THE GROMOV-TISCHLER THEOREM FOR STRATIFIED SPACES

MAHAN MJ AND BALARKA SEN

Abstract. Given a symplectic structure on a stratified space X, we prove that X can be symplectically embedded in some complex projective space equipped with the standard Kähler form. This extends a theorem, due to Gromov and Tischler for manifolds, to stratified spaces.

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1. Introduction

A celebrated theorem due to Gromov [Gro86, Section 3.4.2], [Gro71] and Tischler [Tis77] says the following:

Theorem 1.1. *Let (M, ω) be a closed symplectic manifold. Then there exists $n \in \mathbb{N}$ and an embedding $f : M \to \mathbb{CP}^n$ such that $f^*\omega_n = \omega$, where ω_n is the standard Kähler form on $\mathbb{CP}^n$.*

In this paper, we first generalize Theorem 1.1 to the context of Whitney stratified spaces (see Theorem 4.4). But at the outset, we need to develop the necessary tools to answer the following basic question due to Gromov [Gro86, p. 343]:

Question 1.2. *Can one define singular symplectic (sub) varieties?*

Answers to Question 1.2 have appeared in the context of singular complex varieties and symplectic reduction[SL91, DJ96]. We develop the general theory of symplectic stratified spaces here from a symplectic geometry viewpoint (as opposed to the more algebraic Poisson structure perspective of [LMS93, MS99]). The main theorem of this paper is the following (see Theorem 4.4 for a more general statement):

Theorem 1.3. *Let (X, ω) be a compact stratified symplectic space where $\omega \in \Omega^2(X)$ is integral. There exists $N \geq 1$ and an embedding $f : X \to \mathbb{CP}^N$ such that $f^*\omega_N = \omega$, where ω_N is the standard Kähler form on $\mathbb{CP}^N$.*

There are two basic tools we use to prove Theorem 1.3:

(1) a compression lemma (Proposition 2.11) that allows us to use a smooth stratified map to go between strata
(2) a variant of work of Verona (Proposition 3.10) establishing a chain homotopy equivalence between the chain complex of smooth forms and the chain complex of controlled forms.

A construction going back to Nash (and refined by Gromov and Tischler) can then be reworked in our context (Section 4.3) to complete the proof.

Using Theorem 1.3, we establish an equivalence between the following notions (see Propositions 5.9 and 5.10):

(1) The notion of a Poisson-symplectic stratified space introduced by Sjamaar-Lerman [SL91] in a Poisson algebra setup.
(2) The dual geometric notion of a symplectic stratified space dealt with in this paper.

This allows us to go between the algebra and the geometry interchangeably (we caution only that the usual presence of a dualizing operator going between tangent and cotangent bundles introduces an inverse on individual terms of a matrix representation in Proposition 5.9.) This vindicates a hope expressed by Sjamaar-Lerman in [SL91, p. 412] in the context of Cushman's conjecture [SL91, Conjecture 6.14] predicting universality of $\mathbb{R}^n$ for symplectic reductions:
"Quite possibly the symplectic embedding theorems of Gromov and Tischler could be relevant in this context."
While Cushman's conjecture has turned out to be false in its strongest form, and approaches and partial positive results [LMS93, Ler05] have utilized a primarily algebraic approach, Theorem 1.3 shows that the dual geometric statement is indeed true, if we replace $\mathbb{R}^n$ by $\mathbb{C}P^n$ (see the discussion following Example 5.1).

Finally, we generalize a theorem of Gotay-Tuynman [GT89] to symplectic stratified spaces using a completely different argument from the one in [GT89]. We use Theorem 1.3 in an essential way.

Theorem 1.4. *(see Theorem 6.3) Let (X, ω) be a compact symplectic stratified space. Then there exists $n \geq 1$ and a Whitney stratified set $(Y, \Sigma) \subset \mathbb{R}^{2n}$ equipped with a map $f : Y \to X$ such that*

(1) *(Y, X, f) is a fiber bundle with smooth manifold fibers.*
(2) *For every stratum $S \in \Sigma$, $\ker \iota_S^* \omega_n = \ker df|_S$*

Theorem 1.4 essentially says that $\mathbb{R}^{2n}$ is universal for symplectic reductions in the setup of compact symplectic stratified spaces.

Acknowledgments: We thank Arvind Nair for pointing us to Verona's work [Ver71]. It was the comment of Sjamaar-Lerman in [SL91, p. 412] alluded to above that motivated us to find the equivalence between the algebraic and geometric perspectives in Section 5.

2. Preliminaries on stratified objects

2.1. Abstractly stratified spaces.

Definition 2.1. *Let X be a second countable metric space, and $(I, \leq)$ be a poset. An I-***decomposition** *of X is a locally finite collection of pairwise disjoint, locally closed subsets $\Sigma = \{S_i : i \in I\}$ of X, termed the* **strata** *of X, such that*

(1) *$X = \bigcup_{i \in I} S_i$*
(2) *S_i is a topological manifold for each $i \in I$*
(3) *$S_i \cap \overline{S_j} \neq \emptyset$ if and only if $S_i \subseteq \overline{S_j}$ if and only if $i \leq j$*

We shall carry over the partial order on I along the natural bijection $\Sigma \to I$, $S_i \mapsto i$, to the set of all strata Σ. This canonically makes the I-decomposition on X a Σ-decomposition. Therefore we shall often omit the indexing set I and simply refer to a *decomposition* of X, and call the pair (X, Σ) a *decomposed space*, although we keep the indexing set in the background for notational ease.

Associated to a decomposed space (X, Σ), some useful notions are that of *depth* and *dimension* respectively. Given a stratum $S \in \Sigma$, we define the depth of S:

$$\mathrm{dpt}(S) := \sup\{n : \exists \text{ a chain of strata } S = S_0 < S_1 < \cdots < S_n\}$$

The depth of X itself is defined to be $\mathrm{dpt}(X) := \sup\{\mathrm{dpt}(S) : S \in \Sigma\}$. Likewise, the dimension of X is defined to be $\dim(X) := \sup\{\dim(S) : S \in \Sigma\}$.

For a pair of strata $S, L \in \Sigma$, the condition "$S \cap \overline{L} \neq \emptyset$ if and only if $S \subset \overline{L}$" indicates that "deeper" strata always lie in the topological boundary of the "shallower" strata in X. We shall use the notation $\partial L := \overline{L} \setminus L$ for the topological boundary or the frontier of L in X. Observe that ∂L is naturally decomposed as it is a union of strata of X that lie deeper than L.

