HIGGS BUNDLES ON SASAKIAN MANIFOLDS

INDRANIL BISWAS AND MAHAN MJ

Abstract. We extend the Donaldson-Corlette-Hitchin-Simpson correspondence between Higgs bundles and flat connections on compact Kähler manifolds to compact quasi-regular Sasakian manifolds. A particular consequence is the translation of restrictions on Kähler groups proved using the Donaldson-Corlette-Hitchin-Simpson correspondence to fundamental groups of compact Sasakian manifolds (not necessarily quasi-regular).

Contents

1. Introduction

The aim of this paper is to extend the Donaldson-Corlette-Hitchin-Simpson correspondence between Higgs bundles and flat connections on compact Kähler manifolds to the class of compact quasi-regular Sasakian manifolds. In order to do this, we are led to the natural notions of a Higgs vector bundle on a Sasakian manifold M and that of a holomorphic Sasakian principal $G-$bundle, where G is a connected complex reductive algebraic group. Further, in order to establish a canonical correspondence, we are forced to look at a special class of Sasakian G–Higgs bundles that naturally descend to the base projective variety $M/\mathrm{U}(1)$ of the Sasakian manifold M, at least up to a finite sheeted cover of the

2010 *Mathematics Subject Classification.* 14P25, 57M05, 14F35, 20F65 (Primary); 57M50, 57M07, 20F67 (Secondary).

Key words and phrases. Sasakian manifold, Higgs bundle, flat bundle, Sasakian group.

base. Such Higgs bundles are referred to as virtually basic (Definition 4.2); see Section 5.3 for an example showing the necessity of this hypothesis. The following is the main result proved here (see Theorem 5.1).

Theorem 1.1. *Let G be a connected reductive complex affine algebraic group. Let M be a quasi-regular Sasakian manifold with fundamental group Γ. Any homomorphism*

$$\rho : \Gamma \longrightarrow G$$

with the Zariski closure of $\rho(\Gamma)$ reductive canonically gives a virtually basic polystable principal G–Higgs bundle on M with vanishing rational characteristic classes. Conversely, any virtually basic polystable principal G–Higgs bundle on M with vanishing rational characteristic classes corresponds to a flat principal G–bundle on M with the property that the Zariski closure of the monodromy representation is reductive.

Given any compact Sasakian manifold, its Riemannian structure and the Reeb vector field can be perturbed to make it quasi-regular Sasakian [Ru], [OV]. So the fundamental group of any compact Sasakian manifold is actually the fundamental group of some compact quasi-regular Sasakian manifold. Therefore, in view of Theorem 1.1 all restrictions on Kähler groups proved using the Donaldson-Corlette-Hitchin-Simpson correspondence (see [Si2, p. 53, Lemma 4.7]) are applicable to the fundamental groups of compact Sasakian manifolds.

2. Reductive representations of central extensions

We shall use below a purely group-theoretic description of the Euler class. Let

$$H = \langle g_1, \cdots, g_n \mid r_1, \cdots, r_m \rangle$$

be a finitely presented group with generators g_i, $1 \leq i \leq n$, and relations r_j, $1 \leq j \leq m$, and let X be a 2-complex realizing this presentation. Then a circle bundle $p : E \longrightarrow X$ over X corresponds to an exact sequence

$$\mathbb{Z} \longrightarrow \pi_1(E) \longrightarrow \pi_1(X) \longrightarrow 1$$

and the Euler class gives the obstruction to constructing a continuous section of the projection p. The obstruction class can be evaluated on every 2-cell in X. Group theoretically, if

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(E) \longrightarrow \pi_1(X) \longrightarrow 1$$

is a central extension, then the obstruction can be computed algebraically. Each 2-cell of X corresponds to some relation r_i. Also, if t denotes a generator for the central $\mathbb{Z}$, then we can first construct a trivialization of the circle bundle over the one-skeleton of X. Given this, the restriction of E over a 2-cell σ_i of X induces the obstruction class. Group theoretically this corresponds to $r_i = t^{n_i}$ for some integer n_i which coincides with the Euler class evaluated on the 2-cell σ_i. We use this description in Lemma 2.1.

Lemma 2.1. *Let Γ be a finitely presented central extension of a group Q by $\mathbb{Z}$ with non-zero Euler class. Let G be a connected reductive complex affine algebraic group and $\rho : \Gamma \longrightarrow G$ a homomorphism such that $\rho(\Gamma)$ is Zariski dense in G. Then $\rho(\mathbb{Z})$ is finite.*

Proof. Let

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1 \tag{2.1}$$

be the short exact sequence of the central extension Γ. Since $\rho(\Gamma)$ is Zariski dense in G, and $\rho(\mathbb{Z})$ commutes with $\rho(\Gamma)$, it follows that $\rho(\mathbb{Z})$ lies in the center of G. Let $Z(G) \subset G$ be the center. The natural homomorphism

$$h : G \longrightarrow (G/[G, G]) \times (G/Z(G))$$

is surjective with finite kernel because G is reductive. We note that $G/[G, G]$ is isomorphic to a product of copies of the multiplicative group of nonzero complex numbers, and $G/Z(G)$ is semi-simple with trivial center. Therefore, the homomorphism $h \circ \rho$ sends the central $\mathbb{Z} \subset \Gamma$ into the subgroup $G/[G, G] \times \{1\} \subset (G/[G, G]) \times (G/Z(G))$. Consequently, $h \circ \rho$ descends to a homomorphism

$$h' : Q \longrightarrow G/Z(G)\, .$$

Note that $h \circ \rho(\Gamma)$ is Zariski dense in $(G/[G, G]) \times (G/Z(G))$ because $\rho(\Gamma) \subset G$ is Zariski dense and h is surjective. Hence $h'(Q)$ is Zariski dense in $G/Z(G)$. It should be clarified that we do not exclude the possibility that the latter group is trivial. What we shall use below is the fact that $h'(Q)$ may be regarded as a subgroup of $\{1\} \times (G/Z(G)) \subset (G/[G, G]) \times (G/Z(G))$.

