RELATIVE RIGIDITY, QUASICONVEXITY AND C-COMPLEXES

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Abstract. We introduce and study the notion of relative rigidity for pairs $(X, \mathcal{J})$ where
1) X is a hyperbolic metric space and $\mathcal{J}$ a collection of quasiconvex sets
2) X is a relatively hyperbolic group and $\mathcal{J}$ the collection of parabolics
3) X is a higher rank symmetric space and $\mathcal{J}$ an equivariant collection of maximal flats
Relative rigidity can roughly be described as upgrading a uniformly proper map between two such $\mathcal{J}$'s to a quasi-isometry between the corresponding X's. A related notion is that of a C-complex which is the adaptation of a Tits complex to this context. We prove the relative rigidity of the collection of pairs $(X, \mathcal{J})$ as above. This generalises a result of Schwarz for symmetric patterns of geodesics in hyperbolic space. We show that a uniformly proper map induces an isomorphism of the corresponding C-complexes. We also give a couple of characterizations of quasiconvexity of subgroups of hyperbolic groups on the way.

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Contents

1. Introduction

1.1. **Relative Rigidity and Statement of Results.** In this paper, we study a rigidity phenomenon within the framework of coarse geometry. We call it *relative rigidity*. Much of the work on quasi-isometric rigidity (e.g. Farb-Schwartz [10] Kleiner-Leeb [17] Eskin-Farb [5] and Mosher-Sageev-Whyte [22] [23]) contains a crucial step showing that a self quasi-isometry of a space X coarsely preserves a family $\mathcal{J}$ of distinguished subsets of X. The family $\mathcal{J}$ again has a coarse intersection pattern that may be combinatorially coded and these proofs of quasi-isometric rigidity often show that the intersection pattern is preserved by a quasi-isometry. In this note, we investigate a sort of a converse to this:
When does a uniformly proper map between two families $\mathcal{J}_1$ and $\mathcal{J}_2$ come from a quasi-isometry ϕ between X_1 and X_2? Does such a map preserve intersection patterns?
We show that the answer is affirmative when

(1) X_i's are (Cayley graphs of) hyperbolic groups and $\mathcal{J}_i$'s correspond to cosets of a quasiconvex subgroup

(2) X_i's are (Cayley graphs of) relatively hyperbolic groups and $\mathcal{J}_i$'s correspond to parabolic subgroups

(3) X_i's are symmetric spaces of non-positive curvature and $\mathcal{J}_i$'s correspond to lifts of a maximal torus in a compact locally symmetric space modeled on X_i.

If in addition one can show that a quasi-isometry preserving intersection patterns is close to an isometry, we would be able to conclude that a uniformly proper map between the $\mathcal{J}_i$'s is induced by an isometry. This latter phenomenon has been investigated by Mosher, Sageev and Whyte [23] and has been termed *pattern rigidity*. Thus, in a sense, the notion of *relative rigidity* complements that of *pattern rigidity*.

Some further examples where a family of distinguished subsets of a space and the resulting (combinatorial) configuration yields information about the ambient space are:
1) Collection of flats in a symmetric space of higher rank [24]
2) Collection of maximal abelian subgroups of the mapping class group (Behrstock-Drutu-Mosher [1])
3) Collection of hyperbolic spaces in the Cayley complex of the Baumslag-Solitar groups (Farb-Mosher [7] , [8] ; see also [9])
4) Quasi-isometric rigidity of sufficiently complicated patterns of flats in the universal cover of a Haken 3 manifold (Kapovich-Leeb [15])
5) We were most influenced by a beautiful result of Schwarz [28] which shows that a uniformly proper map from a symmetric pattern of geodesics in $\mathbf{H}^n$

to another symmetric pattern of geodesics in $\mathbf{H}^n$ (for $n > 2$) is induced by an isometry. Again as in Mostow, there are two parts to this. A first step is to construct a quasi-isometry of $\mathbf{H}^n$ inducing the given pairing. Schwarz terms this *ambient extension.* The second is to construct an isometry.

Let us look at a general form of the situation that Schwarz considers. $(X_1, d_1), (X_2, d_2)$ are metric spaces. Let $\mathcal{J}_1, \mathcal{J}_2$ be collections of closed subsets of X_1, X_2 respectively. Then d_i induces a *pseudo-metric* (which, by abuse of notation, we continue to refer to as d_i) on $\mathcal{J}_i$ for $i = 1, 2$. This is just the ordinary (not Hausdorff) distance between closed subsets of a metric space. In [28], $X_1 = X_2 = \mathbf{H}^n$, and $\mathcal{J}_i$ are lifts (to the universal cover) of finite collections of closed geodesics in two hyperbolic manifolds.

Also, the hypothesis in Schwarz's paper [28] is the existence of a *uniformly proper* map ϕ between symmetric patterns of geodesics $\mathcal{J}_1$ and $\mathcal{J}_2$. A uniformly proper map may be thought of as an isomorphism in the so-called *coarse category* in the sense of John Roe [26]. Thus, we can re-interpret the first step of Schwarz's result as saying that an isomorphism ϕ in the coarse category between $\mathcal{J}_i$ implies the existence of a quasi-isometry from $\mathbf{H}^n$ to itself inducing ϕ. In the language of [28], *uniformly proper pairings come from ambient extensions.*

Mostow's approach yields an isomorphism of Tits complexes. We would like to associate to a pair $(X, \mathcal{J})$ some such complex just as a Tits complex is associated to a higher rank locally symmetric space and its collection of maximal parablic subgroups. We propose the notion of a C-complex in this paper as the appropriate generalization of a Tits complex to coarse geometry. Then what we would hope for (as a conclusion) is an isomorphism of these C-complexes. This transition from the existence of a uniformly proper map between $\mathcal{J}_i$'s to the existence of a a quasi-isometry between X_i's inducing an isomorphism of C-complexes is what we term **relative rigidity**. Schwarz proves the relative rigidity of pairs $(X, \mathcal{J})$ where X is hyperbolic space and $\mathcal{J}$ a symmetric collection of geodesics. Much of what he does extends to the case where X is a higher rank symmetric space and $\mathcal{J}$ a symmetric collection of maximal periodic flats or a symmmetric collection of maximal parabolic subgroups in a non-uniform lattice.

The main point of this paper is illustrated first in the context of relative rigidity of the category of pairs $(\Gamma, \mathcal{J})$, where Γ is (the Cayley graph of) a hyperbolic group, and $\mathcal{J}$ the set of cosets of a quasiconvex subgroup. **Throughout this paper we shall assume that the quasiconvex subgroups are of infinite index in the big groups.**

Note that the upgrading of a uniformly proper map between $\mathcal{J}$'s to a quasi-isometry between the Γ's is the most we can hope for in light of the fact that the Cayley graph of a finitely generated group is only determined up to quasi-isometry. (See also Paulin [25].)

We start with a pair of hyperbolic groups G_1, G_2 with Cayley graphs Γ_1, Γ_2, and quasiconvex subgroups H_1, H_2. Let Λ_1, Λ_2 be the limit sets of

H_1, H_2 in $\partial G_1, \partial G_2$ respectively. For convenience we consider the collection $\mathcal{J}_i$ of translates of J_i the join of Λ_i in Γ_i rather than cosets of H_i. Recall that the join of Λ_i is the union of bi-infinite geodesics in Γ_i with end-points in Λ_i. This is a uniformly quasiconvex set (and hence contains the Cayley graph of the subgroup H_i in a bounded neighborhood). The main theorems of this paper are as follows.

