PROPAGATING QUASICONVEXITY FROM FIBERS

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Abstract. Let $1 \to K \longrightarrow G \stackrel{\pi}{\longrightarrow} Q$ be an exact sequence of hyperbolic groups. Let $Q_1 < Q$ be a quasiconvex subgroup and let $G_1 = \pi^{-1}(Q_1)$. Under relatively mild conditions (e.g. if K is a closed surface group or a free group and Q is convex cocompact), we show that infinite index quasiconvex subgroups of G_1 are quasiconvex in G. Related results are proven for metric bundles, developable complexes of groups and graphs of groups.

1. Introduction

The aim of this paper is to provide evidence in favor of the following Scholium.

Scholium 1.1. *For an exact sequence*

$$1 \to K \longrightarrow G \stackrel{\pi}{\longrightarrow} Q \to 1$$

of hyperbolic groups, and more generally for hyperbolic metric bundles, quasiconvexity in fibers propagates to quasiconvexity in subbundles.

The results in this paper are in the same vein as a series of theorems starting with work of Scott-Swarup [SS90], followed by several authors [Mit99, DKL14, KL15, DKT16, DT17, MR18, Gho20]. In all these papers, the setup was as follows: Consider an exact sequence of hyperbolic groups as in Scholium 1.1. Then, under relatively mild conditions, it was shown that an infinite index quasiconvex subgroup of K is quasiconvex in the bigger group G. The purpose of this paper is to extend these results in a different direction. Let $Q_1 < Q$ be a quasiconvex subgroup and let $G_1 = p^{-1}(Q_1)$. The main results of this paper show similarly that, under some conditions, infinite index quasiconvex subgroups of G_1 are quasiconvex in G. In other words the distortion [Gro93, Chapter 4] (see Definition 2.2 below) is entirely captured by the fiber group. The earlier papers cited above all treat the case with $Q_1 = \{1\}$.

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The simplest example that illustrates this is the following. Let M_1, M_2 be two closed hyperbolic 3-manifolds fibering over the circle with the same topological fiber F. Suppose further that the 3-complex $M_1 \cup_F M_2 = M$ obtained by gluing M_1, M_2 along the fiber F has a hyperbolic fundamental group given by $G = \pi_1(M_1) *_{\pi_1(F)} \pi_1(M_1)$. Then a prototypical theorem of this paper shows that an infinite index quasiconvex subgroup of $\pi_1(M_1)$ is quasiconvex in G. Analogous results are also shown for K a free group. Note that, using work of [FM02, KL08, Ham05, DT17], these results can equivalently be formulated in terms of quasiconvex subgroups Q_1 of convex cocompact subgroups of the mapping class group $MCG(\Sigma)$ or the outer automorphism group $Out(F_n)$ of the free group as follows (see Theorems 4.3 and 4.5):

Theorem 1.2. *Let*

$$1 \to K \longrightarrow G \stackrel{\pi}{\longrightarrow} Q \to 1$$

be an exact sequence of hyperbolic groups, where K is either $\pi_1(\Sigma)$, with Σ a closed surface of genus at least 2; or a finitely generated free group F_n, $n > 2$. Let Q be an infinite convex cocompact subgroup of respectively $MCG(\Sigma)$ or $Out(F_n)$. Let $Q_1 < Q$ is a qi embedded subgroup and $G_1 = \pi^{-1}(Q_1)$. Suppose $H < G_1$ is an infinite index, quasiconvex subgroup. Then H is quasiconvex in G.

The main new technical tool used in the proof of Theorem 1.2 is the existence and structure of the Cannon-Thurston map for the pair (G_1, G) established in [KS20]. We show also (Proposition 4.8) that in the absence of convex cocompactness, Theorem 1.2 fails quite dramatically due to the existence of subgroups K_1 of K that are quasiconvex in K and G_1, but not in G. The main theorems of the paper are more general, and make the content of Scholium 1.1 precise in three contexts: exact sequences of groups (Theorem 4.1), complexes of groups (Theorem 4.9), and general graphs of groups (Theorem 4.10).

2. A coarse topological fact

The main aim of this section is to prove Proposition 2.8, which is a generalization in the geometric group theory context of the following simple 3-manifold fact:

Fact 2.1. Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber F. Let $\Sigma \subset M$ be an immersed incompressible quasi-Fuchsian surface. Let $G = \pi_1(M), K = \pi_1(F), H = \pi(\Sigma)$. Let M_F denote the cover of M corresponding to $K < G$, let F_0 denote a lift of F to M_F and Σ_F denote a lift of Σ to M_F. Then, given any finitely generated subgroup $K_1 < K \cap H$, there is a compact subsurface $\Sigma_0 \subset \Sigma_F$, necessarily contained in a finite neighborhood of F_0, such that $K_1 < i_*(\pi_1(\Sigma_0))$, where $i : \Sigma_0 \to M_F$ denotes the inclusion map.

2.1. **Preliminaries.** *All metric spaces in this paper are proper geodesic metric spaces.* A *metric graph* is a simplicial connected graph where all the edges are assigned length 1 and the graph is given the induced length metric [BH99, Chapter I.1, Section 1.9]. For any graph X we shall denote by $V(X)$ and $E(X)$ the vertex set and the edge set of X respectively. For any metric space X, $A \subset X$ and $r \geq 0$ we denote by $N_r(A)$ the set $\{x \in X : d(x,a) \leq r \text{ for some } a \in A\}$ and refer to it as the r-neighborhood of A in X. Also for $x \in X$, $r \geq 0$ we shall denote by $B(x,r)$ the closed ball of radius r in X centered at x. Given $A, B \subset X$ the Hausdorff distance of A, B is given by $Hd(A,B) = \inf\{r \geq 0 : B \subset N_r(A), A \subset N_r(B)\}$. We now recall some basic notions we shall use in this paper. We refer the reader to [BH99, Chapter I.8] for details on quasi-isometries and q(uasi)-i(sometric) embeddings, and to [BH99, Chapter III.H] for details on (Gromov-)hyperbolic groups. If X is a geodesic metric space and $Y \subset X$ is a path-connected subspace we equip Y with the induced path metric [BH99, Chapter I.3] and assume that Y is geodesic metric space with respect to this metric.

If G is a group with a finite generating set S then we shall always identify G with the vertex set of the Cayley graph $\Gamma(G,S)$. The metric on G induced from $\Gamma(G,S)$ will be referred to as the *word metric* on G with respect to the generating set S. Let H be a finitely generated subgroup of a finitely generated group G. Assume further that a finite generating set of G is chosen extending a finite generating set of H, so that the Cayley graph Γ_H embeds in Γ_G.

Definition 2.2. [Gro93, Chapter 4] *Let $i : \Gamma_H \to \Gamma_G$ be the above embedding of Cayley graphs. The* distortion function *of H in G is given by*

$$disto(R) = \frac{1}{R} Diam_H(\Gamma_H \cap B_G(R))$$

where $B_G(R)$ is the $R-$ball about $1 \in \Gamma_G$. If the distortion is bounded below by a super-linear function, we say that H is distorted *in G. Else, we say it is* undistorted.

Note that H is undistorted in G if and only if it is qi-embedded. Given a function $f : \mathbb{N} \to \mathbb{N}$, a family of maps $\phi_i : X_i \to Y_i$, $i \in I$ between metric spaces is called *uniformly metrically proper as measured by f* [MS12] if for all $i \in I$ and for all $x, x' \in X_i$,

$$d_{Y_i}(\phi_i(x), \phi_i(x')) \leq R \Rightarrow d_{X_i}(x, x') \leq f(R)$$

for all $R \geq 0$. We record the following elementary fact for easy reference.

Lemma 2.3. *(1) Suppose $H < G$ are finitely generated groups. Then the inclusion $H \to G$ is uniformly metrically proper with respect to any given word metrics on G, H.*
(2) Suppose X is a geodesic metric space and Y is a subspace such that it is also a geodesic metric space with respect to the induced length metric d_Y. Suppose the inclusion map $Y \to X$ is uniformly metrically proper as measured by $f : \mathbb{N} \to \mathbb{N}$ with respect to these metrics. Then given a k-quasigeodesic α of X contained in Y, it is a quasigeodesic $k' = k'(k, f)$-quasigeodesic of Y with respect to the metric d_Y.

