ON A THEOREM OF SCOTT AND SWARUP

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ABSTRACT. Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by an automorphism ϕ of the free group H. Let $H_1 (\subset H)$ be a finitely generated distorted subgroup of G. Then there exist $N > 0$ and a free factor K of H such that the conjugacy class of K is preserved by ϕ^N and H_1 contains a finite index subgroup of a conjugate of K. This is an analog of a Theorem of Scott and Swarup for surfaces in hyperbolic 3-manifolds.

1. Introduction

In [13] , Scott and Swarup prove the following theorem:

Theorem [13] *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by a pseudo Anosov diffeomorphism of a closed surface with fundamental group H. Let H_1 be a finitely generated subgroup of infinite index in H. Then H_1 is quasiconvex in G.*

In this paper we derive an analogous result for free groups (see Section 2 below or [3] [2] [7] for definitions).

We note at the outset that *hyperbolic* stands for two notions. When qualifying manifolds, they indicate spaces of constant curvature equal to -1. When qualifying groups or metric spaces, we use *hyperbolic* in the sense of Gromov [8]. It will be clear from the context which of these meanings is relevant.

Theorem 3.4 *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by an aperiodic automorphism of the free group H. Let H_1 be a finitely generated subgroup of infinite index in H. Then H_1 is quasiconvex in G.*

In fact we prove the following more general theorem

Theorem 3.7 *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by a hyperbolic automorphism ϕ of the free group H. Let $H_1 (\subset H)$ be a finitely generated distorted subgroup of G. Then there exist $N > 0$ and a free factor K of H such that the*

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conjugacy class of K is preserved by ϕ^N and H_1 contains a finite index subgroup of a conjugate of K.

In fact the methods of this paper can be used to give a new proof of the Theorem of Scott and Swarup mentioned above. We sketch this proof for closed surfaces first. Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber F. Let $\widetilde{F}$ and $\widetilde{M}$ denote the universal covers of F and M respectively. Then $\widetilde{F}$ and $\widetilde{M}$ are quasi-isometric to $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. Now let $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ denote the standard compactifications. In [5] Cannon and Thurston show that the usual inclusion i of $\widetilde{F}$ into $\widetilde{M}$ extends to a continuous map $\hat{i}$ from $\mathbb{D}^2$ to $\mathbb{D}^3$. Cannon and Thurston further show that $\hat{i}$ identifies precisely those pairs of points which are boundary points of an ending lamination. Since a leaf of the stable (or unstable) lamination is dense in in the whole lamination, it cannot be carried by a (perhaps immersed) proper sub-surface (one can see this, for instance, by using the fact that surface groups are LERF [12]). The subgroup corresponding to the fundamental group of such a subsurface must therefore be quasiconvex in G.

This idea goes through for free groups. We give a brief sketch for aperiodic automorphisms. In this case, Bestvina, Feighn and Handel [3] have shown that any leaf of the stable (or unstable) lamination 'fills' H, i.e. it cannot be carried by a finitely generated subgroup H_1 of infinite index in H. We combine this with the description of boundary identifications given in [9] to show that no pair of points on the boundary of H_1 are identified. Thus H_1 must be quasiconvex in G.

2. Ending Laminations

Let G be a hyperbolic group in the sense of Gromov [8] . Let H be a hyperbolic subgroup of G. Choose a finite generating set of G containing a finite generating set of H. Let Γ_G and Γ_H be the Cayley graphs of G, H with respect to these generating sets. Let $i : \Gamma_H \to \Gamma_G$ denote the inclusion map.

Definition : [7] [6] *If $i : \Gamma_H \to \Gamma_G$ be an embedding of the Cayley graph of H into that of G, then the distortion function is given by*

$$disto(R) = Diam_{\Gamma_H}(\Gamma_H \cap B(R)),$$

where $B(R)$ is the ball of radius R around $1 \in \Gamma_G$.

If H is quasiconvex in G the distortion function is linear and we shall refer to H as an undistorted subgroup. Else, H will be termed distorted.

For distorted subgroups, the distortion information is encoded in a certain set of ending laminations defined below.