Theorem 3.5: *Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \to \mathcal{J}_2$. There exists a quasi-isometry q from Γ_1 to Γ_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

The construction of the quasi-isometry q proceeds by constructing a "coarse barycenter" of some infinite diameter sets (reminiscent of the celebrated measure-theoretic barycenter method discovered by Douady and Earle, and extended greatly by Besson, Courtois, Gallot [2]).

We prove an analogous theorem for pairs $(X, \mathcal{J})$ when X is (strongly) hyperbolic relative to the collection $\mathcal{J}$.

Theorem 3.11: *Let X_i be (strongly) hyperbolic relative to collections $\mathcal{J}_i$ ($i = 1, 2$). Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \to \mathcal{J}_2$. There exists a quasi-isometry q from X_1 to X_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

As a Corollary of Theorem 3.11 and work of Hruska and Kleiner [14], we deduce relative rigidity for pairs $(X, \mathcal{J})$ where X is a Cat(0) space with isolated flats and $\mathcal{J}$ is the collection of maximal flats.

The third main theorem of this paper is an analog for higher rank symmetric spaces.

Theorem 3.13: *Let X_i be symmetric spaces of non-positive curvature, and $\mathcal{J}_i$ be equivariant collections of lifts of a maximal torus in a compact locally symmetric space modeled on X_i ($i = 1, 2$). Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \to \mathcal{J}_2$. There exists a quasi-isometry q from X_1 to X_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

In fact, combining Theorem 3.13 with the quasi-isometric rigidity theorem of Kleiner-Leeb [17] and Eskin-Farb [5], we may upgrade the quasi-isometry of Theorem 3.13 to an isometry.

Let $C(G_i, H_i)$ be the C-complexes associated with the pairs (G_i, H_i). (See Section 1.3 for the precise definition.) Roughly speaking, the vertices of $C(G_i, H_i)$ are the translates $g_i^j \Lambda_i$ of Λ_i by distinct coset representatives g_i^j and the $(n-1)$-cells are n-tuples $\{g_1^1 \Lambda, \cdots, g_1^n \Lambda\}$ of distinct translates such that $\cap_1^n g_1^i \Lambda \neq \emptyset$.

Theorem 3.7: *Let $\phi : \mathcal{J}_1 \to \mathcal{J}_2$ be a uniformly proper map. Then ϕ induces an isomorphism of $C(G_1, H_1)$ with $C(G_2, H_2)$.*

On the way towards proving Theorems 3.5 and 3.7, we prove two Propositions characterizing quasiconvexity. These might be of independent interest.

The first is in terms of the Hausdorff topology on the collection $C_c^0(\partial G)$, which is the collection of closed subsets of ∂G having more than one point.

Proposition 2.3:*Let H be a subgroup of a hyperbolic group G with limit set Λ. Let $\mathcal{L}$ be the collection of translates of Λ by elements of distinct cosets of H (one for each coset). Then H is quasiconvex if and only if $\mathcal{L}$ is a discrete subset of $C_c^0(\partial G)$.*

The second characterization is in terms of strong relative hyperbolicity.

Proposition 2.9:*Let G be a hyperbolic group and H a subgroup. Then G is strongly relatively hyperbolic with respect to H if and only if H is a malnormal quasiconvex subgroup.*

The prototypical example is that of (fundamental groups of) a closed hyperbolic manifold with a totally geodesic *embedded* submanifold.

Finally, we give an intrinsic or dynamic reformulation of Theorems 3.5 and 3.7 following Bowditch [4], which makes use of the existence of a cross-ratio on the boundary of a hyperbolic group. The cross-ratio in turn induces a pseudometric on the collection $\mathcal{L}$ of translates of Λ.

Theorem 3.10: *Let G_1, G_2 be uniform convergence (hence hyperbolic) groups acting on compacta M_1, M_2 respectively. Also, let $\mathring{A}_i$ (for $i = 1, 2$) be G_i-invariant annulus systems and let $(..|..)_i$ denote the corresponding annular cross-ratios.*
Let H_1, H_2 be subgroups of G_1, G_2 with limit sets Λ_1, Λ_2. Suppose that the set $\mathcal{L}_i$ of translates of Λ_i (for $i = 1, 2$) by essentially distinct elements of H_i in G_i forms a discrete subset of $C_c^0(M_i)$.
Also assume that there exists a bijective function $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ and that this pairing is uniformly proper with respect to the cross-ratios $(..|..)_1$ and $(..|..)_2$.
Then
1) H_i is quasiconvex in G_i
2) There is a homeomorphism $q : M_1 \to M_2$ which pairs $\mathcal{L}_1$ with $\mathcal{L}_2$ as ϕ does. Further, q is uniformly proper with respect to the cross-ratios $(..|..)_1$ and $(..|..)_2$ on M_1, M_2 respectively.
3) q (and hence also ϕ) induces an isomorphism of C-complexes $C(G_1, H_1)$ with $C(G_2, H_2)$.

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1.2. **Relative Hyperbolicity and Electric Geometry.** We start off by fixing notions and notation. Let G (resp. X) be a hyperbolic group (resp. a hyperbolic metric space) with Cayley graph (resp. a net) Γ equipped with a word-metric (resp. a simplicial metric) d. Let the **Gromov boundary** of Γ be denoted by ∂G. (cf.[11]).

We shall have need for the fact that for hyperbolic metric spaces (in the sense of Gromov [13] the notions of quasiconvexity and qi embeddings coincide [13].

We shall now recall certain notions of relative hyperbolicity due to Gromov [13] and Farb [6].

Let X be a path metric space. A collection of closed subsets $\mathcal{H} = \{H_\alpha\}$ of X will be said to be **uniformly separated** if there exists $\epsilon > 0$ such that $d(H_1, H_2) \geq \epsilon$ for all distinct $H_1, H_2 \in \mathcal{H}$.

The **electric space** (or coned-off space) $\widehat{X}$ corresponding to the pair $(X, \mathcal{H})$ is a metric space which consists of X and a collection of vertices v_α (one for each $H_\alpha \in \mathcal{H}$) such that each point of H_α is joined to (coned off at) v_α by an edge of length $\frac{1}{2}$.

Definition 1.1. [6] [3] *Let X be a geodesic metric space and $\mathcal{H}$ be a collection of mutually disjoint uniformly separated subsets. Then X is said to be* **weakly hyperbolic** *relative to the collection $\mathcal{H}$, if the electric space $\widehat{X}$ is hyperbolic.*

Lemma 1.2. *(See Lemma 4.5 and Proposition 4.6 of Farb* [6]*, see also Klarreich* [16] *and* [21]*) Given δ, C, D there exists Δ such that if X is a δ-hyperbolic metric space with a collection $\mathcal{H}$ of C-quasiconvex D-separated sets. then, the electric space $\widehat{X}$ is Δ-hyperbolic, i.e. X is weakly hyperbolic relative to the collection $\mathcal{H}$.*

Definition 1.3. [6] [3] *Let X be a geodesic metric space and $\mathcal{H}$ be a collection of mutually disjoint uniformly separated subsets such that X is* weakly hyperbolic *relative to the collection $\mathcal{H}$. If any pair of electric quasigeodesics without backtracking starting and ending at the same point have* similar intersection patterns *with* horosphere-like sets *(elements of $\mathcal{H}$) then quasigeodesics are said to satisfy* **Bounded Penetration** *and X is said to be* **strongly hyperbolic** *relative to the collection $\mathcal{H}$.*

Definition 1.4. [21] *A collection $\mathcal{H}$ of uniformly C-quasiconvex sets in a δ-hyperbolic metric space X is said to be* **mutually D-cobounded** *if for all $H_i, H_j \in \mathcal{H}$, $\pi_i(H_j)$ has diameter less than D, where π_i denotes a nearest point projection of X onto H_i. A collection is* **mutually cobounded** *if it is mutually D-cobounded for some D.*

Mutual coboundedness was proven for horoballs by Farb in Lemma 4.7 of [6].

