Math Phys Anal Geom (2018) 21:4
https://doi.org/10.1007/s11040-017-9262-0

Sasakian and Parabolic Higgs Bundles

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Received: 3 July 2017 / Accepted: 20 December 2017

Abstract Let M be a quasi-regular compact connected Sasakian manifold, and let $N = M/S^1$ be the base projective variety. We establish an equivalence between the class of Sasakian G–Higgs bundles over M and the class of parabolic (or equivalently, ramified) G–Higgs bundles over the base N.

Keywords Sasakian manifold · Higgs bundle · Parabolic structure · Ramified bundle

Mathematics Subject Classification (2010) Primary: 14P25, 57M05, 14F35, 20F65 · Secondary: 57M50, 57M07, 20F67

1 Introduction

Let M be a quasi-regular compact Sasakian manifold. The circle group $S^1 = \mathrm{U}(1)$ acts on M; let N be the corresponding quotient, which is a normal complex projective variety. Let G be a complex reductive affine algebraic group. Sasakian Higgs bundles on M with structure group G can be looked at from a number of different points of view:

(1) As a holomorphic Sasakian principal $G-$bundle over M equipped with a Higgs field.

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(2) As a ramified G–Higgs bundle over N with ramification locus a normal crossing divisor D in N.
(3) As a parabolic G–Higgs bundle over N with parabolic structure over D and rational parabolic weights.

The equivalence of (2) and (3) was established in [4]; it should be clarified that the constructions, and methods, in [4] were greatly motivated by [2, 3]. The purpose of this paper is to establish an equivalence between (1) and (3). It should be mentioned that while the Sasakian manifolds are never complex manifolds (their dimension is odd), there is a natural procedure of defining holomorphic objects on them; the Reeb vector field plays a crucial role in this process. The details are recalled in Section 2.2.

Let Γ be the fundamental group of M. In [6] we established a fourth equivalence, the Donaldson-Corlette-Hitchin-Simpson correspondence between representations of Γ and Sasakian $G-$Higgs bundles:
Any homomorphism

$$\rho : \Gamma \longrightarrow G$$

with the Zariski closure of $\rho(\Gamma)$ reductive canonically gives a virtually basic polystable principal G–Higgs bundle on M with vanishing rational characteristic classes. Conversely, any virtually basic polystable principal G–Higgs bundle on M with vanishing rational characteristic classes corresponds to a flat principal G–bundle on M with the property that the Zariski closure of the monodromy representation is reductive [6].

Thus, the equivalence of (1) and (3), proven in this paper, completes the equivalence of these four perspectives.

2 Higgs Bundles on Quasi-regular Sasakian Manifolds

2.1 Sasakian Manifolds

Let M be a quasi-regular compact Sasakian manifold of dimension $2d+1$ (see [7] for definitions and properties). Then $\mathrm{U}(1)$ acts on M and the action is free over a dense open subset of M. The action is free everywhere if and only if M is regular. Without loss of generality, we assume that the action of $\mathrm{U}(1)$ on M is on the left. (This is not important as $\mathrm{U}(1)$ is abelian.) Let

$$N := \mathrm{U}(1)\backslash M$$

be the quotient space; we will refer to N as the base space for M. This base N is a normal complex projective variety. The subset of M over which the action of $\mathrm{U}(1)$ is not free will be denoted by D_M. Let

$$p : M \longrightarrow N \tag{2.1}$$

be the quotient map, so that p gives a C^∞ principal $\mathrm{U}(1)$–bundle over the complement $N - p(D_M)$. Although the structure group acts on the right for a principal bundle, since $\mathrm{U}(1)$ is abelian we do not need to distinguish between left and rights actions.

Assumption 2.1 We assume the following:

(1) The quotient space N is a smooth variety.
(2) The subset $D_N := p(D_M) \subset N$ is a simple normal crossing divisor.

A simple minded example would be the following: Let B_0 be an orbifold surface, and let $p_1 : S \longrightarrow B_0$ a principal U(1) bundle such that the characteristic class of it is nonzero. So S is a Seifert fibered three manifold with base B_0; the characteristic class is an element of $H^2(B_0, \mathbb{Z})$. We can put a Sasakian structure on S such that B_0 is the base of it with p_1 being the projection to the base of the Sasakian manifold. These are precisely the quasi-regular compact Sasakian three-manifolds. Now let Y be a complex projective manifold. Take $M = S \times Y$, $N = B_0 \times Y$ and $p = p_1 \times \mathrm{Id}_Y$. It is easy to see that this they satisfy the two conditions stated above.

