A SURVEY OF LOW DIMENSIONAL (QUASI)PROJECTIVE GROUPS

INDRANIL BISWAS AND MAHAN MJ

Abstract. A brief survey of some recent results on projective and quasiprojective groups of low cohomological dimensions was presented by the second author at the conference in Hyderabad University in March 2015. This is a slightly expanded version of the talk. Part of the survey involves joint work with H. Seshadri and A. J. Parameswaran.

Contents

1. Motivational Questions

There are a couple of strands that have fed into work of several people, in particular the authors, over the last few years.

1.1. **Serre's question and its low-dimensional versions.** Our starting point is the following question of Serre:

Question 1.1. [ABCKT] **(Serre)** *Which finitely presented groups can be realized as fundamental groups of smooth complex projective varieties or more generally of compact Kähler manifolds?*

This question is the theme of the book [ABCKT]. Fundamental groups of compact Kähler manifolds are usually referred to as *Kähler groups*. Similarly fundamental groups of smooth complex projective (respectively, quasiprojective) varieties are referred to as *projective* (respectively, *quasiprojective*) groups. Projective groups are of course Kähler. The converse is a well–known open question.

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A special case of Question 1.1 was posed by Donaldson and Goldman and answered by Dimca and Suciu [DiSu]:

Question 1.2. *Which Kähler groups appear as fundamental groups of closed 3-manifolds?*

More generally, a standard theme in the theory of Kähler groups has been:

Question 1.3. Meta-question: *Take your favorite class of groups. (Our favorite classes occur naturally in geometric group theory or low-dimensional topology.) Which of them are Kähler/projective/quasiprojective?*

Examples of such "favorite" classes include the following:

(1) 3-manifold groups [DiSu, Ko1, BMS, Ko3, DPS, FrSu, BiMj2].
(2) One-relator groups [BiMj3, Ko2].
(3) More generally, groups of cohomological dimension 2 and 3. (See [BiMj1] for preliminary results for cd2 groups.)

A quick look at the above list justifies the term 'low-dimensional' in the title of this paper.

1.2. **Complexifications.** The first part of Hilbert's 16th problem may be thought of as one of the starting points of real algebraic geometry. The main problem can be summarized in the following question.

Question 1.4. *Let X be a variety defined over the reals $\mathbb{R}$. Let $X_{\mathbb{R}}$ denote its set of real points. What is the relationship between the topology of X and that of $X_{\mathbb{R}}$?*

Hilbert's problem deals with the dimension one case of this. Works of Arnold [Ar], Gudkov [Gu], Viro [Vi1, Vi2] deserve special mention in addressing Hilbert's 16th problem in dimension one. In dimension 2, Kharlamov [Kh1, Kh2] made major contributions. The major thrust in dimension three came from Kollár [Kol1, Kol2, Kol3, Kol4]. All these pieces of work dealt with projective varieties, sometimes singular.

Nash [Na] and Tognoli proved that any smooth closed connected manifold may be realized as the real locus $X_{\mathbb{R}}$ of a smooth projective variety. This prompted Nash to ask:

Question 1.5. *Can any smooth closed connected manifold may be realized as the real locus $X_{\mathbb{R}}$ of a rational variety?*

Question 1.5 is often called the Nash Conjecture, though it is very likely the shortest-lived conjecture in mathematical history lasting -38 years. It was disproved in dimension 2 by Commessatti [Co] in 1914. It was disproved in dimension 3 by Kollár in [Kol5].

We shall refer to the complex locus X of $M = X_{\mathbb{R}}$ as a **complexification** of M.

The class of quasiprojective or affine complexifications poses different questions. In this article we shall later have occasion to review a class of affine or quasiprojective complexifications called **minimal** or **good** complexifications [Ku, To, Mc, BiMj2, BMP2].