Lemma 1.5. [21] *Suppose X is a δ-hyperbolic metric space with a collection $\mathcal{H}$ of C-quasiconvex K-separated D-mutually cobounded subsets. Then X is strongly hyperbolic relative to the collection $\mathcal{H}$.*

Gromov gave a different definition of **strong relative hyperbolicity** as follows. Let X be a geodesic metric space with a collection $\mathcal{H}$ of uniformly separated subsets $\{H_i\}$. The hyperbolic cone cH_i is the product of H_i and the non-negative reals $H_i \times \mathcal{R}_+$, equipped with the metric of the type $2^{-t}ds^2 + dt^2$. More precisely, $H_i \times \{n\}$ is given the path metric of H_i scaled by 2^{-n}. The $\mathcal{R}_+$ direction is given the standard Euclidean metric. Let X^h denote X with hyperbolic cones cH_i glued to it along H_i's. X^h will be referred to as the *hyperbolically coned off* X. This is to be contrasted with the coned off space $\hat{X}$ in Farb's definition.

Definition 1.6. *X is said to be strongly hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov if the hyperbolically coned off space X^h is a hyperbolic metric space.*

The equivalence of the two notions of strong relative hyperbolicity was proven by Bowditch in [3].

Theorem 1.7. *(Bowditch* [3] *) X is strongly hyperbolic relative to a collection $\mathcal{H}$ of uniformly separated subsets $\{H_i\}$ in the sense of Gromov if and only if X is strongly hyperbolic relative to the collection $\mathcal{H}$ in the sense of Farb.*

1.3. **Height of Subgroups and C-Complexes.** The notion of height of a subgroup was introduced by Gitik, Mitra, Rips and Sageev in [12] and further developed by the author in [20].

Definition 1.8. *Let H be a subgroup of a group G. We say that the elements $\{g_i | 1 \leq i \leq n\}$ of G are essentially distinct if $Hg_i \neq Hg_j$ for $i \neq j$. Conjugates of H by essentially distinct elements are called essentially distinct conjugates.*

Note that we are abusing notation slightly here, as a conjugate of H by an element belonging to the normalizer of H but not belonging to H is still essentially distinct from H. Thus in this context a conjugate of H records (implicitly) the conjugating element.

Definition 1.9. *We say that the height of an infinite subgroup H in G is n if there exists a collection of n essentially distinct conjugates of H such that the intersection of all the elements of the collection is infinite and n is maximal possible. We define the height of a finite subgroup to be* 0*. We say that the width of an infinite subgroup H in G is n if there exists a collection of n essentially distinct conjugates of H such that the* **pairwise** *intersection of the elements of the collection is infinite and n is maximal possible.*

The main theorem of [12] states:

Theorem 1.10. *If H is a quasiconvex subgroup of a hyperbolic group G, then H has finite height and finite width.*

In this context, a theorem we shall be needing several times is the following due to Short [29].

Theorem 1.11. *(Short [29]) Let G be a hyperbolic group and H_i (for $i = 1 \cdots k$) be quasiconvex subgroups with limit sets Λ_i, $i = 1 \cdots k$. Then $\cap H_i$ is a quasiconvex subgroup with limit set $\cap \Lambda_i$.*

We now proceed to define a complex $C(G, H)$ for a group G and H a subgroup. For G hyperbolic and H quasiconvex, we give below three equivalent descriptions of a complex $C(G, H)$. In this case, let ∂G denote the boundary of G, Λ the limit set of H, and J the join of Λ.
1) Vertices (0-cells) are conjugates of H by essentially distinct elements, and $(n-1)$-cells are n-tuples $\{g_1 H, \cdots, g_n H\}$ of distinct cosets such that $\cap_1^n g_i H g_i^{-1}$ is infinite (in fact by Theorem 1.11 an infinite quasiconvex subgroup of G).
2) Vertices (0-cells) are translates of Λ by essentially distinct elements, and $(n-1)$-cells are n-tuples $\{g_1 \Lambda, \cdots, g_n \Lambda\}$ of distinct translates such that $\cap_1^n g_i \Lambda \neq \emptyset$.
3) Vertices (0-cells) are translates of J by essentially distinct elements, and $(n-1)$-cells are n-tuples $\{g_1 J, \cdots, g_n J\}$ of distinct translates such that $\cap_1^n g_i J$ is infinite.

We shall refer to the complex $C(G, H)$ as the **C-complex** for the pair G, H. (**C** stands for "coarse" or "Cech" or "cover", since $C(G, H)$ is like a coarse nerve of a cover, reminiscent of constructions in Cech cochains.) *Note that if $h(H)$ denote the height of H, then $(h(H)+1)$ is the dimension of the C-complex $C(G, H)$. Also, if $w(H)$ denote the width of H, then $w(H) = w$ is equal to the size of the largest complete graph K_w that is embeddable in $C(G, H)$.* If $C(G, H)$ is connected then its one-skeleton is closely related to the coned off space $\hat{\Gamma}$ with an appropriately chosen set of generators.

This definition is inspired by that of the Tits complex for a non-uniform lattice in a higher rank symmetric space. Related constructs in the context of codimension 1 subgroups also occur in work of Sageev [27] where he constructs cubings.

2. Characterizations of Quasiconvexity

Let G be a hyperbolic group. Let $C_c(\partial G)$ denote the collection of closed subsets of the boundary ∂G equipped with the Hausdorff topology. Let $C_c^0(\partial G) \subset C_c(\partial G)$ denote the subset obtained from $C_c(\partial G)$ by removing the singleton sets $\{\{x\} : x \in \partial G\}$. Next fix a subgroup $H \subset G$ with limit set $\Lambda \subset \partial G$. Consider the G-invariant collection $\mathcal{L} = \{\}\Lambda\} \subset \mathcal{C}'_\rfloor(\partial \mathcal{G})$ with g ranging over *distinct cosets* (one for each coset) of H in G. Note that $\mathcal{L}$ is (strictly speaking) a *multi-set* as distinct elements of $\mathcal{L}$ may denote the same subset of $C_c^0(\partial G)$ in case two distinct translates of Λ coincide. One extreme case is when $\Lambda = \partial G$, though H is of infinite index in G (e.g. if H is normal of infinite index in in G.) Then $\mathcal{L}$ consists of infinitely many copies of Λ.

Definition 2.1. *The* **join** $J(\Lambda)$ *of* Λ *is defined as the union of all bi-infinite geodesics whose end-points lie in* Λ

It is easy to see that $J(\Lambda)$ is 2δ-quasiconvex if G is δ-hyperbolic. In fact this is true for any subset Λ of the boundary of a δ-hyperbolic metric space X (no equivariance is necessary). For Λ the limit set of H, $J(\Lambda)$ is H-invariant. The *visual diameter* $dia_{\partial G}(\Lambda)$ of a subset Λ of ∂G is the same as the diameter in the metric on ∂G obtained from the Gromov inner product.

2.1. **Limit Sets and Quasiconvexity.** The next Lemma follows directly from the fact that sets with visual diameter bounded below contain points with Gromov inner product bounded above and conversely[11].