A geodesic metric space X is *δ-hyperbolic* if for any geodesic triangle Δxyz, $[x,y] \subset N_\delta([y,z] \cup [x,z])$. A metric space X is called *hyperbolic* it is δ-hyperbolic for some $\delta \geq 0$. A finitely generated group is hyperbolic if some (any) of its Cayley graphs (with respect to a finite generating set) is hyperbolic. Given a hyperbolic metric space X its *geodesic boundary or visual boundary* is the set of equivalence classes of geodesic rays where two rays α, β are equivalent if and only if $Hd(\alpha, \beta) < \infty$. The boundary of X is denoted by ∂X. The set $\bar{X} := X \cup \partial X$ comes equipped with a natural topology [BH99, Chapter III.H]. Suppose X is a hyperbolic metric space and $\alpha : [0, \infty) \to X$ is a geodesic ray. Then the equivalence class of α is denoted by $\alpha(\infty)$. Also if $\alpha : [0, \infty) \to X$ is a quasigeodesic ray, then there is a geodesic ray $\beta : [0, \infty) \to X$, unique up to the equivalence relation above, such that $Hd(\alpha, \beta) < \infty$. We shall use the same notation $\alpha(\infty)$ to denote $\beta(\infty)$. If G is a hyperbolic group then we write ∂G to denote the geodesic boundary of any of its

Cayley graphs since any two such boundaries of Cayley graphs are homeomorphic (cf. Lemma 3.12(2)).

If $\alpha : \mathbb{R} \to X$ is a (quasi)geodesic line then by $\alpha(-\infty)$ we shall denote the point of ∂X determined by the ray $t \mapsto \alpha(-t)$. In this case, we say that α joins the pair of points $\alpha(-\infty), \alpha(\infty)$. Suppose ξ_i, $i = 1, 2, 3$ are any three distinct points of ∂X. Then any set of three geodesic lines joining these points in pairs is called an *ideal triangle* with vertices ξ_1, ξ_2, ξ_3. The geodesic lines are called the *sides* of the ideal triangle.

Lemma 2.4. *Suppose X is proper δ-hyperbolic. Then:*
(1) (Visibility [BH99, Chapter III.H, Lemma 3.2]). *Suppose $\xi_1, \xi_2 \in \partial X$ are two distinct points. Then there is a geodesic line in X joining them.*
(2) (Barycenters of ideal triangles) [BH99, Chapter III.H, Lemmas 1.17, 3.3] *There are constant $D = D(\delta), R = R(\delta)$ such that the following holds. For any three distinct points $\xi_1, \xi_2, \xi_3 \in \partial X$ there is a point $x \in X$ which is contained in the D-neighborhood of each of the three sides of any ideal triangle with vertices ξ_1, ξ_2, ξ_3. Moreover, if x' is any other such point then $d(x, x') \leq R$.*

A point x as in Lemma 2.4 is referred to as a *barycenter* of the ideal triangle. Thus we have a coarsely well-defined map $\partial^3 X \to X$ sending distinct triples of points to a barycenter. We call such a map a *barycenter map*. We call X a *non-elementary hyperbolic metric space* if there exists a constant $L \geq 0$ such that the L-neighborhood of the image of a barycenter map is the entire X (the terminology is based on the notion of a non-elementary group of isometries of a hyperbolic metric space).

Definition 2.5. *Suppose X is a hyperbolic metric space and $A \subset X$. Then the* limit set *of A in X, denoted by $\Lambda_X(A)$, is the collection of accumulation points of A in ∂X.*

If G is a hyperbolic group and H is a subgroup then the limit set $\Lambda_G(H)$ of H in G is the limit set of some (any) $H-$orbit. The following is a list of basic properties of the limit set that will be useful for us (see [Thu80, Chapter 8] and [BH99, Chapter III.H.3] for instance).

Lemma 2.6. *(0) If X is a hyperbolic metric space and $A \subset X$ then $\Lambda_X(A)$ is a closed subset of ∂X; $\Lambda_X(A) = \emptyset$ if and only if A is bounded.*
(1) If X is a hyperbolic metric space and $A, B \subset X$ then $\Lambda_X(A) = \Lambda_X(B)$ if $Hd(A, B) < \infty$. If Y is qi-embedded in X, then ∂Y embeds in ∂X.
(2) Suppose $H < G$ are hyperbolic groups. Then the following hold:
(i) If $H \trianglelefteq G$ then $\Lambda_G(H) = \partial G$.
(ii) If H is a finitely generated subgroup of G which is qi embedded and is of infinite index then $\Lambda_G(H)$ is a proper closed subset of ∂G.

We will also need the following elementary lemma.

Lemma 2.7. *Let G be a finitely generated group acting properly and cocompactly by isometries on a metric space X. Let Γ be a Cayley graph of G with respect to a finite generating set. Suppose H, K are two finitely generated subgroups of G and A, B are two subsets of X invariant under H and K respectively. Lastly, assume that A/H, B/K are compact. If $Hd_X(A, B) < \infty$ then $Hd(H, K) < \infty$ where the $Hd(H, K)$ is taken in Γ.*

Proof. Let S be a generating set of G and let Γ be the Cayley graph $\Gamma(G,S)$. On subsets of G we shall use the metric induced from Γ. Let $x \in A, y \in B$. Then $Hd(K.x, K.y) \leq d_X(x,y) < \infty$. Since the actions of H,K on A,B respectively are cocompact, we have $Hd(H.x, A) < \infty, Hd(K.y, B) < \infty$. Since $Hd(A,B) < \infty$, it follows that $Hd(H.x, K.y) < \infty$. Hence $Hd(H.x, K.x) < \infty$. Since the action of G on X is proper and cocompact, the orbit map $G \to X$ given by $g \mapsto g.x$ for all $g \in G$ is a quasi-isometry by the Milnor-Švarc lemma (see [BH99, Proposition 8.19]). Therefore, $Hd(H,K) < \infty$. □

2.2. **Finitely generated subgroups from bounded neighborhoods.** We are now in a position to generalize Fact 2.1.

Proposition 2.8. *Let G be a finitely presented group and H,K be two subgroups where H is finitely generated. Let $S \subset G$, $S_1 \subset H$ be finite generating sets and $\Gamma = \Gamma(G,S)$, $\Gamma_1 = \Gamma(H,S_1)$ be the corresponding Cayley graphs. Let $D > 0$ and suppose that there is an infinite set $A \subset N_D(K) \cap H$ where $N_D(K)$ is the D-neighborhood of K in Γ. Suppose that there exists $r \geq 1$ such that any pair of points of A can be connected by a path lying in the r-neighborhood of A in Γ_1. Then there is a finitely generated infinite subgroup $K_1 < H \cap K$ such that A/K_1 is finite. In particular, A is contained within a finite neighborhood of K_1 in Γ.*

Proof. We start with a finite connected simplicial 2-complex $\mathcal{Y}$ together with a finite 1-subcomplex $\mathcal{Z}$ such that $G = \pi_1(\mathcal{Y})$ and $i^*_{\mathcal{Z},\mathcal{Y}}(\pi_1(\mathcal{Z})) = H$ where $i_{\mathcal{Z},\mathcal{Y}} : \mathcal{Z} \to \mathcal{Y}$ is the inclusion map and $i^*_{\mathcal{Z},\mathcal{Y}} : \pi_1(\mathcal{Z}) \to \pi_1(\mathcal{Y})$ is the induced map. One may metrize $\mathcal{Y}$ in the standard way so that it is a geodesic metric space ([BH99, Chapter I.7, Theorem 7.19]). Fix one such metric on $\mathcal{Y}$ once and for all. Let $p : \mathcal{X} \to \mathcal{Y}$ be the universal cover of $\mathcal{Y}$ endowed with the induced length metric from $\mathcal{Y}$. (See [BH99, Definition 3.24, Chapter I.3].) We note that X is a proper metric space ([BH99, Exercise 8.4(1), Chapter I.8]). Let $x \in \mathcal{X}$ be a point such that $p(x) \in \mathcal{Z}$. G acts on $\mathcal{X}$ by deck transformations so that $\mathcal{X}/G = \mathcal{Y}$. Let $\phi : G \to \mathcal{X}$, $g \mapsto g.x$ be the orbit map. Then, identifying G with $V(\Gamma)$ (equipped with the subspace metric), ϕ gives a Lipschitz map from G to $\mathcal{X}$. Similarly, the inclusion map $i_{H,G} : H \to G$ is also Lipschitz. Therefore, the composition $\phi \circ i_{G,H}$ is Lipschitz. Suppose that ϕ and $\phi \circ i_{H,G}$ are both L-Lipschitz.