We shall use the notion of *abstract stratifications* developed by Thom [Tho69] and Mather [Mat12]. Given a decomposed space (X, Σ), a *tube system* $\mathscr{T}$ associated to the decomposition is a collection of open sets $\{T_i : i \in I\}$ such that each T_i is an open neighborhood of the stratum S_i in X, referred to as the *tubular neighborhood* of S_i in X, along with a retraction map $\pi_i : T_i \to S_i$ called the *tubular projection*,

as well as a continuous function $\rho_i : T_i \to [0, \infty)$ such that $\rho_i^{-1}(0) = S_i$. We shall call ρ the *radial function* associated to the tube.

Given a pair of indices $i \leq j$, one also defines the *tubular neighborhood of S_i in S_j* to be $T_{ij} := T_i \cap S_j$ and the restrictions of the tubular and radial functions are denoted as $\pi_{ij} := \pi_i | T_{ij}$ and $\rho_{ij} := \rho_i | T_{ij}$. Notice that ρ_{ij} is strictly positive for $i < j$. We denote a decomposed space (X, Σ) equipped with a tube system $\mathscr{T}$ by the triple $(X, \Sigma, \mathscr{T})$.

Definition 2.2. *$(X, \Sigma, \mathscr{T})$ is said to be* **abstractly stratified** *if*

1. *Each stratum $S \in \Sigma$ admit a $\mathcal{C}^\infty$-structure*
2. *For any pair $i, j \in I$ of indices with $i \leq j$, the tubular projection and radial functions π_{ij}, ρ_{ij} are smooth*
3. *$(\pi_{ij}, \rho_{ij}) : T_{ij} \to S_i \times (0, \infty)$ is a submersion*
4. *For any triple $i, j, k \in I$ of indices with $i \leq j \leq k$, the following holds:*
   (a) **π-control condition**: *$\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ on $T_{ik} \cap T_{jk} \cap \pi_{jk}^{-1}(T_{ij})$*
   (b) **ρ-control condition**: *$\rho_{ij} \circ \pi_{jk} = \rho_{ik}$ on $T_{ik} \cap T_{jk} \cap \pi_{jk}^{-1}(T_{ij})$*

At first glance this might seem technical, so we elucidate the notion by briefly explaining the purpose of each of these conditions:

Condition (1) imposes a smooth structure on each stratum, and the rest of the conditions are trying to specify how these smooth structure should interact near the boundary of a stratum with the smooth structures on the deeper strata.

Conditions (2) and (3) tell us that (π_{ij}, ρ_{ij}) give *partial cylindrical coordinates* for each of the tubes around the stratum S_i in S_j, much like tubular neighborhoods of smooth submanifolds of a manifold. One should imagine ρ_i as a radial distance function from $S_i \subseteq X$.

Condition (4), or the *control conditions*, demand that the tubular neighborhoods of each strata are so arranged that the projection maps are all compatible with each other, and moreover the projection maps preserve radial co-ordinates.

Remark 2.3. *A pair of abstract stratifications induced by tube systems $\mathscr{T}$ and $\mathscr{T}'$ on a given decomposed space (X, Σ) are said to be* **equivalent** *if π_i, π_i' and ρ_i, ρ_i' agree on an open neighborhood of S contained in $T_i \cap T_i'$ for all $i \in I$. We shall often replace a given tube system by a conveniently chosen tube system equivalent to it, and therefore it is best to think of the tubes as defined up to germinal equivalence, and equipped with a germ of a retraction and a germ of a tubular function.*

Remark 2.4. Given a pair of abstractly stratified spaces $(X, \Sigma, \mathscr{T})$ and $(X', \Sigma', \mathscr{T}')$, a topological embedding $f : X \hookrightarrow X'$ gives rise to a decomposition of X by $f^*\Sigma := \{f^{-1}(S') : S' \in \Sigma'\}$ and a tube system $f^*\mathscr{T}'$ such that $(X, f^*\Sigma', f^*\mathscr{T}')$ is abstractly stratified as well. More explicitly, elements of $f^*\mathscr{T}'$ are of the form $T_i' \cap f(X)$, where $T_i' \in \mathscr{T}'$. The tubular projection functions $f^*\pi_i'$ are given by restrictions of π_i' to strata of $f(X)$, and radial functions $f^*\rho_i'$ are given by $f^*\rho_i' = \rho_i' \circ f$.

We say f is an **isomorphism** if it is a homeomorphism, $f^*\Sigma' = \Sigma$ and the abstract stratifications are equivalent.

Remark 2.5. A similar notion of pullback can be defined if $f : X \hookrightarrow X'$ is a bundle with fiber a smooth manifold F. Then strata (resp. tubular neighborhoods) of X are given by $F-$bundles over strata S_i' (resp. induced $F-$bundles over tubular neighborhoods T_i'). Radial functions $f^*\rho_i'$ are given by $f^*\rho_i' = \rho_i' \circ f$ as before.

Next, to define projection functions $f^*\pi_i'$, let A denote the link of S_i' in X'. Let cA denote the cone over A. Then T_i' is a $cA-$bundle over S_i'. Let $E_i' = f^{-1}(S_i')$ (resp. $E(T_i') = f^{-1}(T_i')$) denote the restriction of the bundle $f : X \hookrightarrow X'$ to $S_i' \subset X'$ (resp. $T_i' \subset X'$). Since $\pi_i' : T_i' \to S_i'$ is a strong deformation retract (with contractible fiber cA), $E(T_i')$ is equivalent to $(\pi_i')^*E_i'$. There is a natural bundle map $(\pi_i')^*E_i' \to E_i'$ covering π_i'. Composing this with the equivalence with $E(T_i')$ furnishes the required projection function $f^*\pi_i'$.

We follow below the convention of [Mat12, Section 6]: $\varepsilon > 0$ will denote a positive function on the stratum S_i. Let

$$T_i(\varepsilon) := \{x \in T_i : \ 0 \leq \rho_i(x) < \varepsilon(\pi_i(x))\}$$

and $T_{ij}(\varepsilon) := T_i(\varepsilon) \cap S_j$ for $i < j$. We shall also denote

$$\partial T_i(\varepsilon) := \{x \in T_i : \ \rho_i(x) = \varepsilon(\pi_i(x))\}.$$

Here, we think of $\partial T_i(\varepsilon)$ as the "unit normal bundle" to the strata $S_i \in \Sigma$. Note that $\partial T_i(\varepsilon)$ admits a natural abstract stratification given by pulling back the stratification $(\Sigma, \mathscr{T})$ on X along the canonical inclusion $\partial T_i(\varepsilon) \hookrightarrow X$.

2.2. **Whitney stratified spaces.** The key reason for introducing abstract stratifications is that it happens to be exactly the right intrinsic definition of *Whitney stratified sets*, which are a universal model for most well-behaved singular spaces. We briefly recall the preliminaries and introduce the definition.

Let (S, L) be a pair of smooth (not necessarily properly) embedded submanifolds of a smooth manifold M such that $S \subset \overline{L}$.