Getting back the general situation, the first of the two conditions means that N is a smooth complex projective variety of complex dimension d. Note that the first condition implies that D_N is a divisor of N. The second condition means that each irreducible component of D_N is a smooth sub-variety of N of dimension $d - 1$, and these irreducible components intersect transversally. The first condition is a strong assumption. As mentioned before, N is a normal variety, so smoothness is a strong assumption. The first condition rules out for example singularities isomorphic to the quotient of $\mathbb{C}^2$ by the involution $(x, y) \longmapsto (-x, -y)$. In view of the first condition, the second condition is rather mild.

2.2 Holomorphic Principal Bundles

We now recall from [6] the definition of a Sasakian Higgs bundle. The Riemannian metric and the Reeb vector field on M will be denoted by g and ξ respectively. The almost complex structure on the orthogonal complement

$$\xi^{\perp} \subset TM$$

with respect to g produces a type decomposition

$$\xi^{\perp} \otimes_{\mathbb{R}} \mathbb{C} = F^{1,0} \oplus F^{0,1}.$$

Let

$$\widetilde{F}^{0,1} := F^{0,1} \oplus (\xi \otimes_{\mathbb{R}} \mathbb{C}) \subset TM \otimes_{\mathbb{R}} \mathbb{C} \tag{2.2}$$

be the distribution. It is known that this distribution $\widetilde{F}^{0,1}$ is integrable [5, p. 550, Lemma 3.2].

Let G be a complex reductive affine algebraic group and $q : E_G \longrightarrow M$ a C^{∞} principal G–bundle on M. Let

$$dq : TE_G \longrightarrow q^*TM$$

be the differential of the projection q. A partial connection on E_G in the direction of ξ is a G–equivariant homomorphism over E_G

$$D_0 : q^*\xi \longrightarrow TE_G$$

such that $(dq) \circ D_0$ coincides with the identity map of the line sub-bundle $q^*\xi \subset q^*TM$. In other words, D_0 is a G–equivariant lift of ξ to the total space of E_G. A

Sasakian principal G–bundle on M is a C^∞ principal G–bundle E_G on M equipped with a partial connection D_0 in the direction of ξ.

Let (E_G, D_0) be a Sasakian principal G–bundle on M as above. A holomorphic structure on E_G is a C^∞ sub-bundle

$$\mathcal{D} \subset TE_G \otimes_{\mathbb{R}} \mathbb{C} \tag{2.3}$$

such that the following four conditions hold:

(1) the action of G on $TE_G \otimes_{\mathbb{R}} \mathbb{C}$ given by the action of G on E_G preserves $\mathcal{D}$,

(2) the complexified differential

$$dq \otimes_{\mathbb{R}} \mathbb{C} : TE_G \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow q^*(TM \otimes_{\mathbb{R}} \mathbb{C}) = (q^*TM) \otimes_{\mathbb{R}} \mathbb{C}$$

is an isomorphism from the sub-bundle $\mathcal{D} \subset TE_G \otimes_{\mathbb{R}} \mathbb{C}$ to the sub-bundle

$$q^*\widetilde{F}^{0,1} \subset q^*(TM \otimes_{\mathbb{R}} \mathbb{C}),$$

where $\widetilde{F}^{0,1}$ is constructed in (2.2),

(3) the sub-bundle $\mathcal{D} \subset TE_G \otimes_{\mathbb{R}} \mathbb{C}$ is closed under the Lie bracket operation of vector fields of E_G, and

(4) $D_0(q^*\xi) \otimes 1 \subset \mathcal{D}$, meaning the image of the homomorphism D_0 is contained in $\mathcal{D}$.

A *holomorphic Sasakian principal G–bundle* is a Sasakian principal G–bundle equipped with a holomorphic structure. When $G = \mathrm{GL}(r, \mathbb{C})$, then a holomorphic Sasakian principal G–bundle is a holomorphic Sasakian vector bundle of rank r (see [5]).

