Pattern Rigidity

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Motivation

Spaces: Geodesic metric spaces

Maps: Quasi-isometries.

$f : X \to Y$ such that

$\frac{1}{k} d_X(x,y) - \epsilon \leq d_Y(f(x), f(y)) \leq k d_X(x,y) + \epsilon$ and

$Y \subset N_k(f(X))$.

Finitely generated groups as geometric objects: Cayley graphs with respect to a finite generating set.

Programme (Gromov): Classify finitely generated groups up to quasi-isometry; understand $QI(X)$

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Collection of groups acting freely, cocompactly and properly discontinuously on some fixed proper metric space **H** are all quasi-isometric to **H**.

Impose additional *combinatorial* restrictions on the quasi-isometries by requiring that they preserve an additional structure given by a 'symmetric pattern' of subsets.

Pattern = G-equivariant collection $\mathcal{J}$ of convex (or uniformly quasiconvex) cocompact subsets in **H**.

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Alternately, go to the boundary. QI $q : \Gamma_{G_1} \rightarrow \Gamma_{G_2}$ induces quasiconformal homeomorphism $\partial q : \partial G_1 \rightarrow \partial G_2$.
$\mathcal{H}_i = \{g.\partial H_i \subset \partial G_i | g \in G_i\}$. Demand that $\partial q(\mathcal{H}_1) = \mathcal{H}_2$.

Question

Is a pattern-preserving quasi-isometry between pairs $(G_i, \mathcal{H}_i)$, $i = 1, 2$ close to an isometry. Does there exist an abstract commensurator I which performs the same pairing?

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History

- Schwartz – G is a non-uniform lattice in a rank one symmetric space. $\exists$ symmetric patterns of horoballs
- Schwartz – G is a uniform lattice in a rank one symmetric space. $\exists$ symmetric patterns of geodesics
- (Biswas, M–) G is a uniform lattice in a rank one symmetric space. $\exists$ symmetric pattern corresponding to certain Duality and PD subgroups of rank one symmetric spaces.
- (Biswas) G a uniform lattice in real hyperbolic space. $\exists$ corresponds to qc subgroup.

Uses *differential calculus of quasiconformal maps*.

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Filling Codimension One Subgroups

Study the groups $PPQI_u(G, H)$–groups of uniform quasi-isometries.

Caveat: Composing quasi-isometries increases qi constants.

One natural situation where such uniform groups arise:

Let H be a codimension one, filling, quasiconvex subgroup of a one-ended hyperbolic group G. There exists $C \geq 0$ such that any geodesic σ in Γ_G of length greater than C is separated by a translate of Γ_H or joins $J(\Lambda_H)$.

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Define a new pseudometric ρ on Γ by declaring $\rho(a, b)$ to be the number of copies of joins $J \in \mathcal{J}$ separating a, b. Then (Γ, ρ) is quasi-isometric to (Γ, d)

Any pattern-preserving quasi-isometry preserves ρ *on the nose*. Hence any pattern-preserving group Q of quasi-isometries is a group of uniform qi's.

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Locally Compact Totally Disconnected

Proposition

$PPQI_u(G, H)$ is locally compact totally disconnected.

Totally disconnected:

Let H – infinite quasiconvex subgroup of a hyperbolic group G of infinite index. Γ–Cayley graph. L – limit set of H and $\mathcal{L}$ – collection of translates of L. Collection $\mathcal{L}$ is discrete in the Hausdorff topology on $C_c(\partial G)$.

Connected component of the identity U is generated by arbitrary small open sets. Hence trivial.

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Local Compactness

Proposition

(M–) There exists a finite collection $L_1, \cdots L_n$ such that:
For any K, ϵ, there exists a C such that if $\phi : \Gamma \to \Gamma$ is a pattern-preserving (K, ϵ)-quasi-isometry of Γ with $\partial\phi(L_i) = L_i$ for $i = 1 \cdots n$, then $d(\phi(1), 1) \leq C$.

Hence there exists $L_1, \cdots L_n \in \mathcal{L}$ such that $Q_0 = \cap_{i=1 \cdots n} Stab(L_i)$ is compact.
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Use structure of Locally Compact Totally Disconnected topological groups.

Goal: To show $PPQI_u(G, H)$ discrete.

- Take compact open neighborhood U of identity.
- P. A. Smith theory implies No small torsion elements *provided* ∂G is a compact manifold (or more generally a homology manifold.)
- True if G is the fundamental group of a closed negatively curved manifold (or more generally a Poincare Duality hyperbolic group).
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Hilbert-Smith Conjecture

Remains to rule out p-adics $Z_{(p)}$.

Conjecture

(Hilbert-Smith) *There is no effective action of $Z_{(p)}$ by homeomorphisms on a compact manifold.*

∂G is an Ahlfors regular metric measure space with Hausdorff dimension $\mathcal{Q} \geq 1$.
Let K be a compact topological group (with Haar measure) acting by uniform C-quasiconformal maps on (X, d, μ).
Let d_K be the average metric on X
Then the Hausdorff dimension of (X, d_K, μ) does not exceed $\mathcal{Q}$.

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Last assumption: (X, d, μ) as above. Further, Hausdorff dimension $<$ homological dimension + 2.

Theorem

(Yang) Let X be a compact homology n-manifold admitting an effective K-action, where $K = Z_{(p)}$ is the group of p-adic integers. Then the homological dimension of X/K is $n + 2$.

With assumption above, (X, d, μ) does not admit an effective $Z_{(p)}$-action by quasiconformal maps.

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Let Q be any pattern-preserving group of quasi-isometries containing G.
Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Then the index of G in Q is finite.

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