Definition : *If λ is a geodesic segment in Γ_H then λ^r, a* **geodesic realization** *of λ, is a geodesic in Γ_G joining the end-points of $i(\lambda)$.*

Now consider sequences of geodesic segments $\lambda_i \subset \Gamma_H$ such that $1 \in \lambda_i$ and $\lambda_i^r \cap B(i) = \emptyset$, where $B(i)$ is the ball of radius i around $1 \in \Gamma_G$. Take all bi-infinite subsequential limits (in the Hausdorff topology) of all such sequences $\{\lambda_i\}$ and denote this set by Σ.

Let t_h denote left translation by $h \in H$. Let $\widehat{\Gamma_H}$ and $\widehat{\Gamma_G}$ denote the Gromov compactifications of Γ_H and Γ_G respectively. Further let $\partial\Gamma_H$ and $\partial\Gamma_G$ denote the boundaries of Γ_H and Γ_G respectively [8] .

Definition : *The set of ending laminations $\Lambda = \Lambda(H,G)$ is given by*

$$\Lambda = \{(p,q) \in \partial\Gamma_H \times \partial\Gamma_H | p \neq q \text{ and } p,q \text{ are the end-points of } t_h(\lambda) \text{ for some } \lambda \in \Sigma\}$$

Lemma 2.1. *H is quasiconvex in G if and only if $\Lambda = \emptyset$*

Proof : Suppose H is quasiconvex in G. Then any geodesic realization λ^r of a geodesic segment $\lambda \subset \Gamma_H$ lies in a bounded neighborhood of Γ_H and hence of λ as H is hyperbolic. Hence $\Lambda = \emptyset$.

Conversely, if H is not quasiconvex in G, there exist $\lambda_i \subset \Gamma_H$ and $p_i \in \lambda_i$ such that $\lambda_i^r \cap B_{p_i}(i) = \emptyset$, where $B_{p_i}(i)$ denotes the ball of radius i around p_i in Γ_G. Translating by p_i^{-1} and taking subsequential limits, we get $\Sigma \neq \emptyset$ and hence $\Lambda \neq \emptyset$. $\square$

Definition : *A* **Cannon-Thurston map** *for the pair (H,G) is a map $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$ which is a continuous extension of $i : \Gamma_H \to \Gamma_G$.*

Note that if such a continuous extension exists, it is unique. We get a simplified collection of ending laminations when a Cannon-Thurston map exists.

Definition : $\Lambda_{CT} = \{(p,q) \in \partial\Gamma_H \times \partial\Gamma_H | p \neq q \text{ and } \hat{i}(p) = \hat{i}(q)\}$.

Lemma 2.2. *If a Cannon-Thurston map exists, $\Lambda = \Lambda_{CT}$.*

Proof: Let $(p,q) \in \Lambda$. After translating by an element of H if necessary assume that a bi-infinite geodesic λ passing through 1 has p,q as its end-points. By definition of Λ there exist geodesic segments $\lambda_i \subset \Gamma_H$ converging to λ in the Hausdorff topology such that $\lambda_i^r \cap B(i) = \emptyset$. Since a Cannon-Thurston map exists, there exists $z \in \partial\Gamma_G$ such that $\lambda_i^r \to z$ in the Hausdorff topology on $\widehat{\Gamma_G}$ and $\hat{i}(p) = z = \hat{i}(q)$. Hence $\Lambda \subset \Lambda_{CT}$.

Conversely, let $(p,q) \in \Lambda_{CT}$. After translating by an element of H if necessary assume that a bi-infinite geodesic λ passing through 1 has p,q as its end-points. Choose $p_i, q_i \in \Gamma_H$ such that $p_i \to p$ and $q_i \to q$. Let λ_i denote the subsegment of λ joining p_i, q_i. Then λ_i^r converges to $\hat{i}(p) = \hat{i}(q)$ in the Hausdorff topology on $\widehat{\Gamma_G}$. Passing to a subsequence if necessary we can assume that $\lambda_i^r \cap B(i) = \emptyset$. Hence $\Lambda_{CT} \subset \Lambda$. $\Box$

