HEIGHT, GRADED RELATIVE HYPERBOLICITY AND QUASICONVEXITY

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Abstract. We introduce the notions of geometric height and graded (geometric) relative hyperbolicity in this paper. We use these to characterize quasiconvexity in hyperbolic groups, relative quasiconvexity in relatively hyperbolic groups, and convex cocompactness in mapping class groups and $Out(F_n)$.

Contents

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1. Introduction

It is well-known that quasiconvex subgroups of hyperbolic groups have finite height. In order to distinguish this notion from the notion of **geometric height** introduced later in this paper, we shall call the former **algebraic height**: Let G be a finitely generated group and H a subgroup. We say that a collection of conjugates $\{g_iHg_i^{-1}\}, i = 1 \cdots n$ are **essentially distinct** if the cosets $\{g_iH\}$ are distinct. We say that H has finite **algebraic height** if there exists $n \in \mathbb{N}$ such that the intersection of any $(n+1)$ essentially distinct conjugates of H is finite. The minimal n for which this happens is called the **algebraic height** of H. This admits a natural (and obvious) generalization to a finite collection of subgroups H_i instead of one H. Thus, if G is a hyperbolic group and H a quasiconvex subgroup (or more generally if $H_1, \cdots H_n$ are quasiconvex), then H (or more generally the collection $\{H_1, \cdots H_n\}$) has finite algebraic height [GMRS97]. (See [Dah03, HW09] for generalizations to the context of relatively hyperbolic groups.) Swarup asked if the converse is true:

Question 1.1. [Bes04] *Let G be a hyperbolic group and H a finitely generated subgroup. If H has finite height, is H quasiconvex?*

An example of an infinitely generated (and hence non-quasiconvex) malnormal subgroup of a finitely generated free group was obtained in [DM15] showing that the hypothesis that H is finitely generated cannot be relaxed. On the other hand, Bowditch shows in [Bow12] (see also [Mj08, Proposition 2.10]) the following positive result:

Theorem 1.2. [Bow12] *Let G be a hyperbolic group and H a subgroup. Then G is strongly relatively hyperbolic with respect to H if and only if H is an almost malnormal quasiconvex subgroup.*

One of the motivational points for this paper is to extend Theorem 1.2 to give a characterization of quasiconvex subgroups of hyperbolic groups in terms of a notion of *graded relative hyperbolicity* defined as follows:

Definition 1.3. *Let G be a finitely generated group, d the word metric with respect to a finite generating set and H a subgroup. Let $\mathcal{H}_i$ be the collection of intersections of i essentially distinct conjugates of H, let $(\mathcal{H}_i)_0$ be a choice of conjugacy representatives, and let $\mathcal{C}\mathbf{H}_i$ be the set of cosets of elements of $(\mathcal{H}_i)_0$. Let d_i be the metric on (G, d) obtained by electrifying[1] the elements of $\mathcal{C}\mathbf{H}_i$. We say that G is* **graded relatively hyperbolic** *with respect to H (or equivalently that the pair $(G, \{H\})$ has graded relative hyperbolicity) if*

1. *H has algebraic height n for some $n \in \mathbb{N}$.*
2. *Each element K of $\mathcal{H}_{i-1}$ has a finite relative generating set S_K, relative to $H \cap \mathcal{H}_i (:= \{H \cap H_i : H_i \in \mathcal{H}_i\})$. Further, the cardinality of the generating set S_K is bounded by a number depending only on i (and not on K).*
3. *(G, d_i) is strongly hyperbolic relative to $\mathcal{H}_{i-1}$, where each element K of $\mathcal{H}_{i-1}$ is equipped with the word metric coming from S_K.*

The following is the main Theorem of this paper (see Theorem 6.4 for a more precise statement using the notion of *graded geometric relative hyperbolicity* defined

[1] The second author acknowledges the moderating influence of the first author on the more extremist terminology *electrocution* [Mj14, Mj11]

later) providing a partial positive answer to Question 1.1 and a generalization of Theorem 1.2:

Theorem 1.4. *Let (G, d) be one of the following:*

(1) *G a hyperbolic group and d the word metric with respect to a finite generating set S.*
(2) *G is finitely generated and hyperbolic relative to $\mathcal{P}$, S a finite relative generating set, and d the word metric with respect to $S \cup \mathcal{P}$.*
(3) *G is the mapping class group $Mod(S)$ and d a metric that is equivariantly quasi-isometric to the curve complex $CC(S)$.*
(4) *G is $Out(F_n)$ and d a metric that is equivariantly quasi-isometric to the free factor complex $\mathcal{F}_n$.*

Then (respectively)

(1) *H is quasiconvex if and only if $(G, \{H\})$ has graded relative hyperbolicity.*
(2) *H is relatively quasiconvex if and only if $(G, \{H\}, d)$ has graded relative hyperbolicity.*
(3) *H is convex cocompact in $Mod(S)$ if and only if $(G, \{H\}, d)$ has graded relative hyperbolicity and the action of H on the curve complex is uniformly proper.*
(4) *H is convex cocompact in $Out(F_n)$ if and only if $(G, \{H\}, d)$ has graded relative hyperbolicity and the action of H on the free factor complex is uniformly proper.*

Structure of the paper:
In Section 2, we will review the notions of hyperbolicity for metric spaces relative to subsets. This will be related to the notion of hyperbolic embeddedness [DGO11]. We will need to generalize the notion of hyperbolic embeddedness in [DGO11] to one of coarse hyperbolic embeddedness in order to accomplish this. We will also prove results on the preservation of quasiconvexity under electrification. We give two sets of proofs: the first set of proofs relies on assembling diverse pieces of literature on relative hyperbolicity, with several minor adaptations. We also give a more self-contained set of proofs relying on asymptotic cones.

In Section 3.1 and the preliminary discussion in Section 4, we give an account of two notions of height: algebraic and geometric. The classical (algebraic) notion of height of a subgroup concerns the number of conjugates that can have infinite intersection. The notion of geometric height is similar, but instead of considering infinite intersection, we consider unbounded intersections in a (not necessarily proper) word metric. This naturally leads us to dealing with intersections in different contexts:

(1) Intersections of conjugates of subgroups in a proper (Γ, d) (the Cayley graph of the ambient group with respect to a finite generating set).
(2) Intersections of conjugates of subgroups in a not necessarily proper (Γ, d) (the Cayley graph of the ambient group with a not necessarily finite generating set).
(3) Intersections of metric thickenings of cosets in a not necessarily proper (Γ, d).

The first is purely group theoretic (algebraic) and the last geometric, whereas the second is a composite, where the ambient space is not necessarily proper, whereas intersections considered are algebraic (i.e. of conjugates). Accordingly, we have three notions of height: algebraic, unparametrized geometric and parametrized geometric. Here the parameter depends on the thickening used in the last item above. In line with this, we investigate three notions of graded relative hyperbolicity in Section 4 (cf. Definition 4.4):

(1) Graded relative hyperbolicity (algebraic)
(2) Unparametrized graded geometric relative hyperbolicity
(3) Parametrized graded geometric relative hyperbolicity

In the fourth section, we also introduce and study a qi-intersection property, a property that ensures that quasi-convexity is preserved under passage to electrified spaces. The property exists in all three variants above.

In the fifth and the sixth sections, we will prove our main results relating height and geometric graded relative hyperbolicity. Again there are the two versions (parametrized and unparametrized). On a first reading, the reader is welcome to keep the simplest (algebraic or group-theoretic) notion in mind. To get a hang of where the paper is headed, we suggest that the reader take a first look at Sections 5 and 6, armed with Section 3.3 and the statements of Proposition 4.6, Theorem 4.7, Theorem 4.11, Theorem 4.14 and Proposition 4.15. This, we hope, will clarify our intent.

2. Relative Hyperbolicity, coarse hyperbolic embeddings

We shall clarify here what it means in this paper for a geodesic space (X, d), to be hyperbolic relative to a family of subspaces $\mathcal{Y} = \{Y_i, i \in I\}$, or to cast it in another language, what it means for the family $\mathcal{Y}$ to be hyperbolically embedded in (X, d). There are slight differences from the more usual context of groups and subgroups (as in [DGO11]), but we will keep the descending compatibility (when these notions hold in the context of groups, they hold in the context of spaces).

We begin by recalling relevant constructions.

2.1. Electrification by cones.

Given a metric space (Y, d_Y), we will endow $Y \times [0, 1]$ with the following product metric: it is the largest metric that agrees with d_Y on $Y \times \{0\}$, and each $\{y\} \times [0, 1]$ is endowed with a metric isometric to the segment $[0, 1]$.

Definition 2.1. [Far98] *Let (X, d) be a geodesic length space, and $\mathcal{Y} = \{Y_i, i \in I\}$ be a collection of subsets of X. The* electrification $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$ *of (X, d) along $\mathcal{Y}$ is defined as the following coned-off space:*
$X^{el}_{\mathcal{Y}} = X \sqcup \{\bigsqcup_{i \in I} Y_i \times [0, 1]\}/ \sim$ where $\sim$ denotes the identification of $Y_i \times \{0\}$ with $Y_i \subset X$ for each i, and the identification of $Y_i \times \{1\}$ to a single cone point v_i (dependent on i).

The metric $d^{el}_{\mathcal{Y}}$ is defined as the path metric on $X^{el}_{\mathcal{Y}}$ for the natural quotient metric coming from the product metric on $Y_i \times [0, 1]$ (defined as above).

Let $Y_i \in \mathcal{Y}$. The **angular metric** $\hat{d}_{Y_i}$ (or simply, $\hat{d}$, when there is no scope for confusion) on Y_i is defined as follows:
For $y_1, y_2 \in Y_i$, $\hat{d}_{Y_i}(y_1, y_2)$ is the infimum of lengths of paths in $X^{el}_{\mathcal{Y}}$ joining y_1 to y_2 not passing through the vertex v_i.

If (X, d) is a metric space, and Y is a subspace, we write $d|_Y$ the metric induced on Y.

Definition 2.2. *Consider a geodesic metric space* (X, d) *and a family of subsets* $\mathcal{Y} = \{Y_i, i \in I\}$. *We will say that* $\mathcal{Y}$ *is* **coarsely hyperbolically embedded** *in* (X, d), *if there is a function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *which is proper (i.e* $\lim_{+\infty} \psi(x) = +\infty$*), and such that*

1. *the electrified space* $X^{el}_{\mathcal{Y}}$ *is hyperbolic,*
2. *the angular metric at each* $Y \in \mathcal{Y}$ *in the cone-off is bounded from below by* $\psi \circ d|_Y$.

Remark 2.3. *This notion originates from Osin's* [Osi06a]*, and was developed further in* [DGO11] *in the context of groups, where one requires that each subset* $Y_i \in \mathcal{Y}$ *is a proper metric space for the angular metric. This automatically implies the weaker condition of the above definition. The converse is not true: if* $\mathcal{Y}$ *is a collection of uniformly bounded subgroups of a group* X *with a (not necessarily proper) word metric, it is always coarsely hyperbolically embedded, but it is hyperbolically embedded in the sense of* [DGO11] *only if it is finite.*

As in the point of view of [Osi06a], we say that (X, d) is **strongly hyperbolic relative to** the collection $\mathcal{Y}$ (in the sense of spaces) if $\mathcal{Y}$ is coarsely hyperbolically embedded in (X, d).

As we described in the remark, unfortunately, it happens that some groups (with a Cayley graph metric) are hyperbolic relative to some subgroups in the sense of spaces, but not in the sense of groups.

Note that there are other definitions of relative hyperbolicity for spaces. Druţu introduced the following definition: a metric space is hyperbolic relative to a collection of subspaces if all asymptotic cones are tree graded with pieces being ultra-translates of asymptotic cones of the subsets.

2.2. **Quasiretractions.**

Definition 2.4. *If* (X, d) *is a metric space, and* $Y \subset X$ *is a subset endowed with a metric* d_Y*, we say that* d_Y *is* λ*-undistorted in* (X, d) *if for all* $y_1, y_2 \in Y$,

$$\lambda^{-1} d(y_1, y_2) - \lambda \leq d_Y(y_1, y_2) \leq \lambda d(y_1, y_2) + \lambda.$$

We say that d_Y *is undistorted in* (X, d) *if it is* λ*-undistorted in* (X, d) *for some* λ.

For the next proposition, define the D-coarse path metric on a subset Y of a path-metric space (X, d) to be the path-metric on Y obtained by taking the infimum of lengths over paths for which any subsegment of length D meets Y.

The next Proposition is the translation (to the present context) of Theorem 4.31 in [DGO11], with a similar proof.

Proposition 2.5. *Let* (X, d) *be a graph. Assume that* $\mathcal{Y}$ *is coarsely hyperbolically embedded in* (X, d). *Then, there exists* D_0 *such that for all* $Y \in \mathcal{Y}$*, the* D_0*-coarse path metric on* $Y \in \mathcal{Y}$ *is undistorted (or equivalently the* D_0*-coarse path metric on* $Y \in \mathcal{Y}$ *is quasi-isometric to the metric induced from* (X, d)*).*

We will need the following Lemma, which originates in Lemma 4.29 in [DGO11]. The proof is the same; for convenience we will briefly recall it. The lemma provides quasiretractions onto hyperbolically embedded subsets in a hyperbolic space.

Lemma 2.6. *Let* (X, d) *be a geodesic metric space. There exists* $C > 0$ *such that whenever* $\mathcal{Y}$ *is coarsely hyperbolically embedded (in the sense of spaces) in* (X, d), *then for each $Y \in \mathcal{Y}$, there exists a map $r : X \to Y$ which is the identity on Y and such that $\hat{d}(r(x), r(y)) \leq Cd(x, y)$.*

Proof. Let p the cone point associated to Y, and for each x, choose a geodesic $[p, x]$ and define $r(x)$ to be the point of $[p, x]$ at distance 1 from p. Then $r(x)$ is in Y, and to prove the lemma, one only needs to check that there is C such that if $d(x, y) = 1$, then $\hat{d}(r(x), r(y)) \leq C$. The constant C will be $10(\delta + 1) + 1$. Assume that x and y are at distance $> 5(\delta + 1)$ from the cone point. By hyperbolicity, one can find two points in the triangle (p, x, y) at distance $2(\delta + 1)$ from $r(x), r(y)$ at distance $\leq 2\delta$ from each other. This provides a path of length at most $6\delta + 4$. Hence $\hat{d}(r(x), r(y)) \leq 6\delta + 4$. If x and y are at distance $\leq 5(\delta + 1)$ from the cone point, then $\hat{d}(r(x), r(y)) \leq d(r(x), x) + d(x, y) + d(y, r(y)) \leq 2 \times 5(\delta + 1) + 1$. □

We can now prove the Proposition.

Proof. Choose $D_0 = \psi(C)$: for all $y_0, y_1 \in Y$ at distance $\leq D_0$, their angular distance is at most C (where C is as given by the Lemma above). Consider any path in X from y_0 to y_1, call the consecutive vertices $z_0, \dots, z_n$, and project that path by r. One gets $r(z_0), \dots, r(z_n)$ in Y, two consecutive ones being at distance at most D_0. This proves the claim. □

Corollary 2.7. *If (X, d) is hyperbolic, and if $\mathcal{Y}$ is coarsely hyperbolically embedded, then there is C such that any $Y \in \mathcal{Y}$ is C-quasiconvex in X.*

2.3. **Gluing horoballs.** Given a metric space (Y, d_Y) , one can construct several models of combinatorial horoballs over it. We recall a construction (similar to that of Groves and Manning [GM08] for a graph).

