# A COMBINATION THEOREM FOR STRONG RELATIVE HYPERBOLICITY

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Abstract. We prove a combination theorem for trees of (strongly) relatively hyperbolic spaces and finite graphs of (strongly) relatively hyperbolic groups. This gives a geometric extension of Bestvina and Feighn's Combination Theorem for hyperbolic groups and answers a question of Swarup. We also prove a converse to the main Combination Theorem.

AMS subject classification = 20F32(Primary), 57M50(Secondary)

Contents



## 1. Introduction

In [BF92], Bestvina and Feighn proved a combination theorem for hyperbolic groups. Motivated by this, Swarup asked the analogous question [Bes04] for relatively hyperbolic groups. Dahmani [Dah03] and Alibegovic [Ali03] have proven combination theorems motivated by applications to convergence groups and limit groups (cf. Sela [Sel01]).

In this paper, we prove a **geometric combination theorem** (as opposed to a dynamical one) for trees of (strong) relatively hyperbolic metric spaces. We use Bestvina and Feighn's Combination Theorem [BF92] directly in deducing the relevant combination Theorem. The conditions we impose are quite different from those of [Dah03] and [Ali03]. Our main Theorems 4.5 and 4.7 are stated below:

**Strong Combination Theorem and converse: Theorems 4.5 , 4.7** Let $X$ be a tree ($T$) of strongly relatively hyperbolic spaces satisfying

1. the qi-embedded condition
2. the strictly type-preserving condition
3. the qi-preserving electrocution condition
4. the induced tree of coned-off spaces satisfies the **hallways flare** condition
5. the **cone-bounded hallways strictly flare** condition

Then $X$ is *strongly hyperbolic* relative to the family $\mathcal{C}$ of *maximal cone-subtrees of horosphere-like spaces*.
Conversely, if $X$ be a tree ($T$) of strongly relatively hyperbolic spaces satisfying conditions (1), (2), (3) such that $X$ is *strongly hyperbolic* relative to the family $\mathcal{C}$ of *maximal cone-subtrees of horosphere-like spaces*, then the tree of spaces satisfies conditions (4), (5).

Of the conditions given in the above theorem, Condition (1) is taken directly from [BF92]. Condition (2) roughly says that the pre-image of a horosphere-like subset (thought of as parabolic) in a vertex space (under the edge-space to vertex-space map) is either empty or a horosphere-like subset in the corresponding edge-space. This condition may be likened to the restriction to strictly type-preserving maps in the theory of Kleinian groups. Condition (2) ensures an induced tree of electrocuted spaces. Condition (3) says that the induced tree of spaces also satisfies the qi-embedded condition. Condition (4) is again taken directly from [BF92]. Condition (5) is the one essential new condition. It says roughly that a pair of geodesics whose vertices consist only of cone-points cannot lie close to each other for long. The notion of *fully quasiconvex subgroups* introduced by Dahmani [Dah03] is related to Condition (3), *the qi-preserving electrocution condition*.

As an immediate consequence of theorem 4.5, we have:

**Strong Combination Theorem for Graphs of Groups: Theorem 4.6** Let $G$ be a finite graph ($\Gamma$) of strongly relatively hyperbolic groups satisfying

1. the qi-embedded condition
2. the strictly type-preserving condition
3. the qi-preserving electrocution condition
4. the induced tree of coned-off spaces satisfies the **hallways flare** condition
5. the **cone-bounded hallways strictly flare** condition

Then $G$ is *strongly hyperbolic* relative to the family $\mathcal{C}$ of *maximal parabolic subgroups*.

All these conditions are satisfied in the classical case of a 3-manifold fibering over the circle with fiber a punctured surface The one condition that needs checking is the *hallways flare condition* for the induced tree (in fact line) of coned-off spaces. This fact is due to Bowditch (see [Bow07] Section 6). The verification involves using the associated singular structure coming from stable and unstable foliations. We shall give a slightly modified version, using an idea of Mosher [Mos98] to show this. (See Section 4.3). In fact we prove the stronger Theorem:

**Theorem 4.9:** Let $\Phi_1 \cdots \Phi_m$ be $m$ pseudo-anosov diffeomorphisms of $\Sigma$ with different sets of stable and unstable foliations. Let $H = \pi_1(\Sigma)$. Then there is an $n \geq 1$ such that the diffeomorphisms $\Phi_1^n, \cdots \Phi_m^n$ generate a free group $F$ and the group $G$ given by the exact sequence:

$$1 \to H \to G \to F \to 1$$

is (strongly) hyperbolic relative to the maximal parabolic subgroups of the form $\mathcal{Z} \times F$.

We remark here that in Dahmani's combination theorem, [Dah03], an essential condition is *acylindricity*. Again, in Alibegovic's combination theorem [Ali03], an essential assumption is the *compact intersection property*. Both acylindricity and the compact intersection property prevent infinite (or even arbitrarily long chains of parabolics from occurring). This, to us, seemed a bit unsatisfactory, as the original motivation for the Bestvina-Feighn result came from Thurston's monster theorem (See [Kap01]), and we wanted a generalization of the Bestvina-Feighn theorem that would cover the case of hyperbolic 3-manifolds with parabolics, particularly hyperbolic 3-manifolds of finite volume fibering over the circle. The hypotheses in the present paper do allow for infinite chains of parabolics and covers the above case. Our emphasis here is geometric and so the main theorem is stated in terms of spaces rather than groups.

## 2. Relative Hyperbolicity

In this section, we shall first recall certain notions of relative hyperbolicity due to Farb [Far98] and Gromov [Gro85].

2.1. **Electric Geometry.** Let $X$ be a path metric space. A collection of closed subsets $\mathcal{H} = \{H_\alpha\}$ of $X$ will be said to be **uniformly separated** if there exists $\epsilon > 0$ such that $d(H_1, H_2) \geq \epsilon$ for all distinct $H_1, H_2 \in \mathcal{H}$.

**Definition 2.1.** *(Farb* [Far98]*) The* **electric space** *(or coned-off space)* $\widehat{X}$ *corresponding to the pair* $(X, \mathcal{H})$ *is a metric space which consists of* $X$ *and a collection of vertices* $v_\alpha$ *(one for each* $H_\alpha \in \mathcal{H}$*) such that each point of* $H_\alpha$ *is joined to (coned off at)* $v_\alpha$ *by an edge of length* $\frac{1}{2}$*. The sets* $H_\alpha$ *shall be referred to as* **horosphere-like sets**.