Remark: Suppose H_1 is a hyperbolic subgroup of H. Let $\hat{j}$ and $\hat{i}$ denote Cannon-Thurston maps for the pairs (H_1, H) and (H,G) respectively. Then the composition $\hat{i} \cdot \hat{j}$ is a Cannon-Thurston map for the pair (H_1, G). Further from Lemma 2.2 it follows that

$$\Lambda(H_1, G) = \Lambda(H_1, H) \cup (\hat{j})^{-1}(\Lambda(H,G)).$$

3. Extensions by Free Groups

Let $1 \to H \to G \to \mathbb{Z} \to 1$ denote an exact sequence of hyperbolic groups arising out of a hyperbolic automorphism ϕ of the hyperbolic group H. The notion of a hyperbolic automorphism was defined in [1] (see below) and shown to be equivalent to requiring that G be hyperbolic.

Definition: *Let ϕ be an automorphism of a hyperbolic group H (equipped with the word metric $|.|$). Let $\lambda > 1$. Let $S(\phi, \lambda) = \{h \in H : |\phi(h)| > \lambda |h|\}$. If $h \in S(\phi, \lambda)$, we say ϕ stretches h by λ. ϕ will be called* **hyperbolic** *if for all $\lambda > 1$ there exists $n > 0$ such that for all $h \in H$, at least one of ϕ^n or ϕ^{-n} stretches h by λ.*

ϕ and ϕ^{-1} induce bijections (also denoted by ϕ and ϕ^{-1}) of the vertices of Γ_H.

A **free homotopy representative** *of a word $w \in H$ is a geodesic $[a, aw_0]$ in Γ_H where w_0 is a shortest word in the conjugacy class of w in H.*

Given $h \in H$ let $\Sigma(h,n,+)$ (resp. $\Sigma(h,n,-)$ be the (H-invariant) collection of all free homotopy representatives of $\phi^n(h)$ (resp. $\phi^{-n}(h)$) in Γ_H. The intersesction with $\partial\Gamma_H \times \partial\Gamma_H$ of the union of all bi-infinite subsequential limits (in the Hausdorff topology on $\widehat{\Gamma_H}$) of elements of $\Sigma(h,n,+)$ (resp. $\Sigma(h,n,-)$ as $n \to \infty$ will be denoted by $\Lambda_+(h)$ (resp. $\Lambda_-(h)$).

Definition: *The* **stable** *and* **unstable ending laminations** *are respectively given by*

$$\Lambda_+ = \bigcup_{h \in H} \Lambda_+(h)$$
$$\Lambda_- = \bigcup_{h \in H} \Lambda_-(h)$$

Theorem 3.1. [11] *Let* $1 \to H \to G \to \mathbb{Z} \to 1$ *denote an exact sequence of hyperbolic groups arising out of a hyperbolic automorphism* ϕ *of the hyperbolic group* H*. Then there exists a Cannon Thurston map for the pair* (H, G)*.*

Theorem 3.2. [9] *Let* $1 \to H \to G \to \mathbb{Z} \to 1$ *denote an exact sequence of hyperbolic groups arising out of a hyperbolic automorphism* ϕ *of the hyperbolic group* H*. Then* $\Lambda_{CT} = \Lambda_+ \cup \Lambda_-$*.*

Further, it is shown in [9] that only finitely many h's need be considered in the definition of Λ_+ or Λ_-.

We turn now to the main focus of this paper, the case where H is free and ϕ is a hyperbolic automorphism [1] . Such automorphisms have been studied in great detail by Bestvina, Feighn and Handel [1] , [3] , [2] .

Definition: A non negative irreducible matrix is *aperiodic* if it has an iterate that is strictly positive.

Definition: Let us assume for a start that transition matrices of ϕ and ϕ^{-1} with respect to train-track representatives [see [4] for definitions] are aperiodic. We shall refer to such automorphisms as *aperiodic.* Note that in this definition, we require transition matrices of both ϕ and ϕ^{-1} to be aperiodic.

In this case, the definitions of ending laminations here and in [3] coincide. To see this and for the sake of completeness we recall definitions from [3] .