Let (E_G, D_0) be a Sasakian principal G–bundle. Note that any vector bundle associated to E_G is a Sasakian vector bundle where the partial connection is given by D_0. In particular, the adjoint vector bundle $\mathrm{ad}(E_G)$ is a Sasakian vector bundle. A holomorphic structure $\mathcal{D}$ on (E_G, D_0) produces a holomorphic structure on any associated vector bundle (associated to E_G by a holomorphic representation of G); see [6]. The trivial line bundle $M \times \mathbb{C}$ has a trivial holomorphic structure. The sub-bundle in (2.3) for this trivial holomorphic structure is the direct summand $q_0^*\widetilde{F}^{0,1}$ of $T(M \times \mathbb{C}) \otimes \mathbb{C} = q_0^*\widetilde{F}^{0,1} \oplus \widetilde{F}^{1,0} \oplus q_1^*T\mathbb{C}$, where q_0 and q_1 are the projections of $M \times \mathbb{C}$ to M and $\mathbb{C}$ respectively (the holomorphic tangent bundle $T\mathbb{C}$ is of course trivial). A holomorphic section of a holomorphic Sasakian vector bundle $((E, D_0), \mathcal{D})$ is a homomorphism from $M \times \mathbb{C}$ to E such that it takes the above direct summand $q_0^*\widetilde{F}^{0,1}$ to $\mathcal{D}$.

A Higgs field on a Sasakian holomorphic principal G–bundle $(E_G, \mathcal{D})$ is a holomorphic section θ of $\mathrm{ad}(E_G) \otimes (F^{1,0})^*$ such that the following two conditions hold:

(1) the section $\theta \bigwedge \theta$ of $\mathrm{ad}(E) \otimes (\bigwedge^2 F^{1,0})^*$ vanishes identically, and

(2) for any point $x \in N - D_N = N - p(D_M)$, where p is the map in (2.1), the restriction of the section θ to the loop $p^{-1}(x)$ is flat with respect to the partial connection D_0 along $p^{-1}(x)$.

The second condition means that the G–Higgs bundle $((E_G, D), \theta)$ is virtually basic (see [6, Definition 4.2] for the definitions of basic and virtually basic G–Higgs bundles).

3 Ramified G–Higgs Bundles

3.1 Ramified G–Bundles

Let X be an irreducible smooth projective variety of dimension d defined over $\mathbb{C}$. Let $D \subset X$ be a simple normal crossing divisor. As before, let G be a complex reductive affine algebraic group. The Lie algebra of G will be denoted by $\mathfrak{g}$.

A *ramified G–bundle* over X with ramification over the divisor D is a smooth complex quasi-projective variety E_G equipped with a right algebraic action of G

$$f : E_G \times G \longrightarrow E_G \tag{3.1}$$

and a surjective algebraic map

$$\psi : E_G \longrightarrow X, \tag{3.2}$$

such that the following five conditions hold:

- $\psi \circ f = \psi \circ p_1$, where p_1 is the projection of $E_G \times G$ to E_G,
- for each point $x \in X$, the action of G on the reduced fiber $\psi^{-1}(x)_{\text{red}}$ is transitive,
- the restriction of ψ to $\psi^{-1}(X - D)$ makes $\psi^{-1}(X - D)$ a principal G–bundle over $X - D$, meaning the map ψ is smooth over $\psi^{-1}(X - D)$ and the map to the fiber product

$$\psi^{-1}(X - D) \times G \longrightarrow \psi^{-1}(X - D) \times_{X-D} \psi^{-1}(X - D)$$

defined by $(z, g) \longmapsto (z, f(z, g))$ is an isomorphism,
- for each irreducible component $D_i \subset D$, the reduced inverse image $\psi^{-1}(D_i)_{\text{red}} \subset \psi^{-1}(D_i)$ is a smooth divisor and

$$\widehat{D} := \sum_{i=1}^{\ell} \psi^{-1}(D_i)_{\text{red}}$$

is a normal crossing divisor on E_G, and
- for any smooth point $z \in \widehat{D}$, the isotropy group $G_z \subset G$, for the action of G on E_G, is a finite cyclic group that acts faithfully on the quotient line $T_z E_G / T_z \psi^{-1}(D)_{\text{red}}$.

(See [2, 3].)

Let Pvect(X) denote the category of parabolic vector bundles over X with parabolic structure over D and rational parabolic weights. Let Rep(G) denote the category of all finite dimensional rational left representations of G. A parabolic G–bundle over X with D as the parabolic divisor is defined to be a functor from Rep(G) to Pvect(X) that is compatible with the operations of taking direct sum, tensor product and dual. See [1, 3, Section 2]. This definition is based on [10].