(1) The pair (S, L) is said to be (a)**-regular** if for any sequence $\{x_n\}$ on L converging to a point $y \in S$, such that the sequence of tangent spaces $T_{x_n}L$ converge to a plane $\tau \subset T_yM$, we have $T_yS \subset \tau$.
(2) The pair (S, L) is said to be (b)**-regular** if moreover for any sequence $\{y_n\}$ on S which also converges to $y \in S$, such that the secant lines $\overline{x_n y_n}$ converge to a line $\ell \subset T_yM$, we have $\ell \subset \tau$.

The notions of convergence of planes and lines mentioned are defined locally, by choosing a coordinate chart $(U, y) \cong (\mathbb{R}^n, 0)$ around M, inside which all but finitely many terms of the sequences $\{x_n\}$ and $\{y_n\}$ belong. The conditions as such are independent of the coordinate charts as can be easily seen using the chain rule, and moreover have invariant coordinate-independent formulations in terms of the Grassmann and secant manifolds (see [Mat12] for further discussion).

Definition 2.6. *A closed, decomposed subset (W, Σ) of a smooth manifold M is said to be* **Whitney stratified** *if Σ consists of smooth (not necessarily properly) embedded submanifolds of M, such that any pair of strata $S, L \in \Sigma$ with $S < L$ satisfy (a)- and (b)-regularity.*

The decomposition of an algebraic variety by its singularities of various depths is not in general a Whitney stratification, a famous counterexample being the Whitney umbrella. However, Whitney proved that a refinement of this stratification does give any algebraic variety the structure of a Whitney stratified set. It was subsequently established by Thom, Lojasiewicz, Hironaka et al that a wide variety of spaces,

including semi-algebraic sets and subanalytic sets over either $\mathbb{R}$ or $\mathbb{C}$ admit Whitney stratifications.

Theorem 2.7. [Mat12] *Let (W, Σ) be a Whitney stratified subset of a smooth manifold M. Associated to each stratum $S \in \Sigma$ there exists a tubular neighborhood $\nu(S) \subset M$ (see discussion following Remark 2.4) with tubular projection $\pi_S : \nu(S) \to S$, and equipped with a fiberwise norm ρ_S induced from a fiberwise Riemannian metric, such that for all $S, L \in \Sigma$, $\nu(S) \cap \nu(L) \neq \emptyset$ only if $S < L$, in which case*

1. *$\pi_S \circ \pi_L = \pi_S$ throughout $\nu(S) \cap \pi_L^{-1}\nu(S)$*
2. *$\rho_S \circ \pi_L = \rho_L$ throughout $\nu(S) \cap \pi_L^{-1}\nu(S)$*

In particular, for any $S \in \Sigma$, define $T_S := \nu(S) \cap W$, equipped with the restrictions $\pi_S|W$ and $\rho_S|W$. This defines a tube system $\mathscr{T}$ on (W, Σ) such that $(W, \Sigma, \mathscr{T})$ is abstractly stratified.

There is a converse to this theorem, which is essentially a Whitney embedding theorem for abstractly stratified spaces. As per [Nat80], a **realization** of an abstractly stratified space $(X, \Sigma, \mathscr{T})$ in a smooth manifold M is an embedding $\iota : X \to M$ such that $Y := \iota(X) \subset M$ admits the structure of a Whitney stratified subset (Y, Σ') of M, and is moreover equipped with a tube system $\mathscr{T}'$ as in Theorem 2.7 so that $(Y, \Sigma', \mathscr{T}')$ is an abstractly stratified space, and the map $\iota : (X, \Sigma, \mathscr{T}) \to (Y, \Sigma', \mathscr{T}')$ is an isomorphism.

Theorem 2.8. [Nat80] *Any abstractly stratified space $(X, \Sigma, \mathscr{T})$ of finite dimension $n := \dim X$ admits a realization in $\mathbb{R}^N$ for any $N \geq 2n + 1$.*

Furthermore, if $N \geq 2n + 2$, any pair of such realizations $\iota_0, \iota_1 : X \to \mathbb{R}^N$ are isotopic, in the following sense: there is a realization

$$\mathcal{I} : (X, \Sigma, \mathscr{T}) \times (-\delta, 1 + \delta) \to \mathbb{R}^N$$

such that $\mathcal{I}(-, t)$ is a realization for all $t \in [0, 1]$ and $\mathcal{I}(-, 0) = \iota_0$, $\mathcal{I}(-, 1) = \iota_1$.

It was conjectured by Whitney that any complex analytic variety admits a stratification such that the neighborhood of any point in a stratum is fibered by links of the stratum. This conjecture was formulated and proved by Thom and Mather using the theory of controlled vector fields, see [Mat12].

Before stating the theorem, we introduce the notion of a *cone* on a stratified space. Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space. Then we define the open cone on X to be the topological space $cX := X \times [0, 1)/X \times \{0\}$, and we denote $v \in cX$ to be the vertex of the cone, given by the image of $X \times \{0\}$. cX admits a natural stratification given by $c\Sigma := \Sigma \times (0, 1) \sqcup \{v\}$ and a tube system $c\mathscr{T} := \{T_i \times (0, 1), \pi_i \times \mathrm{id}, \rho_i, i \in I\} \cup \{B_v, \pi_v, \rho_v\}$ where $B_v \subset X$ is a ball around v, $\pi_v : B_v \to \{v\}$ is the constant map, and $\rho_v : B_v \to [0, \infty)$ is, up to reparametrizations on the codomain, the natural height function $cX \to [0, 1)$ on the cone.

Theorem 2.9. [GM88, p. 41] *Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space. Let $x \in X$ be a point and $S \in \Sigma$ be the unique stratum of X containing x of dimension $n := \dim S$.*

There exists an abstractly stratified space $(L, \Sigma_\ell, \mathscr{T}_\ell)$ called the **link** *of S such that for a sufficiently small neighborhood $U \subset X$ of x equipped with the induced stratification from the canonical inclusion, U is isomorphic to $\mathbb{R}^n \times cL$.*

2.3. **Compression lemma.** In this subsection we describe a technical construction that we shall often use to enforce control on uncontrolled geometric data on abstractly stratified spaces. Consider the following basic motivating example. Let $(M, \partial M)$ be a manifold with boundary and $T := \partial M \times [0, \infty)$ be a collar neighborhood of ∂M inside M, with collar projection $\pi : T \to \partial M$. Consider a smooth function $\psi : [0, \infty) \to [0, 1]$ such that $\psi(t) = 0$ if $t \leq 1/4$, $\psi(t) = 1$ if $t \geq 3/4$, and ψ is strictly increasing on $[1/2, 3/4]$. Then the map $\Pi : (M, \partial M) \to (M, \partial M)$, defined by $\Pi \equiv 1$ on $M \setminus T(3/4)$ and $\Pi(x, t) = (x, \psi(t)t)$ on $T(3/4)$ "compresses" the smaller collar $T(1/4)$ to the boundary ∂M while leaving everything outside a larger collar $T(3/4)$ fixed. Observe $\Pi|T(1/4) \equiv \pi$ is the collar projection.