**Definition 2.2.** • *A path $\gamma : I \to Y$ in a path metric space $Y$ is an* **ambient K-quasigeodesic** *if we have*

$$L(\beta) \leq KL(A) + K$$

*for any subsegment $\beta = \gamma|[a,b]$ and any rectifiable path $A : [a,b] \to Y$ with the same endpoints.*
• *$\gamma$ is said to be an* **electric $K, \epsilon$-quasigeodesic in (the electric space) $\widehat{X}$ without backtracking** *if $\gamma$ is an electric $K$-quasigeodesic in $\widehat{X}$ and $\gamma$ does not return to the neighborhood $N_\epsilon(H_\alpha)$ of any* horosphere-like set *$H_\alpha$ after leaving it.*

In general, a hyperbolic geodesic $\lambda$ may follow a *horosphere-like set $H_\alpha$* for a long time *without/ after/ before/ before and after* intersecting it. This is why in the definition of quasigeodesics without backtracking, we take $N_\epsilon(H_\alpha)$ rather than $H_\alpha$ itself.

We collect together certain facts about the electric metric that Farb proves in [Far98]. $N_R(Z)$ will denote the $R$-neighborhood about the subset $Z$ in the hyperbolic metric. $N^e_R(Z)$ will denote the $R$-neighborhood about the subset $Z$ in the electric metric.

Much of what Farb proved in [Far98] goes through under considerably weaker assumptions than those of [Far98]. In [Far98] the theorems were proven in the particular context of a pair $(X, \mathcal{H})$, where $X$ is a Hadamard space of pinched negative curvature with the interiors of a family of horoballs $\mathcal{H}$ removed. Then $\mathcal{H}$ can be regarded as a collection of horospheres in $X$ separated by a minimum distance from each other. In this situation, $X$ is not a hyperbolic metric space itself, but is hyperbolic relative to a collection of separated *horospheres*. Alternately, $X$ may be regarded as hyperbolic space itself (as we do here), and $\mathcal{H}$ as a uniformly separated collection of *horoballs*. Note that this gives an equivalent description, and moreover, one that is easier to formulate abstractly.

We consider therefore a hyperbolic metric space $X$ and a collection $\mathcal{H}$ of *(uniformly) $C$-quasiconvex uniformly separated subsets*, i.e. there exists $D > 0$ such that for $H_1, H_2 \in \mathcal{H}$, $d_X(H_1, H_2) \geq D$. In this situation $X$ is hyperbolic relative to the collection $\mathcal{H}$. The result in this form is due to Klarreich [Kla99]. However, the property of *Bounded Horosphere Penetration (BHP)* or *Bounded Coset Penetration (BCP)* used by Farb [Far98] was not abstracted out in Klarreich's proof as it was not necessary. What is essential for BCP (or BHP) to go through has been abstracted out by Bowditch [Bow07] [Bow97] in the case that the collection $\mathcal{H}$ is a collection of geodesics or horocycles in a Farey graph. (See also Bumagin [Bum05].) But though these things are available at the level of folklore, an explicit statement seems to be lacking.

The crucial condition can be isolated as per the following definition [Mj05c]:

**Definition:** A collection $\mathcal{H}$ of uniformly $C$-quasiconvex sets in a $\delta$-hyperbolic metric space $X$ is said to be **mutually D-cobounded** if for all $H_i, H_j \in \mathcal{H}$, $\pi_i(H_j)$ has diameter less than $D$, where $\pi_i$ denotes a nearest point projection of $X$ onto $H_i$. A collection is **mutually cobounded** if it is mutually D-cobounded for some $D$.

*Mutual coboundedness* was proven for horoballs by Farb in Lemma 4.7 of [Far98] and by Bowditch in stating that the projection of the link of a vertex onto another [Bow97] has bounded diameter in the link. However, the comparability of intersection patterns in this context needs to be stated a bit more carefully. We give the general version of Farb's theorem below and refer to [Far98] and Klarreich [Kla99] for proofs.

**Lemma 2.3.** *(See Lemma 4.5 and Proposition 4.6 of* [Far98]*, see also* [Mj05c]*) Given $\delta, C, D$ there exists $\Delta$ such that if $X$ is a $\delta$-hyperbolic metric space with a collection $\mathcal{H}$ of $C$-quasiconvex $D$-separated sets. then,*

1. *Electric quasi-geodesics electrically track hyperbolic geodesics: Given $P > 0$, there exists $K > 0$ with the following property: Let $\beta$ be any electric $P$-quasigeodesic from $x$ to $y$, and let $\gamma$ be the hyperbolic geodesic from $x$ to $y$. Then $\beta \subset N_K^e(\gamma)$.*
2. *$\gamma$ lies in a* hyperbolic *$K$-neighborhood of $N_0(\beta)$, where $N_0(\beta)$ denotes the zero neighborhood of $\beta$ in the* electric metric*.*
3. *Hyperbolicity: The electric space $\widehat{X}$ is $\Delta$-hyperbolic.*

The above Lemma motivates:

**Definition 2.4.** [Far98] [Bow97] *Let $X$ be a geodesic metric space and $\mathcal{H}$ be a collection of mutually disjoint uniformly separated subsets. Then $X$ is said to be* **weakly hyperbolic** *relative to the collection $\mathcal{H}$, if the electric space $\widehat{X}$ is hyperbolic.*

We shall need to give a general definition of geodesics and quasigeodesics without backtracking.

**Definitions:** Given a collection $\mathcal{H}$ of $C$-quasiconvex, $D$-separated sets and a number $\epsilon$ we shall say that a geodesic (resp. quasigeodesic) $\gamma$ is a geodesic (resp. quasigeodesic) **without backtracking** with respect to $\epsilon$ neighborhoods if $\gamma$ does not return to $N_\epsilon(H)$ after leaving it, for any $H \in \mathcal{H}$. A geodesic (resp. quasigeodesic) $\gamma$ is a geodesic (resp. quasigeodesic) **without backtracking** if it is a geodesic (resp. quasigeodesic) without backtracking with respect to $\epsilon$ neighborhoods for some $\epsilon \geq 0$.

**Note:** For strictly convex sets, $\epsilon = 0$ suffices, whereas for convex sets any $\epsilon > 0$ is enough.

Item (2) in the above Lemma is due to Klarreich [Kla99], where the proof is given for $\beta$ an electric geodesic, but the same proof goes through for electric quasigeodesics.

**Note:** For the above lemma, the hypothesis is that $\mathcal{H}$ consists of uniformly quasiconvex, mutually separated sets. Mutual coboundedness has not yet been used. We introduce *co-boundedness* in the next lemma.