Lemma 2.2. *For all* $\epsilon > 0$ *there exists* N *such that if the diameter* $dia_{\partial G}(\Lambda) \geq \epsilon$ *for a closed subset* Λ *of* ∂G*, then there exists* $p \in J(\Lambda)$ *such that* $d(p, 1) \leq N$*. Conversely, for all* $N > 0$ *there exists* $\epsilon > 0$ *such that if there exists* $p \in J(\Lambda)$ *with* $d(p, 1) \leq N$*, then* $dia_{\partial G}(\Lambda) \geq \epsilon$*.*

The next Proposition gives our first characterisation of quasiconvex subgroups of a hyperbolic group.

Proposition 2.3. (Characterization of Quasiconvexity I) *Let* H *be a subgroup of a hyperbolic group* G *with limit set* Λ*. Let* $\mathcal{L}$ *be the collection of translates of* Λ *by elements of distinct cosets of* H *(one for each coset). Then* H *is quasiconvex if and only if* $\mathcal{L}$ *is a discrete subset of* $C_c^0(\partial G)$*.*

Proof: Suppose H is quasiconvex. We want to show that $\mathcal{L}$ is a discrete subset of $C_c^0(\partial G)$. Thus it suffices to show that any limit of elements of $\mathcal{L}$ is a singleton set. This in turn follows from the following.

Claim: For all $\epsilon > 0$, $\mathcal{L}_\epsilon = \{L_i \in \mathcal{L} : dia_{\partial G}(L_i) \geq \epsilon\}$ is finite.
Proof of Claim: Let $N = N(\epsilon)$ be as in Lemma 2.2. Since $dia_{\partial G}(L_i) \geq \epsilon$, therefore by Lemma 2.2, there exists $p_i \in J(L_i)$ such that $d_G(p_i, 1) \leq N$. Also, there exists $K > 0$ depending on δ (recall that $J(L_i)$ is 2δ-qc) and the quasiconvexity constant of H such that if $L_i = g_i\Lambda$, then there exists $h_i \in H$ with $d_G(p_i, g_ih_i) \leq K$. Hence, $d_G(1, g_ih_i) \leq K + N$. Since G is finitely generated, the number of such elements g_ih_i is finite. Since g_i are picked from distinct cosets of H, we conclude that the set $\mathcal{L}_\epsilon$ is finite. □

Conversely, suppose that H is not quasiconvex. Then there exist $p_i \in J(\Lambda)$ such that $d_G(p_i, \Gamma_H) \geq i$. Translating by an appropriate element of H, we may assume that $d_G(p_i, \Gamma_H) = d_G(p_i, 1) \geq i$. Further, we may assume (by passing to a subsequence if necessary) that the sequence $d_G(p_i, 1)$ is monotonically increasing. Then $p_i^{-1}J(\Lambda)$ has limit set $p_i^{-1}\Lambda$. Further, as $p_i \in J(\Lambda)$, therefore, $1 \in p_i^{-1}J(\Lambda)$. Since $J(\Lambda)$ is 2δ-qc, so is $p_i^{-1}J(\Lambda)$ for all i. Hence, there exists $\epsilon > 0$ by Lemma 2.2 such that $dia_{\partial G}p_i^{-1}J(\Lambda) \geq \epsilon$. Since $d_G(p_i, 1)$ is monotonically strictly increasing, we conclude that p_i's lie in distinbct cosets of H. Further, since $C_c(\partial G)$ is compact, we conclude that the collection $p_i^{-1}J(\Lambda)$ has a convergent subsequence, converging to a subset of diameter greater than or equal to ϵ. Therefore, the collection $\mathcal{L}$ is *not* a discrete subset (strictly speaking a multiset) of $C^0_c(\partial G)$. $\Box$

We next prove a result about projections of $J(L_i)$ on $J(L_j)$. We start off with an elementary fact about hyperbolic metric spaces. See [19] for a proof.

Lemma 2.4. [19] *Given $\delta > 0$, there exist D, C_1, k, ϵ such that if a, b, c, d are vertices of a δ-hyperbolic metric space (Z, d), with $d(a, [b, c]) = d(a, b)$, $d(d, [b, c]) = d(c, d)$ and $d(b, c) \geq D$ then $[a, b] \cup [b, c] \cup [c, d]$ lies in a C_1-neighborhood of any geodesic joining a, d and is a (k, ϵ)-quasigeodesic.*

Assume that H is quasiconvex and that L_k is the limit set $g_k\Lambda$ of g_kH. Let P_j denote the nearest point projection of Γ_G onto $J(L_j)$. Also, let $H_k = g_k\Gamma_H$ be the left translate of Γ_H by g_k.

Proposition 2.5. *There exists $K > 0$ such that $P_j(\Gamma_{H_i})$ lies in a K-neighborhood of $J(L_i \cap L_j)$ if $(L_i \cap L_j) \neq \emptyset$. Else, $P_j(\Gamma_{H_i})$ has diameter less than K.*

Proof: Since $J(L_i)$ is 2δ-qc and H is quasiconvex, it suffices to show that $P_j(J(L_i))$ lies in a K-neighborhood of $J(L_i \cap L_j)$. By G-equivariance, we may assume that $L_j = \Lambda$ and $g_i = 1$. We represent P_j by P in this case.

First note that by Theorem 1.11 due to Short, $H_i \cap H_j$ is quasiconvex and the limit set of $H_i \cap H_j$ is $L_i \cap L_j$. Also, $J(L_i \cap L_j) \subset J(L_i)$. Let $a, b \in J(L_i)$. Let $P(a) = c, P(b) = d$. Let D, C_1, k, ϵ be as in Lemma 2.4. If $d_G(c, d) \geq D$, then $[a, c] \cup [c, d] \cup [d, b]$ is a (k, ϵ)-quasigeodesic lying in a C_1 neighborhood of $[a, b]$. Since $J(L_i)$, $J(L_j)$ are both 2δ-qc, $[a, b]$ lies in a 2δ -neighborhood of $J(L_i)$, and $[c, d]$ lies in a 2δ -neighborhood of $J(L_j)$. In particular c, d lie in a $(C_1 + 2\delta)$-neighborhood of $J(L_j)$. Translating by an element of H, we may assume that $c = 1$.

We proceed now by contradiction. Suppose there exists a sequence of L_i's and $b_i \in J(L_i)$ such that $P(b_i) = d_i$ lies at a distance greater than i from $J(L_i \cap L_j)$. This shows that the sequence L_i has a limit point on Λ disjoint from $L_i \cap \Lambda$ for all i and further that $J(L_i)$ passes through a bounded neighborhood of 1. Hence the sequence L_i is not discrete in $C^0_c(\partial G)$. This contradicts Proposition 2.3 and proves our claim. $\Box$

2.2. **Quasiconvexity and Relative Hyperbolicity.** As an immediate corollary of Proposition 2.5 in conjunction with Theorem 1.11 of Short [29], we immediately conclude

Corollary 2.6. *Let H be a malnormal quasiconvex subgroup of a hyperbolic group G with Cayley graph Γ and limit set L. Then the set of joins $\mathcal{J}$ of distinct translates of L is a uniformly cobounded collection of uniformly quasiconvex sets in Γ.*

Combining Lemma 1.5 with Corollary 2.6 above, we have

Proposition 2.7. (Characterization of Quasiconvexity II) *Let H be a malnormal quasiconvex subgroup of a hyperbolic group G. Then G is strongly relatively hyperbolic with respect to H.*

In fact the converse to Proposition 2.7 is also true.

Malnormality of strongly relatively hyperbolic subgroups is due to Farb [6]. In fact this does not require G to be hyperbolic.