1.3. **Cohomological Dimension.** Fundamental groups of compact Riemann surfaces are called *surface groups*. A group is said to be *virtually a surface group* if some finite index subgroup of it is a surface group. The following question is basic in the theory of low dimensional projective groups:

Question 1.6. *Let G be a non-trivial Kähler/projective group of (integral cohomological) dimension less than four. Is G the fundamental group of a closed Riemann surface?*

It is easy to see that uniform lattices in $\mathrm{SU}(n,1)$ are projective Poincaré duality groups of dimension $2n$. Toledo, [To], has shown that non–uniform lattices in $\mathrm{SU}(n,1)$ are projective groups if and only if $n > 2$. These are examples of $2n-1$ dimensional duality groups. Thus for every integer $n \geq 4$, there exist projective duality groups of dimension n. But a non–uniform lattice in $\mathrm{SU}(2,1)$ is not Kähler [To].

It was pointed out to us by Delzant, and a complete argument appears already in a paper of Kotschick [Ko1], that the image of the Albanese map for a compact Kähler manifold with fundamental group G of dimension less than four is either a point or a smooth complex projective curve. As observed in [BiMj3] it follows that if in addition G is also coherent (i.e., if every finitely generated subgroup of G is finitely presented) and of cohomological dimension 2, possessing a finite index subgroup with positive first Betti number, then G is in fact a surface group.

2. 3-manifold groups

Question 1.2 was answered positively by Dimca and Suciu [DiSu]:

Theorem 2.1. *Let G be the fundamental group of a closed 3-manifold. Then G is Kähler if and only if G is finite.*

This followed work of Reznikov and different proofs of it were given by Kotschick [Ko1] and Biswas-Mj-Seshadri [BMS]. In fact, Kotschick [Ko1] and Delzant (unpublished) have independently shown that if a Poincaré duality group G of dimension three does not satisfy property T, then G is not Kähler.

In [BMS] with Seshadri we were interested in generalizing Theorem 2.1. We were concerned mainly with the following general set-up:

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1 \quad (1)$$

is an exact sequence of finitely generated groups. Further suppose that Q is infinite and not virtually cyclic. The group G is

(1) either a Kähler group, i.e., the fundamental group of a compact Kähler manifold,
(2) or the fundamental group of a compact complex surface.

In [BMS], together with Seshadri we look at the restrictions that these assumptions impose on the nature of G and Q.

It turns out that if in addition, Q is the fundamental group of a closed 3-manifold, then the existence of the exact sequence (1) with N finitely presented forces Q to be the fundamental group of a Seifert-fibered 3-manifold.

More precisely, let

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be an exact sequence of finitely presented groups, where Q is infinite and not virtually cyclic, and is the fundamental group of some closed 3-manifold.

If G is Kähler, it is shown in [BMS] that Q contains as a finite index subgroup either a finite index subgroup of the 3-dimensional Heisenberg group or the fundamental group of the Cartesian product of a closed oriented surface of positive genus and the circle. As a corollary of the above theorem of [BMS], a new proof of Theorem 2.1 is obtained by setting N to be the trivial group.

If G is the fundamental group of a compact complex surface, it is shown in [BMS] that Q must contain the fundamental group of a Seifert-fibered three manifold as a finite index subgroup, and G contains as a finite index subgroup the fundamental group of an elliptic fibration.

In [BMS], an example is given which shows that the relation of quasi-isometry does not preserve Kähler groups. This gives a negative answer to a question of Gromov (Problem on page 209 of [Gr93]) which asks whether Kähler groups can be characterized by their asymptotic geometry.

The following technical Proposition is the starting point of the proof. Various restrictions are imposed on Q and we deduce that G is virtually a surface group. The above case (i) follows from [De10], [Bru03], case (ii)(a) from [DP10], [CT89] and cases ii(b), (c) from the cut-Kähler Theorem of Delzant-Gromov [DeGr].

Proposition 2.2. *Let G be a Kähler group admitting a short exact sequence*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

where N is finitely generated.
i) Then Q cannot be non-nilpotent solvable.
ii) Suppose in addition that Q satisfies one of the following:
a) Q admits a discrete faithful non-elementary action on $\mathbb{H}^n$ for some $n \geq 2$
b) Q admits a discrete faithful non-elementary minimal action on a simplicial tree with more than two ends.
c) Q admits a (strong-stable) cut R such that the intersection of all conjugates of R is trivial
Then G is virtually a surface group.