We consider inductively on $k \in \mathbb{N} \setminus \{0\}$ the space $\mathcal{H}_k(Y) = Y \times [1, k]$ with the maximal metric d_k that

- induces an isometry of $\{y\} \times [k-1, k]$ with $[0, 1]$ for all $y \in Y$, and all $k \geq 1$,
- is smaller than d_{k-1} on $\mathcal{H}_{k-1}(Y) \subset \mathcal{H}_k(Y)$
- coincides with $2^{-k} \times d$ on $Y \times \{k\}$.

Let (X, d) be a graph, and $\mathcal{Y}$ be a collection of subgraphs (with the induced metric on each of them). The *horoballification* of (X, d) over $\mathcal{Y}$ is defined to be the space $X^h_{\mathcal{Y}} = X \sqcup \{\bigsqcup_{i \in I} \mathcal{H}(Y_i) \times\} / \sim_i$ where $\sim_i$ denotes the identification of the boundary horosphere of $\mathcal{H}(Y_i)$ with $Y_i \subset X$. The metric $d^h_{\mathcal{Y}}$ is defined as the path metric on $X^h_{\mathcal{Y}}$.

One can electrify a horoballification $X^h_{\mathcal{Y}}$ of a space (X, d): one gets a space quasi-isometric to the electrification $X^{el}_{\mathcal{Y}}$ of X. We record this observation in the following.

Proposition 2.8. *Let X be a graph, and $\mathcal{Y}$ be a family of subgraphs. Let $X^{el}_{\mathcal{Y}}$ and $X^h_{\mathcal{Y}}$ be the electrification, and the horoballification as above. Let $(X^h)^{el}_{\mathcal{H}(\mathcal{Y})}$ be the electrification of $X^h_{\mathcal{Y}}$ over the collection of subgraphs $\mathcal{H}(Y_i), i \in I$.*

Then there is a natural injective map $X^{el}_{\mathcal{Y}} \hookrightarrow (X^h)^{el}_{\mathcal{H}(\mathcal{Y})}$ which is the identity on X and sends the cone point of Y_i to the cone point of $\mathcal{H}(Y_i)$.

Consider the map $e : ((X^h)^{el}_{\mathcal{H}(\mathcal{Y})})^{(0)} \to X^{el}_{\mathcal{Y}}$ that

- *is the identity on X,*
- *sends each vertex of $\mathcal{H}(Y_i)$ of depth > 2 to $v_i \in X^{el}_{\mathcal{Y}}$,*
- *sends each vertex $(y, n) \in \mathcal{H}(Y_i)$ of depth $n \leq 2$ to $y \in Y_i \subset X$,*

- *sends the cone point of* $((X^h)^{el}_{\mathcal{H}(\mathcal{Y})})$ *associated to* $\mathcal{H}(Y_i)$ *to the cone point of* $X^{el}_{\mathcal{Y}}$ *associated to* Y_i.

Then e *is a quasi-isometry that induces an isometry on* $X^{el}_{\mathcal{Y}} \subset (X^h)^{el}_{\mathcal{H}(\mathcal{Y})}$.

Proof. First, note that a geodesic in $(X^h)^{el}_{\mathcal{H}(\mathcal{Y})}$ between two points of X never contains an edge with a vertex of depth ≥ 1. If it did, the subpath in the corresponding horoball would be either non-reduced, or would contain at least 3 edges, and could be shortened by substituting a pair of edges through the cone attached to that horoball. Thus the image of such a geodesic under e is a path of the same length. In other words, there is an inequality on the metrics $d_{X^{el}_{\mathcal{Y}}} \leq d_{(X^h)^{el}_{\mathcal{H}(\mathcal{Y})}}$ (restricted to $X^{el}_{\mathcal{Y}}$). On the other hand, there is a natural inclusion $X^{el}_{\mathcal{Y}} \subset (X^h)^{el}_{\mathcal{H}(\mathcal{Y})}$, and therefore on $X^{el}_{\mathcal{Y}}$, $d_{(X^h)^{el}_{\mathcal{H}(\mathcal{Y})}} \leq d_{X^{el}_{\mathcal{Y}}}$. Thus e is an isometry on $X^{el}_{\mathcal{Y}}$. Also, every point in $(X^h)^{el}_{\mathcal{H}(\mathcal{Y})}$ is at distance at most 2 from a point in X, hence also from a point of the image of $X^{el}_{\mathcal{Y}}$. □

2.4. **Relative hyperbolicity and hyperbolic embeddedness.** Recall that we say that a subspace Q of a geodesic metric space (X, d) is C-quasiconvex, for some number $C > 0$, if for any two points $x, y \in Q$, and any geodesic $[x, y]$ in X, any point of $[x, y]$ is at distance at most C from a point of Q.

Definition 2.9. [Mj11, Definition 3.5] *A collection* $\mathcal{H}$ *of (uniformly)* C*-quasiconvex sets in a* δ*-hyperbolic metric space* X *is said to be* **mutually D-cobounded** *if for all* $H_i, H_j \in \mathcal{H}$, *with* $H_i \neq H_j$, $\pi_i(H_j)$ *has diameter less than* D, *where* π_i *denotes a nearest point projection of* X *onto* H_i. *A collection is* **mutually cobounded** *if it is mutually* D*-cobounded for some* D.

The aim of this subsection is to establish criteria for hyperbolicity of certain spaces (electrification, horoballification), and related statements on persistence of quasi-convexity in these spaces. We also show that hyperbolicity of the horoballification implies strong relative hyperbolicity, or coarse hyperbolic embeddedness.

Two sets of arguments are given. In the first set of arguments, the pivotal statement is of the following form: Electrification or de-electrification preserves the property of being a quasigeodesic. The arguments are essentially existent in some form in the literature, and we merely sketch the proofs and refer the reader to specific points in the literature where these may be found.

The second set of arguments uses asymptotic cones (hence the axiom of choice) and is more self-contained (it relies on Gromov-Cartan-Hadamard theorem). We decided to give both these arguments so as to leave it to the the reader to choose according to her/his taste.

2.5. **Persistence of hyperbolicity and quasiconvexity.**

2.5.1. *The Statements.* Here we state the results for which we give arguments in the following two subsubsections.

Proposition 2.10. *Let* (X, d) *be a hyperbolic geodesic space,* $C > 0$, *and* $\mathcal{Y}$ *be a family of* C*-quasiconvex subspaces. Then* $X^{el}_{\mathcal{Y}}$ *is hyperbolic. If moreover the elements of* $\mathcal{Y}$ *are mutually cobounded, then* $X^h_{\mathcal{Y}}$ *is hyperbolic.*

In the same spirit, we also record the following statement on persistence of quasi-convexity.

Proposition 2.11. *Given δ, C, D there exists Δ, C' such that if (X, d_X) is a δ-hyperbolic metric space with a collection $\mathcal{Y}$ of C-quasiconvex, $D-$separated sets. then the following holds:*
If $Q(\subset X)$ is a C-quasiconvex set (not necessarily an element of $\mathcal{Y}$), then Q is C'-quasiconvex in $(X^{el}_{\mathcal{Y}}, d_e)$.

Finally, there is a partial converse. We need a little bit of vocabulary.

If Z is a subset of a metric space (X, d), a (d, R)-coarse path in Z is a sequence of points of Z such that two consecutive points are always at distance at most R for the metric d.

Let H and Y two subsets of X. We will denote by $H^{+\lambda}$ the set of points at distance at most λ from H. We will say that H (Δ, ϵ)-meets Y if there are two points x_1, x_2 of H at distance at least $(1-\epsilon)\Delta$ from each other, and at distance $\leq \epsilon\Delta$ from Y. The two points x_1, x_2 are called a pair of meeting points in H (for Y).

Proposition 2.12. *Let (X, d) be hyperbolic, and let $\mathcal{Y}$ be a collection of uniformly quasiconvex subsets. Let H be a subset of X that is coarsely path connected, and quasi-convex in the electrification $X^{el}_{\mathcal{Y}}$.*

Assume also that there exists $\epsilon \in (0, 1)$, and Δ_0 such that for all $\Delta > \Delta_0$, wherever H (Δ, ϵ)-meets an item Y in $\mathcal{Y}$, there is a path in $H^{+\epsilon\Delta}$ between the meeting points in H that is uniformly a quasigeodesic in the metric (X, d).

Then H is quasi-convex in (X, d).

The quasiconvexity constant of H can be chosen to depend only on the constants involved for $(X, d), \mathcal{Y}, \Delta_0, \epsilon$, the coarse path connection constant, and the quasi-geodesic constant of the last assumption.

2.5.2. *Electroambient quasigeodesics.* We recall here the concept of electroambient quasigeodesics from [Mj11, Mj14].

Let (X, d) be a metric space, and $\mathcal{Y}$ a collection of subspaces. If γ is a path in (X, d), or even in $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$, one can define an **elementary electrification** of γ in $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$ as follows:
For x_1, x_2 in γ, both belonging to some $Y_i \in \mathcal{Y}$, and at distance > 1, replace the arc of γ between them by a pair of edges $(x_1, v_i)(v_i, x_2)$, where v_i is the cone-point corresponding to Y_i.

A **complete electrification** of γ is a path obtained after a sequence of elementary electrifications of subarcs, admitting no further elementary electrifications.

One can *de-electrify* certain paths. Given a path γ in $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$, a **de-electrification** of γ is a path σ in (X, d) such that

(1) γ is a complete electrification of σ,
(2) $(\sigma \setminus \gamma) \cap Y_i$ is either empty or a geodesic in Y_i.

A (λ, μ)**-de-electrification** of a path γ in $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$, is a path in X such that

(1) γ is a complete electrification of σ,
(2) $(\sigma \setminus \gamma) \cap Y_i$ is either empty or a (λ, μ)-quasigeodesic in Y_i.

Observe that, given a path σ in X^{el}, there might be several ways to de-electrify it, but these ways differ only in the choice of the geodesic (or the quasi-geodesic) in the family of subspaces Y_i corresponding to the successive cone points v_{Y_i} on the path σ. It might also happen that there is no way of de-electrifying it, if the spaces in $\mathcal{Y}$ are not quasiconvex.

We say that a path γ in (X, d) is an **electro-ambient geodesic** if it is a de-electrification of a geodesic.

We say that it is a $(\lambda, \mu)-$ **electro-ambient quasigeodesic** if it is the $(\lambda, \mu)-$ de-electrification of a $(\lambda, \mu)-$quasigeodesic in $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$.

We begin by discussing Proposition 2.10.

Proof. The first part is fairly well-known. In some other guise it appears in [Bow12, Proposition 7.4] [Szc98, Proposition 1] [Mj11]. In the first two, the electrification by cones is replaced by collapses of subspaces (identifications to points) which of course requires that the subspaces to electrify are disjoint and separated.

However this is only a technical assumption (as explicated in [Mj11]). Indeed, by replacing (or augmenting) any $Y \in \mathbb{Y}$ by $Y \times [0, D]$ glued along $Y \times \{0\}$, and replacing $\mathcal{Y}$ by the family $\{Y \times \{D\}, Y \in \mathcal{Y}\}$, we achieve a D-separated quasiconvex family. □

Next, we discuss Proposition 2.11.

Proof. The proofs of Lemma 4.5 and Proposition 4.6 of [Far98], Theorem 5.3 of [Kla99] (see also [Bow12]) furnish Proposition 2.11, and even slightly more: under the same assumption, if $\mathcal{Y}$ is D-separated, there exists C_1 depending on δ, C, D such that the following holds:

Let $\Pi_Q(p)$ and $\Pi^e_Q(p)$ denote the nearest point projections of p on Q in (X, d_X) and $(X^{el}_{\mathcal{H}}, d_e)$ respectively. Then $d_e(\Pi_Q(p), \Pi^e_Q(p)) \leq C_1$. □

The rest of this (subsub)section is devoted to discussing Proposition 2.12. Towards doing this, we will obtain an argument for showing a variant of the second point of Proposition 2.10, namely that in a hyperbolic space (X, d), a family $\mathcal{Y}$ of uniformly quasi convex subspaces that is mutually cobounded defines a strong relative hyperbolic structure on (X, d). The second point of 2.10 as it is stated will be however proved in the next subsection.

We shall have need for the following Lemma [Mj11, Lemma3.9] (see also [Kla99, Proposition 4.3] [Mj14, Lemma 2.5]).

Lemma 2.13. *Suppose (X, d) is δ-hyperbolic. Let $\mathcal{H}$ be a collection of C-quasiconvex D-mutually cobounded subsets. There exists $\epsilon_0 = \epsilon_0(C, K, D, \delta)$ such that the following holds:*

Let β be an electric P-quasigeodesic without backtracking and γ a hyperbolic geodesic, both joining x, y. Then, given $\epsilon \geq \epsilon_0$ there exists $D = D(P, \epsilon)$ such that

1. *Similar Intersection Patterns 1: if precisely one of $\{\beta, \gamma\}$ meets an ϵ-neighborhood $N_\epsilon(H_1)$ of an electrocuted quasiconvex set $H_1 \in \mathcal{H}$, then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point to the exit point is at most D.*
2. *Similar Intersection Patterns 2: if both $\{\beta, \gamma\}$ meet some $N_\epsilon(H_1)$ then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point of β to that of γ is at most D; similarly for exit points.*

Note that Lemma 2.13 above is quite general and does not require X to be proper.

Remark 2.14. *In* [Mj11]*, the extra hypothesis of separatedness was used. However, this is superfluous by the same remark on augmentations of elements of $\mathcal{Y}$ that we made in the beginning of the proof of Proposition 2.10. Lemma 2.13 may be stated equivalently as the following (compare with 2.18 below) .*
If X is a hyperbolic metric space and $\mathcal{H}$ a collection of uniformly quasiconvex mutually cobounded subsets, then X is strongly hyperbolic relative to the collection $\mathcal{H}$.

We give a slightly modified version of [Mj11, Lemma 3.15] below by using the equivalent hypothesis of strong relative hyperbolicity (i.e. Lemma 2.13).

Lemma 2.15. *Let (X, d) be a $\delta-$hyperbolic metric space, and $\mathcal{H}$ a family of subsets such that X is strongly hyperbolic relative to $\mathcal{H}$. Then for all $\lambda, \mu > 0$, there exists λ', μ' such that any electro-ambient (λ, μ)-quasi-geodesic is a (λ', μ')-quasi-geodesic in (X, d).*

The proof of Lemma 2.15 goes through mutatis mutandis for strongly relatively hyperbolic spaces as well, i.e. hyperbolicity of X may be replaced by relative hyperbolicity in Lemma 2.15 above. We state this explicitly below:

Corollary 2.16. *Let (X, d) be strongly relatively hyperbolic relative to a collection $\mathcal{Y}$ of path connected subsets. Then, for all $\lambda, \mu > 0$, there exists λ', μ' such that any electro-ambient (λ, μ)-quasi-geodesic is a (λ', μ')-quasi-geodesic in (X, d).*

We include a brief sketch of the proof-idea following [Mj11]. Let γ be an electro-ambient quasigeodesic. By Definition, its electrification $\hat{\gamma}$ is a quasi-geodesic in $X^{el}_{\mathcal{Y}}$. Let σ be the electric geodesic joining the end-points of γ. Hence σ and $\hat{\gamma}$ have similar intersection patters with the sets Y_i [Far98], i.e. they enter and leave any Y_i at nearby points. It then suffices to show that an electro-ambient representative of σ is in fact a quasigeodesic in X. A proof of this is last statement is given in [McM01, Theorem 8.1] in the context of horoballs in hyperbolic space (see also Lemmas 4.8, 4.9 and their proofs in [Far98]). The same proof works after horoballification for an arbitrary relatively hyperbolic space. □

We finally give a proof of Proposition 2.12.