**Lemma 2.5.** [Mj05c] *Suppose $X$ is a $\delta$-hyperbolic metric space with a collection $\mathcal{H}$ of $C$-quasiconvex $K$-separated $D$-mutually cobounded subsets. There exists $\epsilon_0 = \epsilon_0(C, K, D, \delta)$ such that the following holds:*

*Let $\beta$ be an electric $P$-quasigeodesic without backtracking and $\gamma$ a hyperbolic geodesic, both joining $x, y$. Then, given $\epsilon \geq \epsilon_0$ there exists $D = D(P, \epsilon)$ such that*

(1) *Similar Intersection Patterns 1: if precisely one of $\{\beta, \gamma\}$ meets an $\epsilon$-neighborhood $N_\epsilon(H_1)$ of an electrocuted quasiconvex set $H_1 \in \mathcal{H}$, then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$ ) from the entry point to the exit point is at most $D$.*

(2) *Similar Intersection Patterns 2: if both $\{\beta, \gamma\}$ meet some $N_\epsilon(H_1)$ then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$ ) from the entry point of $\beta$ to that of $\gamma$ is at most $D$; similarly for exit points.*

The above Lemma motivates the following definition:

**Definition 2.6.** [Far98] [Bow97] *Let $X$ be a geodesic metric space and $\mathcal{H}$ be a collection of mutually disjoint uniformly separated subsets such that $X$ is* weakly hyperbolic *relative to the collection $\mathcal{H}$. If any pair of electric quasigeodesics without backtracking starting and ending at the same point have* similar intersection patterns *with* horosphere-like sets *(elements of $\mathcal{H}$) then quasigeodesics are said to satisfy* **Bounded Penetration** *and $X$ is said to be* strongly hyperbolic *relative to the collection $\mathcal{H}$.*

We summarise the two Lemmas 2.3 and 2.5 as follows:
• If $X$ is a hyperbolic metric space and $\mathcal{H}$ a collection of uniformly quasiconvex separated subsets, then $X$ is hyperbolic relative to the collection $\mathcal{H}$.

• If $X$ is a hyperbolic metric space and $\mathcal{H}$ a collection of uniformly quasiconvex mutually cobounded separated subsets, then $X$ is hyperbolic relative to the collection $\mathcal{H}$ and satisfies *Bounded Penetration*, i.e. hyperbolic geodesics and electric quasigeodesics have similar intersection patterns in the sense of Lemma 2.5.

2.2. **Partial Electrocution.** In this subsection, we indicate, following [Mj05b], a modification of Farb's [Far98] notion of *strong relative hyperbolicity* and his construction of an electric metric, described earlier. The modification we shall discuss is called *partial electrocution* and will be used in proving the converse to the Strong Combination Theorem. Most of this discussion is taken from [Mj05b].

We start with a few motivating examples:

**Partial Electrocution** of a horosphere $H = \mathbb{R}^{n-1} \times \mathbb{R}$ will be defined as putting the zero metric in the $\mathbb{R}^{n-1}$ direction, and retaining the usual Euclidean metric in the other $\mathbb{R}$ direction.

In the partially electrocuted case, instead of coning all of a horosphere down to a point we cone only horocyclic leaves of a foliation of the horosphere. Effectively, therefore, we have a cone-line rather a cone-point.

Let $Y$ be a convex simpy connected hyperbolic n-manifold. Let $\mathcal{B}$ denote a collection of horoballs. Let $X$ denote $Y$ minus the interior of the horoballs in $\mathcal{B}$. Let $\mathcal{H}$ denote the collection of boundary horospheres.Then each $H \in \mathcal{H}$ with the induced metric is isometric to a Euclidean product $E^{n-2} \times L$ for an interval $L \subset \mathbb{R}$. Partially electrocute each $H$ by giving it the product of the zero metric with the Euclidean metric, i.e. on $E^{n-2}$ give the zero metric and on $L$ give the Euclidean metric. The resulting space is exactly what one would get by gluing to each $H$ the mapping cylinder of the projection of $H$ onto the $L$-factor.

This motivates the following scenario:

**Definition 2.7.** *Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple such that the following holds:*

1. *$X$ is (strongly) hyperbolic relative to a collection of subsets $H_\alpha$, thought of as horospheres (and* not horoballs*).*
2. *For each $H_\alpha$ there is a uniform large-scale retraction $g_alpha : H_\alpha \to L_\alpha$ to some (uniformly) $\delta$-hyperbolic metric space $L_\alpha$, i.e. there exist $\delta, K, \epsilon > 0$ such that for all $H_\alpha$ there exists a $\delta$-hyperbolic $L_\alpha$ and a map $g_\alpha : H_\alpha \to L_\alpha$ with $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \leq K d_{H_\alpha}(x, y) + \epsilon$ for all $x, y \in H_\alpha$. Further, we denote the collection of such $g_\alpha$'s as $\mathcal{G}$.*

*The* **partially electrocuted space** *or* partially coned off space *corresponding to $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ is obtained from $X$ by gluing in the (metric) mapping cylinders for the maps $g_\alpha : H_\alpha \to L_\alpha$.*

In Farb's construction $L_\alpha$ is just a single point. However, the notions and arguments of [Far98] or Klarreich [Kla99] go through even in this setting. The metric, and geodesics and quasigeodesics in the partially electrocuted space will be referred to as the partially electrocuted metric $d_{pel}$, and partially electrocuted geodesics and quasigeodesics respectively. In this situation, we conclude as in Lemmma 2.3:

**Lemma 2.8.** *$(X, d_{pel})$ is a hyperbolic metric space and the sets $L_\alpha$ are uniformly quasiconvex.*

**Note 1:** When $K_\alpha$ is a point, the last statement is a triviality.
**Note 2:** $(X, d_{pel})$ is strongly hyperbolic relative to the sets $\{L_\alpha\}$. In fact the space obtained by electrocuting the sets $L_\alpha$ in $(X, d_{pel})$ is just the space $(X, d_e)$ obtained by (completely) electrocuting the sets $\{H_\alpha\}$ in $X$.

**Note 3:** The proof of Lemma 2.8 and other such results below follow Farb's [Far98] constructions. For instance, consider a hyperbolic geodesic $\eta$ in a convex complete simply connected n-manifold $X$ with pinched negative curvature. Let $H_i$, $i = 1 \cdots k$ be the partially electrocuted horoballs it meets. Let $N(\eta)$ denote the union of $\eta$ and $H_i$'s. Let $Y$ denote $X$ minus the interiors of the $H_i$'s. The first step is to show that $N(\eta) \cap Y$ is quasiconvex in $(Y, d_{pel})$. To do this one takes a hyperbolic $R$-neighborhood of $N(\eta)$ and projects $(Y, d_{pel})$ onto it, using the hyperbolic projection. It was shown by Farb in [Far98] that the projections of all horoballs are uniformly bounded in hyperbolic diameter. (This is essentially mutual coboundedness). Hence, given $K$, choosing $R$ large enough, any path that goes out of an $R$-neighborhood of $N(\eta)$ cannot be a $K$-partially electrocuted quasigeodesic. This is the one crucial step that allows the results of [Far98], in particular, Lemma 2.8 to go through in the context of partially electrocuted spaces.