As a consequence we have the following which we state for completeness:

Theorem 2.3 ([BMS]). *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

be the exact sequence (1) *such that G is a Kähler group and Q is an infinite, not virtually cyclic, fundamental group of some closed 3-manifold. Then there exists a finite index subgroup Q' of Q such that either Q' is a finite index subgroup of the* 3*-dimensional Heisenberg group or $Q' = \pi_1(\Sigma \times S^1)$, where Σ is a closed oriented surface of positive genus.*

(The 3-dimensional Heisenberg group consists of the unipotent upper triangular elements of $\mathrm{GL}(3, \mathbb{Z})$.)

The next theorem deals with the case that G is the fundamental group of a compact complex surface.

Theorem 2.4 ([BMS]). *Let*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{q} Q \longrightarrow 1$$

be the exact sequence (1) *such that G is the fundamental group of a compact complex surface and Q is an infinite, not virtually cyclic, fundamental group of some closed 3-manifold. Then there exists a finite index subgroup Q' of Q such that Q' is the fundamental group of a Seifert-fibered 3-manifold with hyperbolic or flat base orbifold. Also there exists a finite index subgroup G' of G such that G' is the fundamental group of an elliptic complex surface X which is a circle bundle over a Seifert-fibered 3-manifold.*

In Theorem 2.4, setting N to be the trivial group we conclude that Q is not the fundamental group of a compact complex surface if Q is infinite and not virtually cyclic.

Stronger results when X is of Class VII, or admits an elliptic fibration, are given in [BMS].

As a consequence of Theorems 2.3 and 2.4 we also get the following result, the first part of which was proven by J. Hillman [Hi98] based on work of Wall [Wa86]. The second part follows from Theorem 2.3 and the fact that the product of the Heisenberg group with $\mathbb{Z}$ has $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ as its abelianization.

Theorem 2.5. *Let M be a closed orientable 3 manifold. Then*

(i) $M \times S^1$ admits a complex structure if and only if M is Seifert fibered.

(ii) $M \times S^1$ admits a Kähler complex structure if and only if $M = \Sigma \times S^1$ where Σ is a compact surface.

In [Ko3], Kotschick generalized Theorem 1.2 in a different direction, by allowing compact 3-manifolds with boundary:

Theorem 2.6 ([Ko3]). *Let G be the fundamental group of an arbitrary compact three-manifold, possibly with boundary. Then G is a Kähler group, if and only if it is either finite or the fundamental group of a closed orientable surface.*

3. Quasiprojective 3 manifold groups

Quasiprojective 3-manifold groups were explored in [DPS, FrSu, Ko3]. In this paper we characterize quasiprojective 3-manifold groups.

In this Section, we follow the convention that the 3-manifolds in question have **no spherical boundary components** as capping such boundary components off by 3-balls does not change the fundamental group. The following Theorem answers Question 8.3 and Conjecture 8.4 of [FrSu] by Friedl and Suciu:

Theorem 3.1 ([BiMj2]). *Let N be a compact* 3*-manifold (with or without boundary). If $\pi_1(N)$ is a quasiprojective group, then N is either Seifert-fibered or $\pi_1(N)$ is one of the following*

- *virtually free, or*
- *virtually a surface group.*

Finer results leading to a complete characterization are given in [BiMj2]. The following theorem provides an answer to Question 8.1 of [FrSu] under mild hypotheses.