Let $\langle q_1, \cdots, q_m : r_1, \cdots, r_s\rangle$ be a presentation for Q. Then there exists a presentation of Γ of the form

$$\langle t, q_1, \cdots, q_m : [t, q_1], \cdots, [t, q_m], r_1 t^{i_1}, \cdots, r_s t^{i_s}\rangle\, .$$

Since the Euler class of the extension in (2.1) is non-zero, one of the t^{i_j}'s is non-zero (see the discussion on Euler class preceding the lemma). Using this and the fact that $h'(Q)$ is Zariski dense in $G/Z(G)$, it follows that

$$h \circ \rho(t^{i_j}) = h \circ \rho(r_j^{-1}) \in ((G/[G, G]) \times \{1\}) \cap (\{1\} \times (G/Z(G))) = \{1\}\, ,$$

forcing $h \circ \rho(\mathbb{Z})$ to be finite. This implies that $\rho(\mathbb{Z})$ is finite because the kernel of h is finite. □

Corollary 2.2. *Take Γ and ρ as in Lemma 2.1. Then the image $\rho(\Gamma)$ is virtually torsion-free. Hence there exists a finite index subgroup Γ_1 of Γ such that the image $\rho(\mathbb{Z} \cap \Gamma_1)$ is trivial.*

Proof. Since Γ is finitely presented, so is $\rho(\Gamma)$. As $\rho(\Gamma)$ is also linear, it follows from Malcev's theorem, [Ma], that it is residually finite. Further by Selberg's lemma [Se] (see also [Pl]) the image $\rho(\Gamma)$ is virtually torsion-free.

In other words, there exists a finite index subgroup Γ_1 of Γ such that $\rho(\Gamma_1)$ is torsion-free. Since $\rho(\mathbb{Z})$ is finite by Lemma 2.1, it follows that $\rho(\mathbb{Z} \cap \Gamma_1)$ is trivial. □

3. Sasakian groups

We refer to [BG] for definition and basic properties of Sasakian manifolds. Fundamental groups of closed Sasakian manifolds will be called *Sasakian groups*. The Sasakian structure on any Sasakian manifold M can be suitably perturbed producing a quasi-regular Sasakian structure [Ru], [OV, p. 161, Theorem 1.2]. In particular, every Sasakian group is the fundamental group of some closed quasi-regular Sasakian manifold. In view of this, henceforth all Sasakian manifolds considered here will be assumed to be quasi-regular.

Let M be a quasi-regular closed Sasakian manifold. The Kähler orbifold base $M/\mathrm{U}(1)$ of M will be denoted by B [Ru], [BG, p. 208, Theorem 7.1.3]. Let

$$f\,:\,M\,\longrightarrow\,B \tag{3.1}$$

be the quotient map. The Kähler form on B will be denoted by ω. Fix a point $x_0\,\in\,M$ such that the fiber

$$F\,:=\,f^{-1}(f(x_0))$$

of f over $f(x_0)$ is regular, meaning the action of $\mathrm{U}(1)$ on F is free. We denote

$$\Gamma\,:=\,\pi_1(M,\,x_0)\,.$$

Proposition 3.1. *Let G be a reductive complex affine algebraic group, and let*

$$\rho\,:\,\Gamma\,\longrightarrow\,G$$

be a homomorphism whose image is Zariski dense in G. Then $\rho(\pi_1(F,\,x_0))$ is a finite subgroup of G.

Proof. For the map f in (3.1), the Sasakian manifold M is a principal $\mathrm{U}(1)$–bundle over the orbifold B. We have the homotopy exact sequence for this fibration:

$$\pi_2^{orb}(B,\,f(x_0))\,\longrightarrow\,\mathbb{Z}\,=\,\pi_1(F,\,x_0)\,\stackrel{\eta}{\longrightarrow}\,\pi_1(M,\,x_0)\,\longrightarrow\,\pi_1^{orb}(B,\,f(x_0))\,\longrightarrow\,1\,,$$

where $\pi_i^{orb}(B,\,f(x_0))$ is the orbifold homotopy group of B.

If the image of $\pi_2^{orb}(B,\,f(x_0))$ in $\pi_1(F,\,x_0)$ is non-trivial, say $p\mathbb{Z}$, then we have an exact sequence

$$1\,\longrightarrow\,\eta(\pi_1(F,\,x_0))\,=\,\mathbb{Z}/p\mathbb{Z}\,\longrightarrow\,\pi_1(M,\,x_0)\,\longrightarrow\,\pi_1^{orb}(B,\,f(x_0))\,\longrightarrow\,1\,.$$

The lemma follows in this case because $\rho(\pi_1(F,\,x_0))$ is a quotient of $\mathbb{Z}/p\mathbb{Z}$.