As in Lemma 2.5, partially electrocuted quasigeodesics and geodesics without backtracking have the same intersection patterns with *horospheres and boundaries of lifts of tubes* as electric geodesics without backtracking. Further, since electric geodesics and hyperbolic quasigeodesics have similar intersection patterns with *horoballs and lifts of tubes* it follows that partially electrocuted quasigeodesics and hyperbolic quasigeodesics have similar intersection patterns with *horospheres and boundaries of lifts of tubes.* We state this formally below:

**Lemma 2.9.** *Given $K, \epsilon \geq 0$, there exists $C > 0$ such that the following holds:*
*Let $\gamma_{pel}$ and $\gamma$ denote respectively a $(K, \epsilon)$ partially electrocuted quasigeodesic in $(X, d_{pel})$ and a hyperbolic $(K, \epsilon)$-quasigeodesic in $(Y, d)$ joining $a, b$. Then $\gamma \cap X$ lies in a (hyperbolic) $C$-neighborhood of (any representative of) $\gamma_{pel}$. Further, outside of a $C$-neighborhood of the horoballs that $\gamma$ meets, $\gamma$ and $\gamma_{pel}$ track each other.*

## 3. Trees of Hyperbolic Metric Spaces

3.1. **Trees of Spaces: Hyperbolic and Relatively Hyperbolic.** We start with a notion closely related to one introduced in [BF92].

**Definition 3.1.** *A tree (T) of hyperbolic (resp. strongly relatively hyperbolic) metric spaces satisfying the q(uasi) i(sometrically) embedded condition is a metric space $(X, d)$ admitting a map $P : X \to T$ onto a simplicial tree $T$, such that there exist $\delta$,$\epsilon$ and $K > 0$ satisfying the following:*

1. *For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ with the induced path metric $d_v$ is a $\delta$-hyperbolic metric space (resp. a geodesic metric space $X_v$ strongly hyperbolic relative to a collection $\mathcal{H}_{v\alpha}$). Further, the inclusions $i_v : X_v \to X$ are uniformly proper, i.e. for all $M > 0$,*

*$v \in T$ and $x, y \in X_v$, there exists $N > 0$ such that $d(i_v(x), i_v(y)) \leq M$ implies $d_v(x, y) \leq N$.*

(2) *Let $e$ be an edge of $T$ with initial and final vertices $v_1$ and $v_2$ respectively. Let $X_e$ be the pre-image under $P$ of the mid-point of $e$. Then $X_e$ with the induced path metric is $\delta$-hyperbolic (resp. a geodesic metric space $X_e$ strongly hyperbolic relative to a collection $\mathcal{H}_{e\alpha}$).*
(3) *There exist maps $f_e : X_e \times [0,1] \to X$, such that $f_e|_{X_e \times (0,1)}$ is an isometry onto the pre-image of the interior of $e$ equipped with the path metric.*
(4) *$f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ are $(K, \epsilon)$-quasi-isometric embeddings into $X_{v_1}$ and $X_{v_2}$ respectively. $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ will occassionally be referred to as $f_{v_1}$ and $f_{v_2}$ respectively.*
(5) *For a tree of strongly relatively hyperbolic spaces, we demand in addition, that the maps $f_{v_i}$ above ($i = 1, 2$) are* **strictly type-preserving**, *i.e. $f_{v_i}^{-1}(H_{v_i\alpha})$, $i = 1, 2$ (for any $H_{v_i\alpha} \in \mathcal{H}_{v_i\alpha}$) is either empty or some $H_{e\alpha} \in \mathcal{H}_{e\alpha}$.*
(6) *For a tree of strongly relatively hyperbolic spaces, we demand that the coned off spaces are uniformly $\delta$-hyperbolic.*

*$d_v$ and $d_e$ will denote path metrics on $X_v$ and $X_e$ respectively. $i_v$, $i_e$ will denote inclusion of $X_v$, $X_e$ respectively into $X$.*

*For a tree of relatively hyperbolic spaces, the sets $H_{v\alpha}$ and $H_{e\alpha}$ will be referred to as* **horosphere-like vertex sets and edge sets** *respectively.*

When $(X, d)$ is a tree (T) of strongly relatively hyperbolic metric spaces, the *strictly type-preseving condition* (Condition 5 above) ensures that we obtain an induced tree (T) (the same tree T) of *coned-off, or electric spaces.* We demand further that

• **qi-preserving electrocution condition** the induced maps of the electric edge spaces into the electric vertex spaces $\hat{f_{v_i}} : \widehat{X_e} \to \widehat{X_{v_i}}$ $(i = 1, 2)$ are uniform quasi-isometries.

The resulting tree of coned-off spaces will be called the **induced tree of coned-off spaces**. The resulting space will be denoted as $\widehat{X}$.

**Definition:** A finite graph of (strongly) relatively hyperbolic groups is said to satisfy Condition $C$, if the aasociated tree of relatively hyperbolic Cayley graphs satisfies Condition $C$. Here $C$ will be one of the following:

(1) the qi-embedded condition
(2) the strictly type-preserving condition
(3) the qi-preserving electrocution condition
(4) the induced tree of coned-off spaces satisfies the **hallways flare** condition (See below)
(5) the **cone-bounded hallways strictly flare** condition (See below)

**Remark:** Strictly speaking, this induced tree exists for any collection of vertex and edge spaces satisfying the *strictly type-preserving condition*. Hyperbolicity is not essential for the existence of the induced tree of spaces.

The **cone locus** of $\widehat{X}$, the induced tree (T) of coned-off spaces, is the graph (in fact a forest) whose vertex set $\mathcal{V}$ consists of the cone-points in the vertex set snd whose edge-set $\mathcal{E}$ consists of the cone-points in the edge set. The incidence relations are dictated by the incidence relations in $T$.