We shall generalize the above construction to any abstractly stratified space $(X, \Sigma, \mathscr{T})$ to obtain a *compression map* $\Pi : X \to X$ which will "compress" tubular neighborhoods of each stratum down to the stratum itself while leaving everything outside a larger tubular neighborhood fixed pointwise. Moreover, this shall appear as the time-1 map of a homotopy $\mathcal{H} : X \times I \to X$ which progressively shrinks these tubular neighborhoods to the strata. To carry this out, we shall require the *family of lines* machinery developed by Goresky [Gor78].

Theorem 2.10. [Gor78] *Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space. Then there exists $\delta > 0$ and a family of retractions $r_i(\varepsilon) : T_i \setminus S_i \to \partial T_i(\varepsilon)$ for every $0 < \varepsilon < \delta$ which satisfy the following relations:*

1. *$r_i(\varepsilon') \circ r_j(\varepsilon) = r_j(\varepsilon') \circ r_i(\varepsilon)$ for all $i \leq j \in I$, $0 < \varepsilon' < \delta$.*
2. *$\rho_i \circ r_j(\varepsilon) = \rho_i$ and $\rho_j \circ r_i(\varepsilon) = \rho_j$ for all $i \leq j$, $i, j \in I$.*
3. *$\pi_i \circ r_j(\varepsilon) = \pi_i$ for all $i \leq j$, $i, j \in I$.*
4. *$r_i(\varepsilon') \circ r_i(\varepsilon) = r_i(\varepsilon')$ for all $i \in I$, $0 < \varepsilon < \varepsilon' < \delta$.*
5. *$r_i(\varepsilon)|T_i(\varepsilon) \cap S_j : T_i(\varepsilon) \cap S_j \to \partial T_i(\varepsilon) \cap S_j$ is smooth for all $i \leq j$, $i, j \in I$.*

Equipped with such a family of retractions, we can define a stratum-preserving homeomorphism $h_i : T_i \setminus S_i \to \partial T_i(\varepsilon) \times (0, \infty)$ given by $h_i(p) = (r_i(\varepsilon)(p), \rho_i(p))$. This homeomorphism extends to a stratum-preserving homeomorphism $h_i : T_i \to M(\pi_i)$. Here $M(\pi_i) := (\partial T_i(\varepsilon) \times [0, \infty) \sqcup S_i)/(x, 0) \sim \pi_i(x)$ is the semi-infinite mapping cylinder of $\pi_i|S_i(\varepsilon) : \partial T_i(\varepsilon) \to S_i$.

Let $(X, \Sigma, \mathscr{T})$ be a finite-dimensional abstractly stratified space. Choose a smooth function $\psi : [0, \infty) \to [0, 1]$ as before such that $\psi(x) = 0$ for all $x \leq \varepsilon$, $\psi(x) = 1$ for all $x \geq 3\varepsilon$ and ψ is strictly increasing on $[\varepsilon, 3\varepsilon]$. Define

$$\phi_i : \partial T_i(\varepsilon) \times [0, \infty) \times [0, 1] \to \partial T_i(\varepsilon) \times [0, \infty)$$
$$\phi_i((x, t), s) := (x, s\psi(t)t + (1 - s)t)$$

ϕ_i simply performs the compression trick described earlier for a manifold-with-boundary but towards the end $\partial T_i(\varepsilon) \times \{0\}$ of the cylinder, and we keep track of the compression procedure through time, measured by the s-coordinate; at time $s = 0$ the compression process is performed to its completion. Note that this descends to a homotopy $\phi_i : M(\pi_i) \times I \to M(\pi_i)$ on the mapping cylinders. This is the map which we shall use to compress near a single stratum $S_i \in \Sigma$.

Let $\mathcal{H}_i : X \times I \to X$ be defined, for all $s \in I$, by

$$\mathcal{H}_i(-, s) = \begin{cases} \mathrm{id} & \text{throughout } X \setminus T_i \\ h_i^{-1} \circ \phi_i(-, s) \circ h_i & \text{in } T_i(\varepsilon) \end{cases}$$

The collection of these "local compression" homotopies $\{\mathcal{H}_i(-,s) : i \in I_0\}$ commute for each time $s \in I$ due to the choice of our family of retractions from Theorem 2.10. Therefore, we can compose these maps in **any order** to obtain a map $\mathcal{H} : X \times I \to X$ which performs all of these local compressions near every strata at once. This produces the desired compression homotopy $\mathcal{H} : X \times I \to X$.

Observe that $\phi_i(-,1) = \pi_i$ on $\partial T_i(\varepsilon) \times [0,\varepsilon]$, so $\mathcal{H}_i(-,1) = \pi_i$ on $T_i(\varepsilon)$ and $H_i(-,1) \equiv \mathrm{id}$ on $X \setminus T_i(\varepsilon)$ as before. The final map $\Pi := \mathcal{H}(-,1)$ is a composition of the family $\{\mathcal{H}_i(-,1) : i \in I\}$ in any order.

To clarify we exhibit the behavior of a factor of Π given by the composition of these maps in increasing order of depth along a maximal chain in Σ. Let $S_1 < S_2 < \cdots < S_n$ be such a chain. Let us denote

$$\varphi_{p,q} := \mathcal{H}_p(-,1) \circ \cdots \circ \mathcal{H}_q(-,1), \quad 1 \leq p < q \leq n$$

Fix any $x \in X$, let $d = \min\{k : x \in T_k(\varepsilon)\}$ be defined so that S_d is the deepest stratum to which x is ε-close. We claim $\varphi_{1,n}(x) = \varphi_{1,d-1}(\pi_{d,n}(x))$. This follows from induction using the observation that if $x \in T_{k,n}(\varepsilon)$ and $k < \ell$, $\rho_{k,\ell}(\pi_{\ell,n}(x)) = \rho_{k,n}(x) \leq \varepsilon$, implying $\pi_{\ell,n}(x) \in T_{k,\ell}(\varepsilon)$. This establishes that for any $x \in X$, if $D \in \Sigma$ denotes the deepest stratum of X such that x is ε-close to D, $\Pi(x) = \Pi(\pi_D(x))$, since $\Pi|\overline{D}$ is itself a compression map on $(\overline{D}, \Sigma \cap \overline{D}, \mathscr{T} \cap \overline{D})$ obtained from composition of the maps in the family $\{\mathcal{H}_i(-,s) : i \in \Sigma \cap \overline{D}\}$.