Theorem 3.2 ([BiMj2]). *Suppose A and B are groups, such that the free product $G = A * B$ is a quasiprojective group. In addition suppose that both A and B admit nontrivial finite index subgroups, and at least one of A, B has a subgroup of index greater than 2. Then each of A, B are free products of cyclic groups. In particular both A and B are quasiprojective groups.*

4. One Relator groups

In [BiMj3] we prove that infinite one–relator Kähler groups are precisely fundamental groups of (complex) one dimensional orbifolds with at most one cone–point. One–relator groups have rational cohomological dimension two. This provides a case where the answer to Question 1.6 is affirmative. A different proof of the main result of [BiMj3] was given by Kotschick [Ko2] using l^2 cohomology.

In [Ar95], Arapura asks which one-relator groups are Kähler (see [Ar95, p. 12, Section J]). This question was also raised by Amorós. In [BiMj3] we prove:

Theorem 4.1. *Let G be an infinite one-relator group. Then G is Kähler if and only if it is isomorphic to*

$$\langle a_1 , b_1 , \cdots , a_g , b_g \mid (\prod_{i=1}^{g} [a_i , b_i])^n \rangle ,$$

where g and n are some positive integers.

We show that each of the groups

$$\langle a_1 , b_1 , \cdots , a_g , b_g \mid (\prod_{i=1}^{g} [a_i , b_i])^n \rangle , \; g , n > 0,$$

can in fact be realized as the fundamental group of a smooth complex projective variety.

We also prove the following closely related result:

Theorem 4.2. *Let G be a Kähler group such that*

- *it is a coherent group of rational cohomological dimension two, and*
- *the virtual first Betti number of G is positive.*

Then G is virtually a surface group.

We give an overview of the basic strategy of the proof from [BiMj3]:

It follows from the structure theory of one-relator groups that they are described as iterated HNN extensions. The Kähler group G we are interested in therefore acts on the Bass-Serre tree T associated to the HNN splitting. If T is not quasi-isometric to the real line, it must be non-amenable and have infinitely many ends. It follows from a refinement of the theory of stable cuts of Delzant-Gromov [DeGr] using further structure of one-relator groups that G is virtually a surface group in this case.

In case T is quasi-isometric to the real line, then G must be the mapping torus of a free group. These groups are known to be coherent [FH99]. A simple cohomological dimension argument along with the structure of finitely presented normal subgroups of cd 2 groups completes the proof in this case.

The torsion in G is finally handled by further structure theory of one-relator groups.

5. Good Complexifications

A **good complexification** [Ku, To] of a closed smooth manifold M is defined to be a smooth affine algebraic variety U over the real numbers such that M is diffeomorphic to $U(\mathbb{R})$ and the inclusion

$$U(\mathbb{R}) \longrightarrow U(\mathbb{C})$$

is a homotopy equivalence [To], [Ku]. A good complexification comes naturally equipped with a natural antiholomorphic involution A on $U(\mathbb{C})$ whose fixed point set is precisely the set of real points $U(\mathbb{R})$. Kulkarni [Ku] and Totaro [To] investigate the topology of good complexifications using characteristic classes and Euler characteristic.

Theorem 5.1. [Ku, To] *Let M admit a good complexification. Then $\chi(M) \geq 0$. Further, if $\chi(M) > 0$, then $b_{2i+1}(M) = 0$ for all i.*

It should be pointed out here that in the definition of a good complexification, if one replaces "affine" or "quasiprojective" by Stein, then any closed manifold M admits a Stein complexification.

5.1. **3 Manifolds Admitting a Good Complexification.** In [To], Totaro proves that all known examples of manifolds admitting metrics of non-negative sectional curvature do admit good complexifications. He asks for the converse. In [To, p. 69, 2nd para], Totaro asks the following question:

Question 5.2. *If a closed smooth manifold M admits a good complexification, does M also admit a metric of non-negative curvature?*

In [BiMj2] we use the classification of quasiprojective 3-manifold groups to give a positive answer to Question 5.2 for 3-manifolds.

Theorem 5.3 ([BiMj2])**.** *A closed 3-manifold M admits a good complexification if and only if one of the following hold:*

1. *M admits a flat metric,*
2. *M admits a metric of constant positive curvature,*
3. *M is covered by the (metric) product of a round S^2 and $\mathbb{R}$.*

Curiously, the proof of Theorem 5.3 in [BiMj2] is direct and there is virtually no use of the method or results of [Ku, To, DPS, FrSu].