Let $f : X \to X$ be a train-track representative of an outer automorphism with aperiodic transition matrix. Endow X with the structure of a marked $\mathbb{R}$-graph so that f expands lengths of edges by a uniform factor $\lambda > 1$. Let $x \in X$ be an f-periodic point in the interior of some edge. Let $\epsilon > 0$ be small, and let U be the ϵ-neighborhood of x. Then for some $N > 0$, $U \subset f^N(U)$. Choose an isometry $l : (-\epsilon, \epsilon) \to U$ and extend it to the unique locally isometric immersion $l : \mathbb{R} \to X$ such that $l(\lambda^N t) = f^N(l(t))$. We say l is obtained by iterating a neighborhood of x. l will also be termed a *leaf* of the ending lamination.

Definitions : Two isometric immersions $[a, b] \to X$ and $[c, d] \to X$ are said to be equivalent if there is an isometry of $[a, b]$ onto $[c, d]$ making the triangle commute.

A *leaf segment* of an isometric immersion $\mathbb{R} \to X$ is the equivalence class of the restriction to a finite interval.

Two isometric immersions $l, l' : \mathbb{R} \to X$ are *(weakly) equivalent* if every leaf segment of l is a leaf segment of l' and vice versa.

Since f has an aperiodic transition matrix, l is surjective. Using this, Bestvina, Feighn and Handel [3] show that any two leaves of the ending lamination obtained by iterating neighborhoods of f-periodic points are equivalent.

It is now clear (see [3] Proposition 1.6, for instance) that the definition of stable laminations given before Theorem 3.1 above consists of end-points of leaves belonging to the equivalence class obtained by iterating a neighborhood of a periodic point in Bestvina, Feighn and Handel's construction above.

We can now use the results of [3] and [2] in our context.

Remark : Since any two leaves are (weakly) equivalent in the sense of [3] above, the equivalence class can alternately be obtained by translating some (any) leaf by elements of the free group and taking Hausdorff limits. This is analogous to the case for surfaces where the stable lamination of a pseudo anosov diffeomorphism is the closure of some (any) leaf.

The following is a paraphrasing of Proposition 2.4 of [3] . It says roughly that any leaf of the stable (or unstable) lamination of an aperiodic automorphism 'fills' the free group H.

Proposition 3.3. *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by an aperiodic automorphism ϕ of the free group H (i.e. ϕ and ϕ^{-1} have aperiodic transition matrices). If $(p, q) \in \Lambda_+$ or Λ_- lie in the boundary $\partial\Gamma_{H_1} \subset \partial\Gamma_H$ for a finitely generated subgroup H_1 of H, then H_1 is of finite index in H.*

We are now in a position to prove the main theorem of this paper for aperiodic ϕ .

Theorem 3.4. *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by an aperiodic automorphism ϕ of the free group H. Let H_1 be a finitely generated subgroup of infinite index in H. Then H_1 is quasiconvex in G.*

Proof: From Theorem 3.1 the pair (H, G) has a Cannon - Thurston map. Further, from Theorem 3.2

$$\Lambda_{CT}(H, G) = \Lambda(H, G) = \Lambda_+ \cup \Lambda_-.$$

Let $j : H_1 \to H$ and $i : H \to G$ denote inclusions. Since H_1 is quasiconvex in H, $\Lambda(H_1, H) = \emptyset$ (Lemma 2.1). Further from Proposition 3.3 above $j^{-1}(\Lambda(H, G)) = \emptyset$.

Also from the remark following Lemma 2.2 , $\Lambda(H_1, G) = \Lambda(H_1, H) \cup j^{-1}(\Lambda(H, G)) = \emptyset$.

Hence from Lemma 2.1 H_1 is quasiconvex in G. $\Box$

As a second step we deal with automorphisms ϕ of H satisfying the following:
There exists a decomposition $H = K_1 * K_2 * \cdots * K_n$ of H into ϕ-invariant factors K_i such that the restrictions $\phi|_{K_i} = \phi_i$ are aperiodic. Further, assume ϕ_i has a train-track representaive with a fixed point (this is always satisfied by large enough powers of ϕ_i).

Identifying the fixed points of the train-track representatives of ϕ_i we obtain a particular representative of ϕ. Further, let Λ_i denote the ending laminations of ϕ_i. It is easy to see from the representative we have chosen that $\Lambda \cap \partial\Gamma_{K_i} = \Lambda_i$. (See also Remark 1.4 of [3].)