Lemma 2.8. *(Farb [6]) Let G be strongly relatively hyperbolic with respect to H. Then H is malnormal in G.*

It remains to show that H is quasiconvex if a hyperbolic group G be strongly relatively hyperbolic with respect to H. We use Gromov's definition of strong relative hyperbolicity. Attach hyperbolic cones cH to distinct translates of Γ_H in Γ_G to obtain the hyperbolically coned off Cayley graph Γ_G^h. Then Γ_G^h is hyperbolic by Gromov's definition.

If H is not quasi-isometrically embedded in G then for all $i \in \mathbb{N}$, there exist $p_{i1}, p_{i2} \in \Gamma_H$ such that

$$d_H(p_{i1}, p_{i2}) \geq i d_G(p_{i1}, p_{i2})$$

. Also from the metric d_{cH} on cH, we find that $d_{cH}(p_{i1}, p_{i2})$ is of the order of $log_2 d_H(p_{i1}, p_{i2})$. Hence, we can further assume that

$$d_H(p_{i1}, p_{i2}) \geq i d_{cH}(p_{i1}, p_{i2})$$

. Join p_{i1}, p_{i2} by shortest paths α_i, β_i in cH, Γ_G respectively. Then $\alpha_i \cup \beta_i = \sigma_i$ is a closed loop in Γ_G^h with total length $l(\sigma_i) = (d_{cH}(p_{i1}, p_{i2}) + d_G(p_{i1}, p_{i2}))$. Therefore $il(\sigma_i) \leq 2d_H(p_{i1}, p_{i2})$.

Since any (combinatorial) disk D_i in Γ_G^h spanning σ_i must contain a path γ_i in Γ_H joining p_{1i}, p_{2i}, therefore the area $A(D_i)$ of D_i must be at least that of $N_1(\gamma_i)$, the 1-neighborhood of γ_i in D_i.

Therefore there exists $C > 0$ such that for all i,

$$A(D_i) \geq A(N_1(\gamma_i)) \geq \frac{d_H(p_{i1}, p_{i2})}{C} \geq \frac{il(\sigma_i)}{2C}$$

Since i is arbitrary, this shows that Γ_G^h cannot satisfy a linear isoperimetric inequality. Hence Γ_G^h cannot be a hyperbolic metric space. This is a contradiction. Hence H must be quasi-isometrically embedded in G. Hence (see for instance [13]), H is quasiconvex in G. This completes our proof of the following characterisation of strongly relatively hyperbolic subgroups of hyperbolic groups.

Proposition 2.9. *Let G be a hyperbolic group and H a subgroup. Then G is strongly relatively hyperbolic with respect to H if and only if H is a malnormal quasiconvex subgroup.*

3. Relative Rigidity

3.1. **Pairing of Limit Sets by Quasi-isometries.** We now consider two hyperbolic groups G_1, G_2 with quasiconvex subgroups H_1, H_2, Cayley graphs Γ_1, Γ_2. Let $\mathcal{L}_j$ for $j = 1, 2$ denote the collection of translates of limit sets of H_1, H_2 in $\partial G_1, \partial G_2$ respectively. Individual members of the collection $\mathcal{L}_j$ will be denoted as L_i^j. Let $\mathcal{J}_j$ denote the collection $\{J_i^j = J(L_i^j) : L_i^j \in \mathcal{L}_j\}$. Following Schwarz [28], we define:

Definition 3.1. *A bijective map ϕ from $\mathcal{J}_1 \to \mathcal{J}_2$ is said to be uniformly proper if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that*
1) $d_G(J(L_i), J(L_j)) \leq n \Rightarrow d_G(\phi(J(L_i)), \phi(J(L_j))) \leq f(n)$
2) $d_G(\phi(J(L_i)), \phi(J(L_j))) \leq n \Rightarrow d_G(J(L_i), J(L_j)) \leq f(n)$.

Note: We observe that if $\mathcal{J}_i$ is just the collection of singleton sets in Γ_i, then a uniformly proper map between $\mathcal{J}$'s is (almost tautologically) the same as a quasi-isometry between Γ_i's. Hence what is important here is that $\mathcal{J}$'s are infinite diameter sets.

Definition 3.2. *A quasi-isometry q from Γ_1 to Γ_2 is said to pair the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does if there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that*
$d_G(p, J_j^1)) \leq n \Rightarrow d_G(q(p), \phi(J(L_j))) \leq h(n)$.

The following Lemma generalises Lemma 3.1 of Schwarz [28], where the result is proven in the special case of a symmetric pattern of geodesics in $\mathbf{H}^n$.

Lemma 3.3. *For $M, m > 0$, there exists $R > 0$, such that the following holds.*
Let $L_1, \cdots, L_M$ be distinct translates of the limit set of a quasiconvex subgroup H of a hyperbolic group G, such that $d_G(J_i, J_j) \leq m$ for all $i, j = 1 \cdots, M$. Then there exists a ball of radius R meeting J_i for all $i = 1 \cdots, M$.

Proof: If $\cap_1^M L_i \neq \emptyset$, choose any point $p \in J(\cap_1^M L_i)$. Then $B_1(p)$ intersects all J_i and we are through.

Suppose therefore that $\cap_1^m L_i = \emptyset$. We proceed by induction on M. There exists R_{M-1} such that a ball of radius R_{M-1} meets J_i for $i = 1 \cdots M-1$.

We now proceed by contradiction. If no such R exists for M, we have collections $\{L_1^k, \cdots, L_M^k\}, k \in \mathbb{N}$ such that a ball of radius R_{M-1} meets $J_i^k, i = 1 \cdots M-1$ but no ball of radius k meets $J_i^k, i = 1 \cdots M$. In particular, (since $J(\cap_1^{M-1} L_i^k) \subset \cap_1^{M-1} J_i^k)$), if $\cap_1^{M-1} L_i^k \neq \emptyset$, then $N_k(J(\cap_1^{M-1} L_i^k)) \cap J_i^M = \emptyset$.

For all i, j, k, choose points $p_{ij}^k \in J_i^k$ such that $d_G(p_{ij}^k, p_{ji}^k) \leq m$.

Assume by G-invariance of $\mathcal{J}$ that the ball of radius R_{M-1} centered at $1 \in \Gamma_G$ meets $J_i^k, i = 1 \cdots M-1$. Therefore J_M^k lies outside a k-ball about 1.

Since the collection of J_i's through 1 is finite, therefore assume after passing to a subsequence if necessary, that

1) $\{J_i^k\}_k$ is a constant sequence for $i = 1 \cdots M-1$. Hence, $\{L_i^k\}_k$ is a constant sequence L_i (say) for $i = 1 \cdots M-1$.
2) $p_{iM}^k \to p_{iM} \in \partial G$ for $i = 1 \cdots M-1$. Hence $p_{Mi}^k \to p_{iM} \in \partial G$. Further, by (1) above, $p_{iM} \in L_i$.
3) L_M^k converges to a closed set $Z \subset \partial G$. By Proposition 2.3, Z must be a singleton set $\{z\}$.
4) J_M^k lies outside $B_k(1) \cup N_k(J(\cap_1^{M-1} L_i^k))$. If $\cap_1^{M-1} L_i \neq \emptyset$, then assume further by G-invariance, that $1 \in J(\cap_1^{M-1} L_i^k)$. Also, using Theorem 1.11 due to Short [29], and translating by an appropriate element of $\cap_1^{M-1} H_i^k$, we may assume that $1 \in J(\cap_1^{M-1} L_i^k)$ is closest to J_M^k.