The main tools come from recent developments in 3-manifolds:

1. The Geometrization Theorem and its consequences (cf. [AFW]).
2. Largeness of 3-manifold groups [Ag, Wi, CLR, La].

The basic complex geometric tool is a theorem of Bauer, [Bau], regarding existence of irrational pencils for quasiprojective varieties. It is a useful existence result in the same genre as the classical Castelnuovo-de Franchis Theorem and a theorem of Gromov [Gr, ABCKT].

5.2. **Splitting theorem for good complexifications.** In [BMP2], we prove a Cheeger-Gromoll type splitting theorem and initiate a systematic study of fundamental groups of good complexifications:

Theorem 5.4. *Let M be a closed manifold admitting a good complexification. Then M has a finite-sheeted regular covering M_1 satisfying the following:*

1. *M_1 admits a fiber bundle structure with fiber N and base $(S^1)^d$. Here d denotes the (real) Albanese dimension of M_1.*
2. *The first virtual Betti number $vb_1(N) = 0$,*
3. *N admits a good complexification.*

Gromov [Gr81] proves that if a closed smooth manifold M of dimension n admits a metric of non-negative curvature, then there is an upper bound, that depends only on n, on the sum of the Betti numbers of M. He further conjectures, that $b_i(M) \leq b_i((S^1)^n)$. Theorem 5.5 below furnishes positive evidence towards a combination of Question 5.2 with this conjecture of Gromov by giving an affirmative answer for the first Betti number of manifolds admitting good complexifications.

We shall say that a finitely presented group G is a **good complexification group** if G can be realized as the fundamental group of a closed smooth manifold admitting a good complexification (see also [ABCKT]). We deduce from Theorem 5.4 the following critical restriction on good complexification groups:

Theorem 5.5. *Let G be a good complexification group. Then there exists a finite index subgroup G_1 of G such that two following statements hold:*

(1) *There is an exact sequence:*

$$1 \longrightarrow H \longrightarrow G_1 \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

where k can be zero.

(2) *The above H is a finitely presented good complexification group with $vb_1(H) = 0$, where $vb_1(H)$ denotes the virtual first Betti number of H.*

Recall that for a group H, the **virtual first Betti number** $vb_1(H)$ is the supremum of first Betti numbers $b_1(H_1)$ as H_1 runs over finite index subgroups of H.

The following classes of groups are then ruled out as good complexification groups:

(1) Groups with infinite vb_1, in particular large groups.
(2) Hyperbolic CAT(0) cubulated groups.
(3) Solvable groups that are not virtually abelian.
(4) 2- and 3-manifold groups that are not virtually abelian.
(5) any group admitting a surjection onto any of the above.

Question 5.2 has an affirmative answer for 2-manifolds; this is probably classical but follows also from [Ku, To]. An affirmative answer to Question 5.2 for 3-manifolds was given in [BiMj2]. As a consequence of the above restrictions, Question 5.2 has an affirmative answer for 2 and 3-manifold. Thus a new self-contained proof of the main Theorem of [BiMj2] on good complexifications is obtained in [BMP2]. We also give a number of applications to low-dimensional manifolds.

Theorem 5.6.

(1) *Question 5.2 has an affirmative answer for 2-manifolds.*
(2) *Question 5.2 has an affirmative answer for 3-manifolds* [BiMj2].
(3) *Let M be a closed simply connected 4-manifold admitting a symplectic good complexification. Then M admits a metric of non-negative curvature.*
(4) *Let M be a closed 4-manifold admitting a good complexification. Further suppose that $\pi_1(M)$ is infinite, torsion-free abelian. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^d$, where $d = 1, 2$ or 4. Moreover, the manifold M admits a finite-sheeted cover with a metric of non-negative curvature (i.e., Question 5.2 has an affirmative answer up to finite-sheeted covering).*
(5) *Let M be a closed 5-manifold admitting a good complexification. Further suppose that $\pi_1(M)$ is infinite, torsion-free abelian. Then $\pi_1(M)$ is isomorphic to $\mathbb{Z}^d$, where $d = 1, 2, 3$ or 5. Further, if $d = 2, 3$ or 5, then M admits a finite-sheeted cover M_1 homeomorphic to $S^2 \times T^3$ or $S^3 \times T^2$ or T^5. In particular, M_1 admits a metric of non-negative curvature.*