If the image of $\pi_2^{orb}(B,\,f(x_0))$ in $\pi_1(F,\,x_0)$ is trivial, we have a short exact sequence

$$1\,\longrightarrow\,\mathbb{Z}\,\longrightarrow\,\pi_1(M,\,x_0)\,\longrightarrow\,\pi_1^{orb}(B,\,f(x_0))\,\longrightarrow\,1\,, \tag{3.2}$$

and hence $\pi_1(M,\,x_0)$ is a central extension of $\pi_1^{orb}(B)$ by $\mathbb{Z}$.

In view of Lemma 2.1, it suffices to show that the Euler class of the extension in (3.2) is non-zero. This is equivalent to showing that the first Chern class of the principal $\mathrm{U}(1)$–bundle M over B in (3.1) is non-zero.

Consider the connection ∇ on the principal $\mathrm{U}(1)$–bundle $M\,\stackrel{f}{\longrightarrow}\,B$ given by the Sasakian metric on M. So the horizontal distribution for ∇ is given by the orthogonal complement of the Reeb vector field ξ on M associated to the action of $\mathrm{U}(1)$ on it. The curvature of ∇ is the Kähler form ω on B. The cohomology class of this curvature is the first Chern class of the principal $\mathrm{U}(1)$–bundle M over B. The cohomology class $[\omega]\,\in\,H^2(B,\,\mathbb{Q})$ of ω is non-zero because ω is a Kähler form. Hence the Euler class of the extension in (3.2) is non-zero. As noted before, this completes the proof using Lemma 2.1. □

Corollary 3.2. *Let $\rho : \Gamma \longrightarrow G$ be a Zariski dense representation, where G as before is a complex connected reductive affine algebraic group. Then M admits a finite-sheeted unramified cover M_1 such that for any fiber F_1 (not necessarily smooth) of the induced quasi-regular Sasakian structure on M_1, the image $\rho(\pi_1(F_1))$ is trivial. (The image of $\pi_1(F_1)$ in $\pi_1(M, x_0)$ is unique up to a conjugation — it depends on the choice of a base point in F_1 and a homotopy class of path from x_0 to the image of the base point in M.)*

The same conclusion holds for a representation $\rho : \Gamma \longrightarrow G$ such that the Zariski closure of $\rho(\Gamma)$ in G is reductive.

Proof. Take M_1 to be the étale Galois covering of M for the finite index subgroup $\Gamma_1 \subset \Gamma$ in the proof of Corollary 2.2. So $\rho(\pi_1(M_1))$ is torsion-free. The image $\rho(\pi_1(F, x_0))$ is finite by Proposition 3.1, and hence $\rho(\pi_1(F_1))$ is also a finite group. Therefore, $\rho(\pi_1(F_1))$ is trivial because it is also torsion-free.

If the Zariski closure of $\rho(\Gamma)$ in G is reductive, then replace G by the connected component of the Zariski closure of $\rho(\Gamma)$ in G containing the identity element, and also replace M by the étale Galois covering M' of M such that $\rho(\pi_1(M'))$ is contained in the above connected component. Now the second part follows from the first part. □

Let $\rho : \Gamma \longrightarrow G$ be a homomorphism with G as above. The Zariski closure of $\rho(\Gamma)$ in G will be denoted by $\overline{\rho(\Gamma)}$. The connected component of $\overline{\rho(\Gamma)}$ containing the identity element will be denoted by $\overline{\rho(\Gamma)}_0$.

Lemma 3.3. *If $\overline{\rho(\Gamma)}_0$ is not reductive, then $\overline{\rho(\Gamma)}_0$ is contained in some proper parabolic subgroup of G.*

Proof. Since $\overline{\rho(\Gamma)}_0$ is not reductive, the unipotent radical of it is nontrivial. Denote the unipotent radical $R_u(\overline{\rho(\Gamma)}_0)$ by S_0. The normalizer of S_0 in G will be denoted by N_1. The unipotent radical of N_1 will be denoted by S_1. Inductively, define S_i to be the unipotent radical of N_i, and N_{j+1} to be the normalizer of S_j in G. We have

$$\cdots \subset S_i \subset S_{i+1} \subset \cdots \subset N_{j+1} \subset N_j \subset \cdots$$

and $\bigcup_i S_i$ is the unipotent radical of $\bigcap_j N_j$, while $\bigcap_j N_j$ is the normalizer of $\bigcup_i S_i$ in G. Therefore, $\bigcap_j N_j$ is a parabolic subgroup of G; see [Hu, p. 185, § 30.3] for more details. Note that $\bigcap_j N_j$ is a proper subgroup of G because its unipotent radical $\bigcup_i S_i$ is nontrivial as it contains S_0. □

We now give a criterion for $\overline{\rho(\Gamma)}$ to be reductive under the assumption that $G = \mathrm{GL}(r, \mathbb{C})$. For a homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C})$$

the Zariski closure $\overline{\rho(\Gamma)}$ is reductive if and only if $\mathbb{C}^r$ decomposes into a direct sum of irreducible $\rho(\Gamma)$–modules. Indeed, if $\overline{\rho(\Gamma)}$ is reductive, then the $\overline{\rho(\Gamma)}$–module $\mathbb{C}^r$ decomposes into a direct sum of irreducible $\overline{\rho(\Gamma)}$–modules. Hence $\mathbb{C}^r$ decomposes into a direct sum of irreducible $\rho(\Gamma)$–modules. Conversely, if $\mathbb{C}^r$ decomposes into a direct sum of irreducible $\rho(\Gamma)$–modules, then from Lemma 3.3 it follows that $\overline{\rho(\Gamma)}$ is reductive.