Now, $p_{Mi}^k \in J_M^k$ and hence by (3) above, $p_{Mi}^k \to z \in \partial G$. Combining this with (2) above, we get $z = p_{iM}$ for all $i = 1 \cdots M-1$. Therefore, $z \in \cap_1^{M-1} L_i \neq \emptyset$.

But $d_G(1, J_M^k) = d_G(J(\cap_1^{M-1} L_i^k), J_M^k) \geq k$. Let $z_k \in J_M^k$ such that $d_G(1, J_M^k) = d_G(1, z_k) = d_G(J(\cap_1^{M-1} L_i^k), J_M^k) \geq k$.

Then the Gromov inner product $(z_k, p_{iM}^k)_1$ is uniformly bounded above. Therefore $(z_k, p_{iM})_1$ is uniformly bounded above. Hence finally $(z, p_{iM})_1$ is bounded above. In particular $z \neq p_{iM}$. This is the contradiction that proves the Lemma. $\Box$

Definition of q

We are now in a position to define a map $q : \Gamma_1 \to \Gamma_2$ which pairs $\mathcal{J}_1$ with $\mathcal{J}_2$ as ϕ does.

Choose $K > 0$ such that the K neighborhood $B_K(g)$ of $g \in \Gamma_1$ has greater than w_2 (the width of H_2 in G_2) J_i^1's passing through it.

Let $\mathcal{J}_{K,g}^j$ (for $j = 1, 2$) denote the collection of J_i^j's passing through $N_K(g)$ for $g \in \Gamma_j, j = 1, 2$.

By the proof of Proposition 2.3, there exists $M = M(K)$ such that at most M J_i^1's pass through $N_K(g)$. By definition of w_2, and by Theorem 1.11 due to Short [29] at least two of the limit sets of the $\phi(J_i^1)$'s are disjoint. Let L_1^2 and L_2^2 denote these limit sets. Hence, by Corollary 2.6, for any $K_1 \geq f(K)$, there exists D such that the collection of points

$$\{p \in \Gamma_2 : d_2(p, J_2^1) \leq K_1, d_2(p, J_2^2) \leq K_1\}$$

has diameter less than D.

Since ϕ is a bijective pairing, $\phi(\mathcal{J}_{K,g}^j)$ has at least $(w_2 + 1)$ and at most $M(K)$ elements in it.

Also, by uniform properness of ϕ,

$$d_2(\phi(J_m^1), \phi(J_n^1)) \leq f(2K)$$

.

Summarising,
1) L_1^2 and L_2^2 are disjoint.

2) But, by Lemma 3.3, using $m = f(2K)$ and $M = M(K)$, there exists $R = R(K)$ and a ball of radius R meeting each $\phi(J_i^1)$.
3) For any K_1, there exists D, such that $\{p \in \Gamma_2 : d_2(p, J_2^1) \leq K_1, d_2(p, J_2^2) \leq K_1\}$ has diameter less than D.In particular, we may choose $K_1 = R$.

Define $q(g)$ to be the center of the ball of radius R obtained in (2) above. By (3), $q(g)$ is thus defined upto a uniformly bounded amount of discrepancy for all $g \in \Gamma_1$.

Lemma 3.4. *q is uniformly proper with respect to the metrics d_1, d_2.*

Proof: The proof is an almost exact replica of Lemma 3.2 of Schwarz [28] and we content ourselves with reproducing the heuristics of his argument here.

If x, y are close in Γ_1, then the pairwise minimal distances between elements of J^x_{K1} and J^y_{K1} is uniformly bounded above. Hence, by Lemma 3.3, there exists a uniform upper bound to the radius of a minimal radius ball intersecting all elements of $\phi(J^x_{K1})$ as well as $\phi(J^y_{K1})$. Also, since the center w of such a ball is defined upto a bounded amount of discrepancy, it must be at a bounded distance from both $q(x)$ as well as $q(y)$. Hence $d_2(q(x), q(y))$ is uniformly bounded, i.e. close.

Conversely, suppose that $q(x), q(y)$ are close. First, by Lemma 3.3, there exists a uniform upper bound R on radius of minimal radius balls B_1, B_2 centered at $q(x), q(y)$, intersecting all elements of $\phi(\mathcal{J}^x_{K1}), \phi(\mathcal{J}^y_{K1})$ respectively. Then the $(R + d_2(q(x), q(y)))$ ball about $q(x)$ (or $q(y)$) meets every element of $\phi(\mathcal{J}^x_{K1})$ as well as $\phi(\mathcal{J}^y_{K1})$. Since ϕ is uniformly proper, this means that there is a uniform upper bound on the minimal radius of a ball meeting every element of $(\mathcal{J}^x_{K1})$ as well as $(\mathcal{J}^y_{K1})$. As before, $d_1(x, y)$ is uniformly bounded, i.e. x, y are close. □

Similarly, we can construct q^{-1} using the bijective pairing ϕ^{-1} such that q^{-1} is uniformly proper. Also, from Lemma 3.3 q, q^{-1} composed with each other in either direction is close to the identity.

Since ϕ pairs $\mathcal{L}_1$, $\mathcal{L}_2$ bijectively and is uniformly proper from $\mathcal{J}_1$ to $\mathcal{J}_2$, therefore by invariance of $\mathcal{J}_2$ under G_2, every point of Γ_2 lies close to the image of q. Therefore q is uniformly proper, by Lemma 3.4 above, from Γ_1 onto a net in Γ_2. Hence q is a quasi-isometry. This concludes the proof of the main theorem of this subsection.

Theorem 3.5. *Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \rightarrow \mathcal{J}_2$. There exists a quasi-isometry q from Γ_1 to Γ_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

We have thus shown one aspect of **relative rigidity**, viz. upgrading a uniformly proper map between $\mathcal{J}_i$'s to a quasi-isometry between Γ_i's. In the next subsection, we shall deduce the second aspect, viz. isomorphism of C-complexes.

3.2. **C-Complexes.** By Theorem 3.5 we obtain a quasi-isometry q from Γ_1 to Γ_2 which pairs $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does. Since q is a quasi-isometry, it extends to a quasiconformal homeomorphism from ∂G_1 to ∂G_2. Also, for all $\alpha > 0$, there exists $\beta > 0$ such that

$$d_1(x, J_i^1) \leq \alpha \Rightarrow d_2(q(x), \phi(J_i^1)) \leq \beta$$

and conversely,

$$d_2(y, J_i^2) \leq \alpha \Rightarrow d_1(x, \phi^{-1}(J_i^2)) \leq \beta$$

.

In particular, ∂q maps the limit set L_i^1 homeomorphically to the limit set of $\phi(J_i^1))$. Hence, ∂q preserves intersection patterns of limit sets. Since ϕ pairs $\mathcal{J}_1$ with $\mathcal{J}_2$ as q does, summarising we get:

Lemma 3.6. *The following are equivalent.*
1) $\cap_{i=1}^k L_i^1 = \emptyset$
2) $\cap_{i=1}^k \partial q(L_i^1) = \emptyset$
3) $\cap_{i=1}^k \phi(L_i^1) = \emptyset$

Hence by the definition of the C-complexes $C(G_1, H_1)$ and $C(G_2, H_2)$, we find that ∂q induces an isomorphism of $C(G_1, H_1)$ with $C(G_2, H_2)$. We conclude:

Theorem 3.7. *Let $\phi : \mathcal{J}_1 \to \mathcal{J}_2$ be a uniformly proper map. Then ϕ induces an isomorphism of $C(G_1, H_1)$ with $C(G_2, H_2)$.*

Note: In Theorem 3.5 and Theorem 3.7 we start with the assumption that there exists a uniformly proper pairing of the collections $\mathcal{J}_1$ and $\mathcal{J}_2$. This can be translated to a pairing of collections of limit sets $\mathcal{L}_1$ and $\mathcal{L}_2$. Theorem 3.5 then says that the pairing of the $\mathcal{J}_i$'s (or $\mathcal{L}_i$'s) is induced by a quasi-isometry from Γ_1 to Γ_2. Thus, the existence of a uniformly proper pairing implies the existence of a quasi-isometry between the Γ_i's, i.e. an ambient extension (or, equivalently, a quasiconformal homeomorphism between ∂G_i's).