In Theorem 5.6, a manifold $M = X_{\mathbb{R}}$ is said to admit a symplectic good complexification, if its cotangent bundle is symplectomorphic to X.

6. Homological Semistability in Cohomological Dimension 2

In [BiMj1] we do some basic work on the "two-dimensional topology at infinity" of projective groups.

A connected complex manifold M is called *holomorphically convex* if for every sequence of points $\{x_i\}_{i=1}^{\infty}$ of M without any accumulation point, there is a holomorphic function f on M such that the sequence of nonnegative numbers $\{|f(x_i)|\}_{i=1}^{\infty}$ is unbounded. A natural subclass of projective groups is the class of groups that can be realized as fundamental groups of smooth complex projective varieties, of dimension at least two, with holomorphically convex universal covers. We shall call such groups **holomorphically convex groups**. It is easy to see [BiMj1] that any holomorphically convex group is the fundamental group of a smooth complex projective surface with holomorphically convex universal cover. A conjecture of Shafarevich asserts that all smooth projective varieties have holomorphically convex universal covers.

The homological semistability conjecture formulated by Geoghegan, [Gui, Conjecture 5, Section 6.4], is equivalent to the statement that $H^2(G, \mathbb{Z}G)$ is free abelian for every one-ended finitely presented group [Ge, Section 13.7], [GeMi1, GeMi2]. (Geoghegan's conjecture was formulated originally as a question in

1979 [Gui].) This conjecture has been established (in a stronger form) for several special classes of groups arising naturally in the context of geometric group theory: One-relator groups, free products of semistable groups with amalgamation along infinite groups, extensions of infinite groups by infinite groups, (Gromov) hyperbolic groups, Coxeter groups, Artin groups and so on [Mi1, Mi2, Mi3, MT1, MT2]. In [BiMj1] we establish this conjecture for holomorphically convex groups.

Theorem 6.1. *Let $G = \pi_1(X)$ be a torsion-free holomorphically convex group. Then $H^2(G, \mathbb{Z}G)$ is a free abelian group.*

A group is called linear if it is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some n. If X is a smooth complex projective variety such that $\pi_1(X)$ is linear, then the universal cover X is holomorphically convex [EKPR]. Therefore, Theorem 6.1 has the following corollary:

Corollary 6.2. *If G is a linear torsion-free projective group, then $H^2(G, \mathbb{Z}G)$ is a free abelian group.*

It is an open question whether the dualizing module of a duality group G is a free abelian group or not [Br, p. 224]. It follows from Theorem 6.1 that this is indeed the case if G is holomorphically convex of cohomological dimension two.

Proposition 6.3. *Let G be a holomorphically convex group of dimension two. Then G is a duality group with free dualizing module.*

The key ingredients in the proofs of Theorem 6.1, Corollary 6.2 and Proposition 6.3 include

(1) topology (especially second homotopy group) of smooth complex projective surfaces with holomorphically convex universal cover,
(2) a spectral sequence argument for computing group cohomology with local coefficients, which was inspired in part by an argument of Klingler [Kl],
(3) homological group theory of duality, inverse duality and Poincaré duality groups, and
(4) a theorem of Eyssidieux, Katzarkov, Pantev and Ramachandran [EKPR] showing that complex projective manifolds with linear fundamental group have holomorphically convex universal cover.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: `indranil@math.tifr.res.in`

RKM Vivekananda University, Belur Math, WB 711202, India

E-mail address: `mahan.mj@gmail.com, mahan@rkmvu.ac.in`