Finally, we discuss smoothness properties of Π. Let $S < L$ and let A denote the link of S in X. Let $A_L = A \cap L$. Then the tubular neighborhood of S in L is of the form $U \times cA_L$, where cA_L denotes the cone on A_L, and $U \subset S$ is an open Euclidean chart. This follows from Thom's first isotopy lemma (see [MS22, Section 2], [MS22, Lemma 2.15] for further details). Consider local coordinates on $U \times cA_L$ of the form $\mathbb{R}^s \times \mathbb{R}^l \times [0,1]$, where $\mathbb{R}^s \subset S$ is an open chart in S, $\mathbb{R}^l \subset A_L$ is an open chart in L, and $[0,1]$ is the radial (cone) direction in cA. We can choose Π so that it can be given in local charts as

$$\Pi(x,y,t) = (x,y,\psi(t)),$$

where $\psi : [0,1] \to [0,1]$ is a smooth function satisfying the following:

(1) in local chart coordinates, 0 is sent to the cone-point c,
(2) there exists $0 < c < 1$ such that $\psi([0,c]) = \{0\}$,
(3) ψ maps $(c,1)$ diffeomorphically to $(0,1)$.

If Π satisfies the above properties locally, we say henceforth that it is *smooth*.

We summarize the properties of the construction above in the following proposition.

Proposition 2.11. *For an abstractly stratified space $(X,\Sigma,\mathscr{T})$, there exists a map $\Pi : X \to X$, a homotopy $\mathcal{H} : X \times [0,1] \to X$ and $\varepsilon > 0$, such that*

(1) $\mathcal{H}(-,1) = \Pi$
(2) $\mathcal{H}(-,0) = \mathrm{id}$
(3) $\mathcal{H}(-,s) : X \to X$ *is an isomorphism for all* $s \in [0,1)$
(4) *For any pair of strata $S, L \in \Sigma$ with $S < L$, $\Pi \equiv \mathrm{id}$ throughout $L \setminus T_{S,L}(3\varepsilon)$*
(5) *For any $x \in X$, $\Pi(x) = \Pi(\pi_D(x))$ where D is the deepest stratum such that $T_D(\varepsilon)$ contains x.*
(6) Π *is smooth.*

Corollary 2.12. *For any stratum $S \in \Sigma$, $\Pi \circ \pi_S = \Pi$ on $T_S(\varepsilon)$.*

Proof. Indeed, let $x \in T_S(\varepsilon)$, then $\Pi_1(x) = \pi_D(x)$ where $D < S$ is the deepest stratum such that $x \in T_D(\varepsilon)$. Note that $\rho_D(\pi_S(x)) = \rho_D(x) \leq \varepsilon$, therefore $\pi_S(x) \in T_D(\varepsilon)$ as well. If there is a deeper stratum $D' < S$ such that $\pi_S(x) \in T_{D'}(\varepsilon)$, then $\rho_{D'}(x) = \rho_{D'}(\pi_S(x)) \leq \varepsilon$, therefore $x \in T_{D'}(\varepsilon)$ for a stratum $D' < S$ with $\mathrm{dpt}(D') < \mathrm{dpt}(D)$, contradicting the choice of D. Therefore,

$$(\Pi \circ \pi_D)(x) = \Pi(\pi_D(x)) = \pi_D(\pi_S(x)) = \pi_D(x) = \Pi(x)$$

as required. □

Remark 2.13. *Note that the map* Π *stands out as unusual in the theory of stratified spaces so far since it does not preserve strata; indeed, it pushes parts of each stratum down to its boundary. We remark however that* Π *is* **skeleton-preserving** *in the sense that* $\Pi(\overline{S}) \subset \overline{S}$ *for any stratum* $S \subset \Sigma$*. Moreover,* $\Pi|_{\overline{S}}$ *itself is a compression map, therefore we can effectively understand* Π *by induction, using compression maps on stratified spaces of lower depth than that of* X*.*

3. de Rham cohomology on stratified spaces

Given any abstractly stratified space $(X, \Sigma, \mathscr{T})$, the "coarsest" notion of a differential k-form on X is a collection of k-forms $\{\omega_S : S \in \Sigma\}$ defined on each stratum. This defines a complex $\bigoplus_{S\in\Sigma} \Omega^\bullet(S)$ given by the direct sum of the de Rham complexes of each stratum. However, the cochain groups are too large for our purposes, as we have so far not demanded any condition on how the forms on various strata should fit together; the complex is independent of the chosen tube system $\mathscr{T}$. For example, the cohomology groups are $\bigoplus_{S\in\Sigma} H^\bullet(S;\mathbb{R})$. This does not reflect the topology of X. We propose two ways to remedy this:

(1) By equipping X with a realization $X \hookrightarrow M$ and using the pullback of the sheaf of smooth differential forms on M along this embedding. This gives us a way of saying that a collection of forms $\{\omega_S : S \in \Sigma\}$ "fit together", i.e. there is a germ of a smooth form $\widetilde{\omega}$ along the image of X in M such that $\widetilde{\omega}|_S = \omega_S$ for every stratum $S \in \Sigma$.
(2) By defining an appropriate subcomplex of $\bigoplus_{S\in\Sigma} \Omega^\bullet(S)$ which consists of collections $\{\omega_S : S \in \Sigma\}$ such that whenever $S < L$, ω_L is "radially compactly supported" while approaching S, or more precisely, $\pi_{S,L}^* \omega_S = \omega_L$. This notion of "fitting together" is more restrictive than the first notion, but is completely intrinsic to the stratified space X.

We discuss and compare these two notions in this section.

3.1. **Smooth de Rham cohomology.** Let $(W, \Sigma) \subset M$ be a Whitney stratified subset of a smooth manifold M. The smooth structure on M gives a natural way to speak of a smooth structure on W. To be precise, let $\mathcal{C}^\infty_M$ be the sheaf of smooth functions on M. We can restrict this sheaf to the subspace $\iota : W \hookrightarrow M$ to obtain the *sheaf of smooth functions* on W, which we shall denote as $\mathcal{C}^\infty_W := \iota^* \mathcal{C}^\infty_M$. This can be concisely described as the sheaf of germs of smooth functions defined along W. Note that $\mathcal{C}^\infty_M$ is fine by the existence of bump functions on smooth manifolds, and the fact that restriction of fine sheaves over metric space to closed subspaces remains fine. Hence $\mathcal{C}^\infty_W$ is a fine sheaf on W. We shall drop the subscript when the ambient stratified space is understood and simply denote the sheaf as $\mathcal{C}^\infty$.