Suppose H_1 is a finitely generated subgroup of H that is distorted in G. Since H_1 is quasi-convex in H, there exists a pair $(p, q) \in \Lambda = \Lambda_+ \cup \Lambda_-$ lying on the boundary $\partial\Gamma_{H_1} \subset \partial\Gamma_H$. Let l be a leaf of Λ joining p, q. By Theorem 3.2 l lies in the Hausdorff limit of sequences of segments obtained by iterating ϕ or ϕ^{-1} on some $h \in H$. By the pigeon-hole principle there exists arbitrarily long segments of l contained in a (fixed) conjugate of K_j for some i. For ease of exposition let us assume that this is the trivial conjugate of K_i, i.e. K_i itself. Translating by appropriate elements of H_1 and taking a Hausdorff limit we obtain a leaf of the ending lamination Λ whose endpoints lie in the intersection $\partial\Gamma_{H_1} \cap \partial\Gamma_{K_j}$. In particular, there exists a pair of points $s, t \in \partial\Gamma_{H_1} \cap \Lambda_j$ since $\Lambda \cap \partial\Gamma_{K_j} = \Lambda_j$.

Since intersection of quasiconvex subgroups is quasiconvex [14] it follows that $H_1 \cap K_j$ is quasi-convex in H. In particular $H_1 \cap K_j$ is finitely generated. Hence from Theorem 3.4 $H_1 \cap K_j$ is a finite index subgroup of K_j.

We have shown :

Theorem 3.5. *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by an automorphism ϕ of the free group H satisfying the following :*
There exists a decomposition $H = K_1 * K_2 * \cdots * K_n$ of H into ϕ-invariant factors K_i such that the restrictions $\phi|_{K_i} = \phi_i$ are aperiodic. Further, assume ϕ_i and ${\phi_i}^{-1}$ have train-track representatives with fixed points.

Let H_1 be a finitely generated subgroup of infinite index in H such that H_1 is distorted in G. Then there exist $h \in H$ and K_j such that H_1 contains a finite index subgroup of $h^{-1}K_jh$.

We are now in a position to treat a general hyperbolic automorphism ϕ.

We state a special case of a theorem due to Bestvina, Feighn and Handel (Corollary 4.7 of [2]).

Theorem 3.6. [2] *Let ϕ be a hyperbolic automorphism of the free group H. Then there exists an $N > 0$ satisfying the following :*
There exists a decomposition $H = K_1 * K_2 * \cdots * K_n$ of H into ϕ^N-invariant factors K_i such that the restrictions $\phi^N|_{K_i}$ and $\phi^{-N}|_{K_i}$ are aperiodic. Further, each of the restrictions have train-track representatives with fixed points.

Theorem 3.6 above allows us to reduce the general case to the case treated by Theorem 3.5 .

Definition: A subgroup A of a group B is said to be a free factor of B if there exists a group C (perhaps trivial) such that $A * C$ is equal to B.

Theorem 3.7. *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence of hyperbolic groups induced by a hyperbolic automorphism ϕ of the free group H. Let $H_1 (\subset H)$ be a finitely generated distorted subgroup of G. Then there exists $N > 0$ and a free factor K of H such that the conjugacy class of K is preserved by ϕ^N and H_1 contains a finite index subgroup of a conjugate of K.*

Proof: Taking a large enough power of ϕ we obtain an exact sequence $1 \to H \to G_1 \to \mathbb{Z} \to 1$, induced by ϕ^N satisfying the conclusions of Theorem 3.6. Note that G_1 is a finite index subgroup of G. Let $H = K_1 * K_2 * \cdots * K_n$ be a decomposition of H into ϕ^N-invariant factors K_i such that the restrictions $\phi^N|_{K_i}$ and $\phi^{-N}|_{K_i}$ are aperiodic.

Since $H_1 (\subset H)$ is a distorted subgroup of G and G_1 is a finite index subgroup of G, H_1 is distorted in G_1. Then by Theorem 3.5 there exist $h \in H$ and K_j such that H_1 contains a finite index subgroup of $h^{-1} K_j h$. Since K_j is a free factor of H, so are its conjugates. This proves the Theorem. □

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