There is a natural equivalence of categories between parabolic G–bundles and ramified G–bundles. (See [2, 3].)

3.2 Ramified G–Higgs Bundle

Let (E_G, ψ) be a ramified G–bundle as in (3.2). The algebraic tangent bundle on E_G will be denoted by TE_G. Let

$$\mathcal{K} \subset TE_G \tag{3.3}$$

be the sub-bundle defined by the orbits of the action of G on E_G. So for any $z \in E_G$, the fiber $\mathcal{K}_z \subset T_z E_G$ is the image of the differential

$$df_z(e) : \mathfrak{g} \longrightarrow T_z E_G$$

of the map $f_z : G \longrightarrow E_G$, $g \longmapsto f(z, g)$, where f is the map in (3.1). Since $df_z(e)$ is an isomorphism onto its image, which coincides with the vertical tangent space for ψ, we have an algebraic isomorphism of vector bundles

$$\eta : E_G \times \mathfrak{g} \longrightarrow \mathcal{K}. \tag{3.4}$$

This η is an isomorphism of sheaves of Lie algebras; the Lie algebra operation on the sheaf of sections of $\mathcal{K}$ is given by the Lie bracket of vector fields.

Let $\mathcal{Q}$ denote the quotient vector bundle $TE_G/\mathcal{K}$. So we have a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{K} \longrightarrow TE_G \stackrel{q}{\longrightarrow} \mathcal{Q} \longrightarrow 0 \tag{3.5}$$

over E_G. The action of G on E_G induces an action of G on the tangent bundle TE_G. This action of G on TE_G clearly preserves the sub-bundle $\mathcal{K}$. It may be mentioned that the isomorphism η in (3.4) intertwines the action of G on $\mathcal{K}$ and the diagonal action of G constructed using the adjoint action of G on $\mathfrak{g}$. Therefore, we have an induced action of G on the quotient bundle $\mathcal{Q}$.

Let

$$\theta_0 \in H^0(E_G, \mathcal{H}om(\mathcal{Q}, \mathcal{K})) = H^0(E_G, \mathcal{K} \otimes \mathcal{Q}^*) \tag{3.6}$$

be an algebraic section. We note that the actions of G on $\mathcal{K}$ and $\mathcal{Q}$ together define an action of G on the complex vector space $H^0(E_G, \mathcal{H}om(\mathcal{Q}, \mathcal{K}))$.

Combining the exterior algebra structure of $\bigwedge \mathcal{Q}^*$ and the Lie algebra structure on the fibers of the vector bundle $\mathcal{K} = E_G \times \mathfrak{g}$ (see (3.4)), we have a homomorphism

$$\tau : (\mathcal{K} \otimes \mathcal{Q}^*) \otimes (\mathcal{K} \otimes \mathcal{Q}^*) \longrightarrow \mathcal{K} \otimes (\bigwedge^2 \mathcal{Q}^*). \tag{3.7}$$

So $\tau((A_1 \otimes \omega_1) \otimes (A_2 \otimes \omega_2)) = [A_1, A_2] \otimes (\omega_1 \bigwedge \omega_2)$. We will denote $\tau(a, b)$ also by $a \bigwedge b$.

Definition 3.1 A *Higgs field* on a ramified G–bundle E_G is a section

$$\theta_0 \in H^0(E_G, \mathcal{K} \otimes \mathcal{Q}^*)$$

as in (3.6) satisfying the following two conditions:

(1) the action of G on $H^0(E_G, \mathcal{K} \otimes \mathcal{Q}^*)$ leaves θ_0 invariant, and
(2) $\theta_0 \bigwedge \theta_0 = 0$ (see (3.7)).

Definition 3.2 A *ramified Higgs G–bundle* is a pair (E_G, θ_0), where E_G is a ramified G–bundle, and θ_0 is a Higgs field on E_G.