Let $\Omega^\bullet_M$ denote the sheaf of differential forms on M. We can likewise define $\Omega^p_W := \iota^* \Omega^p_M$ to be the *sheaf of smooth differential p-forms* on W, and these are

likewise fine sheaves on W. We once again drop the subscript whenever the ambient stratified space is understood and denote the sheaf simply as $\Omega^\bullet$. The exterior derivative operator $d : \Omega^p_M \to \Omega^{p+1}_M$ restricts to an exterior derivative operator on $\Omega^\bullet$. In particular, it gives rise to a cochain complex $(\Omega^\bullet(W), d)$ at the level of global sections. We shall refer to this as the *smooth de Rham complex* for W. The cohomology of the complex shall be referred to as the *smooth de Rham cohomology* of W, $H^\bullet_{dR}(W) := H^\bullet(\Omega^\bullet(W), d)$.

Remark 3.1. *Here, Ω^p_W is a sheaf of smooth forms obtained by restricting smooth p-forms from a small neighborhood in M. Thus $\Omega^\bullet$ is really the space of smooth forms in a small tubular neighborhood. Since the stratified space is a deformation retract, the cohomologies are preserved by homotopy-invariance. The de Rham cohomology theory we thus obtain is independent of the embedding by Verona's Theorem 3.6 below. Thus, for the purposes of this paper, de Rham cohomology is realized by an actual embedding and taking a* **fixed** *sufficiently small neighborhood in a smooth manifold. We shall implicitly assume such an embedding throughout.*

Recall that the de Rham complex of sheaves,

$$0 \to \underline{\mathbb{R}} \to \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots$$

defines a resolution of the constant $\mathbb{R}$-valued sheaf on M. Therefore, so does

$$0 \to \underline{\mathbb{R}} \to \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots$$

Indeed, it suffices to check this at the level of stalks, and the restriction sheaves on W are indistinguishable from the corresponding sheaves on M at the level of stalks. Moreover, from our comments above this is a resolution by fine — thus acyclic — sheaves. Therefore, the sheaf cohomology of $\underline{\mathbb{R}}$ is isomorphic to the de Rham cohomology of W. Moreover, the sheaf cohomology of $\underline{\mathbb{R}}$ is isomorphic to the singular cohomology of W valued in $\mathbb{R}$, as W is locally contractible. As a consequence,

$$H^\bullet_{dR}(W) \cong H^\bullet_{sing}(W; \mathbb{R})$$

Next, we would like to use the cohomology theory defined above for abstractly stratified spaces. To do so, we introduce the following notion:

Definition 3.2. *A* **smooth structure** *on an abstractly stratified space is a choice of a realization (see Theorem 2.8) in a smooth manifold.*

The rationale behind this definition is as follows. Suppose $(X, \Sigma, \mathscr{T})$ is an abstractly stratified space, equipped with a smooth structure, i.e., a realization $j : X \to M$ in some smooth manifold M. Then the image $W := \iota(X) \subset M$ is a Whitney stratified set, and therefore admits the sheaf of smooth functions $\mathcal{C}^\infty_W$ defined earlier; we can pullback the sheaf to obtain a sheaf $\mathcal{C}^\infty_X := j^*\mathcal{C}^\infty_W$ on X, which we call the *sheaf of smooth functions* on X. This notion depends on the choice of the realization in a crucial way, as the smooth isotopy extension theorem fails for isotopies of embeddings of stratified spaces.

We shall similarly define $\Omega^\bullet_X := j^*\Omega^\bullet_W$ as the *sheaf of smooth differential forms* on X, and drop the subscript from both notations whenever the ambient stratified space along with the smooth structure (X, j) is understood. The discussion above yields a fine resolution $0 \to \underline{\mathbb{R}} \to \Omega^\bullet$, and hence a cochain complex $(\Omega^\bullet(X), d)$ and a cohomology theory $H^\bullet_{dR}(X)$ isomorphic to the real-valued singular cohomology of X, for the abstractly stratified space X.

3.2. **Controlled de Rham cohomology.** Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space. A continuous function $f : X \to \mathbb{R}$ will be said to be *controlled* if for every stratum $S \in \Sigma$, $f|S$ is a smooth function, and for any pair of indices $i, j \in I$, $i < j$, $f \circ \pi_{ij} = f$ throughout T_{ij}. A useful way of thinking of this condition is by the slogan "controlled functions are constant on links".

For any open subset $U \subseteq X$, we obtain an induced abstract stratification on U by pulling back the abstract stratification on X along the inclusion $U \hookrightarrow X$ (cf. Remark 2.4). Let $\mathcal{C}^\infty_{co}(U)$ denote the $\mathbb{R}$-algebra of controlled functions on U. It is clear that the association $U \mapsto \mathcal{C}^\infty_{co}(U)$ defines a sheaf of $\mathbb{R}$-algebras on X; we denote this sheaf by $\mathcal{C}^\infty_{co}$ and call it the *sheaf of controlled functions*.

Theorem 3.3. *$\mathcal{C}^\infty_{co}$ is a fine sheaf on X, i.e., admits a partition of unity.*

Proof. It suffices to construct, for any given point $x \in X$ and any neighborhood $U \subset X$ of x, a controlled function $f : X \to \mathbb{R}$ such that $0 \leq f \leq 1$, $\mathrm{supp}(f) \subset U$ and $f \equiv 1$ throughout a smaller open neighborhood $V \subset X$ of x. We use induction on depth to accomplish this.

Note that an abstractly stratified space of depth 0 is a manifold, in which case this follows from the standard construction of bump functions.

By induction, let us assume that theorem is true for all abstractly stratified spaces of depth less than $\mathrm{dpt}(X)$. Let $S \in \Sigma$ denote the unique stratum of X that contains x. Let $K \subset S$ be a compact subset containing x and choose $g \in \mathcal{C}^\infty(S)$ such that $0 \leq g \leq 1$, $\mathrm{supp}(g) \subseteq K$ and $g \equiv 1$ throughout an open neighborhood of x contained in K. Let T denote the tubular neighborhood of S, and π, ρ be the associated projection and radial function.