Proof. Assuming that the elements of $\mathcal{Y}$ are uniformly quasiconvex in (X, d), we shall show that H is also quasiconvex for a uniform constant.

First, since (X, d) is hyperbolic it follows by Proposition 2.10 that $(X^{el}_{\mathcal{Y}}, d^{el})$ is hyperbolic.

Let $x, y \in H$. By assumption, there exists $C_0 \geq 0$ such that H is $(C_0, C_0)-$qi embedded in (G, d^{el}). Denote by $\mathcal{P}$ the set of cone points corresponding to elements of $\mathcal{Y}$ and let γ be a $(C, C)-$quasi-geodesic without backtracking in (X, d^{el}) with vertices in $H \cup \mathcal{P}$ joining $x, y \in H$. By assumption, the collection $\mathcal{Y}$ is uniformly C-quasiconvex. Further, by assumption, there exists $\epsilon \in (0, 1)$, and Δ_0 such that for all $\Delta > \Delta_0$, wherever H (Δ, ϵ)-meets an item Y in $\mathcal{Y}$, there is a path in $H^{+\epsilon\Delta}$ between the meeting points in H that is uniformly a quasigeodesic in the metric (X, d). Hence, for some uniform constants λ, μ, we may (coarsely) (λ, μ)-de-electrify γ to obtain a (λ, μ)-electro-ambient quasigeodesic γ' in (X, d), that lies close to H. [Note that the meeting points of H with elements of $\mathcal{Y}$ are only coarsely defined. So we are actually replacing pieces of γ by quasigeodesics in $H^{+\epsilon\Delta}$ rather than in H itself.]

By Corollary 2.16, it follows that γ' is a quasi-geodesic in (X, d), for a uniform constant.

Since this was done for arbitrary $x, y \in H_{i,\ell}$, we obtain that H is $D-$quasiconvex in (X, d). This finishes the proof of Proposition 2.12. □

2.5.3. *Proofs through asymptotic cones.* The repeated use of different references coming from different contexts in the previous subsubsection might call for more systematic self-contained proofs of the statements of subsection 2.5.1. This is our purpose in this subsubsection.

In this part we will use the structure of an argument originally from Gromov, and developed by Coulon in particular (see for instance [Cou14, Proposition 5.28]), that uses asymptotic cones in order to show hyperbolicity or quasiconvexity of constructions.

We fix a non-principal ultrafilter ω. We will use the construction of asymptotic cones for this ultrafilter ω. A few observations are in order here.

Recall that if (X_N, x_N) is a sequence of pointed δ_n-hyperbolic spaces, for δ_n converging to 0, then the asymptotic cone $\lim_\omega(X_N, x_N)$ is an $\mathbb{R}$-tree, with a base point. If $\mathcal{Y}_N$ is a family of subsets of X_N, let us define $\lim_\omega(\mathcal{Y}, x_N)$ (denoted by $\lim_\omega(\mathcal{Y})$ for short) in $\lim_\omega((X_N)^{el}_{\mathcal{Y}_N}, x_N)$ to be the collection of limits of sequences of elements of $\mathcal{Y}_N$: a subset $Y^\omega \subset \lim_\omega((X_N)^{el}_{\mathcal{Y}_N}, x_N)$ is in $\lim_\omega \mathcal{Y}$ if there is a sequence (Y_N), with $Y_N \in \mathcal{Y}_N$ for all N, such that $Y^\omega = \{\lim_\omega y_N, (y_N) \in \prod_{N>0} Y_N\}$. Recall that if the subsets of $\mathcal{Y}_N$ are c_n-quasiconvex for c_n going to 0, then all elements of $\lim_\omega \mathcal{Y}$ are convex. If X_n is δ_n-hyperbolic (δ_n going to 0) and if the subsets of $\mathcal{Y}_N$ are c_n-mutually cobounded, then any two elements of $\lim_\omega \mathcal{Y}$ are subsets that share at most one point.

Let us prove Proposition 2.10. For convenience of the reader we repeat the statement.

Proposition. *2.10 Let (X, d) be a hyperbolic geodesic space, $C > 0$, and $\mathcal{Y}$ be a family of C-quasiconvex subspaces. Then $X^{el}_{\mathcal{Y}}$ is hyperbolic. If moreover the elements of $\mathcal{Y}$ are mutually cobounded, then $X^h_{\mathcal{Y}}$ is hyperbolic.*

Proof. We claim that, for all ρ, there exists $\delta_0 < \rho/10^{14}$ and $C_0 < \rho/10^{14}$ such that if X is δ_0-hyperbolic, and if $\mathcal{Y}$ is a collection of C_0 quasi-convex subsets, then every ball of radius ρ of $X^{el}_{\mathcal{Y}}$ is 10-hyperbolic.

For proving the claim, assume it false, and consider a sequence of counterexamples $(X_N, \mathbb{Y}_N)$ for $\delta_0 = C_0 = \frac{1}{N}$, $N = 1, 2, \ldots$. This means that $(X_N)^{el}_{\mathcal{Y}_N}$ fails to be 10-hyperbolic. There are four points $x_N, y_N z_N, t_N$, all at distance at most 2ρ from x_N, such that $(x_N, z_N)_{t_N} \leq \inf\{(x_N, y_N)_{t_N}, (y_N, z_N)_{t_N}\} - 10$. We pass to the ultralimit for ω. In $\lim_\omega(X_N, x_N)$, each sequence x_N, y_N, z_N, t_N converges, since these points stay at bounded distance from x_N, and the inequality persists. Hence one gets four points falsifying the 10-hyperbolicity condition in a pointed space $\lim_\omega((X_N)^{el}_{\mathcal{Y}_N}, x_N)$.

But the asymptotic cone $\lim_\omega((X_N)^{el}_{\mathcal{Y}_N}, x_N)$ is easily seen to be the electrification (of parameter 1) of the real tree $\lim_\omega(X_N, x'_N)$, for some base point x'_N at distance ≤ 1 from x_N, over the family $\lim_\omega \mathcal{Y}$, which consists of *convex* subsets (*i.e.* of subtrees).

This space $(\lim_\omega(X_N, x'_N))^{el}_{\lim_\omega \mathcal{Y}}$ has 2-thin geodesic triangles, hence is 10-hyperbolic, contradiction. The claim hence holds: $X^{el}_{\mathcal{Y}}$ is ρ-locally 10-hyperbolic.

We now claim that, under the same hypothesis, it is $(2 + 10C_0 + 10\delta_0)$-coarsely simply-connected, that is to say that any loop in it can be homotoped to a point by a sequence of substitutions of arcs of length $< (2+10C_0+10\delta_0)$ by its complement in a loop of length $< (2+10C_0+10\delta_0)$. Indeed, any time such a loop passes through a cone point associated to some $Y \in \mathcal{Y}$, one can consider a geodesic in X between its entering and exiting points in Y, which stays in the C_0 neighborhood of Y. Therefore, a $(2+10C_0)$-coarse homotopy of the loop transforms it into a loop in X, which is δ-hyperbolic. Since a δ- hyperbolic space is 10δ-coarsely simply-connected, the second claim follows.

The final ingredient is the Gromov-Cartan-Hadamard theorem [Cou14, Theorem A.1], stating that, if ρ is sufficiently large compared to μ, any ρ-locally 10-hyperbolic space which is μ-coarsely simply connected is (globally) δ'-hyperbolic, for some δ'. We thus get that there exists $\delta_0 < \rho/10^{14}$ and $C_0 < \rho/10^{14}$ such that if X is δ_0-hyperbolic, and if $\mathcal{Y}$ is a collection of C_0 quasi-convex subsets, then $X^{el}_{\mathcal{Y}}$ is δ'-hyperbolic.

Now let us argue that this implies the first point of the proposition. If X and $\mathcal{Y}$ are given as in the statement, one may rescale X by a certain factor $\lambda > 1$, so that it is δ_0-hyperbolic, and such that $\mathcal{Y}$ is a collection of C_0-quasiconvex subsets. Let us define $X^{el_\lambda}_{\mathcal{Y}}$ to be

$$X^{el_\lambda}_{\mathcal{Y}} = X \sqcup \{\bigsqcup_{i\in I} Y_i \times [0,\lambda]\}/\sim$$

where $\sim$ denotes the identification of $Y_i \times \{0\}$ with $Y_i \subset X$ for each i, and the identification of $Y_i \times \{1\}$ to a single cone point v_i (dependent on i), and where $Y_i \times [0,\lambda]$ is endowed with the product metric as defined in the first paragraph of 2.1 except that $\{y\} \times [0,n]$ is isometric to $[0,\lambda]$. The claim ensures that $X^{el_\lambda}_{\mathcal{Y}}$ is hyperbolic. However, it is obviously quasi-isometric to $X^{el}_{\mathcal{Y}}$. We have the first point.

For the second part, one can proceed with a similar proof, with horoballs.

The claim is then that for all ρ, there exists δ_0, C_0 and D_0 such that if X is δ_0-hyperbolic, and if $\mathcal{Y}$ is a collection of C_0 quasi-convex subsets, D_0-mutually cobounded, then any ball of radius ρ of the horoballification $X^h_{\mathcal{Y}}$ is 10-hyperbolic.

The proof of the claim is similar. Consider a sequence of counterexample $X_N, \mathcal{Y}_N$, for the parameters $\delta = C = D = 1/N$ for N going to infinity, with the four points x_N, y_N, z_N, t_N in $(X_N)^h_{\mathcal{Y}}$, in a ball of radius ρ, falsifying the hyperbolicity condition.

There are two cases. Either x_N (which is in $(X_N)^h_{\mathcal{Y}}$) escapes from X_N, i.e. its distance from some basepoint in X_N tends to ∞ for the ultrafilter ω, or it does not. In the case it escapes from X_N, then, when it is larger than ρ all four points x_N, y_N, z_N, t_N are in a single horoball, but such a horoball is 10-hyperbolic hence a contradiction.

The other case is therefore when there is $x'_N \in X_N$ whose distance to x_N remains bounded (for the ultrafilter ω). Note that $\{x_N, y_N, z_N, t_N\}$ converges in the asymptotic cone $\lim_\omega((X_N)^h_{\mathcal{Y}_N}, x'_N)$ of the sequence of pointed spaces $((X_N)^h_{\mathcal{Y}_N}, x'_N)$. It is also immediate by definition of $\lim_\omega \mathcal{Y}_N$ that $\lim_\omega((X_N)^h_{\mathcal{Y}_N}, x'_N)$ is the horoballification of the asymptotic cone of the sequence (X_N, x'_N) over the family $\lim_\omega(\mathcal{Y}_N, x'_N)$ defined above.

This family $\lim_\omega(\mathcal{Y}_N, x'_N)$ consists of *convex* subsets (hence subtrees), such that any two share at most one point. This horoballification is therefore a tree-graded space in the sense of [DS05], with pieces being the combinatorial horoballs over the subtrees constituting $\lim_\omega \mathcal{Y}_N$. As a tree of 10-hyperbolic spaces, this space is 10-hyperbolic, contradicting the inequalities satisfied by the limits $\lim_\omega\{x_N, y_N, z_N, t_N\}$. Therefore, $X^h_{\mathcal{Y}}$ is ρ-locally 10-hyperbolic.

As before, one may check that (under the same assumptions) $X^h_{\mathcal{Y}}$ is $(2+10C_0+10\delta_0)$-coarsely simply connected, and again this implies by the Gromov-Cartan-Hadamard theorem that (under the same assumptions) $X^h_{\mathcal{Y}}$ is hyperbolic.

This implies the second point. Indeed, let us denote by $\frac{1}{\lambda}X$ the space X with metric rescaled by $\frac{1}{\lambda}$.

The previous claim shows that, under the assumption of the second point of the proposition, there exists $\lambda > 1$ such that $\lambda(\frac{1}{\lambda}X)^h_{\frac{1}{\lambda}\mathcal{Y}}$ is hyperbolic. Consider the map η between $X^h_{\mathcal{Y}} \to \lambda(\frac{1}{\lambda}X)^h_{\frac{1}{\lambda}\mathcal{Y}}$ that is identity on X and that sends $\{y\}\times\{n\}$ to $\{y\}\times\{\lambda\times(n+\lfloor\log_2\lambda\rfloor)\}$ for all $y\in Y_i$ and all Y_i (and all n). All paths in $X^h_{\mathcal{Y}}$ that have only vertical segments in horoballs have their length expanded by the map η by a factor between 1 and $\lambda+\log_2\lambda$. But the geodesics in $X^h_{\mathcal{Y}}$ and $\lambda(\frac{1}{\lambda}X)^h_{\frac{1}{\lambda}\mathcal{Y}}$ are paths whose components in horoballs consist of a vertical (descending) segment, followed by a single edge, followed by a vertical ascending segment (see [GM08]). Hence η is a quasi-isometry, and the space $X^h_{\mathcal{Y}}$ is hyperbolic. □

We continue with the persistence of quasi-convexity.

Proposition. *(2.11) Given δ, C, D there exists Δ, C' such that if (X, d_X) is a δ-hyperbolic metric space with a collection $\mathcal{Y}$ of C-quasiconvex, sets. Then the following holds:*
If $Q(\subset X)$ is some (any) C-quasiconvex set (not necessarily an element of $\mathcal{Y}$), then Q is C'-quasiconvex in $(X^{el}_{\mathcal{Y}}, d_e)$.

Proof. The strategy is similar to that in the previous proposition. The main claim is that for all ρ, there is $\delta_0 < 1, C_0 < 1$ such that if (X, d_X) is δ_0-hyperbolic, if $\mathcal{Y}$ is a collection of C_0-quasiconvex subsets and if Q is another C_0-quasiconvex subset of X, then Q is ρ-locally 10-quasiconvex in $X^{el}_{\mathcal{Y}}$ (of course δ_0, C_0 will be very small).

To prove the claim, again, by contradiction, consider a sequence $X_N, \mathcal{Y}_N, Q_N$ of counter examples for $\delta_0 = C_0 = N$ for $N = 1, 2, \dots$ going to infinity. There exists two points x_N, y_N in Q_N, at distance $\leq \rho$ form each other, and a geodesic $[x_N, y_N]$ in $X^{el}_{\mathcal{Y}}$ with a point z_N on it at distance > 10 from Q (we record z'_N a point in Q at minimal distance ($\leq \rho$ in any case) from z_N.

With a non principal ultrafilter ω, we may take the asymptotic cone of the family of pointed spaces (X_N, x_N). In this limit $\lim_\omega((X_N)^{el}_{\mathcal{Y}}, x_N)$, the sequences (y_N) ($[x_N, y_N]$) and (z_N) have limits, for which the distance inequalities persist, and we get that $\lim_\omega(Q_N, x_N)$ is not ρ-locally 10-quasiconvex in $\lim_\omega((X_N)^{el}_{\mathcal{Y}}, x_N)$. But as we noticed, $\lim_\omega((X_N)^{el}_{\mathcal{Y}}, x_N)$ is $(\lim_\omega(X_N))^{el}_{\lim_\omega \mathcal{Y}}, x_N)$, which is the electrification of an $\mathbb{R}$-tree $\lim_\omega(X_N)$ over a family of convex subsets (*i.e* subtrees). But in this space, $\lim_\omega Q_N$ is also a subtree of $\lim_\omega(X_N)$. Hence it is 2-quasiconvex in $(\lim_\omega(X_N))^{el}_{\lim_\omega \mathcal{Y}}, x_N)$, and this contradicts the inequalities satisfied by $\lim_\omega\{x_N, y_N, z_N, z'_N\}$. The claim is established for all ρ.