4. Sasakian Higgs bundles

4.1. **Partial connection.** Take a connected C^∞ manifold X. Take any C^∞ subbundle of positive rank

$$S \subset TX \otimes_{\mathbb{R}} \mathbb{C}$$

which is closed under the operation of Lie bracket of vector fields; such a subbundle is called integrable. We have the dual of the inclusion map of S in $TX \otimes_{\mathbb{R}} \mathbb{C}$

$$q_S : T^*X \otimes_{\mathbb{R}} \mathbb{C} = (TX \otimes_{\mathbb{R}} \mathbb{C})^* \longrightarrow S^* . \tag{4.1}$$

A *partial connection on E in the direction of S* is a C^∞ differential operator

$$D : E \longrightarrow S^* \otimes E$$

satisfying the Leibniz condition, which says that

$$D(fs) = fD(s) + q_S(df) \otimes s$$

for a smooth section s of E and a smooth function f on X, where q_S is the projection in (4.1).

Since the distribution S is integrable, the smooth sections of ideal subbundle of the exterior algebra bundle $\bigwedge T(^*X \otimes_{\mathbb{R}} \mathbb{C})$ generated by kernel(q_S) is closed under the exterior derivation. Therefore, we have an induced exterior derivation acting on the smooth sections of S^*

$$\widehat{d} : S^* \longrightarrow \bigwedge^2 S^* \tag{4.2}$$

which is a differential operator of order one.

Let D be a partial connection on E in the direction of S. Consider the differential operator

$$D_1 : S^* \otimes E \longrightarrow (\bigwedge^2 S^*) \otimes E$$

defined by

$$D_1(\theta \otimes s) = \widehat{d}(\theta) \otimes s - \theta \wedge D(s) ,$$

where $\widehat{d}$ is constructed in (4.2). The composition

$$E \xrightarrow{D} S^* \otimes E \xrightarrow{D_1} (\bigwedge^2 S^*) \otimes E \tag{4.3}$$

is $C^\infty(X)$–linear. Therefore, the composition in (4.3) defines a C^∞ section

$$\mathcal{K}(D) = C^\infty(X, (\bigwedge^2 S^*) \otimes E \otimes E^*) = C^\infty(X, (\bigwedge^2 S^*) \otimes \mathrm{End}(E)) . \tag{4.4}$$

The section $\mathcal{K}(D)$ in (4.4) is called the *curvature* of D. If

$$\mathcal{K}(D) = 0 ,$$

then the partial connection D is called *flat*.

4.2. **Holomorphic hermitian vector bundles.** Let M be a compact quasi-regular Sasakian manifold. The Riemannian metric and the Reeb vector field on M will be denoted by g and ξ respectively. The almost complex structure on the orthogonal complement

$$\xi^{\perp} \subset TM$$

for g produces a type decomposition

$$\xi^{\perp} \otimes_{\mathbb{R}} \mathbb{C} \,=\, F^{1,0} \oplus F^{0,1}\, .$$

Define $F^{p,q} \,:=\, (\bigwedge^p F^{1,0}) \otimes (\bigwedge^q F^{0,1})$. Let

$$\widetilde{F}^{0,1} \,:=\, F^{0,1} \oplus (\xi \otimes_{\mathbb{R}} \mathbb{C}) \,\subset\, TM \otimes_{\mathbb{R}} \mathbb{C} \tag{4.5}$$

be the distribution. It is known that this distribution $\widetilde{F}^{0,1}$ is integrable [BSc, p. 550, Lemma 3.2].

A *Sasakian complex vector bundle* on the Sasakian manifold $(M,\, g,\, \xi)$ is a pair $(E,\, D_0)$, where E is a C^∞ complex vector bundle on M, and D_0 is a partial connection on E in the direction ξ.

A *hermitian structure* on a Sasakian complex vector bundle $(E,\, D_0)$ is a C^∞ hermitian structure on the complex vector bundle E preserved by the partial connection D_0.

Definition 4.1. A *holomorphic structure* on a Sasakian complex vector bundle $(E,\, D_0)$ is a flat partial connection D on E in the direction of $\widetilde{F}^{0,1}$ (constructed in (4.5)) satisfying the compatibility condition that D_0 coincides with the partial connection on E, in the direction of ξ, defined by D.

A *Sasakian holomorphic vector bundle* is a Sasakian complex vector bundle equipped with a holomorphic structure.

Let $(E,\, D)$ be a Sasakian holomorphic vector bundle on M equipped with a hermitian metric h. There is a unique connection ∇ on the complex vector bundle E satisfying the following two conditions:

(1) ∇ preserves h, and
(2) the partial connection on E in the direction of $\widetilde{F}^{0,1}$ induced by ∇ coincides with D.

The second condition is equivalent to the condition that the curvature of ∇ is a smooth section of $(F^{1,1})^* \otimes E \otimes E^*$.

Let $\mathcal{K}(E, h) \,:=\, \mathcal{K}(\nabla)$ be the curvature of the above connection ∇; as mentioned above, it is a section of $(F^{1,1})^* \otimes E \otimes E^*$.

Let $(E,\, D_E)$ and $(E',\, D_{E'})$ be two Sasakian holomorphic vector bundles on $(M,\, g,\, \xi)$. A fiberwise $\mathbb{C}$–linear C^∞ map

$$\Psi \,:\, E' \,\longrightarrow\, E''$$

is called *holomorphic*, if Ψ intertwines D_E and $D_{E'}$. A holomorphic section of $(E',\, D_{E'})$ is a holomorphic homomorphism to it from the trivial complex line bundle on M equipped with the trivial Sasakian holomorphic structure given by the trivial partial connection on it in the direction of $\widetilde{F}^{0,1}$.