Define $\tilde{g} \in \mathcal{C}^\infty_{co}(T)$ by $\tilde{g} := g \circ \pi$. Note $\mathrm{dpt}(T \setminus S) < \mathrm{dpt}(X)$. Therefore, by induction hypothesis, there exists $h \in \mathcal{C}^\infty_{co}(T \setminus S)$ with $0 \leq h \leq 1$, $\mathrm{supp}(h) \subset T(\varepsilon)$ and $h \equiv 1$ throughout $T(\varepsilon/2) \setminus S$. Define $\tilde{h} \in \mathcal{C}^\infty_{co}(T)$ by $\tilde{h}|(T \setminus S) = h$ and $\tilde{h}|S = 1$. We choose $\varepsilon > 0$ and $K \subset S$ such that $\pi^{-1}(K) \cap T(\varepsilon) \subset V$, and define $f \in \mathcal{C}^\infty_{co}(X)$ by $f|T = \tilde{g}\tilde{h}$ and $f|(X \setminus T) = 0$. This is our desired function. □

In [Ver71], Verona introduced a controlled notion of differential forms on abstractly stratified spaces. We recall the definition:

Definition 3.4. *A* **controlled differential p-form** *on an abstractly stratified space $(X, \Sigma, \mathscr{T})$ is a collection $\omega := \{\omega_i : i \in I\}$ such that for each $i \in I$, ω_i is a p-form on the stratum $S_i \in \Sigma$, such that whenever $i, j \in I$ is a pair of indices, $i < j$, the following control condition holds:*

$$\pi^*_{ij}\omega_j = \omega_i \text{ on } T_{ij}$$

As before, we define a sheaf Ω^p_{co} of controlled differential p-forms on X. Note that controlled 0-forms are simply controlled functions, thus $\Omega^0_{co} = \mathcal{C}^\infty_{co}$. For each open set $U \subset X$, $\Omega^p_{co}(U)$ is naturally a $\mathcal{C}^\infty_{co}(U)$-module, hence Ω^p_{co} is a sheaf of $\mathcal{C}^\infty_{co}$-modules. Therefore, Ω^p_{co} is also a fine sheaf.

On a manifold M, a p-form can be written in a neighborhood of any point $x \in M$ as a linear combination of *primitive p-forms* of the form $f dg_1 \wedge \cdots \wedge dg_p$ where $f, g_1, \cdots, g_p \in \mathcal{C}^\infty_x$. We give an analogous local description for controlled p-forms on an abstractly stratified space $(X, \Sigma, \mathscr{T})$.

Suppose $x \in X$ is a point, then there is a unique stratum $S \in \Sigma$ containing x, and any neighborhood of x in X must contain a further neighborhood of the form

$(\pi_S)^{-1}(B) \cap T_S(\varepsilon) = \bigsqcup_{L \geq S} (\pi_{S,L})^{-1}(B) \cap T_{S,L}(\varepsilon)$ where $B \subset S$ is a neighborhood of x in S.

By the last paragraph, ω_S is a linear combination of primitive forms $f dg_1 \wedge \cdots \wedge dg_p$ where $f, g_1, \cdots, g_p \in \mathcal{C}^\infty(B)$ on the ball $B \subset S$, after shrinking the ball if necessary. The functions $f, g_1, \cdots, g_p$ have a unique controlled extension on $(\pi_S)^{-1}(B)$ by setting $\tilde{f} := f \circ \pi_{S,L}$ and $\tilde{g}_i := g_i \circ \pi_{S,L}$ on $\pi_{S,L}^{-1}(B)$, for $1 \leq i \leq p$, for every stratum $L \in \Sigma$ containing S as a boundary stratum. Therefore, we obtain the controlled p-form $\tilde{f}_i d\tilde{g}_1 \wedge \cdots \wedge d\tilde{g}_p$ defined in a neighborhood of x in X.

For every stratum $L \geq S$ of X, $\omega_L = (\pi_{S,L})^* \omega_S$ which is a linear combination of $(\pi_{S,L})^*(f dg_1 \wedge \cdots \wedge dg_p) = \tilde{f} d\tilde{g}_1 \wedge \cdots \wedge d\tilde{g}_p$ in a neighborhood of x in X. Therefore, $\Omega^p_{co,x}$ is generated by germs of *controlled primitive p-forms* $f dg_1 \wedge \cdots \wedge dg_p$ where $f, g_1, \cdots, g_p \in \mathcal{C}^\infty_{co,x}$ are germs of controlled function around x.

Notice that if $\omega = \{\omega_S : S \in \Sigma\}$ is a controlled p-form, $d\omega := \{d\omega_S : S \in \Sigma\}$ is a controlled $(p+1)$-form, as π_i^* commutes with d. Therefore, the exterior derivative gives a sheaf homomorphism $d : \Omega^p_{co} \to \Omega^{p+1}_{co}$.

Theorem 3.5 (Controlled Poincaré lemma)**.** *The complex of sheaves*

$$\Omega^{p-1}_{co} \xrightarrow{d} \Omega^p_{co} \xrightarrow{d} \Omega^{p+1}_{co}$$

is exact.

Proof. It suffices to prove exactness at the level of stalks. That is, suppose $\alpha \in \Omega^p_{co,x}$ is the germ of a closed controlled p-form around a neighborhood of 0, i.e, $\alpha \in \Omega^p_{co}(U)$ for some neighborhood $U \subset X$ of x such that $d\alpha = 0$. Let $S \in \Sigma$ be the unique stratum of X containing U. Then, in particular, α_S defines a closed form in the neighborhood $S \cap U$ of x in S. Therefore, by the Poincaré lemma for manifolds, there exists a $(p-1)$-form β_S on a ball neighborhood $B \subset S \cap U$ of x such that $\alpha_S = d\beta_S$.

Once again, we provide a controlled extension of β_S by defining $\beta_L = \pi^*_{S,L} \beta_S$ on $\pi^{-1}_{S,L}(B)$ for all $L \in \Sigma$, $L \geq S$. Then $\beta := \{\beta_L : L \in \Sigma\}$ defines a germ of a controlled $(p-1)$-form around a neighborhood of x such that $d\beta = \alpha$, since $d\beta_S = \alpha_S$ and d commutes with $\pi^*_{S,L}$. This concludes the proof. □

Let $\Omega^p_{co}(X)$ be the $\mathbb{R}$-vector space of controlled differential p-forms on X. These define a *controlled de Rham complex* $(\Omega^\bullet_{co}(X), d)$.

We obtain from Theorem 3.5 that the complex of sheaves

$$0 \to \underline{\mathbb{R}} \xrightarrow{d} \Omega^0_{co} \xrightarrow{d} \Omega^1_{co} \xrightarrow{d} \cdots$$

is exact. Moreover, each Ω^p_{co} is a fine sheaf and therefore acyclic. Hence the above is a resolution of the constant sheaf $\underline{\mathbb{R}}$. Therefore, the sheaf cohomology of $\underline{\mathbb{R}}$ computes the cohomology of the controlled de Rham complex $(\Omega^\bullet_{co}(X), d)$. On the other hand, the sheaf cohomology of $\underline{\mathbb{R}}$ is the $\mathbb{R}$-valued Čech cohomology of X. By Theorem 2.9, X is locally contractible, therefore sheaf cohomology is isomorphic to singular cohomology. Combining these, we obtain:

Theorem 3.6 (Verona's de Rham theorem)**.** *For any abstractly stratified space $(X, \Sigma, \mathscr{T})$, the cohomology of the complex of controlled differential forms computes the singular cohomology of X with real coefficients:*

$$H^\bullet(\Omega^\bullet_{co}(X), d) \cong H^\bullet_{sing}(X; \mathbb{R})$$

We shall henceforth call the cohomology of the complex of controlled differential forms as the *controlled de Rham cohomology* of the abstractly stratified space $(X, \Sigma, \mathscr{T})$ and denote it as $H^\bullet_{co}(X) := H^\bullet(\Omega^\bullet_{co}(X), d)$.