Now there exists ρ_0 such that, in any 1-hyperbolic space, any subset that is ρ_0-locally 10-quasiconvex is 10^{14}-globally quasiconvex (this classical fact, perhaps found elsewhere with other (better !) constants, follows also from Gromov-Cartan-Hadamard theorem for instance). So, by choosing an appropriate ρ, we have proven that there is $\delta_0 < 1, C_0 < 1$ and C_1, such that if (X, d_X) is δ_0-hyperbolic, if $\mathcal{Y}$ is a collection of C_0-quasiconvex subsets and if Q is another C_0-quasiconvex subset of X, then Q is C_1-quasiconvex in $X^{el}_{\mathcal{Y}}$.

Coming back to the statement of the proposition, by rescaling our space, we have proven that if (X, d_X) is a δ-hyperbolic metric space with a collection $\mathcal{Y}$ of C-quasiconvex sets, and if Q is C-quasiconvex, then Q is λC_1-quasiconvex in $X^{el_\lambda}_{\mathcal{Y}}$ (as defined in the previous proof) for $\lambda = \max\{\delta/\delta_0, C/C_0\}$. Since $X^{el_\lambda}_{\mathcal{Y}}$ is quasi-isometric to $X^{el}_{\mathcal{Y}}$, by a (λ,λ)-quasi-isometry, it follows that Q is C'-quasiconvex in $X^{el}_{\mathcal{Y}}$ for C' depending only on δ, C.

□

Finally, we consider the proposed converse.

Proposition. *2.12*

Let (X, d) be hyperbolic, and let $\mathcal{Y}$ be a collection of uniformly quasiconvex subsets. Let H be a subset of X that is coarsely path connected, and quasi-convex in the electrification $X^{el}_{\mathcal{Y}}$.

Assume also that there exists $\epsilon \in (0,1)$, and Δ_0 such that for all $\Delta > \Delta_0$, wherever H (Δ,ϵ)-meets an item Y in $\mathcal{Y}$, there is a path in $H^{+\epsilon\Delta}$ between the meeting points in H that is uniformly a quasigeodesic in the metric (X, d).

Then H is quasi-convex in (X, d).

The quasiconvexity constant of H can be chosen to depend only on the constants involved for $(X, d), \mathcal{Y}, \Delta_0, \epsilon$, the coarse path connection constant, and the quasi-geodesic constant of the last assumption.

We use the same strategy again. The claim is now the following lemma.

Lemma 2.17. *Fix C^{el}_H, $R > 0$, $Q > 1$, $\epsilon > 0$, and $\Delta > 10\epsilon$.*

Then there exists δ_0, C_0, m_0, positive, such that the following holds. Assume that (X, d) is geodesic, δ_0-hyperbolic, with a collection $\mathcal{Y}$ of C_0-quasiconvex, and if H is a subset of X which is m_0-coarsely connected, and C^{el}_H-quasiconvex in the electrification $X^{el}_{\mathcal{Y}}$. Equip $H^{+\epsilon\Delta}$ by its coarse path metric d_H.

Assume also that whenever H (Δ,ϵ)-meets a set $Y \in \mathcal{Y}$, then there is a (Q, C_0)-quasigeodesic path, with mesh $< m_0/10$, in $H^{+\epsilon\Delta}$ joining the meeting points in H.

Then for all $a, b \in H^{+\epsilon\Delta}$ at d_H-distance at most R from each other, any m_0-coarse geodesic between them is $(\Delta \times C^{el}_H)$-close to a geodesic of X.

Proof. Assume the claim is false. For all choice of δ, C, m there is a counterexample. Set $\delta_N = C_N = 1/N$.

For each ϵ, there exists N such that, in a $\frac{1}{N}$-hyperbolic space, for any two points x, y, and any $(Q, 1/N)$-quasigeodesic p with mesh $\leq 1/N$ between these two points, the ϵ-neighborhood of p contains the geodesics $[x, y]$. (This is for instance visible on an asymptotic cone).

Thus, it is possible to choose a sequence $m_N > 10/N$ decreasing to zero, such that pairs of $\frac{9\Delta}{10}$-long $(Q, 1/N)$-quasigeodesics with mesh $\leq 1/N$ in a $\frac{1}{N}$-hyperbolic spaces, with starting points at distance $\leq \Delta/10$ from each other, and ending points at distance $\leq \Delta/10$ from each other, necessarily lie at distance $(m_N/10)$ from one another.

Let then $X_N, H_N, \mathcal{Y}_N$ be a counterexample to our claim for these values: for each N there is a_N, b_N in $H^{+\Delta}_N$, R-close to each other for d_{H_N}, and a point $c_N \in H^{+\Delta}_N$ in a coarse geodesic in $[a_N, b_N]_{d_{H_n}}$ at distance at least $(\Delta \times C^{el}_H)$ from a geodesic

$[a_N, b_N]$ in X_N. However c_N is C_H^{el}-close to a geodesic $[a_N, b_N]_{el}$ in X^{el}. Passing to an asymptotic cone, we find a map p_ω from an interval to a continuous path from a^ω to b^ω (which can be equal to a^ω) that passes through a point c^ω at distance $\geq (\Delta \times C_H^{el})$ from the arc $[a^\omega, b^\omega]$ of the $\mathbb{R}$-tree X^ω. However, it is at distance $\leq C_H^{el}$ in the electrification of X^ω by $\mathcal{Y}^\omega$.

It follows that on the path from $[a^\omega, b^\omega]$ to c^ω, there must exist a segment of length $\geq \Delta$ belonging to the same $Y^\omega \in \mathcal{Y}^\omega$. Let us say that Y^ω is the limit of a sequence Y_N. Note that the limit path p_ω crosses at least twice this segment (once in either direction).

Thus, for N large enough, H_N (Δ, ϵ)-meets Y_N, with two pairs of meeting points $(r_1, r_2), (s_1, s_2)$ in H_N, where $d(r_1, r_2) \geq 9\Delta/10$ (and $(s_1, s_2) \geq 9\Delta/10$) and $d(r_1, s_1) \leq 3\Delta/10$ and $d(r_2, s_2) \leq 3\Delta/10$. By assumption, there is a $(Q, \frac{1}{N})$-quasi-geodesic path in $H^{+\epsilon\Delta}$ from s_1 to s_2 and another from r_1 to r_2, with mesh $< m_N/10$. They thus have to fellow travel on a large subpath, and pass at distance $\leq m_N/10$ from each other, by choice of m_N. One can therefore find a shortcut of p_n that is still a path in $H^{+\Delta}$ of mesh $\leq m_N$, a contradiction. □

From the claim, we can prove the statement of the Proposition.

Consider a situation as in the statement. We may choose the coarse path connection constant of H to be more than 10 times the quasigeodesic constant of the last assumption.

Take ϵ, given by the assumption of the Proposition, and $\Delta > \max\{100\epsilon, \Delta_0\}$. Let Q be the quasi-geodesic constants given by the the assumption of the proposition on (Δ, ϵ)-meetings, and C_H^{el} as given by the assumption.

Take R larger (it will be made clear how large in the proof).

Rescale the space X so that the hyperbolicity constant, the quasiconvexity constant of items of $\mathcal{Y}$, and the constant of coarse path connection of H are respectively smaller than δ_0, C_0, m_0 from the Lemma.

Note that the assumption of the proposition on (Δ, ϵ)-meetings is invariant under rescaling (except for the value of Δ_0). Thus, this assumption still holds, with the same ϵ, and for the specified Δ chosen above.

Then the Lemma applies, and $H^{+\epsilon\Delta}$ is R-locally quasi-convex for the rescaled metric. By the local to global principle (in δ_0 hyperbolic spaces), with a suitable preliminary choice of R it is then globally quasiconvex. After rescaling back to the original metric of X, $H^{+\lambda}$ is still quasiconvex for some λ (depending on $\epsilon\Delta$, and the coefficient of rescaling), hence H is quasiconvex.

By construction, we also have the statement on the dependency of the quasiconvexity constant. □

2.5.4. *Coarse hyperbolic embeddedness and Strong Relative Hyperbolicity.* The following Proposition establishes the equivalence of Coarse hyperbolic embeddedness and Strong Relative Hyperbolicity in the context of this paper.

Proposition 2.18. *Assume that (X, d) is a metric space, and that $\mathcal{Y}$ is a collection of subspaces.*

If the horoballification $X^h_{\mathcal{Y}}$ of X over $\mathcal{Y}$ is hyperbolic, then $\mathcal{Y}$ is coarsely hyperbolically embedded in the sense of spaces.

If X is hyperbolic and if $\mathcal{Y}$ is coarsely hyperbolically embedded in the sense of spaces, then $X^h_{\mathcal{Y}}$ is hyperbolic.

We remark here parenthetically that the converse should be true without the assumption of hyperbolicity of X. However, this is not necessary for this paper.

Proof. Assume that the horoballification $X^h_{\mathcal{Y}}$ of X over $\mathcal{Y}$ is $\delta-$hyperbolic. The horoballs Y^h (corresponding to Y) are thus 10δ-quasi-convex. Therefore, by Proposition 2.10, the electrified space obtained by electrifying (coning off) the horoballs Y^h is hyperbolic.

Since by Proposition 2.8, this space is quasi-isometric to $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$, it follows that the later is hyperbolic. This proves the first condition of Definition 2.2.

We want to prove the existence of a proper increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, such that the angular metric at each cone point v_Y (for $Y \in \mathcal{Y}$) of $(X^{el}_{\mathcal{Y}}, d^{el}_{\mathcal{Y}})$ is bounded below by $\psi \circ d|_Y$. Define

$$\psi(r) = \inf_{Y\in\mathcal{Y}} \inf_{y_1,y_2\in Y, d(y_1,y_2)\leq r} \hat{d}(y_1, y_2).$$

Of course, the angular metric at v_Y is bounded below by $\psi \circ d|_Y$. The function ψ is obviously increasing. We need to show that it is proper, i.e. that it goes to $+\infty$.

If ψ is not proper, then there exists $\theta_0 > 0$ such that for all D, there exist $Y \in \mathcal{Y}$ and $y, y' \in Y$ at $d-$distance greater than D but $\hat{d}(y, y') \leq \theta_0$ (where $\hat{d}$ is the angular metric on Y). We choose $D >> \theta_0\delta$ (for instance $D = \exp(100(\theta_0 + 1)(\delta + 1))$).

Consider a path in $(X^h_{\mathcal{Y}})^{el}$ of length less than θ_0 from y to y' avoiding the cone point of Y. Because $D >> \theta_0$, this path has to pass through other cone points. It can thus be chosen as a concatenation of $N + 1$ geodesics whose vertices are y, y' and some cone points $v_1, \cdots, v_N$ (corresponding to $Y_1, \cdots Y_N$ with $N < \theta_0$). Adjoining the (geodesic) path $[y, v_Y] \cup [v_Y, y']$ (where v_Y corresponds to the cone point for Y), we thus have a geodesic $(N + 2)$-gon σ. Next replace each passage of σ through a cone point (v_i or v_Y) in $X^{el}_{\mathcal{Y}}$ by a geodesic (μ_i or μ_Y respectively) in the corresponding horoball (Y^h_i or Y^h respectively) in $X^h_{\mathcal{Y}}$ to obtain a geodesic $(2N+2)$-gon P in $X^h_{\mathcal{Y}}$. The geodesic segments μ_i or μ_Y comprise $(n+1)$ alternate sides of this geodesic $(2N + 2)$-gon.

Since $X^h_{\mathcal{Y}}$ is $\delta-$hyperbolic, it follows that the mid-point m of μ_Y is at distance $\leq (2N + 2)\delta$ from another edge of P. Note that m is in the horoball of Y, and because the distance in Y between y and y' is larger than D, we have that $d^h(m, Y)$ is at least $\log(D)/2$.

Since no other edge of P enters the horoball Y^h, this forces $\log(D)$ (and hence D) to be bounded in terms of θ_0 and δ: $D \leq \exp(4(N + 1)\delta)$. Since $N \leq \theta_0$, this is a contradiction with the choice of D. We can conclude that ψ is proper, and we have the first statement.

Let us consider the second statement. If X is hyperbolic and if $\mathcal{Y}$ is coarsely hyperbolically embedded in the sense of spaces, then elements of $\mathcal{Y}$ are uniformly quasiconvex in (X, d) by 2.7, and, by the property of the angular distance on any $Y \in \mathcal{Y}$, they are mutually cobounded. The statement then follows by Proposition 2.10. □

3. Algebraic Height and Intersection Properties

3.1. **Algebraic Height.** We recall here the general definition for height of finitely many subgroups.

Definition 3.1. *Let G be a group and $\{H_1, \cdots H_m\}$ be a finite collection of subgroups. Then the* **algebraic height** *of this collection is n if $(n+1)$ is the smallest number with the property that for any $(n+1)$ distinct left cosets $g_1 H_{\alpha_1}, \ldots, g_{n+1} H_{\alpha_{n+1}}$, the intersection $\bigcap_{1 \leq i \leq n+1} g_i H_{\alpha_i} g_i^{-1}$ is finite.*

We shall describe this briefly by saying that algebraic height is the largest n for which the intersection of n **essentially distinct** conjugates of $H_1, \cdots H_m$ is infinite. Here 'essentially distinct' refers to the cosets of $H_1, \cdots H_m$ and not to the conjugates themselves.

For hyperbolic groups, one of the main Theorems of [GMRS97] is the following:

Theorem 3.2. [GMRS97] *Let G be a hyperbolic group and H a quasiconvex subgroup. Then the algebraic height of H is finite. Further, there exists R_0 such that if $H \cap gHg^{-1}$ is infinite, then g has a double coset representative with length at most R_0.*

The same conclusions hold for finitely many quasiconvex subgroups $\{H_1 \cdots H_n\}$ of G.

We quickly recall a proof of Theorem 3.2 for one subgroup H in order to generalize it to the context of mapping class groups and $Out(F_n)$.

Proof. Let G be hyperbolic, $X (= \Gamma_G)$ a Cayley graph of G with respect to a finite generating set (assumed to be $\delta-$hyperbolic), and H a C_0-quasiconvex subgroup of G. Suppose that there exist N essentially distinct conjugates $\{H^{g_i}\}, i = 1 \cdots N$, of H that intersect in an infinite subgroup. The N left-cosets $g_i H$ are disjoint and share an accumulation point p in the boundary of G (in the limit set of $\cap_i H^{g_i}$). Since all $g_i H$ are C_0-quasi-convex, there exist N disjoint quasi-geodesics $\sigma_1, \cdots, \sigma_N$ (with same constants λ, μ depending only on C_0, δ) converging to p. Since X is $\delta-$hyperbolic, there exists $R (= R(\lambda, \mu, \delta) = R(C_0, \delta))$ and a point p_0 sufficiently far along σ_1 such that all the quasi-geodesics $\sigma_1, \cdots, \sigma_N$ pass through $B_R(p_0)$. Hence $N \leq \#(B_R(p_0))$ giving us finiteness of height.