Let $\tilde{\mathcal{Z}}$ be the connected component of $p^{-1}(\mathcal{Z})$ containing x. It follows from the hypothesis that if $R = (D + r)L$ then $K.B(x;R)$ contains a connected set $\mathcal{F} \subset \tilde{\mathcal{Z}}$ such that $\phi(A) \subset \mathcal{F}$. Now consider the quotient map $q : \mathcal{X} \to \mathcal{X}/K$. Since $\mathcal{X}$ is a proper metric space, $B(x,R)$ is compact. Hence, $q(\mathcal{F})$ is contained in the compact set $q(B(x;R))$. Thus $\overline{q(\mathcal{F})}$ is compact. We note that $q(\tilde{\mathcal{Z}})$ is a closed subcomplex of $\mathcal{X}/K$. Hence, there is a finite subcomplex, say $\mathcal{W}$, of $q(\tilde{\mathcal{Z}})$ containing $\overline{q(F)}$. Let $i : \mathcal{W} \to q(\tilde{\mathcal{Z}})$, and $j : q(\tilde{\mathcal{Z}}) \to \mathcal{X}/K$ be the inclusion maps. Let K_1 be the image of $j^* \circ i^*$. Clearly this is a finitely generated subgroup of $H \cap K$. Since $q^{-1}(\mathcal{W})$ contains the set $\mathcal{F}$ it follows that there is a connected component, say $\tilde{\mathcal{W}}$, of $q^{-1}(\mathcal{W})$ containing $\mathcal{F}$. Since $\tilde{\mathcal{W}}/K_1$ is compact and $\phi(A) \subset \mathcal{F} \subset \tilde{\mathcal{W}}$, it follows that $\phi(A)/K_1$ is finite.

Finally, we note that the orbit map $\phi : G \to \mathcal{X}$ is a G-equivariant quasi-isometry by the Milnor-Švarc lemma ([BH99, Proposition 8.19]) since the G-action on $\mathcal{X}$ is proper and cocompact. It follows that A/K_1 is finite. Since A is an infinite set, it follows that K_1 is also infinite. □

3. Graphs of spaces and Cannon-Thurston Maps

We shall recall some material from [MS12, MR18, KS20] and deduce some consequences. Informally a graph of metric spaces is a 1-Lipschitz surjective map $\pi : \mathcal{X} \to \mathcal{B}$ from $\mathcal{X}$ to a metric graph $\mathcal{B}$ satisfying some additional conditions. We refer to $\mathcal{X}$ as the *total space* and $\mathcal{B}$ as the *base*. We shall need two specific instances of this: metric graph bundles and trees of metric spaces. The common feature in both these cases is that $\mathcal{X}$ is a graph, the map π is simplicial and the fibers are uniformly properly embedded in the total space. As usual, we shall use $V(\mathcal{G})$ to denote the vertex set of a graph $\mathcal{G}$.

3.1. Metric graph bundles.

Definition 3.1. [MS12, Definition 1.2] *Suppose $\mathcal{X}$ and $\mathcal{B}$ are metric graphs and $f : \mathbb{N} \to \mathbb{N}$ is a function. We say that $\mathcal{X}$ is an f-*metric graph bundle *over $\mathcal{B}$ if there exists a surjective simplicial map $\pi : \mathcal{X} \to \mathcal{B}$ such that the following hold.*

(1) *For all $b \in V(\mathcal{B})$, $\mathcal{F}_b := \pi^{-1}(b)$ is a connected subgraph of $\mathcal{X}$. Moreover, the inclusion maps $\mathcal{F}_b \to \mathcal{X}$, $b \in V(\mathcal{B})$ are uniformly metrically proper as measured by f.*
(2) *For all adjacent vertices $b_1, b_2 \in V(\mathcal{B})$, any $x_1 \in V(\mathcal{F}_{b_1})$ is connected by an edge to some $x_2 \in V(\mathcal{F}_{b_2})$.*

For all $b \in V(B)$ we shall refer to F_b as the *fiber* over b and its path metric by d_b. Condition (2) of Definition 3.1 immediately gives:

Lemma 3.2. *If $\pi : \mathcal{X} \to \mathcal{B}$ is a metric graph bundle then for any points $v, w \in V(\mathcal{B})$ we have $Hd(F_v, F_w) < \infty$.*

Definition 3.3. *Suppose $\mathcal{X}$ is an f-*metric graph bundle *over $\mathcal{B}$. Given $k \geq 1$ and a connected subgraph $\mathcal{A} \subset \mathcal{B}$, a k-*qi section *over $\mathcal{A}$ is a map $s : \mathcal{A} \to \mathcal{X}$ such that s is a k-qi embedding and $\pi \circ s$ is the identity map on $\mathcal{A}$.*

We shall only need qi sections over geodesics in this paper. We next discuss the main examples of metric graph bundles that we shall refer to later.

Lemma 3.4. (Restriction of metric graph bundles [KS20, Lemma 3.17]) *Suppose $\pi : \mathcal{X} \to \mathcal{B}$ is an f-metric graph bundle and $\mathcal{B}_1$ is a connected subgraph of $\mathcal{B}$. Let $\mathcal{X}_1 = \pi^{-1}(\mathcal{B}_1)$. Then clearly $\mathcal{X}_1$ is a connected subgraph of $\mathcal{X}$. Let $\pi_1 : \mathcal{X}_1 \to \mathcal{B}_1$ denote the restriction of π to $\mathcal{X}_1$. Then $\pi_1 : \mathcal{X}_1 \to \mathcal{B}_1$ is also an f-metric graph bundle.*

Example 3.5. (Metric graph bundles from short exact sequences [MS12, Example 1.8], [KS20, Example 5].) Suppose we have a short exact sequence of finitely generated groups

$$1 \to K \stackrel{i}{\to} G \stackrel{\pi}{\to} Q \to 1.$$

Suppose S is a finite generating set of G such that S contains a generating set A of K. Let $\mathcal{X} = \Gamma(G, S)$ be the Cayley graph of G with respect to the generating set S. Let $B = (\pi(S) \setminus \{1\})$ and $\mathcal{B} := \Gamma(Q, B)$ be the Cayley graph of the group Q with respect to the generating set B. Then the map π naturally induces a simplicial map $\pi : \mathcal{X} \to \mathcal{B}$ between Cayley graphs. This is a metric graph bundle. The fibers are the translates of $\Gamma(K, A)$ under left multiplication by elements of G.

Moreover, suppose $Q_1 < Q$ is a finitely generated subgroup and $G_1 = \pi^{-1}(Q_1)$. Suppose B contains a generating set B_1 of Q_1. Let $S_1 = S \cap G_1$, $\mathcal{X}_1 = \Gamma(G_1, S_1)$, and $\mathcal{B}_1 = \Gamma(Q_1, B_1)$. Then π restricts to a metric graph bundle map $\pi_1 : \mathcal{X}_1 \to \mathcal{B}_1$ by Lemma 3.4.

Example 3.6. (Metric graph bundles from complexes of groups [KS20, Example 3]) We refer to [BH99] and [Hae92] for basics on developable complexes of (finitely generated) groups. Suppose $\mathcal{Y}$ is a finite connected simplicial complex and $\mathbb{G}(\mathcal{Y})$ is a developable complex of finitely generated groups over $\mathcal{Y}$ with the following additional condition:

(†) For all faces $\sigma \subset \tau$ of $\mathcal{Y}$ the corresponding homomorphism $G_\sigma \to G_\tau$ is an isomorphism onto a finite index subgroup.

Let $G = \pi_1(\mathbb{G}(\mathcal{Y}))$ be the fundamental group of the complex of groups. (See [BH99, Definitions 3.1, 3.5, Chapter III.$\mathcal{C}$]).

