on M are finite cyclic groups. The local structure of B follows from this (see [BoGa, p. 146, Theorem 4.7.7]). \square

Let $i_{\mathcal{S}} : U_{\mathcal{S}} \rightarrow B$ be the inclusion map between orbifolds, and let $i_{\mathcal{S},*}$ denote the homomorphism between orbifold fundamental groups induced by $i_{\mathcal{S}}$.

Lemma 4.6. *Suppose $i_{\mathcal{S},*}(\pi_1^{orb}(U_{\mathcal{S}}))$ is trivial for all components \mathcal{S} . Then the group G is projective. In particular, if G is torsion-free, it is projective.*

Proof. We may blow-up the orbifold B and obtain another orbifold

$$\varpi : B' \rightarrow B$$

such that the underlying topological space for B' is a smooth projective variety. We note that the underlying topological space for an orbifold B'' is a smooth projective variety if the locus in B'' of points with nontrivial inertia is a normal crossing divisor. In view of the given condition, we may choose B' such that $\pi_1(B)$ coincides with the fundamental group of the underlying topological space for B' . But the underlying topological space for B' is a smooth projective variety, so $\pi_1(B)$ is a projective group.

If G is torsion-free, then $i_{\mathcal{S},*}(\pi_1^{orb}(U_{\mathcal{S}}))$ must be trivial for all components \mathcal{S} . The conclusion of the lemma follows. \square

Lemma 4.7. *Suppose $i_{\mathcal{S},*}(\pi_1^{orb}(U_{\mathcal{S}}))$ is non-trivial for some \mathcal{S} . Then any torsion element of G has a non-trivial power that is conjugate to an element of $i_{\mathcal{S},*}(\pi_1^{orb}(U_{\mathcal{S}}))$.*

Proof. This is an immediate consequence of Proposition 4.4. \square

We next analyze the case where the underlying topological space for B is smooth.

Lemma 4.8. *Suppose that the underlying topological space for the projective orbifold B is smooth. Then G is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V_1 with at most one cone-point. Furthermore, V_1 is orientable.*

Proof. Let

$$G = \langle x_1, \dots, x_k \mid w^n \rangle, \quad n > 1.$$

Then, replacing the orbifold B by its underlying topological space \mathcal{B} , we note by Proposition 4.4 and Lemma 4.7, that $\pi_1(\mathcal{B})$ is the quotient of $\pi_1(B)$ by some powers of w . Hence $G_1 = \pi_1(\mathcal{B})$ is of the form

$$G_1 = \langle x_1, \dots, x_k \mid w^r \rangle, \quad r \geq 1.$$

Since \mathcal{B} is a smooth projective variety, G_1 is a one-relator Kähler group. Hence, by the classification of one-relator Kähler groups [BM1] (see also [Ko]), G_1 is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V with at most one cone-point y_0 . If $r = 1$, then G_1 is torsion-free. Else the loop that goes around y_0 represents the conjugacy class of w . Replacing r by n , we note that G is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V_1 , where V_1 differs from V only in the order of the cone-point y_0 . Thus, G is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V with at most one cone-point. \square

Lemma 4.5 furnishes the local structure of singularities of B . We now describe the blowup. We first note that a lens space cL on a lens space L of dimension $2k - 1$ with fundamental group $\mathbb{Z}/q\mathbb{Z}$ can be realized as the topological boundary of the total space E_q of a twisted line bundle (more precisely, the q -th tensor power of the tautological line bundle) over $\mathbb{C}P^{k-1}$. Hence, the blow up of cL may be obtained by removing an open neighborhood of the cone point $y_0 \in cL$ and attaching a copy of E_q along the resulting boundary L . Let $BU(L)$ denote the blowup of cL , and let $ED(L) \subset BU(L)$ be the exceptional divisor, which is homeomorphic to $\mathbb{C}P^{k-1}$. Then we note the following two properties of $BU(L)$:

Remark 4.9.

- (1) If cL has an orbifold cone-point of order q then $BU(L)$ is also an orbifold so that the exceptional divisor $ED(L)$ is an orbifold locus of ramification order q .
- (2) The underlying topological space of $BU(L)$ is a smooth manifold.

Next, we describe the local structure of a singularity of B as obtained in Lemma 4.5 along with the $U(1)$ bundle over it. Let E_S denote the circle bundle over U_S induced by the inclusion of U_S into B . Thus, we have $E_S \subset M$. Then we have the local structure of the Sasakian manifold M given by the following commutative diagram:

$$\begin{array}{ccc} D^{2k} \times U(1) & \longrightarrow & E_S \\ \downarrow & & \downarrow \\ D^{2k} & \longrightarrow & U_S = cL \end{array} \quad (4.2)$$

Theorem 4.10. *Let*

$$G = \langle x_1, \dots, x_n \mid w^k \rangle, \quad k > 1,$$

be a one-relator Sasakian group. Then G is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V with at most one cone-point. Further, the underlying manifold of V is orientable.

Proof. As discussed at the beginning of Section 4.2, it remains only to deal with the case that $n = 2$. If $k = 1$, then G is torsion-free by Theorem 4.4, and the hypothesis of Lemma 4.6 is satisfied, forcing G to be the fundamental group of a smooth projective variety. The main theorem of [BM1] now furnishes the result.

Else, the local structure of singularities is given by Lemma 4.5. Blowing up and using Remark 4.9, we obtain an orbifold B_1 satisfying the following:

- (1) The underlying topological space \mathcal{B}_1 of B_1 is a smooth projective variety.
- (2) The orbifold locus of B_1 is a divisor.
- (3) Any loop that goes around a component of the divisor represents an element of $\pi_1^{orb}(B) = G$ that is conjugate to w^r for some r that divides k .

Hence, by Lemma 4.8, $G_1 = \pi_1(\mathcal{B}_1)$ is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V with at most one cone-point. The rest of the argument is a replica of the last part of the proof of Lemma 4.8 forcing G to be isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V_1 , where V_1 differs from V only in the order of the cone-point. \square

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