Further, any such σ_i furnishes a double coset representative g_i' of g_i (say by taking a word that gives the shortest distance between H and the coset $g_i H$) of length bounded in terms of R. This furnishes the second conclusion. □

Remark 3.3. A word about generalizing the above argument to a family $\mathcal{H}$ of finitely many subgroups is necessary. The place in the above argument where $\mathcal{H}$ consists of a singleton is used essentially is in declaring that the N left-cosets $g_i H$ are disjoint. This might not be true in general (e.g. $H_1 < H_2$ for a family having two elements). However, by the pigeon-hole principle, choosing N_1 large enough, any N_1 distinct conjugates $\{H_i^{g_i}\}, i = 1 \cdots N, H_i \in \mathcal{H}$ must contain N essentially distinct conjugates $\{H^{g_i}\}, i = 1 \cdots N$ of some $H \in \mathcal{H}$ and then the above argument for a single $H \in \mathcal{H}$ goes through.

Remark 3.4. A number of other examples of finite algebraic height may be obtained from certain special subgroups of Relatively Hyperbolic Groups, Mapping Class Groups and Out(F_n). These will be discussed after we introduce geometric height later in the paper.

3.2. **Geometric i-fold intersections.** Given a finite family of subgroups of a group we define collections of geometric $i-$fold intersections.

Definition 3.5. *Let G be endowed with a left invariant word metric d. Let $\mathcal{H}$ be a family of subgroups of G. For $i \in \mathbb{N}, i \geq 2$, define $\mathcal{H}_i$ to be the set of subsets J of G for which there exist $H_1, \ldots, H_i \in \mathcal{H}$ and $g_1, \ldots, g_i \in G$, $\Delta, C_0 \geq 0$ satisfying:*

- *the cosets $g_j H_j$ are pairwise distinct*
- *J is the intersection $\bigcap_j (g_j H_j)^{+\Delta}$.*
- **Minimal thickening condition:** $\Delta = \inf\{D : \mathrm{Diam}_{(G,d)}(J) \geq 10(D+1)\} + C_0$ *and so* $\mathrm{Diam}_{(G,d)}(J) \geq 10(\Delta + 1)$.

We shall call $\mathcal{H}_i$ the family of **geometric $i-$fold intersections** *with parameter C_0 or $i-$fold C_0-intersections for short.*

Definition 3.5 is somewhat technical in nature. Let us try to explain where the various conditions come from in analogy with the previous algebraic case. Notice first that when a collection of i-essentially disjoint conjugates of H intersect along a subgroup of infinite diameter, then there is a Δ such that the Δ-neighborhoods of the associated left cosets intersect in an infinite diameter set, which is in a bounded neighborhood of the intersection of the conjugates. We choose here to (partly) geometrize the situation. Thus we do not work directly with the intersection of the conjugates, but the intersection of thickenings of cosets. This way, we may also find intersections that do not correspond to subgroups (in proper settings, though, they will never be very far from subgroups, but we will also work in non proper settings). In order to discard trivial meaningless situations (since any large enough thickenings will always intersect) we add the ad-hoc condition that the diameter of the intersection must be ten times the magnitude of the thickening. This is one of the features ensured by the *minimal thickening* condition. The *minimal thickening* condition also avoids counting several times the same intersection of thickening. Indeed, if an intersection of Δ-thickenings of H_1 and H_2 is of infinite diameter, then the intersection of the $(\Delta + k)$-thickening would still be of infinite diameter. We would not want to count all these intersections separately. Hence we select the relevant intersection among them by taking Δ minimal. Elementary precaution on its existence tells us to take the infimum plus 1.

Remark 3.6. In order to establish minimality, we could have imposed in addition the following *stabilization condition* that ensures that 'plusification' of intersections is (coarsely) intersection of plusifications:

S: (Stabilization condition) There exists $C_1 \geq 0$ such that $\bigcap_j (g_j H_j)^{+D+k} \subset (\bigcap_j (g_j H_j)^{+D})^{+k+C_0}$ for all $D \geq C_1$.

It is conceivable that in our geometric situation, we have some thickening for which an intersection is infinite but not maximal: for instance two quasiconvex subsets might intersect along a line, while their thickenings could intersect along a whole tree. The stabilization condition prevents some pathologies and hence we need an extra parameter C_1.

We shall see in Corollary 3.8 below that this Condition automatically follows in the situation that (G, d) is hyperbolic with respect to a (not necessarily finite) generating set and $H_1, \cdots, H_n$ are quasiconvex subgroups. This renders the parameter C_0 in Definition 3.5 unimportant provided it is sufficiently large.

Let (G, d) be a group with a word metric and $H < G$ a subgroup. The subspace metric on H induced from (G, d) will be called the **induced metric** on H from G.

Proposition 3.7. *Let (G,d) be a group with a δ-hyperbolic word metric (not necessarily locally finite).*

Assume that $A_1,\dots,A_n$ are C-quasiconvex subsets of G. Then for all $\Delta > C+20\delta$, $\bigcap A_i^{+\Delta}$ is (4δ)-quasiconvex in (G,d).

Moreover, if A and B are C-quasiconvex subsets of G, and if $\Pi_B(A)$ denotes the set of nearest points projections of A on B, then, either $\Pi_B(A) \subset A^{+3C+10\delta}$ or $\mathrm{Diam}\Pi_B(A) \le 4C+20\delta$.

Proof. Consider $x,y \in A^{+\Delta} \cap B^{+\Delta}$ and take a_i, b_i some nearest point projection on A_i. On a geodesic $[x,y]$ take p at distance greater than 4δ from x and y. Hyperbolicity applied to the quadrilateral (x,a_i,a_i,y) tells us that x is 4δ-close to $[x'_i,a_i]\cup[a_i,b_i]\cup[b_i,y'_i]$, where x'_i and y'_i are the points of, respectively $[x,a_i]$ and $[y,b_i]$, at distance 4δ from, respectively, x and y.

Let us call $[x'_i,a_i],[b_i,y'_i]$ the approaching segment, and $[a_i,b_i]$ the traveling segments. Hence for each i, p is closed to either an approaching segment, or the traveling segment, with subscript i.

If p is close to an approaching segment of index i, then it is in $A_i^{+\Delta}$.

If x is close to the traveling segment of index i, then it is at distance at most $C+10\delta$ from A_i, hence in $A_i^{+\Delta}$ because $\Delta > C+10\delta$.

We thus obtain that $[x,y]$ remains at distance 4δ from $A^{+\Delta}\cap B^{+\Delta}$.

To prove the second statement, take a_0, b_0 in A and B respectively realizing the distance (up to δ if necessary). Let $b \in \Pi_B(A)$, and assume that it is the projection of a. In the quadrilateral a,a_0,b,b_0, the geodesic $[a,a_0]$ stays in A^{+C} and $[b,b_0]$ is in B^{+C}. Since b is a projection, $[b,a]$ fellow-travels $[b,b_0]$ for less than $2C+10\delta$, and similarly for $[b_0,a_0]$ with $[b_0,b]$. By hyperbolicity $[b,b_0]$ thus stays 10δ close to $[a,a_0]$ except for the part $(2C+10\delta)$-close to either b or b_0. It follows that either $b \in A^{+(3C+10\delta)}$ or b is at distance $\le 4C+20\delta$ from b_0. Thus $\Pi_B(A)\subset A^{+3C+10\delta}$ or $\mathrm{Diam}\Pi_B(A)\le 4C+20\delta$.

□

Choosing $C_1 = 4C+20\Delta$ and $C_0 \ge 4\Delta$ we have the following immediate corollary:

Corollary 3.8. *Let (G,d) be a group with a δ-hyperbolic word metric (not necessarily locally finite). Assume that $H_1,\dots,H_n$ are C-quasiconvex subgroups of G. Then, for $C_1 = 4C+20\delta$ we have*

$$\bigcap_j (g_jH_j)^{+D+k} \subset \Big(\bigcap_j (g_jH_j)^{+D}\Big)^{+k+4\delta},$$

for all $D \ge C_1$

In short, Condition **S:**, *the* **(Stabilization condition)** *in Remark 3.6 is automatic.*

Remark 3.9. *In the rest of the paper we shall mostly be interested in the setup where (G,d) is a group with a δ-hyperbolic word metric (not necessarily locally finite) and $H_1,\dots,H_n$ are C-quasiconvex subgroups of G. We reiterate that Corollary 3.8 ensures that the constants C_0, C_1 as parameters in Definition 3.5 of geometric $i-$fold intersections are unimportant provided they are sufficiently large.*

3.3. **Algebraic i-fold intersections.** We provide now a more algebraic (group theoretic) treatment of the preceding discussion. This is in keeping with the more well-known setup of intersections of subgroups and their conjugates cf. [GMRS97]. Given a finite family of subgroups of a group we first define collections of $i-$fold conjugates or algebraic i-fold intersections.

Definition 3.10. *Let G be endowed with a left invariant word metric d. Let $\mathcal{H}$ be a family of subgroup of G. For $i \in \mathbb{N}, i \geq 2$, define $\mathcal{H}_i$ to be the set of subgroups J of G for which there exists $H_1, \ldots, H_i \in \mathcal{H}$ and $g_1, \ldots, g_i \in G$ satisfying:*

- *the cosets $g_j H_j$ are pairwise distinct (and hence as in* [GMRS97] *we use the terminology that the conjugates $\{g_j H_j g_j^{-1}, j = 1, \ldots, i\}$ are* **essentially distinct***)*
- *J is the intersection $\bigcap_j g_j H_j g_j^{-1}$.*
- *J is unbounded in (G, d).*

We shall call $\mathcal{H}_i$ the family of **algebraic i-fold intersections** *or simply, $i-$***fold conjugates**.

The second point in the following definition is motivated by the behavior of nearest point projections of cosets of quasiconvex subgroups of hyperbolic groups on each other. Let (G, d) be hyperbolic and H_1, H_2 be quasiconvex. Let aH_1, bH_2 be cosets and $c = a^{-1}b$. Then the nearest point projection of bH_2 onto aH_1 is the (left) $a-$translate of the nearest point projection of cH_2 onto H_1. Let $\Pi_B(A)$ denote the (nearest-point) projection of A onto B. Then $\Pi_{H_1}(cH_2)$ lies in a bounded neighborhood (say D_0-neighborhood) of $H_2^c \cap H_1$ and so $\Pi_{aH_1} bH_2$ lies in a D_0-neighborhood of $bH_2 b^{-1} a \cap aH_1$. The latter does lie in a bounded neighborhood of $(H_2)^b \cap (H_1)^a$, but this bound depends on a, b and is not uniform. Hence the somewhat convoluted way of stating the second property below. The language of nearest-point projections below is in the spirit of [Mj11, Mj14] while the notion of geometric i-fold intersections discussed earlier is in the spirit of [DGO11].

Definition 3.11. *Let G be a group and d a word metric on G.*

A finite family $\mathcal{H} = \{H_1, \cdots, H_m\}$ of subgroups of G, each equipped with a word-metric d_i is said to have the **uniform qi-intersection property** *if there exist $C_1, \cdots, C_n, \cdots$ such that*

1. *For all n, and all $H \in \mathcal{H}_n$, H has a conjugate H' such that if d' is any* **induced metric** *on H' from some $H_i \in \mathcal{H}$, then (H', d') is $(C_1, C_1)-$qi-embedded in (G, d).*
2. *For all n, let $(\mathcal{H}_n)_0$ be a choice of conjugacy representatives of elements of $\mathcal{H}_n$ that are C_1-quasiconvex in (G, d). Let $\mathcal{CH}_n$ denote the collection of left cosets of elements of $(\mathcal{H}_n)_0$.*

   For all $A, B \in \mathcal{CH}_n$ with $A = aA_0, B = bB_0$, and $A_0, B_0 \in (\mathcal{H}_n)_0$, $\Pi_B(A)$ either has diameter bounded by C_n for the metric d, or $\Pi_B(A)$ lies in a (left) $a-$translate translate of a C_n-neighborhood of $A_0 \cap B_0^c$, where $c = a^{-1}b$.

In keeping with the spirit of the previous subsection, we provide a geometric version of the above definition below.

Definition 3.12. *Let G be a group and d a word metric on G.*

A finite family $\mathcal{H} = \{H_1, \cdots, H_m\}$ of subgroups of G, each equipped with a word-metric d_i is said to have the **uniform geometric qi-intersection property** *if there exist $C_1, \cdots, C_n, \cdots$ such that*

(1) *For all n, and all $H \in \mathcal{H}_n$, (H, d) is C_n-coarsely path connected, and $(C_1, C_1)-$qi-embedded in (G, d) (for its coarse path metric).*
(2) *For all $A, B \in \mathcal{H}_n$ either* $\mathrm{diam}_{G,d}(\Pi_B(A)) \leq C_n$, *or* $\Pi_B(A) \subset A^{+C_n}$ *for d.*

Remark 3.13. The second condition of Definition 3.12 follows from the first condition if d is hyperbolic by Proposition 3.7. Further, the first condition holds for such (G, d) so long as Δ is taken of the order of the quasiconvexity constants (again by Proposition 3.7).

Note further that if G is hyperbolic (with respect to a not necessarily locally finite word metric) and H is C-quasiconvex, then for any parameter $C_0 > C + 20\delta$, the collection of geometric n-fold intersections $\mathcal{H}_n$ is mutually cobounded for the metric of $(G, d)^{el}_{\mathcal{H}_{n+1}}$ (as in Definition 2.9).

3.4. **Existing results on algebraic intersection properties.** We start with the following result due to Short.

Theorem 3.14. [Sho91, Proposition 3] *Let G be a group generated by the finite set S. Suppose G acts properly on a uniformly proper geodesic metric space (X, d), with a base point x_0. Given C_0, there exists C_1 such that if H_1, H_2 are subgroups of G for which the orbits $H_i x_0$ are C_0-quasiconvex in (X, d) (for $i = 1, 2$) then the orbit $(H_1 \cap H_2)x_0$ is C_1-quasiconvex in (X, d).*

We remark here that in the original statement of [Sho91, Proposition 3], X is itself the Cayley graph of G with respect to S, but the proof there goes through without change to the general context of Proposition 3.14.

In particular, for G (Gromov) hyperbolic, or $G = Mod(S)$ acting on Teichmuller space $Teich(S)$ and $Out(F_n)$ acting on Outer space cv_N, the notions of (respectively) quasiconvex subgroups or convex cocompact subgroups of $Mod(S)$ or $Out(F_n)$ are independent of the finite generating sets chosen. Hence we have the following.

Theorem 3.15. *Let G be either $Mod(S)$ or $Out(F_n)$ equipped with some finite generating set. Given C_0, there exists C_1 such that if H_1, H_2 are C_0-convex cocompact subgroups of G, then $H_1 \cap H_2$ is C_1-convex cocompact in G.*

The corresponding statement for relatively hyperbolic groups and relatively quasiconvex groups is due to Hruska. For completeness we recall it.

Definition 3.16. [Osi06b, Hru10] *Let G be finitely generated hyperbolic relative to a finite collection $\mathcal{P}$ of parabolic subgroups. A subgroup $H \leq G$ is* **relatively quasiconvex** *if the following holds.*
Let S be some (any) finite relative generating set for $(G, \mathcal{P})$, and let P be the union of all $P_i \in \mathcal{P}$. Let $\overline{\Gamma}$ denote the Cayley graph of G with respect to the generating set $S \cup P$ and d the word metric on G. Then there is a constant $C_0 = C_0(S, d)$ such that for each geodesic $\gamma \subset \overline{\Gamma}$ joining two points of H, every vertex of γ lies within C_0 of H (measured with respect to d).