Note that connected components of the cone-locus can be naturally identified with sub-trees of $T$. Each such connected component of the cone-locus will be called a **maximal cone-subtree**. The collection of *maximal cone-subtrees* will be denoted by $\mathcal{T}$ and elements of $\mathcal{T}$ will be denoted as $T_\alpha$. Further, each maximal cone-subtree $T_\alpha$ naturally gives rise to a tree $T_\alpha$ of horosphere-like subsets depending on which cone-points arise as vertices and edges of $T_\alpha$. The metric space that $T_\alpha$ gives rise to will be denoted as $C_\alpha$ and will be referred to as a **maximal cone-subtree of horosphere-like spaces**. The collection of $C_\alpha$'s will be denoted as $\mathcal{C}$.
**Note:** Each $T_\alpha$ thus appears in two guises:

(1) as a subset of $\widehat{X}$
(2) as the underlying tree of $C_\alpha$

We shall have need for both these interpretations.

3.2. **The Bestvina-Feighn Flare Condition.** Next, we would like to recall the essential condition (due to Bestvina and Feighn [BF92]) ensuring hyperbolicity of a tree of spaces. We retain the terminology.

**Definition 3.2.** *A disk $f : [-m, m]\times I \to X$ is a hallway of length $2m$ if it satisfies:*

(1) $f^{-1}(\cup X_v : v \in T) = \{-m, \cdots, m\}\times I$
(2) *$f$ maps $i\times I$ to a geodesic in $X_v$ for some vertex space.*
(3) *$f$ is transverse, realtive to condition (1) to $\cup_e X_e$.*

**Definition 3.3.** *A hallway is $\rho$-thin if $d(f(i, t), f(i + 1, t)) \leq \rho$ for all $i, t$.*
*A hallway is* $\lambda$-hyperbolic *if*

$\lambda l(f(\{0\} \times I)) \leq max\ \{l(f(\{-m\} \times I)), l(f(\{m\} \times I))$

*A hallway is* essential *if the edge path in $T$ resulting from projecting $X$ onto $T$ does not backtrack (and is therefore a geodesic segment in the tree $T$).*

*An essential hallway of length $2m$ is* **cone-bounded** *if $f(i \times \partial I)$ lies in the cone-locus for $i = \{-m, \cdots, m\}$.*

**Definition 3.4. Hallways flare condition:** *The tree of spaces, $X$, is said to satisfy the* hallways flare *condition if there are numbers $\lambda > 1$ and $m \geq 1$ such that for all $\rho$ there is a constant $H(\rho)$ such that any $\rho$-thin essential hallway of length $2m$ and girth at least $H$ is $\lambda$-hyperbolic.*

**Definition 3.5. Cone-bounded hallways strictly flare condition:** *The tree of spaces, $X$, is said to satisfy the* hallways flare *condition if there are numbers $\lambda > 1$ and $m \geq 1$ such that any cone-bounded hallway of length $2m$ is $\lambda$-hyperbolic.*

The main theorem of Bestvina and Feighn follows. Bowditch [Bow07] notes the equivalence of the hallways flare condition with the hyperbolicity of the tree of spaces.

**Theorem 3.6.** [BF92] [Bow07] *Let $X$ be a tree of hyperbolic metric spaces satisfying the* q.i.-embedded condition *and the* hallways flare condition. *Then $X$ is hyperbolic.*
*Conversely, if $X$ is hyperbolic, then hallways flare.*

Apart, from Theorem 3.6 above, we shall need one more simple observation.

**Lemma 3.7.** *Suppose that $\widehat{X}$ is hyperbolic. Then maximal cone-subtrees $T_\alpha$ are uniformly quasi-convex in $\widehat{X}$.*

**Proof:** Let $P : X \to T$ be the natural projection map of the tree of spaces to the underlying sub-tree. Then $P$ induces $P' : \widehat{X} \to T$ as $\widehat{X}$ may be regarded as (the same) tree (T) of coned-off spaces. $P'$ is distance non-increasing. Further, restricted to each $T_\alpha$, $P'$ is an isometry.

Also note that any path from $x \in H_{v1}$ to $y \in H_{v2}$ in $\widehat{X}$ has length not less than $d_T(P'(x), P'(y))$, where $d_T$ is the natural metric on $T$ and $v_1, v_2 \in T_\alpha$.

Now suppose that $x, y \in T_\alpha \subset \widehat{X}$. Let $\gamma \subset T_\alpha \widehat{X}$ be the geodesic in $T_\alpha$ joining $x, y$. It therefore follows that for any $x, y \in T_\alpha \subset \widehat{X}$ and any path $A$ joining $x, y \in \widehat{X}$,

$$l(A) \leq d_T(P'(x), P'(y)) = l(\gamma)$$

Hence $\gamma$ is quasi-isometrically (in fact isometrically) embedded in $\widehat{X}$ and hence a geodesic in $\widehat{X}$. The lemma follows. $\Box$

## 4. The Combination Theorem

### 4.1. **Weak Combination Theorem.** We start with the following

**Theorem 4.1. Weak Combination Theorem** *Let $X$ be a tree ($T$) of strongly relatively hyperbolic spaces satisfying*

1. *the qi-embedded condition*
2. *the strictly type-preserving condition*
3. *the qi-preserving electrocution condition*
4. *the induced tree of coned-off spaces satisfies the* hallways flare *condition*

*Then $X$ is* weakly hyperbolic *relative to the family $\mathcal{C}$ of* maximal cone-subtrees of horosphere-like spaces.

**Proof:** As usual let $\widehat{X}$ denote the induced tree ($T$) of coned-off spaces, $\mathcal{T}$ denote the family of maximal cone-subtrees $T_\alpha \subset \widehat{X}$. Let $\widehat{\widehat{X}}$ denote $\widehat{X}$ with the family of sets $\mathcal{T}$ coned off (i.e. vertices $v_\alpha$ are introduced, one each for each $T_\alpha$, and joined to points of the corresponding $T_\alpha$ by edges of length $\frac{1}{2}$.)

Since vertex and edge-spaces are strongly relatively hyperbolic, then by item (6) in the definition of a tree of strongly relatively hyperbolic spaces, $\widehat{X}$ is a tree of (uniformly) hyperbolic metric spaces.

By the *qi-preserving electrocution condition*, the induced tree of coned-off spaces satisfies the qi-embedded condition.

By the *hallways flare condition* and Theorem 3.6, $\widehat{X}$ is a hyperbolic metric space.

By Lemma 3.7, the sets $T_\alpha \in \mathcal{T}$ are uniformly quasiconvex and uniformly separated.

Hence by Lemma 2.3, $\widehat{X}$ is weakly hyperbolic relative to the sets $T_\alpha \in \mathcal{T}$, i.e. $\widehat{\widehat{X}}$ is a hyperbolic metric space.