Proposition 3.7. [KS20, section 3] *There is a metric graph bundle $\pi : \mathcal{X} \to \mathcal{B}$ where G acts on both $\mathcal{X}$, $\mathcal{B}$ by isometries such that the following hold.*
(1) The map π is G-equivariant.
(2) The G-action is proper and cocompact on $\mathcal{X}$. The G-action is cocompact (but not necessarily proper) on $\mathcal{B}$.
(3) There is an isomorphism of graphs $p : \mathcal{B}/G \to \mathcal{Y}^{(1)}$ such that for all $\sigma_0 \in \mathcal{Y}^{(0)}$, and $v \in p^{-1}(\sigma_0)$, G_v is a conjugate of G_{σ_0} in G.
(4) For all $v \in V(\mathcal{B})$, the G_v-action on $V(\mathcal{F}_v)$ is transitive but the action on $E(\mathcal{F}_v)$ has a uniformly bounded number of orbits. Thus, the G_v-action on $\mathcal{F}_v$ is proper and cocompact. In particular if all the groups G_σ are hyperbolic then the fibers of the metric graph bundle $\pi : \mathcal{X} \to \mathcal{B}$ are uniformly hyperbolic.
(5) Suppose moreover that $\mathcal{Y}_1 \subset \mathcal{Y}$ is a connected subcomplex such that the inclusion morphism $\mathbb{G}(\mathcal{Y}_1) \to \mathbb{G}(\mathcal{Y})$ induces an injective homomorphism at the level of fundamental groups (see [BH99, Proposition 3.6, Chapter III.$\mathcal{C}$]*). Let G_1 be the image of $\pi_1(\mathbb{G}(\mathcal{Y}_1))$ in G. Then we can construct a metric graph bundle $\pi : \mathcal{X} \to \mathcal{B}$ as above along with a connected subgraph $\mathcal{B}_1 \subset \mathcal{B}$ invariant under G_1 such that $p(\mathcal{B}_1/G_1) = \mathcal{Y}_1$, and G_1 acts properly and cocompactly on $\mathcal{X}_1 = \pi^{-1}(\mathcal{B}_1)$.*

We shall refer to a developable complex of groups satisfying the property (†) of Example 3.6 as a **developable complex of groups with qi condition**. In this paper we are particularly interested in developable complexes of groups where all face groups are non-elementary hyperbolic. Such a complex of groups will be called a **developable complex of hyperbolic groups with qi condition**.

Remark 3.8. For developable complexes of hyperbolic groups $\mathbb{G}(\mathcal{Y})$ with qi condition we shall consider connected subcomplexes $\mathcal{Y}_1 \subset \mathcal{Y}$ such that the inclusion morphism $\mathbb{G}(\mathcal{Y}_1) \to \mathbb{G}(\mathcal{Y})$ induces an injective homomorphism at the level of fundamental groups. In addition to this we shall require, with the notation of Proposition 3.7 that $\mathcal{B}_1$ *is qi embedded in* $\mathcal{B}$.

We have he following corollary.

Corollary 3.9. *Suppose $\mathbb{G}(\mathcal{Y})$ is a developable complex of infinite hyperbolic groups with qi condition over a finite connected simplicial complex $\mathcal{Y}$ and suppose $G = \pi_1(\mathbb{G}(\mathcal{Y}))$ is also hyperbolic. Then for all $y \in V(\mathcal{Y})$, $g \in G$ we have $\Lambda_G(gG_yg^{-1}) = \partial G$.*

Proof. Let $\pi : \mathcal{X} \to \mathcal{B}$ be a metric graph bundle satisfying the properties of Proposition 3.7. Let $K = gG_vg^{-1}$. Then for all $x \in G$, K and xKx^{-1} fixes two vertices of $\mathcal{B}$, say v, w respectively. Also $\mathcal{F}_v, \mathcal{F}_w$ are invariant under K, xKx^{-1} respectively and these two induced actions are cocompact. By Lemma 3.2 $Hd(\mathcal{F}_v, \mathcal{F}_w) < \infty$. Since the G-action on $\mathcal{X}$ is proper and cocompact, Lemma 2.7 then implies that $Hd(K, xKx^{-1}) < \infty$. Clearly, $Hd(xK, xKx^{-1}) \leq d(1, x) < \infty$. Thus $Hd(K, xK) < \infty$ for all $x \in G$. Hence, $\Lambda_G(K) = \Lambda_G(xK)$ for all $x \in G$ by Lemma 2.6(1). Also, since K is an infinite subgroup of G, $\Lambda_G(K) \neq \emptyset$ by Lemma 2.6(0). Finally, by Lemma 3.12(2) we have $\Lambda_G(K) = \partial G$ because $\Lambda_G(K) = \Lambda_G(xK) = x\Lambda_G(K)$ for all $x \in G$ whence $\Lambda_G(K)$ is a nonempty closed G-invariant subset of ∂G. □

Definition 3.10. (Trees of hyperbolic metric spaces [BF92]) *Suppose $\mathcal{T}$ is a tree and $\mathcal{X}$ is a metric space. Then a map $\pi : \mathcal{X} \to \mathcal{T}$ is called a tree of hyperbolic metric spaces with qi embedded condition if there are constants $\delta \geq 0$, $K \geq 1$ and function $f : \mathbb{N} \to \mathbb{N}$ with the following properties:*

(1) For all $v \in V(\mathcal{T})$, $\mathcal{X}_v = \pi^{-1}(v)$ is a geodesic metric space with the induced path metric d_v, induced from $\mathcal{X}$. Moreover, with respect to these metrics the inclusion maps $\mathcal{X}_v \to \mathcal{X}$ are uniformly metrically proper as measured by f.

(2) Suppose e is an edge of $\mathcal{T}$ joining $v, w \in V(\mathcal{T})$ and $m_e \in \mathcal{T}$ is the midpoint of this edge. Then $\mathcal{X}_e = \pi^{-1}(m_e)$ is a geodesic metric space with respect to the induced path metric d_e from $\mathcal{X}$. Let $[v, w]$ denote the edge e from v to w. Then moreover, there is a map $\phi_e : \mathcal{X}_e \times [v, w] \to \pi^{-1}(e) \subset \mathcal{X}$ such that
(i) $\pi \circ \phi_e$ is the projection map onto $[v, w]$.
(ii) ϕ_e restricted to $(0, 1) \times \mathcal{X}_e$ is an isometry onto $\pi^{-1}(\mathring{e})$ where $\mathring{e}$ is the interior of e.
(iii) ϕ_e restricted to $\mathcal{X}_e \times \{v\}$ and $\mathcal{X}_e \times \{w\}$ are K-qi embeddings from X_e into $\mathcal{X}_v$ and $\mathcal{X}_w$ respectively with respect to the induced path metric d_e on $\mathcal{X}_e$, and d_v, d_w on $\mathcal{X}_v, \mathcal{X}_w$ respectively.

Given a tree of hyperbolic metric spaces with qi embedding condition it is convenient to replace the vertex and edge spaces by quasi-isometric metric graphs and glue them using the maps ϕ_e's to get a tree of hyperbolic metric graphs. This is an example of a tree of metric graphs obtained by discretizing the classical Bass-Serre tree of spaces (see [SW79] for a topological exposition of Bass-Serre theory and [Sar18, Section 3] for the discretized version). The universal cover of a finite graph of spaces is a source of examples for a tree of metric spaces [SW79].

3.2. **Cannon-Thurston maps.** In this subsection, we collect together various existence theorems for Cannon-Thurston maps along with properties of CT laminations

Definition 3.11. [CT07, Mit98a, Mit98c] *Suppose $f : Y \to X$ is a map between hyperbolic metric spaces. We say that f admits a Cannon-Thurston map (or a CT map for short) if f induces a continuous map $\partial f : \partial Y \to \partial X$. Equivalently, for all $\xi \in \partial Y$, there exists $\partial f(\xi) \in \partial X$ such that for any sequence $\{y_n\}$ in Y converging to ξ, $\{f(y_n)\}$ converges to $\partial f(\xi)$. Further, ∂f is required to be continuous.*

We refer to [CT07] for the origin of CT maps and to [Mj19] for a survey.

Suppose $H < G$ are hyperbolic groups. Suppose Γ_G, Γ_H are Cayley graphs of G, H respectively with respect to some finite generating sets. Since G is identified with $V(\Gamma_G)$, we have a natural map $i : H \to \Gamma_G$. This map can be extended to a coarsely well-defined map $\Gamma_H \to \Gamma_G$ by sending any point on an edge joining

$h_1, h_2 \in H$ to $i(h_1)$ or $i(h_2)$. If the map $\Gamma_H \to \Gamma_G$ admits a CT map then we say that the inclusion map $H \to G$ admits a CT map.

The first and fourth parts of the following lemma are standard and follow from the definitions of CT maps and limit sets. For the second part see [BH99, Chapter III.H] for instance. The third part follows exactly as in [Mit99, Lemma 2.1].