Using the Levi–Civita connection on M for g, the vector bundle $(F^{1,0})^*$ gets a flat partial connection along $\widetilde{F}^{0,1}$, thus making $(F^{1,0})^*$ a Sasakian holomorphic vector bundle. Let E be a Sasakian holomorphic vector bundle on $(M,\, g,\, \xi)$. Then $\text{End}(E)$ also has the structure of a Sasakian holomorphic vector bundle. Hence $\text{End}(E)\otimes(F^{1,0})^*$ is a Sasakian holomorphic vector bundle.

A Higgs field on a Sasakian holomorphic vector bundle $(E,\, D_E)$ is a holomorphic section θ of $\text{End}(E)\otimes(F^{1,0})^*$ such that the section $\theta\wedge\theta$ of $\text{End}(E)\otimes(F^{2,0})^*$ vanishes identically. A *Higgs vector bundle* on the Sasakian manifold M is a Sasakian holomorphic vector bundle on M equipped with a Higgs field.

Let H be a Lie group and $q\,:\, E_H\,\longrightarrow\, M$ a C^∞ principal H–bundle on M. Let

$$dq\,:\, TE_H\,\longrightarrow\, q^*TM$$

be the differential of q. A partial connection on E_H in the direction of ξ is a H–equivariant homomorphism

$$D_0\,:\, q^*\xi\,\longrightarrow\, q^*TE_H$$

such that $(dq)\circ D_0$ coincides with the identity map of $q^*\xi$. In other words, D_0 is a H–equivariant lift of ξ to the total space of E_H. A *Sasakian principal H–bundle* on M is a C^∞ principal H–bundle E_H on M equipped with a partial connection in the direction of ξ. Let H be a complex Lie group and $(E_H,\, D_0)$ a Sasakian principal H–bundle on M. A holomorphic structure on E_H is an H–invariant lift D of the subbundle $\widetilde{F}^{0,1}\,\subset\, TM\otimes_{\mathbb R}\mathbb C$ (see (4.5)) to $TE_H\otimes_{\mathbb R}\mathbb C$ such that restriction of D to ξ coincides with the given lift D_0. A *holomorphic Sasakian principal H–bundle* is a Sasakian principal H–bundle equipped with a holomorphic structure.

Let $(E_H,\, D)$ be a holomorphic Sasakian principal H–bundle on M, and let H_1 be a complex Lie subgroup of H. A C^∞ reduction of structure group $E_{H_1}\,\subset\, E_H$ to H_1 is called *holomorphic* if for every $z\,\in\, E_{H_1}$, the image of $\widetilde{F}^{0,1}$ in $T_zE_H\otimes\mathbb C$ under D is contained in $T_zE_{H_1}\otimes\mathbb C$.

Let G be a connected complex reductive affine algebraic group. Fix a maximal compact subgroup

$$K_G\,\subset\, G\,.$$

Let $(E_G,\, D)$ be a Sasakian holomorphic principal G–bundle. A Hermitian structure on it is a C^∞ reduction of structure group

$$E_G\,\supset\, E_K\,\stackrel{q}{\longrightarrow}\, M$$

such that for any $z\,\in\, E_K$, the image $D(\xi(q(z)))\,\in\, T_zE_G$ is contained in T_zE_K. Note that this condition is equivalent to the condition that D induces a Sasakian principal K–bundle structure on E_K.

For any Hermitian structure E_K as above, there is a unique connection ∇ on the principal K–bundle E_K such that for every $z\,\in\, E_K$, the lift of $\widetilde{F}^{0,1}(z)$ to $T_zE_K\otimes_{\mathbb R}\mathbb C$ given by ∇ coincides with the one given by D. This condition is equivalent to to the condition that the curvature of ∇ is a section of $(F^{1,1})^*\otimes\text{ad}(E_G)$, where

$$\text{ad}(E_G)\,=\, E_G\times^G\mathfrak g$$

is the vector bundle on M associated to E_G for the adjoint action of G on its Lie algebra $\mathfrak g$; this $\text{ad}(E_G)$ is also called the adjoint vector bundle for E_G.

Let $(E_G,\, D_0)$ be a Sasakian principal G–bundle. Note that any vector bundle associated to E_G has the structure of a Sasakian vector bundle using D_0. In particular, the adjoint vector bundle $\text{ad}(E_G)$ is a Sasakian vector bundle. A holomorphic structure D on $(E_G,\, D_0)$ produces a holomorphic structure on any associated vector bundle.

A Higgs field on a Sasakian holomorphic principal G–bundle $(E_G,\, D)$ is a holomorphic section of $\text{ad}(E_G) \otimes (F^{1,0})^*$ such that the section $\theta \wedge \theta$ of $\text{ad}(E) \otimes (\bigwedge^2 F^{1,0})^*$ vanishes identically.

Let $((E_G,\, D),\, \theta)$ be a Sasakian G–Higgs bundle, and let $P \subsetneq G$ be a maximal parabolic subgroup. A reduction E_P of $((E_G,\, D),\, \theta)$ to P over a big open subset U means the following:

- $U \subset M$ is an open subset preserved by the action of $\text{U}(1)$ such that the complement $B \setminus f(U)$ (the map f is defined in (3.1)) is a complex analytic subset of complex codimension at least two,
- $E_P \subset E_G|_U$ is a holomorphic reduction of structure group over U, and
- the Higgs field θ is a section of the subbundle $\text{ad}(E_P)\otimes(F^{1,0})^* \subset \text{ad}(E_G)\otimes(F^{1,0})^*$.