Let $\widehat{X_1}$ denote the space obtained from $X$ by coning off *maximal cone-subtrees of horosphere-like sets*. Then $\widehat{X_1}$ is quasi-isometric to $\widehat{\widehat{X}}$. (To see this, one notes that $\widehat{\widehat{X}}$ is obtained from $X$ by first coning-off or partially electrocuting $C_\alpha$'s, the maximal cone-subtrees of horosphere-like sets, to maximal cone-subtrees $T_\alpha$. This gives rise to $\hat{X}$. Further coning off the $T_\alpha$'s gives $\widehat{\widehat{X}}$. On the other hand, $\widehat{X_1}$ is obtained from $X$ by coning off or completely electrocuting the $C_\alpha$'s to points in one step. The two constructions clearly give quasi-isometric spaces.)

Hence $\widehat{X_1}$ is hyperbolic, i.e. $X$ is weakly hyperbolic relative to the collection of sets $C_\alpha \in \mathcal{C}$. $\square$

4.2. **Strong Combination Theorem.** Under the additional *cone-bounded hallways strictly flare condition*, we would now like to prove a stronger version of the combination Theorem 4.1, i.e. $X$ is **strongly hyperbolic** relative to the collection of $C_\alpha \in \mathcal{C}$:

By Lemma 2.5 and Theorem 4.1, it suffices to show that the sets $T_\alpha \subset \widehat{X}$ are mutually co-bounded. Most of the rest of this subsection is devoted to proving *mutual coboundedness*.

The next lemma follows easily from stability of quasigeodesics [GdlH90] [Gro85]. (See, for instance Lemma 4.1.1 of [Mit97].)

**Lemma 4.2.** *Give $\delta, C$, there exist $D, K, \epsilon$ such that the following holds: Let $(X,d)$ be a $\delta$-hyperbolic metric space and $Y$ a $C$-quasiconvex subset. Let $\pi$ be a nearest-point retraction of $X$ onto $Y$. Let $x, y \in X$ such that $d(\pi(x),\pi(y)) \geq D$. Then $[x,\pi(x)] \cup [\pi(x),\pi(y)] \cup [\pi(y),y]$ is a $(K,\epsilon)$-quasigeodesic.*

We use Lemma 4.2 below:

**Corollary 4.3.** *Given $\delta$, $C$, there exist $D, M$ such that the following holds: Suppose that $Y, Z$ are $C$-quasiconvex subsets of a $\delta$-hyperbolic metric space $(X, d)$. Let $\pi$ denote nearest point projection onto $Y$. If $\pi(Z)$ has diameter greater than $D$, then $\pi_Y(Z)$ lies in a $M$-neighborhood of $Z$.*

**Proof:** Let $x, y \in Z$. By Lemma 4.2, there exist $D_0, K, \epsilon$ (depending on $\delta$, $C$) such that if $d(\pi(x), \pi(y)) \geq D_0$, then $[x, \pi(x)] \cup [\pi(x), \pi(y)] \cup [\pi(y), y] = \gamma$ is a $(K, \epsilon)$-quasigeodesic. Since $Z$ is $C$-quasiconvex, $\gamma$ lies in an $M_0 = M_0(K, \epsilon, C, \delta)$-neighborhood of $Z$.

Now, choose $a, b$ in $Y, Z$ respectively, such that $d(a, b) = d(Y, Z)$. From the previous paragraph, we deduce that if $z \in Z$ such that $d(\pi(z), a) \geq D_0$, then $\pi(z)$ lies in an $M_0$-neighborhood of $Z$. Taking $D = 2D_0$ and $M = M_0 + D_0$, we are through. □

**Proposition 4.4.** *Suppose that the tree of coned-off spaces $\widehat{X}$ is hyperbolic and that the* **cone-bounded hallways strictly flare** *condition is satisfied. Then there exists $D \geq 0$ such that the family of maximal cone-subtrees $T_\alpha$ in $\widehat{X}$ is $D$-cobounded.*

**Proof:** Suppose not. Then, by Corollary 4.3, there exists $M \geq 0$ such that for any $D \geq 0$, there exist maximal cone-subtrees $T_1, T_2$ and a connected subtree $T_3 \subset T_1$ such that $T_3$ has diameter greater than $D$ and lies in an $M$-neighborhood of $T_2$. Hence, there exist geodesic edge-paths $\gamma_1 \subset T_1$ and $\gamma_2 \subset T_2$ such that each $\gamma_i$ has length greater than $(D - 2M)$ and such that they lie in an $M$-neighborhood of each other in $\widehat{X}$.

Next, $P(\gamma_i) \subset T$ (where $T$ is the tree underlying $\widehat{X}$, the tree (T) of coned-off spaces). Also, $P(\gamma_i)$ $(i = 1, 2)$ is abstractly isomorphic to $\gamma_i$ as an edge-path, since $P : \widehat{X} \to T$ is an isometry restricted to each $T_i$. Then $P(\gamma_i)$ may be regarded as geodesic paths in $T$ having lengths greater than $(D - 2M)$ and lying in an $M$-neighborhood of each other (since $P$ does not increase distances). This means that $P(\gamma_i)$ must overlap for at least a length of $(D - 4M)$.

Let $\alpha_i \subset \gamma_i$ be paths having length at least $(D-4M)$ with $P(\alpha_1) = P(\alpha_2)$. Then there exists $M_1 = M_1(M)$ and a *cone-bounded hallway* $\Delta : [-m, m] \times I \to \widehat{X}$ with $2m \geq (D - 4M)$ such that

(1) $\Delta([-m, m] \times \{0\} = \alpha_1$
(2) $\Delta([-m, m] \times \{1\} = \alpha_2$
(3) each $\Delta(j \times I)$ has length less than $M_1$

Since $D$, and hence $m$ can be arbitrarily large, while $M$ (and hence $M_1$) are fixed,it follows that for any given $\lambda > 1$, there exists a hallway $\Delta$, which is not $\lambda$-hyperbolic. This violates the *cone-bounded hallways strictly flare* condition. Hence, by contradiction, there exists $D \geq 0$ such that the family of maximal cone-subtrees $T_\alpha$ in $\widehat{X}$ is $D$-cobounded. □

We are now in a position to prove:

**Theorem 4.5. Strong Combination Theorem** *Let $X$ be a tree $(T)$ of strongly relatively hyperbolic spaces satisfying*

(1) *the qi-embedded condition*
(2) *the strictly type-preserving condition*
(3) *the qi-preserving electrocution condition*
(4) *the induced tree of coned-off spaces satisfies the* **hallways flare** *condition*
(5) *the* **cone-bounded hallways strictly flare** *condition*

*Then $X$ is* strongly hyperbolic *relative to the family $\mathcal{C}$ of* maximal cone-subtrees of horosphere-like spaces.