Lemma 3.12. *(Properties of CT maps) (1) Suppose X, Y, Z are hyperbolic metric spaces and there exist maps $g : Z \to Y, f : Y \to X$ admitting CT maps $\partial g : \partial Z \to \partial Y$, $\partial f : \partial y \to \partial X$. Then the composition $f \circ g : Z \to X$ admits a CT map and $\partial(f \circ g) = \partial f \circ \partial g$.*
(2) If $f : Y \to X$ is a qi embedding then there is an injective CT map $\partial f : \partial Y \to \partial X$. A CT map induced by a quasi-isometry is a homeomorphism. In particular the action of G on its Cayley graph induces an action of G on ∂G by homeomorphisms. This action is minimal, i.e. there is no proper nonempty closed subset of ∂G invariant under G, provided G is non-elementary.
(3) Suppose that a hyperbolic group G acts by isometries on a hyperbolic metric space X and the action is properly discontinuous. Suppose $x \in X$ and that the orbit map $h : G \to X$ given by $h(g) = gx$ admits a CT map. If h is not a qi embedding then there are points $\xi_1 \neq \xi_2 \in \partial G$ such that $\partial h(\xi_1) = \partial h(\xi_2)$. In particular this is true for a hyperbolic subgroup H of a hyperbolic group G if the inclusion $H \to G$ admits a CT map.
(4) If $H < G$ are hyperbolic groups and the inclusion $i : H \to G$ admits a CT map $\partial i : \partial H \to \partial G$ then $\partial i(\partial H) = \Lambda_G(H)$.

The following is an immediate consequence of the Milnor-Švarc lemma and the above Lemma.

Corollary 3.13. *Suppose G is a hyperbolic group acting on a hyperbolic metric space X properly and cocompactly by isometries. Suppose that H is a finitely generated subgroup of G and that there is a subset Y of X invariant under the H-action with the following properties:*
(1) Y is a hyperbolic metric space with respect to the induced length metric from X.
(2) The inclusion $Y \to X$ admits a CT map.
(3) The H-action on Y is proper and cocompact.
Then H is hyperbolic and the inclusion $H \to G$ admits a CT map.

The non-injectivity of CT maps motivates the following definition which will be crucial in this paper, cf. [Mit97, MR18].

Definition 3.14. *Suppose $f : Y \to X$ is a map between hyperbolic metric spaces which admits a CT map. Then the* Cannon-Thurston lamination *(*or CT lamination *for short) for this map is given by*

$$\mathcal{L}_X(Y) = \{(\xi_1, \xi_2) \in \partial Y \times \partial Y : \xi_1 \neq \xi_2, \, \partial f(\xi_1) = \partial f(\xi_2)\}.$$

If α is a (quasi)geodesic line in Y such that $(\alpha(-\infty), \alpha(\infty)) \in \mathcal{L}_X(Y)$ then we say that α is a leaf *of the CT lamination $\mathcal{L}_X(Y)$.*

Let $f : Y \to X$ be a map between hyperbolic metric spaces admitting a CT map. Let $Z \subset Y$ be a qi-embedded subset, so that Z is hyperbolic and ∂Z embeds in ∂Y (Lemma 2.6).

Definition 3.15. *We say that a leaf $\alpha \subset Y$ of $\mathcal{L}_X(Y)$ is carried by Z if α lies in a bounded neighborhood of Z. Equivalently (by quasiconvexity of Z), $\alpha(\pm\infty) \in \partial Z(\subset \partial Y)$.*

A consequence of Proposition 2.8 is the following.

Corollary 3.16. *Suppose $G_1 < G$ is a hyperbolic subgroup of a hyperbolic group such that the inclusion $G_1 \to G$ admits a CT map. Further let H, K be hyperbolic subgroups of G_1 with the following properties:*
(i) The inclusion $K \to G_1$ admits a CT map.
(ii) $\Lambda_{G_1}(K) = \partial G_1$.
(iii) H is a qi embedded subgroup of G_1 with $[G_1 : H] = \infty$.
(iv) H is not qi embedded in G.
Then a CT map for the pair (H, G) exists and for any leaf α of the CT lamination $\mathcal{L}_G(H)$ which is contained in a finite neighborhood of K there exists a finitely generated subgroup K_1 of $H \cap K$, such that the following hold:
(1) α is contained in a finite neighborhood of K_1 (in a Cayley graph of G).
(2) $\Lambda_K(K_1) \neq \partial K$. In particular, $[K : K_1] = \infty$.
Hence,
(3) K_1 is not qi embedded in G, i.e. K_1 is distorted in G; and
(4) if K_1 is qi embedded in K then K_1 supports a leaf of the CT lamination $\mathcal{L}_G(K)$.

Proof. We first note that since the inclusion $G_1 \to G$ admits a CT map and H is qi embedded in G_1, the inclusion $H \to G$ admits a CT map. Hence, by Lemma 3.12, $\mathcal{L}_G(H)$ is non-empty. We now apply Proposition 2.8. Since G_1 is hyperbolic it is finitely presented. See for instance [Gro85, Corollary 2.2A]. Using Proposition 2.8 with G_1 in place of G and α in place of A we have a finitely generated subgroup K_1 of $H \cap K$ such that conclusion (1) of the Corollary holds.

To prove (2), suppose $\Lambda_K(K_1) = \partial K$. Since the inclusion $i : K \to G_1$ admits a CT map $\partial i : \partial K \to \partial G_1$, it follows that $\Lambda_{G_1}(K_1) = \partial i(\Lambda_K(K_1)) = \partial i(\partial K)$. However, by Lemma 3.12(4) $\Lambda_{G_1}(K) = \partial i(\partial K) = \partial G_1$. Hence, $\Lambda_{G_1}(K_1) = \partial G_1$. On the other hand $\Lambda_{G_1}(K_1) \subset \Lambda_{G_1}(H)$. Hence, $\Lambda_{G_1}(H) = \partial G_1$, contradicting hypothesis (iii). Hence, $\Lambda_K(K_1) \neq \partial K$, forcing $[K : K_1] = \infty$ and proving (2).

To prove (3), we again argue by contradiction. Suppose K_1 is qi embedded in G. Then $\mathcal{L}_G(K_1)$ is empty by Lemma 3.12. Further, K_1 is qi embedded in H as well. Thus α is a leaf of the CT lamination $\mathcal{L}_G(H)$ supported by K_1, forcing $\mathcal{L}_G(K_1)$ to be non-empty. This contradiction proves (3).

The inclusions $K \to G_1$ and $G_1 \to G$ admit CT maps. Hence, the inclusion $K \to G$ admits a CT map by Lemma 3.12(1). Now suppose K_1 is qi embedded in K. Since α is contained in a finite neighborhood of K_1 then there is a (bi-infinite) quasigeodesic line β of K contained in K_1 with $Hd(\alpha, \beta) < \infty$. To see this, suppose $\alpha \subset N_D(K_1)$ where the neighborhood is taken in a Cayley graph of G. Then for all $t \in [0, \infty)$ one may choose $\beta(t)$ to be a point of K_1 such that $d(\alpha(t), \beta(t)) \leq D + 1$. Thus, β is a quasigeodesic in (a Cayley graph of) K by Lemma 2.3. Therefore, β is a leaf of the CT lamination $\mathcal{L}_G(K)$ carried by K_1. This finishes the proof of (4). □

We now recall some existence theorems for CT maps:

Theorem 3.17. ([Mit98b, Theorem 3.10, Corollary 3.11]) *Suppose $\pi : \mathcal{X} \to \mathcal{T}$ is a tree of uniformly hyperbolic metric spaces with qi embedded condition. If the total* *space $\mathcal{X}$ is hyperbolic then for all $v \in V(\mathcal{T})$ the inclusion map $\mathcal{X}_v \to \mathcal{X}$ admits CT map.*

In particular given a finite graph of hyperbolic groups $(\mathcal{G}, \mathcal{Y})$ with qi embedded condition, if $G = \pi_1(\mathcal{G}, \mathcal{Y})$ is hyperbolic then for all $v \in V(\mathcal{Y})$, the inclusion map $G_v \to G$ admits a CT map.

Theorem 3.18. ([Mit98a]) *Suppose we have a short exact sequence of hyperbolic groups*

$$1 \to K \xrightarrow{i} G \xrightarrow{\pi} H \to 1.$$

Then a CT map exists for the inclusion $K \to G$.

Theorem 3.19. ([MS12, Theorem 3.2]) *Suppose $\pi : \mathcal{X} \to \mathcal{B}$ is a metric graph bundle where $\mathcal{X}$, $\mathcal{B}$ and all the fibers are uniformly hyperbolic. Also we assume that the fibers are non-elementary. Then for any fiber $\mathcal{F}_b, b \in V(\mathcal{B})$, the inclusion $\mathcal{F}_b \to \mathcal{X}$ admits a CT map.*

We note that hyperbolicity of $\mathcal{B}$ is not an assumption for Theorem 3.19 since it follows from the hypotheses ([Mos96] and [MS12, Proposition 2.10]). We have the following corollary.