A Sasakian G–Higgs bundle $((E_G,\, D),\, \theta)$ is called *stable* (respectively, *semistable*) if for all maximal parabolic subgroup $P \subsetneq G$ and every reduction E_P of $((E_G,\, D),\, \theta)$ to P over every big open subset U,

$$\text{degree}(\text{ad}(E_P)) \,<\, 0 \quad (\text{respectively, } \;\text{degree}(\text{ad}(E_P)) \,\leq\, 0)\,.$$

A Sasakian G–Higgs bundle $((E_G,\, D),\, \theta)$ is called *polystable* if the following conditions hold:

1. $((E_G,\, D),\, \theta)$ is semistable, and
2. there is a Levi subgroup L of some parabolic subgroup of G and a holomorphic reduction $E_L \subset E_G$ to L, such that
   - θ is a section of the subbundle $\text{ad}(E_L) \otimes (F^{1,0})^* \subset \text{ad}(E_G) \otimes (F^{1,0})^*$, and
   - $((E_L,\, D),\, \theta)$ is stable.

Definition 4.2. A Sasakian G–Higgs bundle $(V,\, \theta)$ on M will be called *basic* if $(V,\, \theta)$ descends to the projective variety $M/\text{U}(1)$. In other words, there is a Higgs G–Higgs bundle $(W,\, \theta')$ on $M/\text{U}(1)$ such that $(f^*W,\, f^*\theta')$ equipped with the natural Sasakian G–Higgs bundle structure is isomorphic $(V,\, \theta)$.

The characteristic classes of a basic Sasakian G–Higgs bundle $(V,\, \theta)$ are defined to be the corresponding characteristic classes of the above principal G–bundle W.

A Sasakian G–Higgs bundle $(V,\, \theta)$ on M will be called *virtually basic* if there is finite étale Galois covering $\widetilde{M} \longrightarrow M$ such that the pullback of $(V,\, \theta)$ to $\widetilde{M}$ is basic. Note that $\widetilde{M}$ is also a quasi-regular Sasakian manifold.

The characteristic classes of a virtually basic Sasakian G–Higgs bundle $(V,\, \theta)$ on M are defined to be the corresponding characteristic classes of the basic Sasakian G–Higgs bundle on $\widetilde{M}$.

Note that the condition that the rational characteristic classes of a virtually basic Sasakian G–Higgs bundle on M vanish does not depend on the choice of the covering $\widetilde{M}$. This condition can be interpreted in terms of equivariant cohomology.

An example in Section 5.3 shows why the above class of Higgs bundles is relevant.

5. The Flat Connection-Higgs Bundle Correspondence

The purpose of this Section is to establish the following main Theorem of this paper:

Theorem 5.1. *Let G be a connected reductive complex affine algebraic group. Let M be a quasi-regular Sasakian manifold with fundamental group Γ. Any homomorphism*

$$\rho : \Gamma \longrightarrow G$$

with the Zariski closure of $\rho(\Gamma)$ reductive canonically gives a virtually basic polystable principal G–Higgs bundle on M with vanishing rational characteristic classes. Conversely, any virtually basic polystable principal G–Higgs bundle on M with vanishing rational characteristic classes corresponds to a flat principal G–bundle on M with the property that the Zariski closure of the monodromy representation is reductive.

Section 5.1 proves the forward direction and Section 5.2 establishes the converse direction of the above Theorem.

5.1. **From flat connections to Higgs bundles.** As before, G is a connected reductive complex affine algebraic group. Let

$$\rho : \Gamma \longrightarrow G$$

be a homomorphism such that the Zariski closure $\overline{\rho(\Gamma)}$ is reductive.

Let

$$\varphi : \widetilde{M} \longrightarrow M \tag{5.1}$$

be a finite étale Galois covering. Then $\widetilde{M}$ is also a quasi-regular Sasakian manifold. The Reeb vector field on $\widetilde{M}$, which is simply the inverse image of ξ, will be denoted by $\widetilde{\xi}$; note that the differential

$$d\varphi : T\widetilde{M} \longrightarrow \varphi^* TM$$

is an isomorphism. Fix a point $\widetilde{x}_0 \in \widetilde{M}$ over x_0. Each orbit of $\widetilde{\xi}$ defines a conjugacy class in $\pi_1(\widetilde{M}, \widetilde{x}_0)$; however, all regular orbits define the same conjugacy class. The normal subgroup of $\pi_1(\widetilde{M}, \widetilde{x}_0)$ generated by all the conjugacy classes given by the orbits of $\widetilde{\xi}$ will be denoted by Γ_0.

From Corollary 3.2 we know that there is a finite étale Galois covering φ as above such that the composition

$$\Gamma_0 \hookrightarrow \pi_1(\widetilde{M}, \widetilde{x}_0) \xrightarrow{\varphi_*} \Gamma \xrightarrow{\rho} G$$

is the trivial homomorphism. Fix such a covering φ.

Let

$$\rho' : \pi_1(\widetilde{M}, \widetilde{x}_0) \longrightarrow G$$

be the composition $\pi_1(\widetilde{M}, \widetilde{x}_0) \xrightarrow{\varphi_*} \Gamma \xrightarrow{\rho} G$. For notational convenience, the Galois group $\mathrm{Gal}(\varphi)$ for the covering φ will henceforth be denoted by Π.