**Proof:** By Theorem 4.1, we know that $X$ is weakly hyperbolic relative to the family $\mathcal{C}$ of *maximal cone-subtrees of horosphere-like spaces.*

This is equivalent to saying that $\widehat{X}$ is weakly hyperbolic relative to the family $\mathcal{T}$ of *maximal cone-subtrees* $T_\alpha \subset \widehat{X}$.

By the **cone-bounded hallways strictly flare** condition and Proposition 4.4, we see that the family $\mathcal{T}$ is mutually cobounded.

Hence by Lemma 2.5, we conclude that $\widehat{X}$ is strongly hyperbolic relative to the family $\mathcal{T}$ of *maximal cone-subtrees* $T_\alpha \subset \widehat{X}$. Equivalently, $X$ is strongly hyperbolic relative to the family $\mathcal{C}$ of *maximal cone-subtrees of horosphere-like spaces.* $\Box$

Recall that a finite graph of (strongly) relatively hyperbolic groups is said to satisfy a Condition $C$, if the associated tree of relatively hyperbolic Cayley graphs also satisfies Condition $C$. The resulting group will be denoted as $G$. A quotient of *maximal cone-subtrees of horosphere-like spaces* in this case, is called a *maximal cone-subgraph of horosphere-like subgroups.* Note that such a subgraph gives rise to a subgroup of $G$. We shall refer to such subgroups as **maximal parabolic subgroups** As an immediate consequence of Theorem 4.5, we have the following:

**Theorem 4.6. Strong Combination Theorem for Graphs of Groups** *Let $G$ be a finite graph $(\Gamma)$ of strongly relatively hyperbolic groups satisfying*

(1) *the qi-embedded condition*
(2) *the strictly type-preserving condition*
(3) *the qi-preserving electrocution condition*
(4) *the induced tree of coned-off spaces satisfies the* **hallways flare** *condition*
(5) *the* **cone-bounded hallways strictly flare** *condition*

*Then $G$ is* strongly hyperbolic *relative to the family $\mathcal{C}$ of* maximal parabolic subgroups.

4.3. **Converse to the Strong Combination Theorem.** Recall that the **partially electrocuted space** or *partially coned off space* corresponding to a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ is obtained from $X$ by gluing in the (metric) mapping cylinders for the maps $g_\alpha : H_\alpha \to L_\alpha$. Note that from Theorem

4.5, it follows that $\widehat{X}$ is obtained from $X$ by partially electrocuting each $C_\alpha$. Here

(1) $H_\alpha = C_\alpha$ and $\mathcal{H} = \mathcal{C}$
(2) $L_\alpha = T_\alpha$ and $\mathcal{L} = \mathcal{T}$
(3) $g_\alpha : C_\alpha \to T_\alpha$ collapses $C_\alpha$, the tree of horosphere-like spaces to the underlying tree $T_\alpha$.

**Theorem 4.7. Converse to Strong Combination Theorem** *Let $X$ be a tree $(T)$ of strongly relatively hyperbolic spaces satisfying*

(1) *the qi-embedded condition*
(2) *the strictly type-preserving condition*
(3) *the qi-preserving electrocution condition*
(4) *$X$ is* strongly hyperbolic *relative to the family $\mathcal{C}$ of* maximal cone-subtrees *of horosphere-like spaces*

*Then the induced tree of coned-off spaces satisfies the* **hallways flare** *condition and the* **cone-bounded hallways strictly flare** *condition.*

**Proof:** As usual let $C_\alpha$ denote maximal cone sub-trees $(T_\alpha)$ of horosphere-like sets. By Lemma 2.8, the induced tree of coned-off spaces $\widehat{X}$, obtained by partially electrocuting each $C_\alpha$ to $T_\alpha$. Then by the converse part of Theorem 3.6, hallways, including cone-bounded hallways flare.

It remains to show that cone-bounded hallways *strictly* flare. Suppose not. Then there exists $D_0$ such that for all $N \in \mathbb{N}$, there exist cone-bounded hallways of length greater than $N$, bounded by "vertical" (parametrized) geodesics $\lambda_1, \lambda_2$ in distinct cone-subtrees $T_1, T_2$ respectively such that $d(\lambda_1(i), \lambda_2(i)) \leq D_0$ for all $i = 0, \cdots N$. Let $\mu_0, \mu_N$ denote "horizontal" paths in the hallway joining $\lambda_1(i), \lambda_2(i)$ for $i = 0, N$. Hence, there exist points $a_j, b_j$ $(j = 0, N)$ lying on the corresponding cones $C_1 \cap \mu_j, C_2 \cap \mu_j$, respectively such that $d(a_j, b_j) \leq D_0$. Then we have two paths:
• $\sigma_1$ starts at $a_0$, moves to $\lambda_1(0)$ (by a cone-edge of length $\frac{1}{2}$), proceeds to $\lambda_1(N)$ and exits to $a_N$ (again by a cone-edge of length $\frac{1}{2}$)
• $\sigma_2$ starts at $a_0$, moves to $b_0$ by a path of length $\leq D_0$, then to $\lambda_2(0)$ (by a cone-edge of length $\frac{1}{2}$), proceeds to $\lambda_2(N)$, exits to $b_N$ (again by a cone-edge of length $\frac{1}{2}$) and then goes to $a_N$ by a path of length $\leq D_0$.

$\sigma_1$ has length 1 in the electric metric on $\widehat{\widehat{X}}$ and $\sigma_2$ has length $\leq 2D_0 + 1$. Since $D_0$ is fixed, both $\sigma_1$ and $\sigma_2$ are uniform quasigeodesics beginning and ending at the same point, but have manifestly different intersection patterns with $C_1, C_2$. This contradicts Lemma 2.5 and hence $X$ cannot be strongly hyperbolic relative to the collection $C_\alpha$. This final contradiction proves the theorem. □

4.4. **Examples.** The first class of examples are hyperbolic 3-manifolds fibering over the circle with fiber a punctured hyperbolic surface $\Sigma$. All the conditions of Theorem 4.6 are satisfied in this case of a 3-manifold fibering over the circle with fiber a punctured surface. The one condition that needs checking is the *hallways flare condition* for the induced tree (in fact line) of coned-off spaces. This fact is due to Bowditch (see [Bow07] Section 6). We give here a somewhat different argument based on work of Mosher [Mos98].