Let

$$\mathcal{A}_{E_G} := (\psi_*(\mathcal{K} \otimes \mathcal{Q}^*))^G \subset \psi_*(\mathcal{K} \otimes \mathcal{Q}^*) \tag{3.8}$$

be the invariant direct image, where ψ is the projection in (3.2). Therefore,

$$H^0(X, \mathcal{A}_{E_G}) = H^0(E_G, \mathcal{K} \otimes \mathcal{Q}^*)^G. \tag{3.9}$$

For $i \geq 0$, let

$$\widetilde{\mathcal{K}}_i := \left(\psi_*\left(\mathcal{K} \otimes \left(\bigwedge^i \mathcal{Q}^*\right)\right)\right)^G \subset \psi_*\left(\mathcal{K} \otimes \left(\bigwedge^i \mathcal{Q}^*\right)\right) \tag{3.10}$$

be the invariant direct image. So, $\widetilde{\mathcal{K}}_1 = \mathcal{A}_{E_G}$. The homomorphism τ in (3.7) yields a homomorphism

$$\widetilde{\tau} : \widetilde{\mathcal{K}}_1 \otimes \widetilde{\mathcal{K}}_1 \longrightarrow \widetilde{\mathcal{K}}_2. \tag{3.11}$$

See [4, Lemma 2.3] for the following lemma.

Lemma 3.3 *A Higgs field on E_G is a section*

$$\theta \in H^0(X, \mathcal{A}_{E_G})$$

such that

$$\widetilde{\tau}(\theta, \theta) = 0,$$

where $\widetilde{\tau}$ is constructed in (3.11).

3.3 The Adjoint Vector Bundle

We noted earlier that there is a natural equivalence of categories between parabolic G–bundles and ramified G–bundles (see [2, 3]). Let E_G^P denote the parabolic G–bundle corresponding to a ramified G–bundle E_G. We also recall that E_G^P associates a parabolic vector bundle over X to each object in $\text{Rep}(G)$. Let

$$\text{ad}(E_G) := E_G^P(\mathfrak{g}) \tag{3.12}$$

be the parabolic vector bundle over X associated to the parabolic G–bundle E_G^P for the adjoint action of G on its Lie algebra $\mathfrak{g}$. This parabolic vector bundle $\text{ad}(E_G)$ will be called the *adjoint vector bundle* of E_G. The vector bundle underlying the parabolic vector bundle $\text{ad}(E_G)$ will also be denoted by $\text{ad}(E_G)$. From the context it will be clear which one is being referred to.

Consider the vector bundle $\mathcal{K} \longrightarrow E_G$ constructed in (3.3). We noted that $\mathcal{K}$ is equipped with a natural action of G. It is straight forward to check that the invariant direct image $(\psi_*\mathcal{K})^G$ is identified with the vector bundle underlying the parabolic vector bundle $\text{ad}(E_G)$ constructed in (3.12). Indeed, this follows from the fact that for a usual principal bundle, its adjoint vector bundle coincides with the invariant direct image of the relative tangent bundle. Therefore, we have

$$\text{ad}(E_G) = (\psi_*\mathcal{K})^G \subset \psi_*\mathcal{K}. \tag{3.13}$$

There is a natural $\mathcal{O}_X$–linear homomorphism

$$\mathrm{ad}(E_G) \otimes \Omega^1_X \longrightarrow \mathcal{A}_{E_G}, \tag{3.14}$$

where $\mathcal{A}_{E_G}$ is constructed in (3.8) [4, (2.13)]. This homomorphism is an isomorphism over the complement $X - D$, but it is not an isomorphism over X in general. In fact, $\mathcal{A}_{E_G}$ is the vector bundle underlying the parabolic tensor product of $\mathrm{ad}(E_G)$ and Ω^1_X, so the usual tensor product $\mathrm{ad}(E_G) \otimes \Omega^1_X$ is a sub-sheaf of $\mathcal{A}_{E_G}$.

There is a natural homomorphism

$$\tau' : \bigwedge^2 \left(\mathrm{ad}(E_G) \otimes \Omega^1_X\right) \longrightarrow \mathrm{ad}(E_G) \otimes \Omega^2_X.$$

Consider the isomorphism in (3.14). This takes $\widetilde{\tau}$ in (3.11) to the above homomorphism τ'. Therefore, from Lemma 3.3 it follows that a Higgs field on E_G is a holomorphic section θ of $\mathrm{ad}(E_G) \otimes \Omega^1_X$ such that $\tau'(\theta \bigwedge \theta) = 0$.

Henceforth, a ramified bundle (respectively, ramified Higgs bundle) will also be called a parabolic bundle (respectively, parabolic Higgs bundle).