Corollary 3.20. *Suppose $\mathbb{G}(\mathcal{Y})$ is a developable complex of non-elementary hyperbolic groups with qi condition over a finite connected simplicial complex $\mathcal{Y}$ and suppose $G = \pi_1(\mathbb{G}(\mathcal{Y}))$ is also hyperbolic. Then for all $y \in V(\mathcal{Y})$, $g \in G$ the inclusion $gG_yg^{-1} \to G$ admits a CT map.*

Proof. By Proposition 3.7 there is a metric graph bundle $\pi : \mathcal{X} \to \mathcal{B}$ along with simplicial actions of G on $\mathcal{X}$ and $\mathcal{B}$ such that π is G-equivariant. Further, all the fibers are uniformly hyperbolic and $\mathcal{X}$ is hyperbolic. Also, there is an isomorphism of graphs $p : \mathcal{B}/G \to \mathcal{Y}^{(1)}$ such that for all $y \in \mathcal{Y}^{(0)}$, $\{G_v : v \in p^{-1}(y)\}$, is the set of all conjugates of G_y. Moreover, each subgroup G_v acts properly and cocompactly on $\mathcal{F}_v$. By Theorem 3.19 inclusion of each fiber in $\mathcal{X}$ admits a CT map. Therefore, we are done by Corollary 3.13. □

The next two theorems [KS, KS20] extend the above theorems to trees of metric spaces and metric graph bundles. For trees of metric spaces, we have the following.

Theorem 3.21. (CT maps for subtrees of spaces [KS]) *Suppose $\pi : \mathcal{X} \to \mathcal{T}$ is a tree of hyperbolic metric spaces with qi embedded condition and such that the total space $\mathcal{X}$ is hyperbolic. Suppose $\mathcal{T}_1 \subset T$ is a subtree and $\mathcal{X}_1 = \pi^{-1}(\mathcal{T}_1)$. Then $\mathcal{X}_1$ is hyperbolic and the inclusion $\mathcal{X}_1 \to \mathcal{X}$ admits a CT map.*

One then immediately obtains:

Corollary 3.22. (CT maps for subgraph of groups, [KS]) *Given a finite graph of hyperbolic groups $(\mathcal{G}, \mathcal{Y})$ with qi embedded condition, and a connected subgraph $\mathcal{Y}_1 \subset \mathcal{Y}$, if $G = \pi_1(\mathcal{G}, \mathcal{Y})$ is hyperbolic then so is $G_1 = \pi_1(\mathcal{G}, \mathcal{Y}_1)$ and the inclusion map $G_1 \to G$ admits a CT map.*

For a metric graph bundle, we have:

Theorem 3.23. (CT maps for restriction bundles [KS20]) *Suppose $\pi : \mathcal{X} \to \mathcal{B}$ is a metric graph bundle where $\mathcal{X}$, $\mathcal{B}$ and all the fibers are uniformly hyperbolic. Also we assume that the fibers are non-elementary. Suppose $\mathcal{B}_1 \subset \mathcal{B}$ is a connected subgraph such that the inclusion map $\mathcal{B}_1 \to \mathcal{B}$ is a qi embedding. Let $\mathcal{X}_1 = \pi^{-1}(\mathcal{B}_1)$. Then $\mathcal{X}_1$ is hyperbolic and the inclusion map $\mathcal{X}_1 \to \mathcal{X}$ admits a CT map.*

We note that the conclusion about hyperbolicity of $\mathcal{X}_1$ in the above theorem was a consequence of [MS12, Remark 4.4]. The theorem immediately implies the following.

Corollary 3.24. *(1) Suppose we have a developable complex of non-elementary hyperbolic groups $\mathbb{G}(\mathcal{Y})$ with qi condition over a finite connected simplicial complex $\mathcal{Y}$, such that $G = \pi_1(\mathbb{G}(\mathcal{Y}))$ is hyperbolic. Suppose $\mathcal{Y}_1 \subset \mathcal{Y}$ is a connected subcomplex such that the induced homomorphism $G_1 = \pi_1(\mathbb{G}(\mathcal{Y}_1)) \to G = \pi_1(\mathbb{G}(\mathcal{Y}))$ is injective. Further assume that, with the notation of Proposition 3.7, $\mathcal{B}_1$ is qi embedded in $\mathcal{B}$. Then G_1 is hyperbolic and the inclusion $G_1 \to G$ admits a CT map.*
(2) Given an exact sequence of infinite hyperbolic groups

$$1 \to K \stackrel{i}{\to} G \stackrel{\pi}{\to} H \to 1$$

if $H_1 < H$ is qi embedded then $G_1 = \pi^{-1}(H_1)$ is hyperbolic and the inclusion $G_1 \to G$ admits a CT map.

We also have the following analogous statements for the leaves of the CT lamination. The CT lamination for a subtree of spaces satisfies the following.

Theorem 3.25. (CT lamination for subtree of spaces, [KS]) *Assume the hypotheses of Theorem 3.21. Suppose that α is a leaf of $\mathcal{L}_{\mathcal{X}}(\mathcal{X}_1)$. There exist $v \in V(\mathcal{T}_1)$ and a geodesic line $\beta \subset \mathcal{X}_v$ such that β is also a quasi-geodesic in $\mathcal{X}_1$ and $Hd(\alpha, \beta) < \infty$. Moreover, there is a point $\xi \in \partial\mathcal{T} \setminus \partial\mathcal{T}_1$ and a quasi-isometric lift γ of the geodesic in T joining v to ξ such that $\lim_{n\to\infty} \alpha(n) = \gamma(\infty)$.*

One then immediately obtains:

Corollary 3.26. (CT lamination for subgraph of groups, [KS]) *Assume the hypotheses of Corollary 3.22. Given a leaf α of the CT lamination $\mathcal{L}_G(G_1)$, there exist $v \in V(\mathcal{Y}_1)$ and $g \in G_1$ such that α is contained in a finite neighborhood of gG_vg^{-1}. More precisely there is a geodesic β in gG_vg^{-1} which is also a quasi-geodesic in G_1 and $Hd(\alpha, \beta) < \infty$.*

Similar statements hold also for the restriction bundle of a metric graph bundle:

Theorem 3.27. (CT lamination for restriction bundle, [KS20]) *Assume the hypotheses of Theorem 3.23. Suppose that α is a leaf of $\mathcal{L}_{\mathcal{X}}(\mathcal{X}_1)$. Then for all $v \in V(\mathcal{B}_1)$ there is a geodesic line $\beta \subset \mathcal{X}_v$ such that β is also a quasi-geodesic in $\mathcal{X}_1$ and $Hd(\alpha, \beta) < \infty$. Moreover, there exists $\xi \in \partial\mathcal{B} \setminus \partial\mathcal{B}_1$ and a qi section γ over a geodesic in $\mathcal{B}$ converging to ξ such that $\lim_{n\to\infty} \alpha(n) = \gamma(\infty)$.*

This immediately gives:

Corollary 3.28. (CT lamination for subcomplexes of groups, [KS20])

(1) Assume the hypotheses of Corollary 3.24(1). Then for any $v \in V(\mathcal{Y}_1)$ and $g \in G$, any leaf α of the CT lamination $\mathcal{L}_{G_1}(G)$ is contained in a finite neighborhood of gG_vg^{-1}.

(2) Assume the hypotheses of Corollary 3.24(2). Then given any leaf α of the CT lamination $\mathcal{L}_G(G_1)$ there is a geodesic line β in K such that it is a quasi-geodesic of G_1 and $Hd(\alpha, \beta) < \infty$.

4. From fibers to subbundles

In this section, we prove the main theorems of this paper and provide precise statements in support of Scholium 1.1.

4.1. **Exact sequences of hyperbolic groups.** Let

$$1 \to K \to G \stackrel{\pi}{\to} Q \to 1 \quad (*)$$

be an exact sequence of infinite hyperbolic groups and $Q_1 < Q$ be a qi embedded subgroup. Let $G_1 = \pi^{-1}(Q_1)$. Then G_1 is a hyperbolic group by [MS12, Remark 4.4]. Suppose H is a qi embedded, infinite index subgroup of G_1. The following theorem asserts that if H is distorted in G (cf. Definition 2.2; [Gro93, Chapter 4]), then this is due to the presence of a finitely generated subgroup $K_1 < K \cap H$ such that K_1 is distorted in G. In other words, the cause of distortion lies in finitely generated subgroups of the fiber group K.