Since $\rho'|_{\Gamma_0}$ is the trivial homomorphism, the homomorphism ρ' descends to a homomorphism, to G, from the fundamental group of the quotient space $\widetilde{B} := \widetilde{M}/\mathrm{U}(1)$.

It should be clarified that here we consider $\widetilde{B}$ just as a quotient space and not as an orbifold. Note that the necessary and sufficient condition for the homomorphism ρ' to descend to a homomorphism to G from the fundamental group of the orbifold $\widetilde{M}/\mathrm{U}(1)$ is that the restriction of ρ' to the normal subgroup of $\pi_1(\widetilde{M},\, \widetilde{x}_0)$ generated by the regular orbits of $\widetilde{\xi}$ is trivial. The stronger condition that $\rho'|_{\Gamma_0}$ is trivial ensures that ρ' descends to a homomorphism, to G, from the fundamental group of the quotient space $\widetilde{B}$.

The image of $\widetilde{x}_0$ in $\widetilde{B}$ will be denoted by y_0. Let

$$\rho'_0 \,:\, \pi_1(\widetilde{B},\, y_0) \longrightarrow G$$

be the homomorphism given by ρ'. Let

$$\beta \,:\, (E'_G,\, \nabla) \longrightarrow \widetilde{B} \tag{5.2}$$

be the associated flat principal G–bundle.

The map φ in (5.1) descends to a map

$$\widetilde{\varphi} \,:\, \widetilde{M}/\mathrm{U}(1) \,=\, \widetilde{B} \longrightarrow M/\mathrm{U}(1) \,=\, B\,. \tag{5.3}$$

The action of the Galois group Π on $\widetilde{M}$ descends to $\widetilde{B}$. Consequently, $\widetilde{\varphi}$ in (5.3) is an étale Galois covering with Galois group Π.

Let

$$\widetilde{f} \,:\, \widetilde{M} \longrightarrow \widetilde{B} \tag{5.4}$$

be the quotient map. Since the flat principal G–bundle

$$(\widetilde{f}^* E'_G,\, \widetilde{f}^*\nabla) \longrightarrow \widetilde{M}$$

(see (5.2)) is the pull back of the flat principal G–bundle on M associated to ρ. Therefore, $(\widetilde{f}^* E'_G,\, \widetilde{f}^*\nabla)$ is Π–equivariant, meaning it is equipped with a lift of the Galois action of Π on $\widetilde{M}$. This action of Π on $(\widetilde{f}^* E'_G,\, \widetilde{f}^*\nabla)$ descends to an action of Π on $(E'_G,\, \nabla)$, because the map $\widetilde{f}$ is Π–equivariant. Consequently, $(E'_G,\, \nabla)$ is a Π–equivariant flat principal G–bundle.

Take a desingularization

$$\delta \,:\, Z \longrightarrow \widetilde{B}\,.$$

Let $(V,\, \theta)$ be the polystable G–Higgs bundle on Z corresponding to the flat principal G–bundle $(\delta^* E'_G,\, \delta^*\nabla)$ on Z [Co], [Do], [Si2], [BSu]. All the rational characteristic classes of V vanish because the principal G–bundle V is topologically isomorphic to $\delta^* E'_G$. The restriction of $(\delta^* E'_G,\, \delta^*\nabla)$ to any fiber of δ is trivial. Hence $(V,\, \theta)$ descends to $\widetilde{B}$; this descended G–Higgs bundle on $\widetilde{B}$ will be denoted by $(V_0,\, \theta_0)$ (see [ES]). The action of the Galois group Π on $(E'_G,\, \nabla)$ produces an action of Π on $(V_0,\, \theta_0)$.

Therefore, the pullback $(\widetilde{f}^* V_0,\, \widetilde{f}^*\theta_0)$ by the map $\widetilde{f}$ in (5.4) is a Π–equivariant principal G–Higgs bundle. Hence, $(\widetilde{f}^* V_0,\, \widetilde{f}^*\theta_0)$ descends to M.

Note that the above constructed G–Higgs bundle on M is virtually basic; see Definition 4.2.All the rational characteristic classes of this virtually basic G–Higgs bundle on M vanish because all the rational characteristic classes of V vanish (see Definition 4.2).

5.2. **From Higgs bundles to flat connections.** Let $(V,\, \theta)$ be a Sasakian virtually basic G–Higgs bundle on M such that all the rational characteristic classes of V vanish. Fix a Galois étale covering

$$\varphi\, :\, \widetilde{M}\, \longrightarrow\, M$$

such that $(\varphi^*V,\, \varphi^*\theta)$ is basic. Let $(V',\, \theta')$ be the G–Higgs bundle on $\widetilde{B}\, :=\, \widetilde{M}/\mathrm{U}(1)$. As before, the Galois group $\mathrm{Gal}(\varphi)$ will be denoted by Π.

Fix a desingularization

$$\delta\, :\, Z\, \longrightarrow\, \widetilde{B}\, .$$

The Higgs G–bundle $(\delta^*V',\, \delta^*\theta')$ corresponds to a flat principal G–bundle $(F_G,\, \nabla)$ on Z [Si1], [Si2], [Hi], [BSu]. The Zariski closure of the monodromy representation for ∇ is reductive. Since

$$\delta_*\, :\, \pi_1(Z)\, \longrightarrow\, \pi_1(\widetilde{B})$$

is an isomorphism [Ko, p. 203, Theorem (7.5.2)], the above flat principal G–bundle $(F_G,\, \nabla)$ descends to a flat principal G–bundle $(F'_G,\, \nabla')$ on $\widetilde{B}$; see [ES].