4 Parabolic Higgs Bundles and Sasakian Higgs Bundles

In this section, we shall establish the equivalence between Sasakian $G-$Higgs bundles over Sasakian manifolds M with base N and parabolic $G-$Higgs bundles over the base N. Section 4.1 will give us a way of going from a parabolic Higgs bundle to a Sasakian Higgs bundle while Section 4.2 will give us the reverse path.

4.1 From Parabolic Higgs Bundles to Sasakian Higgs Bundles

Let Y be a complex projective variety and Γ a finite group acting on Y through algebraic automorphisms. So we have a homomorphism

$$h : \Gamma \longrightarrow \mathrm{Aut}(Y), \tag{4.1}$$

where $\mathrm{Aut}(Y)$ is the group of all automorphisms of the variety Y. A Γ–*linearized* principal G–bundle over Y is a principal G–bundle

$$\phi : F_G \longrightarrow Y \tag{4.2}$$

and an action of Γ on the left of F_G

$$\rho : \Gamma \times F_G \longrightarrow F_G$$

such that the following two conditions hold:

- the actions of Γ and G on F_G commute, and
- $\phi \circ \rho(\gamma, z) = h(\gamma)(\phi(z))$ for all $(\gamma, z) \in \Gamma \times F_G$, where h is the homomorphism in (4.1) and ϕ is the projection in (4.2).

Let $\psi : E_G \longrightarrow N$ be a parabolic principal Higgs G–bundle. Using the "Covering lemma" of Kawamata (see [9, Ch. 1.1, p. 303–305]) it can be shown that there is a finite Galois covering

$$\varphi : Y \longrightarrow N \tag{4.3}$$

and a Γ–linearized principal G–bundle F_G over Y, where $\Gamma := \mathrm{Gal}(\varphi)$ is the Galois group, such that

$$E_G = \Gamma \backslash F_G \tag{4.4}$$

[3, Section 4.1], [4, Section 4.1].

Now let θ be a Higgs field on the parabolic G–bundle E_G. There is a natural linear isomorphism between the Higgs fields on E_G and the Γ–invariant Higgs fields on F_G [4, Proposition 4.1]. Let θ' be the Γ–invariant Higgs field on F_G corresponding to the Higgs field θ on E_G.

Fix an ample holomorphic line bundle L_0 on Y. Define the tensor product

$$L := \bigotimes_{\gamma \in \Gamma} \gamma^* L_0,$$

which is an ample holomorphic line bundle on Y. Note that the action of Γ on Y has a natural lift to an action of Γ on L. Take a Hermitian structure h_0 on L_0 such that the curvature $\mathrm{Curv}(L_0, h_0)$ of (L_0, h_0) is positive. Let

$$h := \bigotimes_{\gamma \in \Gamma} \gamma^* h_0$$

be the Hermitian structure on L. The action of Γ on L clearly preserves h. Note that the curvature $\mathrm{Curv}(L, h)$ of (L, h) coincides with $\sum_{\gamma \in \Gamma} \gamma^* \mathrm{Curv}(L_0, h_0)$, hence the $(1, 1)$–form $\mathrm{Curv}(L, h)$ is positive.

Let

$$L \supset \{v \in L \mid h(v) = 1\} := M_1 \xrightarrow{p} Y \tag{4.5}$$

be the principal $\mathrm{U}(1)$–bundle over Y. Using h, and the positive form $\mathrm{Curv}(L, h)$ on Y, there is a regular Sasakian structure on M_1 [7]. Since h is preserved by the action of Γ on L, the action of Γ on L preserves M_1. The quotient $M := M_1/\Gamma$ is a quasi-regular Sasakian manifold with $\mathrm{U}(1)\backslash M = N$.

The pullback $(p^* F_G, p^*\theta')$ is a G–Higgs bundle on the Sasakian manifold M_1, where p is the projection in (4.5) and F_G is the principal G–bundle in (4.4). The action of Γ on (F_G, θ') pulls back to an action of Γ on $(p^* F_G, p^*\theta')$. Consequently, $(p^* F_G, p^*\theta')$ produces a G–Higgs bundle on M.

Therefore, we have the following:

Proposition 4.1 *Given a ramified G–Higgs bundle (E_G, θ) on N, there is a Sasakian manifold M over N and a G–Higgs bundle on M.*

4.2 From Sasakian Higgs Bundles to Parabolic Higgs Bundles

Let $((E_G, \mathcal{D}), \theta)$ be a G–Higgs bundle on the Sasakian manifold M in Section 2.1.