Theorem 3.17. [Hru10] *Let G be finitely generated hyperbolic relative to $\mathcal{P}$. Given C_0, there exists C_1 such that if H_1, H_2 are C_0-relatively quasiconvex subgroups of G, then $H_1 \cap H_2$ is C_1-relatively quasiconvex in G.*

4. Geometric height and graded geometric relative hyperbolicity

We are now in a position to define the geometric analog of height. There are two closely related notions possible, one corresponding to the geometric notion of $i-$fold intersections and one corresponding to the algebraic notion of $i-$fold conjugates. The former is relevant when one deals with subsets and the latter when one deals with subgroups.

Definition 4.1. *Let G be a group, with a left invariant word metric $d(= d_G)$ with respect to some (not necessarily finite) generating set. Let $\mathcal{H}$ be a family of subgroups of G.*

The **parametrized geometric height for the parameter** C_0, *(or C_0-geometric height) of $\mathcal{H}$ in (G, d) (for d) is the minimal number $i \geq 0$ so that the collection $\mathcal{H}_{i+1}$ of $(i + 1)-$fold C_0-intersections consists of uniformly bounded sets.*

The **unparametrized geometric height**, *of $\mathcal{H}$ in (G, d) (for d) is the minimal number $i \geq 0$ so that the collection $\mathcal{H}_{i+1}$ of $(i + 1)-$fold conjugates consists of uniformly bounded sets.*

If H is a single subgroup, its geometric height is that of the family $\{H\}$.

Remark 4.2. *As noted before, the parameter C_0 will be irrelevant for us provided it is sufficiently large, so long as G is hyperbolic (with respect to a not necessarily finite generating set) and $\mathcal{H}$ consists of quasiconvex subgroups.*

Remark 4.3. Comparing notions of height:

- *Unparametrized geometric height is related to algebraic height, but is more flexible, since in the former, we allow the group G to have an infinite generating set. We are then free to apply the operations of electrification, horoballification in the context of non-proper graphs.*
- *In the case of a locally finite word metric, unparametrized geometric height and algebraic height coincide.*
- *Since $i-$fold conjugates (occurring in the notion of unparametrized geometric height) are necessarily contained in $i-$fold intersections (occurring in the notion of parametrized geometric height), one always has that the unparametrized geometric height is less than the parametrized geometric height for sufficiently large parameters. In particular, finiteness of parametrized geometric height for all large parameters implies finiteness of unparametrized geometric height.*
- *Next, note that finiteness of algebraic height implies that $i-$fold conjugates are finite (and hence bounded in any metric) for all sufficiently large i. Hence finiteness of unparametrized geometric height also follows from finiteness of algebraic height.*

We generalize Definition 1.3 of graded relatively hyperbolicity to the context of geometric height as follows. There are two versions, one parametrized, relevant for subsets, the other unparametrized (but still geometric), relevant for subgroups.

Definition 4.4. *Let G be a group, d the word metric with respect to some (not necessarily finite) generating set and $\mathcal{H}$ a finite collection of subgroups.*

- *Let $\mathcal{H}_i$ be the collection of all $i-$fold intersections of $\mathcal{H}$. Let d_i be the metric on (G, d) obtained by electrifying the elements of $\mathcal{H}_i$. Let $\mathcal{H}_\mathbb{N}$ be the graded family $(\mathcal{H}_i)_{i\in\mathbb{N}}$.*

We say that G is parametrized **graded geometric relatively hyperbolic** *with respect to $\mathcal{H}_{\mathbb{N}}$ for the parameter C_0, (or equivalently that $(G, \mathcal{H}_{\mathbb{N}}, d)$ has C_0-graded geometric relative hyperbolicity) if*

(1) *$\mathcal{H}$ has parametrized geometric height n for some $n \in \mathbb{N}$.*
(2) *For all $i \leq n+1$, $\mathcal{H}_{i-1}$ is coarsely hyperbolically embedded in (G, d_i).*
(3) *There is D_i such that all items of $\mathcal{H}_i$ are D_i-coarsely path connected in (G, d).*

- *Let $\mathcal{H}_i$ be the collection of all $i-$fold conjugates of $\mathcal{H}$. Let $(\mathcal{H}_i)_0$ be a choice of conjugacy representatives, and $\mathcal{CH}_i$ the set of left cosets of elements of $(\mathcal{H}_i)_0$ Let d_i be the metric on (G, d) obtained by electrifying the elements of $\mathcal{CH}_i$. Let $\mathcal{CH}_{\mathbb{N}}$ be the graded family $(\mathcal{CH}_i)_{i \in \mathbb{N}}$.*

  We say that G is unparametrized **graded geometric relatively hyperbolic** *with respect to $\mathcal{CH}_{\mathbb{N}}$ if*

  (1) *$\mathcal{H}$ has unparametrized geometric height n for some $n \in \mathbb{N}$.*
  (2) *For all $i \leq n+1$, $\mathcal{CH}_{i-1}$ is coarsely hyperbolically embedded in (G, d_i).*
  (3) *There is D_i such that all items of $\mathcal{CH}_i$ are D_i-coarsely path connected in (G, d).*

Remark 4.5. Comparing geometric and algebraic graded relative hyperbolicity:
Note that, in both cases, the second condition of Definition 4.4 is equivalent, by Proposition 2.18, to saying that (G, d_i) is strongly hyperbolic relative to the collection $\mathcal{H}_{i-1}$. This is exactly the third (more algebraic) condition in Definition 1.3. Also, again in both cases, the third condition of Definition 4.4 is the analog of (and follows from) the second (more algebraic) condition in Definition 1.3.

Thus finite (parametrized or unparametrized) geometric height along with (algebraic) graded relative hyperbolicity implies (parametrized or unparametrized) graded geometric relative hyperbolicity.

Finally, note that if d is locally finite, the unparametrized graded geometric relative hyperbolicity is nothing else than the graded relative hyperbolicity of Definition 1.3.

The rest of this section furnishes examples of finite height in both its geometric and algebraic incarnations.

4.1. Hyperbolic groups.

Proposition 4.6. *Let (G, d) be a hyperbolic group with a locally finite word metric, and let H be a quasi-convex subgroup of G. Then H has finite (parametrized or unparametrized) geometric height.*

More precisely, if C be the quasi-convexity constant of H in (G, d), and if δ be the hyperbolicity constant in (G, d), and if N is the cardinality of a ball of (G, d) of radius $2C + 10\delta$, and if $g_0H, \ldots g_kH$ are distinct cosets of H for which there exists Δ such that the total intersection $\bigcap_{i=0}^{k} (g_iH)^{+\Delta}$ has diameter more than 10Δ, and more that 100δ, then there exists $x \in G$ such that each g_iH intersects the ball of radius N around x.

First note that the second statement implies the first in the parametrized version, for all parameters, and also (by the third point of Remark 4.3), the unparametrized version. We will directly prove the second. The proof is similar to the finiteness of the algebraic height. Also note that the second statement can be rephrased in terms of double cosets representatives of the g_i: under the assumption on the total intersection, and if $g_0 = 1$, there are double coset representatives of the g_i of length at most $2(2C + 10\delta)$.

Proof. Assume that there exists $\Delta > 0$, and elements $1 = g_0, g_1, \dots, g_k$ for which the cosets $g_i H$ are distinct, and $\bigcap_{i=0}^{k} (g_i H)^{+\Delta}$ has diameter larger than 10Δ and than 100δ.

First we treat the case $\Delta > 5\delta$.

Pick $y_1, y_2 \in \bigcap_{i=0}^{k} (g_i H)^{+\Delta}$ at distance 10Δ from each other, and pick $x \in [y_1, y_2]$ at distance larger than $\Delta + 10\delta$ from both y_i. For each i an application of hyperbolicity and quasi-convexity tells us that x is at distance at most $2C + 10\delta$ from each of $g_i H$. The ball of radius $2C + 10\delta$ around x thus meets each coset $g_i H$.

If $\Delta \leq 5\delta$, we pick $y_1, y_2 \in \bigcap_{i=0}^{k} (g_i H)^{+\Delta}$ at distance 100δ from each other, and take x at distance greater than 10δ from both ends. The end of the proof is the same.

□

4.2. **Relatively hyperbolic groups.** If G is hyperbolic relative to a collection of subgroups $\mathcal{P}$, then Hruska and Wise defined in [HW09] the relative height of a subgroup H of G as n if $(n+1)$ is the smallest number with the property that for any $(n+1)$ elements $g_0, \dots, g_n$ such that the $g_i H$ are $(n+1)$- distinct cosets, the intersections of conjugates $\bigcap_i g_i H g_i^{-1}$ is finite or parabolic.

The notion of relative algebraic height is actually the unparametrized geometric height for the relative distance, which is given by a word metric over a generating set that is the union of a finite set and a set of conjugacy representatives of the elements of $\mathcal{P}$. Indeed, in a relatively hyperbolic group, the subgroups that are bounded in the relative metric are precisely those that are finite or parabolic.

The notion of relative height can actually be extended to define the height of a collection of subgroups $H_1, \dots, H_k$, as in the case for the algebraic height.

Hruska and Wise proved that relatively quasiconvex subgroups have finite relative height. More precisely:

Theorem 4.7. [HW09, Theorem 1.4, Corollaries 8.5-8.7] *Let $(G, \mathcal{P})$ be relatively hyperbolic, let S be a finite relative generating set for G and Γ be the Cayley graph of G with respect to S. Then for $\sigma \geq 0$, there exists $C \geq 0$ such that the following holds.*
Let $H_1, \cdots, H_n$ be a finite collection of $\sigma-$relatively quasiconvex subgroups of $(G, \mathcal{P})$. Suppose that there exist distinct cosets $\{g_m H_{\alpha_m}\}$ with $\alpha_m \in \{1, \cdots, n\}$, $m = 1 \cdots n$, such that $\cap_m g_m H_{\alpha_m} g_m^{-1}$ is not contained in a parabolic $P \in \mathcal{P}$. Then there exists a vertex $z \in G$ such that the ball of radius C in Γ intersects every coset $g_m H_{\alpha_m}$.

Further, for any $i \in \{1, \cdots, n\}$, there are only finitely many double cosets of the form $H_i g_i H_{\alpha_i}$ such that $H_i \cap \bigcap_i g_i H_{\alpha_i} g_i^{-1}$ is not contained in a parabolic $P \in \mathcal{P}$.

Let G be a relatively hyperbolic group, and let H be a relatively quasiconvex subgroup. Then H has finite relative algebraic height.

This allows us to give an example of geometric height in our setting.

Proposition 4.8. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group, and (G, d) a relative word metric d (i.e. a word metric over a generating set that is the union of a finite set and of a set of conjugacy representatives of the elements of $\mathcal{P}$, and hence, in* *general, not a finite generating set). Let H be a relatively quasiconvex subgroup. Then, H has finite (parametrized or unparametrized) geometric height for d.*

The unparametrized case is just a rephrasing of Hruska and Wise's result Theorem 4.7. For the parametrized case, the proof is similar to that in the hyperbolic groups case, using for instance cones instead of balls.

4.3. **Mapping Class Groups.** Another source of examples arise from convex-cocompact subgroups of Mapping Class Groups, and of $Out(F_n)$. We establish finiteness of both algebraic and geometric height of convex cocompact subgroups of Mapping Class Groups in this subsubsection. $Teich(S)$ and $CC(S)$ will denote respectively the Teichmuller space and Curve Complex of S.

Definition 4.9. [FM02] *A finitely generated subgroup H of the mapping class group $Mod(S)$ for a surface S (with or without punctures) is $\sigma-$convex cocompact if for some (any) $x \in Teich(S)$, the Teichmuller space of S, the orbit $Hx \subset Teich(S)$ is $\sigma-$quasiconvex with respect to the Teichmuller metric.*

Kent-Leininger [KL08] and Hamenstadt [Ham08] prove the following:

Theorem 4.10. *A finitely generated subgroup H of the mapping class group $Mod(S)$ is convex cocompact if and only if for some (any) $x \in CC(S)$, the curve complex of S, the orbit $Hx \subset CC(S)$ is qi-embedded in $CC(S)$.*

One important ingredient in Kent-Leininger's proof of Theorem 4.10 is a lifting of the limit set of H in $\partial CC(S)$ (the boundary of the curve complex) to $\partial Teich(S)$ (the boundary of Teichmuller space). What is important here is that $Teich(S)$ is a proper metric space unlike $CC(S)$. Further, they show using a Theorem of Masur [Mas80], that any two Teichmuller geodesics converging to a point on the limit set Λ_H (in $\partial Teich(S)$) of a convex cocompact subgroup H are asymptotic. An alternate proof is given by Hamenstadt in [Ham10]. With these ingredients in place, the proof of Theorem 4.11 below is an exact replica of the proof of Theorem 3.2 above:

Theorem 4.11. *Let G be the mapping class group of a surface S. Then for $\sigma \geq 0$, there exists $C \geq 0$, and $D \geq 0$ such that the following holds.*

Let $H_1, \cdots, H_n$ be a finite collection of $\sigma-$convex cocompact subgroups of G. Suppose that there exist distinct cosets $\{g_m H_{\alpha_m}\}$ with $\alpha_m \in \{1, \cdots, n\}$, $m = 1 \cdots n$, such that, for some Δ, $\cap_m (g_m H_{\alpha_m})^{+\Delta}$ is larger than $\max\{10\Delta, D\}$. Then there exists a vertex $z \in Teich(S)$ such that the ball of radius C in $Teich(S)$ intersects every coset $g_m H_{\alpha_m}$.

Further, for any $i \in \{1, \cdots, n\}$, there are only finitely many double cosets of the form $H_i g_i H_{\alpha_i}$ such that $H_i \cap \bigcap_i g_i H_{\alpha_i} g_i^{-1}$ is infinite.

The collection $\{H_1, \cdots, H_n\}$ has finite algebraic height.

A more geometric strengthening of Theorem 4.11 can be obtained as follows using recent work of Durham and Taylor [DT14b], who have given an intrinsic quasi-convexity interpretation of convex cocompactness, by proving that convex cocompact subgroups of Mapping Class Groups are stable: in a word metric, they are undistorted, and quasi geodesics with end points in the subgroup remain close to each other [DT14b].

Theorem 4.12. *Let G be the mapping class group of a surface S and d the word metric with respect to a finite generating set. Then for $\sigma \geq 0$, and any subgroup*

H that is σ-quasiconvex, the group H has finite (parametrized or unparametrized) geometric height in (G, d).

Moreover, any σ-quasiconvex subgroup H has finite geometric height in (G, d_1), where d_1 is the word metric with respect to any (not necessarily finite) generating set.

Proof. Assume that the theorem is false: there exists σ such that for all k, and all D, there exists a σ-convex cocompact subgroup H, with a collection of distinct cosets $\{g_mH, m = 0, \dots k\}$ (with $g_0 = 1$), satisfying the property that $\cap_m(g_mH)^{+\Delta}$ has diameter larger than $\max\{10\Delta, D\}$.

Let a, b be two points in $\cap_m(g_mH)^{+\Delta}$ such that $d(a, b) \geq \max\{10\Delta, D\}$. For each i, let a_i, b_i in g_iH be at distance at most Δ from a and b respectively. Consider γ_i geodesics in H from $g_i^{-1}a_i$ to $g_i^{-1}b_i$. Consider also a'_i and b'_i- nearest point projections of a_0 and b_0 on $g_i\gamma_i$. Finally, denote by $g_i\gamma'_i$ the subpath of $g_i\gamma_i$ between a'_i and b'_i

By [DT14b, Prop. 5.7], H is quasi-convex in G (for a fixed chosen word metric), and for each i, $g_i\gamma_i$ is a $f(\sigma)$-quasi-geodesic (for some function f).