Also Theorem 3.7 shows that a uniformly proper pairing induces an isomorphism of the C-complexes $C(G_i, H_i)$. This is reminiscent of the initial step in the proof of rigidity theorems for higher rank symmetric spaces, where Tits complexes replace C-complexes.

3.3. **Cross Ratios, Annular Systems and a Dynamical Formulation.** In this subsection, we give a more intrinsic formulation of Theorems 3.5 and 3.7. The hypothesis of these theorems is given in terms of distances between elements of $\mathcal{J}_i$. A more intrinsic way of formulating this hypothesis would be in terms of the action of G_i on ∂G_i, $i = 1, 2$. In this case, the distance between J_l^i, J_m^i can be approximated by the hyperbolic cross-ratio of their limit sets. This was described in detail by Bowditch [4]. We give the relevant definitions and Theorems below and then dynamically reformulate Theorems 3.5 and 3.7.

Let M be a compactum.

Definition 3.8. *An* **annulus** $\mathcal{A}$ *is an ordered pair* (A^-, A^+) *of disjoint closed subsets of* M *such that* $M \setminus (A^- \cup A^+) \neq \emptyset$. *An* **annulus system** *is a collection of such annulii. If* $A = (A^-, A^+)$*, then* $-A = (A^+, A^-)$. *An annulus system is symmetric if* $A \in \mathcal{A} \Rightarrow -A \in \mathcal{A}$.
Given a closed set $K \subset M$ *and an annulus* A*, we say that* $K < A$ *if* $K \subset intA^-$. *Also,* $A < K$ *if* $K < -A$.
If A, B *are annulii, we say that* $A < B$ *if* $M = intA^- \cup intB^+$.
Fix an annulus system $\mathcal{A}$. *Given closed sets* $K, L \subset M$*, we say that the* **annular cross-ratio** $(K|L)_{\mathcal{A}} \in \mathbb{N} \cup \infty$ *for the maximal number* $n \in \mathbb{N}$ *such that we can find annulii* $A_1, \cdots A_n \in \mathcal{A}$ *such that*

$$K < A_1 < \cdots < A_n < L$$

. *We set* $(K|L)_{\mathcal{A}} = \infty$ *if there is no such bound.*

Thus $(K|L)_{\mathcal{A}}$ is the length of the maximal chain of nested annulii sepatrating K, L. For two point sets $\{x, y\} = K$ and $\{z, w\} = L$, we write $(K|L)_{\mathcal{A}}$ as $(xy|zw)_{\mathcal{A}}$.

One of the crucial results of [4] is:

Theorem 3.9. *(Bowditch [4]) Suppose a group* G *acts as a uniform convergence group on a perfect metrizable compactum* M. *Then there exists a symmetric* G*-invariant annulus system* $\mathcal{A}$ *such that if* x, y, z, w *are distinct elements in* M*, then the theree quantities* $(xy|zw)_{\mathcal{A}}$, $(xz|yw)_{\mathcal{A}}$, $(xw|zy)_{\mathcal{A}}$ *are all finite and at least two of them are zero. Also, if* $x \neq y$*, then* $(x|y)_{\mathcal{A}} > 0$. *Further,* G *is hyperbolic, and* $d_G(J(K), J(L))$ *differs from* $(K, L)_{\mathcal{A}}$ *upto bounded additive and multiplicative factors.*

Combining Theorems 3.5 , 3.7 with Proposition 2.3 and Theorem 3.9, we get the dynamical formulation we promised. Let $C_c^0(M)$ denote the collection of closed subsets of M containing more than one point.

Theorem 3.10. *Let* G_1, G_2 *be uniform convergence (hence hyperbolic) groups acting on compacta* M_1, M_2 *respectively. Also, let* $\mathcal{A}_i$ *(for* $i = 1, 2$ *) be* G_i*-invariant annulus systems and let* $(..|..)_i$ *denote the corresponding annular cross-ratios.*
Let H_1, H_2 *be subgroups of* G_1, G_2 *with limit sets* Λ_1, Λ_2. *Suppose that the set* $\mathcal{L}_i$ *of translates of* Λ_i *(for* $i = 1, 2$*) by essentially distinct elements of* H_i *in* G_i *forms a discrete subset of* $C_c^0(M_i)$.
Also assume that there exists a bijective function $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ *and that this pairing is uniformly proper with respect to the cross-ratios* $(..|..)_1$ *and* $(..|..)_2$. *Then*

1. H_i *is quasiconvex in* G_i
2. *There is a homeomorphism* $q : M_1 \to M_2$ *which pairs* $\mathcal{L}_1$ *with* $\mathcal{L}_2$ *as* ϕ *does. Further,* q *is uniformly proper with respect to the cross-ratios* $(..|..)_1$ *and* $(..|..)_2$ *on* M_1, M_2 *respectively.*
3. q *(and hence also* ϕ*) induces an isomorphism of* C*-complexes* $C(G_1, H_1)$ *with* $C(G_2, H_2)$.

Thus from a uniformly proper map with respect to the pseudometrics on $\mathcal{L}_i$'s induced by cross-ratios we infer a quasi-isometry that is an ambient extension as also a (simplicial) isomorphism of C-complexes.

3.4. **Axiomatisation, Relative Hyperbolicity.** For classes of pairs $(X, \mathcal{J})$, what did we really require to ensure relative rigidity? Assume (X, d) is a metric space and let the induced pseudometric on $\mathcal{J}$ be also denoted by d.
1) For all $k > 0$ there exists $M \in \mathbb{N}$ such that for all $x \in X$, $N_k(x)$ meets at most M of the J's in $\mathcal{J}$. *(This is a coarsening of the notion of height.)*
2) For all $K \in \mathbb{N}$, there exists $k = k(K) > 0$ such that for all $x \in X$, $N_k(x)$ meets at least K of the J's in $\mathcal{J}$. *(This is the converse condition to (1).)*
3) For all $k > 0, n \in \mathbb{N}$ there exists $K > 0$ such that for any collection $J_1, \cdots, J_n \in \mathcal{J}$ with $d(J_i, J_j) \leq k$, there exists a ball of radius at most K meeting all the J_i's.
4) There exists $N \in \mathbb{N}$ such that for all $k > 0$ there exists $K = K(k) > 0$ such that the following holds.
For all $n \geq N$ and $J_1, \cdots, J_n \in \mathcal{J}$, the set of points $\{x \in X : N_k(x) \cap J_i \neq \emptyset, i = 1 \cdots n\}$ is either empty or has diameter bounded by K.

Given (1)-(4), the construction of $q : X_1 \to X_2$ from a uniformly proper pairing $\phi : \mathcal{J}_1 \to \mathcal{J}_2$ goes through as in Theorem 3.5. In short, pick N from (4). From (2), pick $k = k(N)$. Now for all $x \in X_1$, consider the collection of J's in $\mathcal{J}_1$ that meet $N_k(x)$. By (1) there is an upper bound $M = M(k)$ on the number of such J's. Map these over to $\mathcal{J}_2$. Any two of these are at a distance of at most m apart where m depends on ϕ and k. From (3) choose $K = K(M, m)$ such that a ball of radius K meets all these. Set $q(x)$ to be the center of such a ball. By (4), $q(x)$ is defined upto a uniformly bounded degree of discrepancy. The rest of the proof goes through as before. Hence (1)-(4) define sufficient conditions for relative rigidity for a class of pairs $(X, \mathcal{J})$.