Proposition 4.2 *The quotient* $\mathrm{U}(1)\backslash E_G$ *is a ramified holomorphic principal G–bundle on* $\mathrm{U}(1)\backslash M = N$.

The Higgs field θ produces a Higgs field on $\mathrm{U}(1)\backslash E_G$.

Proof Consider the smooth complex projective variety N and the simple normal crossing divisor $D_N = p(D_M)$ on it. Let $\{D_i\}_{i=1}^n$ be the irreducible components of D_N. For each irreducible component D_i, let $m_i > 0$ be the multiplicity of D_i associated to the projection p from M. So m_i is the order of the isotropy subgroup of a general point of $p^{-1}(D_i)$ for the action of U(1) on M (this uses Assumption 2.1). Given this collection of pairs $\{(D_i, m_i)\}_{i=1}^n$ the "covering lemma" of Kawamata says the following:

There is a smooth projective variety Z and a (ramified) Galois covering

$$\beta : Z \longrightarrow N \tag{4.6}$$

such that

(1) the reduced divisor $\beta^{-1}(D_N)_{\rm red}$ is a simple normal crossing divisor on Z, and
(2) $\beta^{-1}(D_i) = k_i m_i \beta^{-1}(D_i)_{\rm red}$ for all $1 \leq i \leq n$, where k_i are positive integers.

(See [9, Theorem 1.1.1], [8, Theorem 17].) The Galois group Gal(β) for the covering in (4.6) will be denoted by Γ.

Let

$$\beta_1 := \beta|_{\beta^{-1}(N-D_N)} : \beta^{-1}(N - D_N) \longrightarrow N - D_N$$

be the restriction of β. Consider the principal U(1)–bundle $M' := M - D_M \longrightarrow N - D_N$ in (2.1). The pulled back principal U(1)–bundle

$$\beta_1^* M' \longrightarrow \beta^{-1}(N - D_N)$$

extends to a principal U(1)–bundle $\widehat{Z}$ on Z. Indeed, this follows from the fact that $\beta^{-1}(D_i) = k_i m_i \beta^{-1}(D_i)_{\rm red}$. This $\widehat{Z}$ is a regular Sasakian manifold with U(1)$\backslash\widehat{Z} = Z$, and it fits in a commutative diagram

$$\begin{array}{ccc} \widehat{Z} & \stackrel{\delta}{\longrightarrow} & M \\ \downarrow\mu & & \downarrow p \\ Z & \stackrel{\beta}{\longrightarrow} & N \end{array}$$

where p and β are in the maps in (2.1) and (4.6) respectively. The map δ is a Galois covering with Galois group $\Gamma = \mathrm{Gal}(\beta)$.

Consider the pulled back G–Higgs bundle $((\delta^* E_G, \delta^*\mathcal{D}), \delta^*\theta)$ on $\widehat{Z}$. Since δ is a Galois covering with Galois group Γ. The Galois group Γ acts on $((\delta^* E_G, \delta^*\mathcal{D}), \delta^*\theta)$. The actions of U(1) and Γ on $\delta^* E_G$ commute. Since $\widehat{Z}$ is a regular Sasakian manifold, and U(1)$\backslash\widehat{Z} = Z$, we conclude that there is a G–Higgs bundle (E'_G, θ') on Z such that $((\delta^* E_G, \delta^*\mathcal{D}), \delta^*\theta)$ is the pullback of (E'_G, θ') to $\widehat{Z}$. The action of Γ on $((\delta^* E_G, \delta^*\mathcal{D}), \delta^*\theta)$ produces an action of Γ on the G–Higgs bundle (E'_G, θ'). Hence (E'_G, θ') produces a ramified G–Higgs bundle on $Z/\Gamma = N$ (see [4, Proposition 4.1]). Let (E''_G, θ'') be the ramified G–Higgs bundle on N defined by (E'_G, θ').

Now it is straight-forward to check that E''_G coincides with the quotient U(1)$\backslash E_G$. Furthermore, the Higgs field θ'' coincides with θ on $p^{-1}(N - D_N)$. □

The construction in Section 4.1 of a Sasakian G–Higgs bundle from a parabolic G–Higgs bundle and the construction in Section 4.2 of a parabolic G–Higgs bundle from a Sasakian G–Higgs bundle are evidently inverses of each other.

Acknowledgements We thank the referees for their helpful comments. The authors acknowledge the support of their respective J. C. Bose Fellowships.

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