We thus obtain from a_0 to b_0 a family of $k+1$ paths, namely γ_0 and (one for all i), the concatenation $\eta_i = [a_0, a'_i] \cdot g_i\gamma'_i \cdot [b'_i, b_0]$.

For D large enough, the paths η_i are $2f(\sigma)$-quasigeodesics in G.

Stability of H ([DT14b, Thm. 1.1]) implies that, there exists $R(\sigma)$ such that in G, the paths remain at mutual Hausdorff distance at most $R(\sigma)$. This is thus also true in the Teichmuller space by the orbit map. Hence it follows that all the subpaths $g_i\gamma'_i$ are at distance at most $2R(\sigma)$ from each other, but are disjoint, and all lie in a thick part of the Teichmuller space, where the action is uniformly proper. This leads to a contradiction.

Since the diameter of intersections can only go down if the generating set is increased, the last statement follows. □

4.4. **Out**(F_n)**.** Following Dowdall-Taylor [DT14a], we say that a finitely generated subgroup H of $Out(F_n)$ is $\sigma-$**convex cocompact** if

(1) all non-trivial elements of H are atoroidal and fully irreducible.
(2) for some (any) $x \in cv_n$, the (projectivized) Outer space for F_n, the orbit $Hx \subset cv_n$ is $\sigma-$quasiconvex with respect to the Lipschitz metric.

The following Theorem gives a characterization of convex cocompact subgroups in this context and is the analog of Theorem 4.10.

Theorem 4.13. [DT14a] *Let H be a finitely generated subgroup of $Out(F_n)$ all whose non-trivial elements are atoroidal and fully irreducible. Then H is convex cocompact if and only if for some (any) $x \in \mathcal{F}_n$ (the free factor complex of F_n), the orbit $Hx \subset \mathcal{F}_n$ is qi-embedded in $\mathcal{F}_n$.*

Dowdall and Taylor also show [DT14a, Theorem 4.1] that any two quasi-geodesics in cv_n converging to the same point p on the limit set Λ_H (in ∂cv_n) of a convex cocompact subgroup H are asymptotic. More precisely, given λ, μ and $p \in \Lambda_H$ there exists $C_0 (= C_0(\lambda, \mu, p)$ such that any two $(\lambda, \mu)-$quasi-geodesics in cv_n converging to p are asymptotically C_0-close. As observed before in the context of Theorem 4.11, this is adequate for the proof of Theorem 4.11 to go through:

Theorem 4.14. *Let $G = Out(F_n)$. Then for $\sigma \geq 0$, there exists $C \geq 0$ such that the following holds.*

Let $H_1, \cdots, H_n$ be a finite collection of $\sigma-$convex cocompact subgroups of G. Suppose that there exist distinct cosets $\{g_m H_{\alpha_m}\}$ with $\alpha_m \in \{1, \cdots, n\}$, $m = 1 \cdots n$, such that $\cap_m g_m H_{\alpha_m} g_m^{-1}$ is infinite. Then there exists a point $z \in cv_n$ such that the ball of radius C in cv_n intersects every coset $g_m H_{\alpha_m}$.

Further, for any $i \in \{1, \cdots, n\}$, there are only finitely many double cosets of the form $H_i g_i H_{\alpha_i}$ such that $H_i \cap \bigcap_i g_i H_{\alpha_i} g_i^{-1}$ is infinite.

The collection $\{H_1, \cdots, H_n\}$ has finite algebraic height.

Since an analog of the stability result of [DT14b] in the context of $\mathrm{Out}(F_n)$ is missing at the moment, we cannot quite get an analog of Theorem 4.12.

4.5. **Algebraic and geometric qi-intersection property: Examples.** In the Proposition below we shall put parentheses around (geometric) to indicate that the statement holds for both the qi-intersection property as well as the geometric qi-intersection property.

Proposition 4.15.
(1) *Let H be a quasi-convex subgroup of a hyperbolic group G, endowed with a locally finite word metric. Then, $\{H\}$ satisfies the uniform (geometric) qi-intersection property.*
(2) *Let H be a relatively quasi-convex subgroup of a relatively hyperbolic group $(G, \mathcal{P})$. Let $\mathcal{P}_0$ be a set of conjugacy representatives of groups in $\mathcal{P}$, and d a word metric on G over a generating set $S = S_0 \cup \mathcal{P}_0$, where S_0 is finite. Then $\{H\}$ satisfies the uniform (geometric) qi-intersection property with respect to d.*
(3) *Let H be a convex-cocompact subgroup of the Mapping Class Group $Mod(\Sigma)$ of an oriented closed surface Σ of genus ≥ 2. If d is a word metric on $Mod(\Sigma)$ that makes it quasi-isometric to the curve complex of Σ, then H satisfies the uniform (geometric) qi-intersection property with respect to d.*
(4) *Let H be a convex-cocompact subgroup of $Out(F_n)$ for a some $n \geq 2$. If d is a word metric on $Out(F_n)$ that makes it quasi-isometric to the free factor complex of F_n, then H satisfies the uniform (geometric) qi-intersection property with respect to d.*

Proof. All four cases have similar proofs. Consider the first point.

Case 1: G hyperbolic, H quasiconvex.
The first conditions of Definition 3.11 and Definition 3.12 follow from Theorems 3.2, Proposition 4.6 and 3.14.

The second condition of Definition 3.12 follows from Proposition 3.7.

To prove the second condition of Definition 3.11, let A, B be in $\mathcal{H}_n$. Both are C'_n-quasiconvex in G.

We first note that, by Theorem 4.6, there exist finitely many double coset representatives g_i for which $H \cap g_i H g_i^{-1}$ is infinite. For each such representative g_i, there exists $C(g_i) \geq 0$ such that $\Pi_H(g_i H)$ lies in a $C(g_i)-$neighborhood of $H \cap g_i H g_i^{-1}$ (since G is hyperbolic). Let $C_1 = max_i C(g_i)$ (where the maximum is taken over a finite set). By equivariance, the projection of any element of $\{gH : g \in G\}$ on any other lies in a C_1-neighborhood of a translate of one of the (finitely many) $H \cap g_i H g_i^{-1}$'s. This proves the second point.

Case 2: G relatively hyperbolic, H relatively quasiconvex.

The first points of Definition 3.11 follows from Theorem 3.17 and the second point of Definition 3.11 follows as in Case 1 using Theorem 4.7.

Cases 3 and 4: $G = Mod(\Sigma)$ **or** $Out(F_n)$**,** H **convex cocompact.**
Case 3 and 4 are slightly different from Cases 1 and 2 in the sense that convex cocompactness is defined in terms of actions on non-hyperbolic spaces: $(Teich(\Sigma), d_T)$ and (cv_n, d_S) respectively for $Mod(\Sigma)$ and $Out(F_n)$, where d_T is the Teichmuller metric and d_S the (symmetrization of the) Lipschitz metric on Outer space (cf. [BF14]). Note however, that though $Teich(\Sigma)$ and cv_n are non-hyperbolic, they are proper metric spaces. On the other hand, in order to ensure that the nearest-point projection of gH on H lies in a bounded neighborhood of $H^g \cap H$ we required (in Case 1 above) that G be hyperbolic. This statement generalizes to the case when (G, d) is hyperbolic with respect to any (not necessarily finite) generating set.

The proof that follows is a hybrid. Most of the arguments in Cases 1 and 2 are done in proper spaces except for the preceding projection statement, which requires hyperbolicity. This statement will be used at the end of the proof.

For the mapping class group $Mod(\Sigma)$, let X be the curve complex $CC(S)$. Hyperbolicity of X was established by Masur-Minsky [MM99], and, as elucidated in [MM99], X is quasi-isometric to $(Mod(\Sigma), d)$, where d is the word-metric on $Mod(\Sigma)$ obtained by taking as generating set a finite generating set of $Mod(\Sigma)$ along with *all* elements of sub-mapping class groups. Similarly for $Out(F_n)$, let X be the free factor complex $\mathcal{F}_n$ Hyperbolicity of $\mathcal{F}_n$ was established by Bestvina-Feighn [BF14]). It was also explained in [BF14], that certain subgroups of $Out(F_n)$ may be electrified to obtain a word-metric quasi-isometric to $\mathcal{F}_n$. $Out(F_n)$ equipped with this word metric will be denoted for the purposes of this proof as $(Out(F_n), d)$. This establishes that the hypotheses in the statements of Cases 3 and 4 are not vacuous.

We now indicate the modifications necessary to the argument in Case 1. Finiteness of height of convex cocompact subgroups follows from Theorems 4.12 and 4.14 for $G = Mod(\Sigma)$ and $Out(F_n)$ respectively. The first condition of Definition 3.11 now follows from Theorems 3.15.

We now proceed with proving the second condition of Definition 3.11. The existence of finitely many double coset representatives g_i for which $H \cap g_i H g_i^{-1}$ is infinite again follows from Theorems 4.12 and 4.14 for $G = Mod(\Sigma)$ and $Out(F_n)$ respectively. By Theorems 4.10 and 4.13 respectively for Cases 3 and 4, the orbit of H on (G, d) (the electric metrics on $Mod(\Sigma)$ or $Out(F_n)$ making them quasi-isometric to $CC(\Sigma)$ or $\mathcal{F}_n$ respectively) is quasi-isometrically embedded. Hence (by the first paragraph of the proof for Cases 3 and 4 above, using hyperbolicity of $CC(S)$ and $\mathcal{F}_n$), the nearest-point projection of $g_i H$ onto H lies in a $C(g_i)-$neighborhood of $H^{g_i} \cap H$ in (G, d). The rest of the proof is as in Case 1.

□

5. From quasiconvexity to graded relative hyperbolicity

Recall that we defined parametrized/unparametrized graded geometric relative hyperbolicity in Definition 4.4.

5.1. Ensuring geometric graded relative hyperbolicity.

Proposition 5.1. *Let G be a group, d a word metric on G with respect to some (not necessarily finite) generating set, such that (G, d) is hyperbolic. Let H be a subgroup of G.*

Assume that H is C-quasiconvex in (G, d). Let C_0 be a parameter satisfying $C_0 > C + 20\delta$.

If $\{H\}$ has parametrized finite geometric height for d, for the parameter C_0, then $(G, \{H\}, d)$ has C_0-graded geometric relative hyperbolicity.

Proof. Let $\mathcal{H}_n$ denote the collection of n-fold intersections for distinct cosets of H, as in Definition 3.5. Let d_n be the metric of $G^{el}_{\mathcal{H}_n}$ restricted to G.

By Proposition 3.7 for all n, all elements of $\mathcal{H}_n$ are coarsely path connected. Thus the last point of Definition 4.4 (of geometric graded relative hyperbolicity) is satisfied. Of course the first point of Definition 4.4 is also satisfied by assumption of finite geometric height. It remains to see that the second point of 4.4 holds, namely that, for all n, $\mathcal{H}_n$ is coarsely hyperbolically embedded in (G, d_{n+1}).

By Proposition 3.7, for all n, all elements of $\mathcal{H}_n$ and of $\mathcal{H}_{n+1}$ are C_1-quasiconvex in (G, d). Therefore, by Proposition 2.11, all elements of $\mathcal{H}_n$ are C_1'-quasiconvex in (G, d_{n+1}) for some C_1' depending on the hyperbolicity constant of d, and on C_1.

Also by Proposition 3.7, elements of $\mathcal{H}_n$ are mutually cobounded in the metric d_{n+1}. Proposition 2.10 now shows that the horoballification of (G, d_{n+1}) over $\mathcal{H}_n$ is hyperbolic, for all n. Proposition 2.18 then guarantees that $\mathcal{H}_n$ is coarsely hyperbolically embedded in (G, d_{n+1}). Thus, $(G, \{H\}, d)$ has graded geometric relative hyperbolicity as in Definition 4.4. □

The next Proposition gives the unparametrized (more group-theoretic) version of Proposition 5.1 above.

Proposition 5.2. *Let G be a group, d a word metric on G with respect to some (not necessarily finite) generating set, such that (G, d) is hyperbolic. Let H be a subgroup of G. If $\{H\}$ has finite unparametrized geometric height for d and has the uniform qi-intersection property, then $(G, \{H\}, d)$ has unparametrized graded geometric relative hyperbolicity.*

Proof. As in Definition 3.11, $\mathcal{H}_n$ denotes the collection of intersections of n essentially distinct conjugates of H. Let $(\mathcal{H}_n)_0$ denote a set of conjugacy representatives of $(\mathcal{H}_n)$ that are C_1-quasiconvex, and let $\mathcal{CH}_n$ denote the collection of cosets of elements of $(\mathcal{H}_n)_0$. Let d_n be the metric on $X = (G, d)$ after electrifying the elements of $\mathcal{CH}_n$.

By Definition 3.11, for all n, all elements of $\mathcal{CH}_n$ and of $\mathcal{CH}_{n+1}$ are C_1-quasiconvex in (G, d). Therefore, by Proposition 2.11, all elements of $\mathcal{CH}_n$ are C_1'-quasiconvex in (G, d_{n+1}) for some C_1' depending on the hyperbolicity of d, and on C_1.

By Definition 3.11, $\mathcal{CH}_n$ is mutually cobounded in the metric d_{n+1}. Proposition 2.10 now shows that the horoballification of (G, d_{n+1}) over $\mathcal{CH}_n$ is hyperbolic, for all n. Proposition 2.18 then guarantees that $\mathcal{CH}_n$ is coarsely hyperbolically embedded in (G, d_{n+1}). Since H is assumed to have finite unparametrized geometric height, $(G, \{H\}, d)$ has unparametrized graded geometric relative hyperbolicity. □

5.2. **Graded relative hyperbolicity for quasiconvex subgroups.**

Proposition 5.3. *Let H be a quasi-convex subgroup of a hyperbolic group G, with a word metric d (with respect to a finite generating set). Then the pair $(G, \{H\})$*

has parametrized graded geometric relative hyperbolicity for all sufficiently large parameter values.

The pair $(G, \{H\})$ also has unparametrized graded geometric relative hyperbolicity, and graded relative hyperbolicity.

Proof. We first prove the parametrized version.

By Proposition 4.6, H has finite parametrized geometric height. Therefore, by Proposition 5.1, the pair $(G, \{H\})$ has parametrized geometric graded relative hyperbolicity.

We now prove the unparametrized version. For the word metric d with respect to a finite generating set, unparametrized graded geometric relative hyperbolicity agrees with the notion of graded relative hyperbolicity (Definition 1.3).

By Theorem 3.2, H has finite height. By Proposition 4.15 it satisfies the uniform qi-intersection property 3.11. Therefore, by Proposition 5.2, the pair $(G, \{H\})$ has unparametrized graded relative hyperbolicity.

Finally, note that since the word metric we use is locally finite, by the last comment of Remark 4.5, graded relative hyperbolicity is the same property as the one just established. □

Proposition 5.4. *Let $(G, \mathcal{P})$ be a finitely generated relatively hyperbolic group. Let H be a relatively quasi-convex subgroup. Let S be a finite relative generating set of G (relative to $\mathcal{P}$) and let d be the word metric with respect to $S \cup \mathcal{P}$. Then $(G, \{H\}, d)$ has graded relative hyperbolicity as well as (parametrized and unparametrized) graded geometric relative hyperbolicity.*

Proof. The proof is similar to that of Proposition 5.3. By Theorem 4.7, H has finite relative height, hence it has finite (parametrized and unparametrized) geometric height for the relative metric (see Example 4.8). The parametrized version follows from Proposition 5.1.