Theorem 4.1. *With the above assumptions and notation, suppose H is not qi embedded in G. Then there is a finitely generated, infinite subgroup K_1 of $K \cap H$ such that $\Lambda_K(K_1) \neq \partial K$, $[K : K_1] = \infty$ and K_1 is distorted in G.*

Proof. The inclusion $G_1 \to G$ admits a CT map by the last part of Corollary 3.24(1). Since H is qi embedded in G_1, there is a CT map for the inclusion $H \to G_1$ by Lemma 3.12(2). By Lemma 3.12(1) the inclusion $i_{H,G} : H \to G$ also admits a CT map $\partial i_{H,G} : \partial H \to \partial G$. Since H is not qi embedded in G, by Lemma 3.12(3) there is a geodesic line γ in H whose end points are identified by the CT map $\partial i_{H,G}$. By Corollary 3.28(2) there exists a geodesic line $\alpha \subset K$ within a finite Hausdorff distance (in G_1, and hence in G) of γ such that

(1) α is a quasigeodesic in G_1 and hence in H;
(2) α is a leaf of the CT lamination $\mathcal{L}_G(H)$.

Recall that $\Lambda_G(K) = \partial G$ by Lemma 2.6(2)(i). Also the inclusion $K \to G_1$ admits a CT map by Theorem 3.18. The theorem now follows immediately from Corollary 3.16. □

Surface group fibers: We specialize the exact sequence $(*)$ above to the case where $K = \pi_1(\Sigma)$, with Σ a closed orientable surface of genus at least 2. It follows [Ham05, KL08, MS12] that Q is an infinite convex cocompact subgroup of the mapping class group $MCG(\Sigma)$. We shall need the following Theorem by Dowdall, Kent and Leininger [DKL14, Theorem 1.3] (see also [MR18, Theorem 4.7] for a different proof).

Theorem 4.2. *Consider the exact sequence $(*)$ with $K = \pi_1(\Sigma)$ (Σ closed) and Q convex cocompact. Let K_1 be a finitely generated infinite index subgroup of K. Then K_1 is quasi-convex in G.*

Suppose $Q_1 < Q$ is a qi embedded subgroup and $G_1 = \pi^{-1}(Q_1)$. Then Q_1 is also convex cocompact [Ham05, KL08]. Further, by Corollary 3.24(2), G_1 is also hyperbolic. We continue using the notation before Theorem 4.1.

Theorem 4.3. *Suppose $H < G_1$ is an infinite index, quasiconvex subgroup. Then H is quasiconvex in G.*

Proof. Suppose not. Then, by Theorem 4.1, there exists a finitely generated, infinite subgroup K_1 of $K \cap H$ such that

(1) $[K : K_1] = \infty$.
(2) K_1 is not quasiconvex in G.

This contradicts Theorem 4.2. □

Free group fibers: Next, instead of specializing to $K = \pi_1(\Sigma)$, we specialize the exact sequence $(*)$ to the case where $K = F_n$, the free group on n generators $(n > 2)$, and continue with the rest of the notation before Theorem 4.1. We recall from [DT17, HH18] that a subgroup $Q < Out(F_n)$ is said to be *convex cocompact* if some (and hence any) orbit of Q in the free factor complex $\mathcal{F}_n$ is qi embedded. Also, $Q < Out(F_n)$ is *purely atoroidal* if every element of Q is hyperbolic. We shall need the following:

Theorem 4.4. [DT17, Theorem 7.9][MR18, Theorem 5.14] *Consider the exact sequence $(*)$ with $K = F_n$, $(n > 1)$ and Q purely atoroidal and convex cocompact in $Out(F_n)$. Let K_1 be a finitely generated infinite index subgroup of K. Then K_1 is quasi-convex in G.*

Next, suppose $Q_1 < Q$ is a qi embedded subgroup and $G_1 = \pi^{-1}(Q_1)$. Then Q_1 is purely atoroidal convex cocompact [DT17, DT18, HH18]. By Corollary 3.24(2), G_1 is also hyperbolic. We continue with the notation before Theorem 4.1.

Theorem 4.5. *Under the hypotheses of Theorem 4.4, let $Q_1 < Q$ be qi embedded and $G_1 = \pi^{-1}(Q_1)$. Suppose $H < G_1$ is an infinite index, qi embedded subgroup. Then H is qi embedded in G.*

Proof. The proof is an exact replica of the proof of Theorem 4.3 with the use of Theorem 4.2 replaced by Theorem 4.4. □

A counter-example: We give an example to show that the conclusion of Theorem 4.5 fails if Q is not assumed to be convex cocompact even if G is hyperbolic. It follows from Theorem 4.1 that the failure is due to the existence of an infinite index quasiconvex subgroup K_1 of the normal free subgroup K such that K_1 is distorted in G. In fact, the examples below show that the corresponding results of [DT17, MR18] fail without the convex cocompactness assumption. We start with the following construction due to Uyanik [Uya19, Corollary 1.5] and Ghosh [Gho18, Theorem 5.6].

Definition 4.6. *Two automorphisms $\phi, \psi \in Out(F_n)$ are said to be* commensurable *if there exist $m, l \neq 0$ such that $\phi^m = \psi^l$.*

Theorem 4.7. [Uya19, Gho18] *Let $\phi \in Out(F_n)$ be a fully irreducible and atoroidal outer automorphism. Then, for any (not necessarily fully irreducible) atoroidal outer automorphism $\psi \in Out(F_n)$ which is not commensurable with ϕ, there exists $N \in \mathbb{N}$ such that, for all $m, l \geq N$, the subgroup $Q = \langle \phi^m, \psi^l \rangle < Out(F_n)$ is purely atoroidal and the corresponding extension G of F_n by Q is hyperbolic.*

Now, let $F_n = A * B$ be the product of two free factors, each of rank 3 or more. Let ψ_A and ψ_B be fully irreducible purely atoroidal automorphisms of A, B respectively, and let $\psi = \psi_A * \psi_B$. Let ϕ be a fully irreducible purely atoroidal automorphism of F_n. Theorem 4.7 furnishes a family of purely atoroidal subgroups $Q = \langle \phi^m, \psi^l \rangle < Out(F_n)$ for all m, l large enough such that the corresponding extension G of F_n by Q is hyperbolic. Now, let $Q_1 = \langle \phi^m \rangle < Q$ and $G_1 < G$ be given by $\pi^{-1}(Q_1)$, where $\pi : G \to Q$ is the natural quotient map. Similarly, let $Q_2 = \langle \psi^l \rangle < Q$ and $G_2 < G$ be given by $\pi^{-1}(Q_2)$. Then $A < F_n < G_1$ is quasiconvex by Theorem 4.4. But $A < F_n < G_2$ is not quasiconvex, since A is distorted in $(A \rtimes_{\psi_A^l} \mathbb{Z}) < G_2$. Hence A is not quasiconvex in G.

In fact A (or B) can be used for building more complicated examples as well. Pick any element $\gamma \in G_1 \setminus A$. Then by the usual ping-pong argument $A_2 = \langle A, \gamma^m \rangle$ is quasiconvex in G_1 for all large enough m. But A_2 is not quasiconvex in G since it contains the distorted subgroup A. Thus, the example above can be refined to prove the following.

Proposition 4.8. *Let F_n, ϕ, G_1 be as above. Let $H < G_1$ be an infinite quasiconvex subgroup of infinite index such that H is not virtually cyclic. Then there exists a finite index subgroup $G_1' < G_1$, so that*

(1) *$G_1' = F_m \rtimes \mathbb{Z}$, where the semi-direct product is given by a positive power ϕ_1 of a lift of ϕ to a finite index subgroup F_m of F_n,*
(2) *a purely atoroidal automorphism ψ of F_m such that $Q = \langle \psi, \phi_1 \rangle < Out(F_m)$ is free, and $G = F_m \rtimes Q$ is hyperbolic.*
(3) *$G_1' \cap H$ is of finite index in H and is not quasiconvex in G.*

Proof. Since $H < G_1$ is an infinite quasiconvex subgroup of infinite index that is not virtually cyclic, $H \cap F_n$ is infinite, free of rank greater than one. Let $K_1 < H \cap F_n$ be a free subgroup of finite rank greater than 2. Note that K_1 is quasiconvex in G_1 by Theorem 4.5. Now, by Hall's theorem, there exists a finite index subgroup F_k of F_n, containing K_1, such that K_1 is a free factor of F_k (this is essentially the LERF property for free groups [Sco78]). By passing to a further finite index subgroup F_m of F_n, we can assume that

(1) ϕ lifts to an automorphism ϕ' of F_m.
(2) $K_1 \cap F_m = K_2$ is a free factor of F_m.