In [Mos98], Mosher constructs examples of exact sequences of hyperbolic groups of the form:

$$1 \to H \to G \to F \to 1$$

where $H$ is a closed surface group, $G$ is hyperbolic and $F$ is free. (This construction is modified by Bestvina, Feighn and Handel [BFH97] to the case where $H$ is a free group.)

We shall modify Mosher's argument slightly to make it work for punctured surfaces.

Let $\Phi$ be a pseudoanosov diffeomorphism of a punctured hyperbolic surface $\Sigma$. Taking a suitable power of $\Phi$ if necessary, we may assume that $\Phi$ fixes all the punctures. The stable and unstable foliations of $\Phi$ give rise to a piecewise Euclidean metric on $\Sigma$. This metric is incomplete at the punctures. Complete it to get a surface with boundary $\Sigma_B$, which may be thought of as the blow-up of $\Sigma$ at the punctures. Equip $\Sigma_B$ with a pseudo-metric which is zero on all the boundary components and equal to the piecewise Euclidean metric elsewhere. This metric is discontinuous at the boundary, but this is not important. This is essentially the *electric metric* on $\Sigma_B$.

Then any electric geodesic $\lambda$ in $\Sigma_B$ is the union of two types of segments:

(1) Geodesics in the piecewise Euclidean metric meeting the boundary at right angles. Let $\lambda_{eu}$ denote this union.
(2) segments lying along the boundary.

The total length of such an electric geodesic is the sum of the lengths of the Euclidean pieces. The projection of the union of the Euclidean pieces $\lambda_{eu}$ onto the stable and unstable foliations will be denoted by $\lambda_{eus}$ and $\lambda_{euu}$ respectively. Abusing notation slightly, we assume that $\lambda_{eu}$, $\lambda_{eus}$, $\lambda_{euu}$ denote the respective lengths also. Then $max(\lambda_{eus}, \lambda_{euu}) \geq \frac{1}{2}\lambda_{eu}$. Let $\Phi(\lambda_{eu})$ denote the image of $\lambda_{eu}$ under $\Phi$. If we assume that the stable and unstable foliations meet the boundary components of $\Sigma_B$ at right angles, then it can be easily shown that for any given $k > 1$, there is an $n$ (depending on the stretch factor of $\Phi$) such that
$max(\Phi(\lambda_{eus}), \Phi^{-1}(\lambda_{euu})) \geq k\lambda_{eu}$ and hence,
$max(\Phi(\lambda_{eu}), \Phi^{-1}(\lambda_{eu})) \geq k\lambda_{eu}$.

This proves the one condition that needed checking, viz. the *hallways flare condition* for the induced tree (in fact line) of coned-off spaces.

More generally, we may take any $m$ (equal to two below for concreteness) pseudoanosov diffeomorphisms $\Phi, \Psi$ with different stable and unstable foliations. Then generalising the above construction, we can prove the following generalisation of an essential Lemma of Mosher:*3 out of 4 stretch.*

**Lemma 4.8.** *For any $k > 1$, there exists $n > 0$ such that for any electric geodesic $\lambda$ in $\Sigma_B$, at least three of the four elements $\Phi^n, \Phi^{-n}, \Psi^n, \Psi^{-n}$ stretch $\lambda$ by a factor of $\lambda$.*

Thus we get an exact sequence of groups of the form:

$$1 \to H \to G \to F \to 1$$

where $H$ is a punctured surface group, and and $F$ is free.

The Cayley graph of $G$ may thus be regarded as a tree $T$ of hyperbolic spaces, where $T$ arises as the Cayley graph of $F$. Also, the parabolics here correspond to the peripheral subgroups. Maximal cone-subtrees are each isometric to $T$. Lemma 4.8 shows that the induced tree of coned off spaces is hyperbolic.

Thus, we obtain from Theorem 4.6

**Theorem 4.9.** *Let $\Phi_1 \cdots \Phi_m$ be $m$ pseudo-anosov diffeomorphisms of $\Sigma$ with different sets of stable and unstable foliations. Let $H = \pi_1(\Sigma)$. Then there is an $n \geq 1$ such that the diffeomorphisms $\Phi_1^n, \cdots \Phi_m^n$ generate a free group $F$ and the group $G$ given by the exact sequence:*

$$1 \to H \to G \to F \to 1$$

*is (strongly) hyperbolic relative to the maximal parabolic subgroups of the form $\mathbb{Z} \times \mathbb{F}$.*

For $n = 1$, we get back Bowditch's theorem [Bow07].

4.5. **Applications, Consequences and Problems.** Theorem 4.5 and Theorem 4.6 open up the possibility of generalizing several theorems about hyperbolic groups to (strongly) relatively hyperbolic groups.

**1) Cannon-Thurston Maps**: In [Mit98], the first author proved the existence of Cannon-Thurston maps for trees of hyperbolic metric spaces. In [Mj05a], he generalized this theorem to the relatively hyperbolic case under the additional assumption that the tree of spaces gives rise to a hyperbolic 3-manifold of bounded geometry whose core is incompressible away from cusps. In [MP07], Mj-Pal prove the existence of Cannon-Thurston maps for the situation discussed in this paper, viz. trees of (strongly) relatively hyperbolic trees of metric spaces that are (strongly) relatively hyperbolic.

**2) Strongly relatively hyperbolic Extensions of Groups:** A Theorem of Mosher [Mos96] says that if an exact sequence of groups of the form

$$1 \to H \to G \to Q \to 1$$

exists, where $H$ is hyperbolic, then there exists a quasi-isometric section of $\Gamma_Q$ into $\Gamma_G$ exists. In particular, if $G$ is hyperbolic, so is $Q$. The essential technique is to use the action of $Q$ on the boundary $\partial H$ of $H$.A fact (due to Gromov [Gro85]) that is used is that the space of triples of points on the boundary of a hyperbolic group $H$ is quasi-isometric to $\Gamma_H$. An analogous result is shown by Pal in [Pal09].

**3) Heights of Groups:** In [GMRS97], Gitik, Mitra, Rips and Sageev show that quasiconvex subgroups of hyperbolic groups have finite height and finite width. A partial converse was obtained by the first author in [Mit04] for groups splitting over subgroups. This converse was used by Swarup in [Swa00] to prove a weak hyperbolization theorem. All three theorems should have analogues in the (strongly) relatively hyperbolic world. Hruska and Wise [HW06] have already shown the finiteness of height and width of quasiconvex subgroups of relatively hyperbolic groups.

**Added subsequently:** After the submission of this paper, we learnt of the paper [Gau07] which has substantial thematic overlap with the present paper.

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