The above flat principal G–bundle $(F'_G,\, \nabla')$ on $\widetilde{B}$ is Π–equivariant, because $(V',\, \theta')$ is Π–equivariant. Hence the pullback $(\widetilde{f}^*F'_G,\, \widetilde{f}^*\nabla')$ is also Π–equivariant, where $\widetilde{f}$, as before, is the quotient map $\widetilde{M}\, \longrightarrow\, \widetilde{M}/\mathrm{U}(1)$. Consequently, this flat principal G–bundle $(\widetilde{f}^*F'_G,\, \widetilde{f}^*\nabla')$ descends to a flat principal G–bundle on M. The monodromy representation for this flat principal G–bundle on M is reductive because the monodromy representation for ∇ is reductive.

5.3. **An example.** We now give an example which shows that the condition "virtually basic" in Definition 4.2 is essential.

Consider the unit three-sphere

$$M\, =\, S^3\, :=\, \{(z_1,\, z_2)\, \in\, \mathbb{C}^2\, \mid\, |z_1|^2+|z_2|^2\, =\, 1\}$$

equipped with the standard Sasakian structure. The Riemannian metric is the one obtained by restricting the standard metric on $\mathbb{C}^2$. The Reeb vector field is given by the diagonal action of $\mathrm{U}(1)$ on $\mathbb{C}^2$ for the standard action of it on $\mathbb{C}$. So the action of any $c\, \in\, \mathrm{U}(1)$ sends any $(z_1,\, z_2)\, \in\, \mathbb{C}^2$ to $(c\cdot z_1,\, c\cdot z_2)$. So in (3.1), the surface B is $\mathbb{CP}^1$, and f defines the Hopf fibration on it.

Take any integer $n\, \neq\, 0$, and consider the nontrivial line bundle $L\, :=\, \mathcal{O}_{\mathbb{CP}^1}(n)$ on $\mathbb{CP}^1$ of degree n. The pullback f^*L has a tautological action of $\mathrm{U}(1)$ simply because it is pulled back from $\mathbb{CP}^1$. Let $\mathbb{L}$ denote this holomorphic line bundle on the Sasakian manifold M. Let $\mathbb{L}_0$ denote the trivial holomorphic line bundle on M equipped with the trivial $\mathrm{U}(1)$–action. We note that $\mathbb{L}$ is not holomorphically isomorphic to $\mathbb{L}_0$ because $\mathbb{L}$ descends to L on $\mathbb{CP}^1$, while $\mathbb{L}_0$ descends to the trivial holomorphic line bundle on $\mathbb{CP}^1$. Also note that the complex line bundle f^*L is topologically trivial because $H^2(M,\, \mathbb{Z})\, =\, 0$. Therefore, $(\mathbb{L},\, 0)$ is a nontrivial Higgs line bundle on M of vanishing characteristic classes; being of rank one $(\mathbb{L},\, 0)$ is polystable.

On the other hand, there is no nontrivial flat line bundle on S^3 as it is simply connected. So Theorem 5.1 is not valid for S^3 if we drop the condition "virtually basic" in the statement.

Acknowledgements

We thank Carlos Simpson for a useful discussion. The authors acknowledge the support of their respective J. C. Bose Fellowships.

References

[BSc] I. Biswas and G. Schumacher, Vector bundles on Sasakian manifolds, *Adv. Theo. Math. Phy.* **14** (2010), 541–561.

[BSu] I. Biswas and S. Subramanian, Flat holomorphic connections on principal bundles over a projective manifold, *Trans. Amer. Math. Soc.* **356** (2004), 3995–4018.

[BG] C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[Co] K. Corlette, Flat G-bundles with canonical metrics, *Jour. Diff. Geom.* **28** (1988), 361–382.

[Do] S. K. Donaldson, Twisted harmonic maps and the self-duality equations, *Proc. London Math. Soc.* **55** (1987), 127–131.

[ES] P. Eyssidieux and F. Sala, Instantons and framed sheaves on Kähler Deligne-Mumford stacks, arXiv:1404.3504 [math.AG].

[Hi] N. J. Hitchin, The self–duality equations on a Riemann surface, *Proc. London Math. Soc.* **55** (1987), 59–126.

[Hu] J. E. Humphreys, *Linear algebraic groups,* Graduate Texts in Mathematics, Vol. 21, Springer-Verlag, New York, Heidelberg, Berlin, 1987.

[Ko] J. Kollár, Shafarevich maps and plurigenera of algebraic varieties, *Invent. Math.* **113** (1993), 177–216.

[Ma] A. I. Malcev, On isomorphic matrix representations of infinite groups of matrices, *Mat. Sb.* **8** (1940), 405–422, & *Amer. Math. Soc. Transl.* **45** (1965), 1–18.

[OV] L. Ornea and M. Verbitsky, Sasakian structures on CR-manifolds, *Geom. Dedicata* **125** (2007), 159–173.

[Pl] V. P. Platonov, A certain problem for finitely generated groups, *Dokl. Akad. Nauk BSSR* **12** (1968), 492–494.

[Ru] P. Rukimbira, Chern-Hamilton's conjecture and K-contactness, *Houston Jour. Math.* **21** (1995), 709–718.

[Se] A. Selberg, On discontinuous groups in higher-dimensional symmetric spaces, *Contributions to Function Theory*, pp. 147–164, Tata Institute of Fundamental Research, Bombay, 1960.

[Si1] C. T. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, *Jour. Amer. Math. Soc.* **1** (1988), 867–918.

[Si2] C. T. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: mahan@math.tifr.res.in