Let $F_m = K_2 * B$ be a free factor decomposition. Let ψ_A, ψ_B be fully irreducible purely hyperbolic automorphisms of K_2, B respectively as in the construction of the counterexample preceding Proposition 4.8. (Note that we can assume, without loss of generality that B has rank greater than 2 by passing to finite index subgroups if necessary.) Let $\psi_0 = \psi_A * \psi_B$ as before.

By work of Reynolds [Rey11] summarized in [MR18, Theorem 4.10], a lift of a fully irreducible automorphism is fully irreducible. Then Theorem 4.7 furnishes a positive integer N such that $Q = \langle (\phi')^l, \psi_0^m \rangle$ is a purely atoroidal free subgroup of $Out(F_m)$, and $G = F_m \rtimes Q$ is hyperbolic. Choosing $\psi = \psi_0^m$, $\phi_1 = (\phi')^l$ and $G_1' = F_m \rtimes \phi_1$, it follows that K_2 is distorted in G as in the construction of the counterexample described before the present proposition. Since H contains K_2, and K_2 (being a finite index subgroup of K_1) is quasiconvex in G', it follows that $G_1' \cap H$ is not quasiconvex in G. We note in conclusion that since $[G_1 : G_1'] < \infty$, therefore, $[H : G_1' \cap H] < \infty$. □

4.2. **Complexes of hyperbolic groups with qi condition.** Suppose $\mathcal{Y}$ is a finite connected simiplicial complex and $\mathbb{G}(\mathcal{Y})$ is a developable complex of hyperbolic groups with qi condition over $\mathcal{Y}$. Suppose $\mathcal{Y}_1 \subset \mathcal{Y}$ is a connected subcomplex, $G = \pi_1(\mathbb{G}(\mathcal{Y}))$ and $G_1 = \pi_1(\mathbb{G}(\mathcal{Y}_1))$. Finally suppose that we have the hypotheses of Corollary 3.24(1). Then we have the following.

Theorem 4.9. *Suppose H is an infinite index, qi embedded subgroup of G_1 which is distorted in G. Then the inclusion $H \to G$ admits a CT map. Further, for any $g \in G_1$ and any $v \in V(\mathcal{Y}_1)$, and any leaf α of the CT lamination $\mathcal{L}_G(H)$, α is contained in a finite neighborhood of gG_vg^{-1}.*

Moreover, given a leaf α of $\mathcal{L}_G(H)$, $g \in G_1$ and $v \in V(\mathcal{Y})$ there is a finitely generated subgroup $K_1 < H \cap gG_vg^{-1}$ such that the following hold:
(1) α is contained in a finite neighborhood of K_1.
(2) $\Lambda_{gG_vg^{-1}}(K_1) \neq \partial(gG_vg^{-1})$. In particular, $[gG_vg^{-1} : K_1] = \infty$.
(3) K_1 is a distorted subgroup of G.
(4) If K_1 is qi embedded in gG_vg^{-1} then it supports a leaf of the CT lamination $\mathcal{L}_G(gG_vg^{-1})$.

Proof. By Corollary 3.24(1) the inclusion $G_1 \to G$ admits a CT map. By Lemma 3.12(1) the inclusion $H \to G_1$ admits a CT map since H is a qi embedded subgroup of G_1. Hence, the inclusion $H \to G$ admits a CT map too. Also by lemma 3.12(4) $\mathcal{L}_G(H) \neq \emptyset$ since H is distorted in G. Next, we note that α is a quasigeodesic in G_1 too since H is qi embedded in G_1. Thus it is a leaf of the CT lamination $\mathcal{L}_G(G_1)$. Hence, by Corollary 3.28(1) it follows that for all $g \in G_1$ and $v \in \mathcal{Y}$, α is contained in a finite neighborhood of gG_vg^{-1}.

The remaining parts of the theorem follow immediately from Corollary 3.16. We quickly check the various hypotheses of Corollary 3.16. We have already noted that the inclusion $G_1 \to G$ admits a CT map. Let $K = gG_vg^{-1}$. Clearly, H, K are hyperbolic groups. By Corollary 3.20 the inclusion $K \to G_1$ admits a CT map. By Corollary 3.9 $\Lambda_{G_1}(K) = \partial G_1$. Lastly H is given to be qi embedded in G_1 with $[G_1 : H] = \infty$. Thus, the hypotheses of Corollary 3.16 are satisfied, completing the proof of this theorem. □

To conclude this subsection, we mention a construction due to Min [Min11, Theorem 1.1] of a graph of groups where

(1) Each edge and vertex group is a closed hyperbolic surface group.
(2) The inclusion map of an edge group into a vertex group takes the edge group injectively onto a subgroup of finite index in the vertex group.
(3) The resulting graph of groups is hyperbolic.

By choosing the maps in the above construction carefully, Min furnishes examples [Min11, Section 5] that are not abstractly commensurable to a surface-by-free group. In particular, the resulting groups are not commensurable to those occurring in the exact sequence $(*)$ in Section 4.1.

4.3. **Graphs of groups.** Finally we look at graphs of hyperbolic groups. We note that in this case the homomorphisms $G_e \to G_v$ from the respective edge groups into the vertex groups are only qi-embeddings, and not necessarily quasi-isometries. Thus the corresponding tree of metric spaces is not a metric bundle. Consequently Theorem 4.10 below is not a consequence of any of the results proved so far.

Now, suppose $(\mathcal{G}, \mathcal{Y})$ is a finite graph of groups and suppose $(\mathcal{G}, \mathcal{Y}_1)$ is a subgraph of groups satisfying the hypotheses of Corollary 3.26 As before, let $G = \pi_1(\mathcal{G}, \mathcal{Y})$, and $G_1 = \pi_1(\mathcal{G}, \mathcal{Y}_1)$.

Theorem 4.10. *Suppose H is a qi-embedded, infinite index subgroup of G_1 such that H is not qi embedded in G. Then there exists a CT map for the pair (H, G). Let α be a leaf of the CT lamination for the inclusion $H \to G$. Then there exists $y \in V(\mathcal{Y}_1)$, $g \in G_1$ and a finitely generated subgroup $K_1 < gG_yg^{-1} \cap H$ such that the following hold:*
(1) α is contained in a finite neighborhood of K_1.
(2) K_1 is distorted in G.

(3) If K_1 is qi embedded in gG_yg^{-1} then it supports a leaf of the CT lamination $\mathcal{L}_G(gG_yg^{-1})$.

Proof. The proof runs along the line of the proof of Corollary 3.16 using Proposition 2.8. By Corollary 3.22 the group G_1 is hyperbolic and the inclusion $G_1 \to G$ admits a CT map. Also since H is qi embedded in G the inclusion $H \to G_1$ admits a CT map. Hence, by Lemma 3.12(1) the inclusion $H \to G$ admits a CT map. Since H is not qi-embedded in G (by hypothesis), $\mathcal{L}_G(H)$ is non-empty by Lemma 3.12.

Since H is qi embedded in G_1, α is a quasigeodesic in G_1 and therefore it is a leaf of the CT lamination $\mathcal{L}_G(G_1)$. Hence, by Corollary 3.26 there is $y \in V(\mathcal{Y}_1)$, $g \in G_1$ and a geodesic line β in gG_yg^{-1} such that $Hd(\beta, \alpha) < \infty$ and β is also a quasigeodesic in G_1.

We can now apply Proposition 2.8 with $A = \alpha$. The triple of groups (G_1, H, gG_yg^{-1}) plays the role of (G, H, K) in Proposition 2.8. It follows that there is a finitely generated infinite group $K_1 < gG_yg^{-1} \cap H$ such that α is contained in a finite neighborhood of K_1. This verifies (1).

Conclusion (2), (3) follow from (1) as in the proof of Corollary 3.16. We include the argument for the sake of completeness. We argue by contradiction by assuming K_1 is qi embedded in G. Then K_1 is qi embedded in H as well. Hence α is a leaf of $\mathcal{L}_G(H)$ supported by K_1. Hence K_1 is distorted in G, contradicting our assumption.

Finally, if K_1 is qi embedded in K then it is hyperbolic. Then K_1 supports β, which is a leaf of the CT lamination $\mathcal{L}_G(gG_yg^{-1})$. This completes the proof of the theorem. □

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