With these conditions, it is easy to extend Theorem 3.5 to pairs $(X, \mathcal{J})$ where X is (strongly) hyperbolic relative to the collection $\mathcal{J}$. Conditions (1) and (2) are trivial.

Condition (3) follows from "bounded penetration" (see Farb [6]). Suppose we have an electric triangle for triples $J_1, J_2, J_3 \in \mathcal{J}$ of horosphere-like sets, such that the hyperbolic geodesic γ_{ij} joining J_i, J_j has length bounded by C_0. Then γ_{ij} and γ_{ik} meet J_i at a uniformly bounded distance from each other by "bounded penetration". Hence, all of the geodesic γ_{ik} lies near γ_{ij} for all k. Since γ_{ij} has length bounded by C_0, condition (3) follows.

Condition (4) follows from the fact that for a pair of distinct J_i, J_j, $N_k(J_i) \cap N_k(J_j)$ is either empty or has diameter bounded by some $C(k)$.

We have thus shown:

Theorem 3.11. *Let X_i be (strongly) hyperbolic relative to collections $\mathcal{J}_i$ $(i = 1, 2)$. Let ϕ be a uniformly proper (bijective, by definition) map from*

$\mathcal{J}_1 \to \mathcal{J}_2$. There exists a quasi-isometry q from X_1 to X_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.

By work of Hruska and Kleiner [14], CAT(0) spaces with isolated flats are (strongly) hyperbolic relative to maximal flats. Hence we have from Theorem 3.11 above:

Corollary 3.12. *Let X_i be CAT(0) spaces with isolated flats and let $\mathcal{J}_i$ denote the collections of maximal flats ($i = 1, 2$). Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \to \mathcal{J}_2$. There exists a quasi-isometry q from X_1 to X_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

3.5. **Symmetric Spaces of Higher Rank.** We now consider CAT(0) spaces which are at the other end of the spectrum. Let M be a compact locally symmetric space and T a totally geodesic torus with rank = rank(M). Take $X = \widetilde{M}$ and $\mathcal{J}$ to be the lifts of T to $\widetilde{M}$. As these are all equivariant examples, it is enough to check (1)-(4) at a point.

(1) and (2) are clear. To prove condition (4), we consider $\cap_i N_k(F_i)$ and it is easy to bound from below the N appearing in Condition (4) (Section 3.4) in terms of the size of the Weyl group and rank. In that case, $\cap_i N_k(F_i)$ has bounded diameter or is empty.

Finally, to prove (3), we proceed as in Lemma 3.3. As in Lemma 3.3 we assume by induction that any k flats $\{F_1, cdots, F_k\}$ that "coarsely pairwise intersect at scale D" (i.e. $N_D(F_i) \cap N_D(F_j) \neq \emptyset$) intersect coarsely (i.e. $\cap_{i=1\cdots k} N_{D'}(F_i) \neq \emptyset$ for some $D' = D'(D, k)$). To get to the inductive step, we suppose that for $i = k + 1$, we have collections of worse and worse counterexamples. Consider a maximal collection $\mathcal{F} = \{F_1, \cdots, F_k\}$ of maximal flats whose "coarse intersection at scale D" $\cap_i N_D(F_i) = F$ is non-null. Translate the collection by a group element so that a fixed point 0 (thought of as the origin) lies on the intersection F. Now take a sequence of maximal flats F^j whose D-neighborhoods $N_D(F^j)$ intersect each $N_D(F_i)$, but $d_j = d(F^j, F) = d(0, F) \geq j$. We scale the metric on (X, d) by a factor of d_j to obtain a sequence of metric spaces $(X, \frac{d}{d_j})$ converging (via a non-principal ultrafilter) to a Euclidean building X^∞ (this fact is due to Kleiner and Leeb [17], but we shall only mildly need the exact nature of X_∞). F_i's converge to flats $F_i^\infty \subset X^\infty$ and F^j's converge to a flat $G^\infty \subset X^\infty$. Then the collection $\mathcal{G} = F_i^\infty, G^\infty$ satisfy the following:
$(P1)$ Each element of $\mathcal{G}$ is a flat in X^∞
$(P2)$ By induction, the intersection of any i elements of $\mathcal{G}$ is non-empty and convex for $i \leq k$
$(P3)$ The intersection of all the $(k+1)$ elements of $\mathcal{G}$ is empty.

Consider the subcomplex $K = G^\infty \bigcup_i F_i^\infty$ of X^∞. Then K is a union of r-flats, where $r = \text{rank}(X)$. In particular, the homology groups $H_n(K) = 0$ for $n > r$. On the other hand, if we consider the nerve of the covering of K by the sets G^∞, F_i^∞, then using the properties $(P1), (P2), (P3)$ to compute

Cech homology groups, we conclude that K has the same homology groups as the boundary of a k-simplex. In particular, $H_k(K) = \mathbb{Z}$. For $k > r$ this is a contradiction, finally proving Condition (3). Thus we conclude:

Theorem 3.13. *Let X_i be symmetric spaces of non-positive curvature, and $\mathcal{J}_i$ be equivariant collections of lifts of a maximal torus in a compact locally symmetric space modeled on X_i ($i = 1, 2$). Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \to \mathcal{J}_2$. There exists a quasi-isometry q from X_1 to X_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

Combining Theorem 3.13 with the quasi-isometric rigidity theorem of Kleiner-Leeb [17] and Eskin-Farb [5] we can upgrade the quasi-isometry q to an isometry i.

Corollary 3.14. *Let X_i be symmetric spaces of non-positive curvature, and $\mathcal{J}_i$ be equivariant collections of lifts of a maximal torus in a compact locally symmetric space modeled on X_i ($i = 1, 2$). Let ϕ be a uniformly proper (bijective, by definition) map from $\mathcal{J}_1 \to \mathcal{J}_2$. There exists an isometry i from X_1 to X_2 which pairs the sets $\mathcal{J}_1$ and $\mathcal{J}_2$ as ϕ does.*

Remark 3.15. *The technique of using asymptotic cones and the nerve of the covering by flats can be generalised easily to equivariant flats of arbitrary (not necessarily maximal) rank.*

We conclude this paper with two (related) questions:

Question 1: In analogy with a Theorem of Ivanov, Korkmaz, Luo (see for instance [18]), regarding the automorphism group of the curve complex, we ask:

If the C-Complex $C(G, H)$ of a pair (G, H) (for G a hyperbolic group and H a quasiconvex subgroup) is connected, is the automorphism group of $C(G, H)$ commensurable with G?

Question 2: Consider the pair (G, H), with G a hyperbolic group and H a quasiconvex subgroup. Let $(\mathcal{J}, d)$ be the collection of joins as in Lemma 3.3 with the induced pseudometric. For a uniformly proper map ϕ from $(\mathcal{J}, d)$ to itself, is there an isometry pairing the elements of $\mathcal{J}$ as ϕ? We have proved in Theorem 3.5 that a quasi-isometry q exists pairing the $\mathcal{J}$ as ϕ does. The question is whether q may be upgraded to an isometry, or better, to an element of G? This question is related to the notion of *pattern rigidity* introduced by Mosher, Sageev and Whyte in [23].

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