Next, by Proposition 4.15, H satisfies the uniform qi-intersection property for a relative metric, and the unparametrized version follows from Proposition 5.2.

Again, since G has a word metric with respect to a finite relative generating set, and H and all $i-$fold intersections are relatively quasiconvex as well, the above argument furnishes graded relative hyperbolicity as well. □

Similarly, replacing the use of Theorem 3.2 by Theorems 4.11 and 4.14, one obtains the following.

Proposition 5.5. *Let G be the mapping class group $Mod(S)$ (respectively $Out(F_n)$). Let d be a word metric on G making it quasi-isometric to the curve complex $CC(S)$ (respectively the free factor complex $\mathcal{F}_n$). Let H be a convex cocompact subgroup of G. Then $(G, \{H\}, d)$ has graded relative hyperbolicity.*

Again, replacing the use of Theorem 3.2 by Theorem 4.12, we obtain:

Proposition 5.6. *Let G be the mapping class group $Mod(S)$. Let d be a word metric on G making it quasi-isometric to the curve complex $CC(S)$. Let H be a convex cocompact subgroup of G. Then $(G, \{H\}, d)$ has (parametrized and unparametrized) graded geometric relative hyperbolicity.*

Remark 5.7. *Since we do not have an exact (geometric) analog of Theorem 4.12 for Out(F_n) (more precisely an analog of the stability result of* [DT14b]*) as of now, we have to content ourselves with the slightly weaker Proposition 5.5 for Out(F_n).*

6. From graded relative hyperbolicity to quasiconvexity

6.1. A Sufficient Condition.

Proposition 6.1. *Let G be a group and d a hyperbolic word metric with respect to a (not necessarily finite) generating set. Let H be a subgroup such that $(G, \{H\}, d)$ has (parametrized or unparametrized) graded geometric relative hyperbolicity. Then H is quasiconvex in (G, d).*

Also, if the generating set of G is finite and H is a subgroup such that $(G, \{H\}, d)$ has graded (algebraic) relative hyperbolicity, then H is quasiconvex in (G, d).

Proof. Assume $(G, \{H\}, d)$ has (parametrized or unparametrized) graded geometric relative hyperbolicity as in Definition 4.4. Then H has finite geometric height in (G, d). Let k be this height. Thus, $\mathcal{H}_{k+1}$ is a collection of uniformly bounded subsets, and d_{k+1} is quasi-isometric to d. It follows that (G, d_{k+1}) is hyperbolic.

Further, by Definition 4.4, $\mathcal{H}_k$ is hyperbolically embedded in (G, d_{k+1}). This means in particular that the electrification $(G, d_{k+1})^{el}_{\mathcal{H}_k}$ is hyperbolic. Since (G, d_k) is quasi-isometric to $(G, d_{k+1})^{el}_{\mathcal{H}_k}$ (being the restriction of the metric on G) it follows that (G, d_k) is hyperbolic as well. Further, by Corollary 2.7 the elements of $\mathcal{H}_k$, are uniformly quasiconvex in (G, d_{k+1}).

We now argue by descending induction on i.

The inductive hypothesis for $(i+1)$*:* We assume that d_{i+1} is a hyperbolic metric on G, and that there is a constant c_{i+1} such that, for all $j \geq 1$ the elements of $\mathcal{H}_{i+j}$ are uniformly c_{i+1}-quasiconvex in (G, d).

We assume the inductive hypothesis for $i + 1$ (*i.e.* as stated), and we now prove it for i.

Of course, we also assume, as in the statement of the Proposition, that $\mathcal{H}_i$ is coarsely hyperbolically embedded in (G, d_{i+1}). Hence d_i is a hyperbolic metric on G.

We will now check that the assumptions of Proposition 2.12 are satisified for $(X, d) = (G, d_{i+1})$, $\mathcal{Y} = \mathcal{H}_{i+1}$, and H arbitrary in $\mathcal{H}_i$.

Elements of $\mathcal{H}_i$ in (G, d_{i+1}) are uniformly quasiconvex in (G, d_{i+1}): this follows from Corollary 2.7. We will write C_i for their quasiconvexity constant.

A second step is to check that, for some uniform Δ_0 and ϵ, for all $\Delta > \Delta_0$, when an element $H_{i,\ell}$ of $\mathcal{H}_i$ (Δ, ϵ)-meets an item of $\mathcal{H}_{i+1}$, then $H^{+\epsilon\Delta}$ contains a quasigeodesic between the meeting points in H. Thus, fix $\epsilon < 1/100$, and take $\Delta_0 > 1/(13\epsilon)$, so that for all $\Delta > \Delta_0$, one has $(1 - 3\epsilon)\Delta > 10 \times \epsilon\Delta$. Assume $H_{i,\ell}$ (Δ, ϵ)-meets $Y \in \mathcal{H}_{i+1}$. Then, by definition of n-fold intersections, either $Y^{+\epsilon\Delta} \cap (H_{i,\ell})^{+\epsilon\Delta}$ counts as an item of $\mathcal{H}_{i+2}$, or $Y \subset H_{i,\ell}$. In both cases, by the inductive assumption, there is a path in $H_{i,\ell}^{+\epsilon\Delta}$ between the meeting points in $H_{i,\ell}$ that is a quasigeodesic for d. Hence the second assumption of Proposition 2.12 is satisfied.

We can thus conclude that $H_{i,\ell}$ is quasiconvex in (G, d) for a uniform constant, and therefore the inductive assumption holds for i.

By induction it is then true for $i = 0$, hence the first statement of the Proposition holds, i.e. quasiconvexity follows from (parametrized or unparametrized) graded geometric relative hyperbolicity.

Finally, since (algebraic) graded relative hyperbolicity coincides with unparametrized graded geometric relative hyperbolicity, when G, H are finitely generated (cf. Remarks 4.3 and 4.5) the last statement follows. □

We shall deduce various consequences of Proposition 6.1 below. However, before we proceed, we need the following observation since we are dealing with spaces/graphs that are not necessarily proper.

Observation 6.2. *Let X be a (not necessarily proper) hyperbolic graph. For all $C_0 \geq 0$, there exists $C_1 \geq 0$ such that the following holds:*
Let H be a hyperbolic group acting uniformly properly on X, i.e. for all D_0 there exists N such that for any $x \in X$, any D_0 ball in X contains at most N orbit points of Hx. Then a C_0-quasiconvex orbit of H is $(C_1, C_1)-$quasi-isometrically embedded in X.

Combining Proposition 6.1 with Observation 6.2 we obtain the following:

Proposition 6.3. *Let G be a group and d a hyperbolic word metric with respect to a (not necessarily finite) generating set. Let H be a subgroup such that*

1. *$(G, \{H\}, d)$ has graded geometric relative hyperbolicity.*
2. *The action of H on (G, d) is uniformly proper.*

Then H is hyperbolic and H is qi-embedded in (G, d).

Proof. Hyperbolicity of (G, d) was established while proving Proposition 6.1. Qi-embeddedness of H follows from Observation 6.2. Hyperbolicity of H is an immediate consequence. □

6.2. **The Main Theorem.** We assemble the pieces now to prove the following main theorem of the paper. A word about the convention followed in the statement in order to distinguish between (the more algebraic) graded relative hyperbolicity and graded geometric relative hyperbolicity. We shall put parentheses around (geometric) to indicate that the statement holds for both graded relative hyperbolicity and graded geometric relative hyperbolicity. Observe that Condition (3) of Definition 4.4 of graded geometric relative hyperbolicity follows from Condition (2) of Definition 1.3 of graded relative hyperbolicity.

Theorem 6.4. *Let (G, d) be one of the following:*

1. *G a hyperbolic group and d the word metric with respect to a finite generating set S.*
2. *G is finitely generated and hyperbolic relative to $\mathcal{P}$, S a finite relative generating set, and d the word metric with respect to $S \cup \mathcal{P}$.*
3. *G is the mapping class group $Mod(S)$ and d the metric obtained by electrifying the subgraphs corresponding to submapping class groups so that (G, d) is quasi-isometric to the curve complex $CC(S)$.*
4. *G is $Out(F_n)$ and d the metric obtained by electrifying the subgroups corresponding to subgroups that stabilize proper free factors so that (G, d) is quasi-isometric to the free factor complex $\mathcal{F}_n$.*

Then (respectively)

1. *H is quasiconvex if and only if $(G, \{H\})$ has graded (geometric) relative hyperbolicity.*
2. *H is relatively quasiconvex if and only if $(G, \{H\}, d)$ has graded (geometric) relative hyperbolicity.*
3. *H is convex cocompact in $Mod(S)$ if and only if $(G, \{H\}, d)$ has graded geometric relative hyperbolicity and the action of H on the curve complex is uniformly proper.*

(4) *H is convex cocompact in $Out(F_n)$ if and only if $(G, \{H\}, d)$ has graded relative hyperbolicity and the action of H on the free factor complex is uniformly proper.*

Proof. The forward implications of quasiconvexity to graded (geometric) relative hyperbolicity in the first 3 cases are proved by Propositions 5.3, 5.4, 5.5 and 5.6 and case 4 by Proposition 5.5. In cases (3) and (4) properness of the action of H on the curve complex follows from convex cocompactness.

We now proceed with the reverse implications. Again, the reverse implications of (1) and (2) are direct consequences of Proposition 6.1.

The proofs of the reverse implications of (3) and (4) are similar. Proposition 6.3 proves that any orbit of H on either the curve complex $CC(S)$ or the free factor complex $\mathcal{F}_n$ is qi-embedded. Convex cocompactness now follows from Theorems 4.10 and 4.13. □

6.3. **Examples.** We give a couple of examples below to show that finiteness of geometric height does not necessarily follow from quasiconvexity.

Example 6.5. Let $G_1 = \pi_1(S)$ and $H =< h >$ be a cyclic subgroup corresponding to a simple closed curve. Let $G_2 = H_1 \oplus H_2$ where each H_i is isomorphic to $\mathbb{Z}$. Let $G = G_1 *_{H=H_1} G_2$. Let d be the metric obtained on G with respect to some finite generating set along with all elements of H_2. Then G_1 is quasiconvex in (G, d), but G_1 does not have finite geometric height.

Note however, that the action of G_1 on (G, d) is not acylindrical. We now furnish another example to show that graded geometric relative hyperbolicity does not necessarily follow from quasiconvexity even if we assume acylindricity.

Example 6.6. Let $G = \langle a_i, b_i : i \in \mathbb{N}, a_{2i}^{b_i} = a_{2i-1} \rangle$ and let F be the (free) subgroup generated by $\{a_i\}$. Then $F^{b_i} \cap F =< a_{2i-1} >$ for all i. Let d be the word metric on G with respect to the generators a_i, b_i. Then the action of F on (G, d) is acylindrical and F is quasiconvex. However there are infinitely many double coset representatives corresponding to b_i such that $F^{b_i} \cap F$ is infinite.

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References

[Bes04] M. Bestvina. Geometric group theory problem list. *M. Bestvina's home page: http:math.utah.edu∼bestvina*, 2004.

[BF14] M. Bestvina and M. Feighn. Hyperbolicity of the complex of free factors. *Adv. Math. 256*, pages 104–155, 2014.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra and Computation. 22, 1250016, 66pp*, 2012.

[Cou14] Remi Coulon. On the Geometry of Burnside Quotients of Torsion Free Hyperbolic Groups. *Internat. J. Algebra and Computation*, 24(no.3):251–345, 2014.

[Dah03] F. Dahmani. Combination of convergence groups. *Geom. Topol. 7*, pages 933–963, 2003.

[DGO11] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *preprint, arXiv:1111.7048, to appear in Mem. Amer. Math. Soc.*, 2011.

[DM15] S. Das and M. Mj. Controlled Floyd Separation and Non Relatively Hyperbolic Groups. *J. Ramanujan Math. Soc. 30, no. 3*, pages 267–294, 2015.

[DS05] Cornelia Druţu and Mark Sapir. Tree graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005.

[DT14a] S. Dowdall and S. J. Taylor. Hyperbolic extensions of free groups. *preprint, arXiv:math.1406.2567*, 2014.

[DT14b] M. Durham and S. Taylor. Convex cocompactness and stability in mapping class groups. *preprint, arXiv:1404.4803*, 2014.

[Far98] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal. 8*, pages 810–840, 1998.

[FM02] B. Farb and L. Mosher. Convex cocompact subgroups of mapping class groups. *Geom. Topol., 6*, pages 91–152, 2002.

[GM08] D. Groves and J. Manning. Dehn filling in relatively hyperbolic groups. *Israel Journal of Mathematics 168*, pages 317–429, 2008.

[GMRS97] R. Gitik, M. Mitra, E. Rips, and M. Sageev. Widths of Subgroups. *Trans. AMS*, pages 321–329, Jan. '97.

[Ham08] U. Hamenstadt. Word hyperbolic extensions of surface groups. *preprint, arxiv:0807.4891v2*, 2008.

[Ham10] U. Hamenstadt. Stability of quasi-geodesics in teichmuller space. *Geom. Dedicata, 146*, pages 101–116, 2010.

[Hru10] G. Christopher Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. *Algebr. Geom. Topol. 10 no. 3*, pages 1807–1856, 2010.

[HW09] G. Christopher Hruska and D. T. Wise. Packing Subgroups in Relatively Hyperbolic Groups. *Geom. Topol. 13, no. 4*, pages 1945–1988, 2009.

[KL08] R. P. Kent and C. J. Leininger. Shadows of mapping class groups: capturing convex cocompactness. *Geom. Funct. Anal., 18(4)*, pages 1270–1325, 2008.

[Kla99] E. Klarreich. Semiconjugacies between Kleinian group actions on the Riemann sphere. *Amer. J. Math. 121*, pages 1031–1078, 1999.

[Mas80] H. Masur. Uniquely ergodic quadratic differentials. *Comment. Math. Helv. 55 (2)*, pages 255–266, 1980.

[McM01] C. T. McMullen. Local connectivity, Kleinian groups and geodesics on the blow-up of the torus. *Invent. math.*, 97:95–127, 2001.

[Mj08] M. Mj. Relative Rigidity, Quasiconvexity and C-Complexes. *Algebraic and Geometric Topology 8*, pages 1691–1716, 2008.

[Mj11] M. Mj. Cannon-Thurston Maps, i-bounded Geometry and a Theorem of McMullen. *Actes du séminaire Théorie spectrale et géométrie, Grenoble, vol 28, 2009-10, arXiv:math.GT/0511104*, pages 63–108, 2011.

[Mj14] M. Mj. Cannon-Thurston Maps for Surface Groups. *Ann. of Math., 179(1)*, pages 1–80, 2014.

[MM99] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Invent. Math.138*, pages 103–139, 1999.

[Osi06a] Denis Osin. Elementary subgroups of relatively hyperbolic groups and bounded generation. *Internat. J. Algebra and Computation*, 16(1):99–118, 2006.

[Osi06b] D.V. Osin. Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc. 179(843)*, pages 1–100, 2006.

[Sho91] H. Short. Quasiconvexity and a theorem of Howson's. *Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky eds.)*, 1991.

[Szc98] Andrzej Szczepanski. Relatively hyperbolic groups. *Michigan Math. J.*, 45(3):611–618, 1998.

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