3.3. **Isomorphism of cohomology theories.** We have thus obtained three abstractly isomorphic cohomology theories: $H^\bullet_{dR}, H^\bullet_{co}$ and $H^\bullet_{sing}$. In this section we give an explicit isomorphism between the three theories.

If $(X, \Sigma, \mathscr{T})$ is an abstractly stratified space, we can always replace the tube system $\mathscr{T}$ by an equivalent tube system $\mathscr{T}'$ and consider a homotopy $\mathcal{H} : X \times I \to X$ compressing the new tube system $\mathscr{T}'$ using Proposition 2.11 on the equivalent abstract stratification $(X, \Sigma, \mathscr{T}')$ instead. Often we shall need to choose $\mathscr{T}'$ such that the tubes are of sufficiently small radius, cf. Remark 2.3. In such cases, we shall indicate such a choice by declaring the compression map *sufficiently thin*.

Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space equipped with a smooth structure $j : X \to M$. By a slight abuse of notation we identify X with $j(X)$ and treat it as a subset of M. We equip M with an abstract stratification as follows: let $\Sigma' := \Sigma \cup \{M \setminus X\}$ and $\mathscr{T}'$ consist of the tubular neighborhood system on the strata of Σ as in Theorem 2.7. We declare the tube around the open top dimensional stratum $M \setminus X$ to be itself. This defines a valid abstract stratification $(M, \Sigma', \mathscr{T}')$ on M by Theorem 2.8 and Theorem 2.7.

Let $\Pi := \mathcal{H}(-, 1) : M \to M$ be a sufficiently thin compression map for the abstractly stratified space $(M, \Sigma', \mathscr{T}')$. We choose $U := \bigcup_{S \in \Sigma} \nu(S)$ to be a union of tubular neighborhoods of strata as in Theorem 2.7 satisfying $\Pi(U) = X$. We shall think of U as an analog of the manifold normal neighborhood for the stratified subset $X \subset M$, and $\Pi : U \to X$ as an analog of the normal projection.

First, we show that a controlled differential form on X has a natural smooth extension to germ of a smooth differential form along $X \subset M$. This will give a canonical embedding of subcomplexes $\Omega^\bullet_{co}(X) \subset \Omega^\bullet(X)$.

Proposition 3.7. *Let $\omega = \{\omega_S : S \in \Sigma\}$ be a controlled differential form on X. Then the collection of forms $\{\pi_S^* \omega_S \in \Omega^\bullet(\nu(S)) : S \in \Sigma\}$ patch to give a germ of a smooth differential form $\widetilde{\omega}$ on U.*

Proof. Let $S, L \in \Sigma$ be a pair of strata, with $S < L$. Then,

$$\pi_L^* \omega_L = \pi_L^* \pi_{S,L}^* \omega_S = (\pi_{S,L} \circ \pi_L)^* \omega_S = \pi_S^* \omega_S$$

throughout $\nu(S) \cap \pi_L^{-1}(T_{S,L}) = \nu(S) \cap \pi_L^{-1} \nu(S)$. Now, choose a smaller tubular neighborhood $\nu'(S) \subset \nu(S)$ if necessary such that $\nu'(S) \cap \nu(L) \subset \pi_L^{-1}(\nu(S))$. Then $\pi_L^* \omega_L = \pi_S^* \omega_S$ on $\nu'(S) \cap \nu(L)$.

Therefore, by adjusting tubular neighborhoods downward along strata if necessary, we can choose a collection of tubular neighborhoods $\{\nu(S) : S \in \Sigma\}$ such that $\pi_S^* \omega_S \in \Omega^\bullet(\nu(S))$ patch to a well-defined smooth differential form $\widetilde{\omega}$ on the open neighborhood $\bigcup_{S \in \Sigma} \nu(S)$ of $X \subset M$, as required. □

Let $\eta = \{\eta_S : S \in \Sigma\}$ be a smooth differential form on X. We choose an extension $\widetilde{\eta}$ to a smooth differential form in an open neighborhood U of $X \subset M$. U inherits an induced stratification from $(M, \Sigma', \mathscr{T}')$ as usual; we treat $\widetilde{\eta} = \{\eta_S : S \in \Sigma'\}$ as collection of differential forms on strata of U, where $\eta_{U \setminus X} := \widetilde{\eta} | (U \setminus X)$.

Proposition 3.8. $\Pi^* \widetilde{\eta} = \{\iota_S^* \Pi^* \widetilde{\eta} : S \in \Sigma'\}$ *is a controlled differential form on U.*

Proof. Let $S, L \in U \cap \Sigma'$ be a pair of strata, with $S < L$. Let $\iota_S, \iota_L : S, L \to U$ denote the canonical inclusion maps and $\pi_S : \nu(S) \to S \subset U$ be the tubular projection. Then, as $\pi_{S,L} = \pi_S \circ \iota_L$, we have:

$$\pi_{S,L}^*(\iota_S^*\Pi^*\widetilde{\eta}) = \iota_L^*\pi_S^*\iota_S^*\Pi^*\widetilde{\eta} = \iota_L^*(\Pi \circ \iota_S \circ \pi_S)^*\widetilde{\eta} = \iota_L^*(\Pi \circ \pi_S)^*\widetilde{\eta} = \iota_L^*\Pi^*\widetilde{\eta}$$

throughout $T_{S,L}$, since $\Pi \circ \pi_S \equiv \Pi$ by Corollary 2.12. □

Corollary 3.9. *The germ of the smooth form $\Pi^*\widetilde{\eta}$ along X does not depend on the choice of the extension $\widetilde{\eta}$ of η.*

Proof. We proceed by upward induction on strata. Let $S \in \Sigma$ be a deepest stratum of X. Then as $\partial S = \emptyset$, $\Pi \equiv \pi_S$ on $T_S(\varepsilon)$. Thus, $\iota_S^*\Pi^*\widetilde{\eta} = \iota_S^*\widetilde{\eta} = \eta_S$ defined on S is independent of the choice of $\widetilde{\eta}$.

Let $L \in \Sigma$ be a stratum and by induction assume that for all boundary strata $S \in \Sigma$, $S \subset \partial L$ of L, $\iota_S^*\Pi^*\widetilde{\eta}$ defined on S is independent of the choice of $\widetilde{\eta}$. Then $\iota_L^*\Pi^*\widetilde{\eta} = \pi_{S,L}^*\iota_S^*\Pi^*\widetilde{\eta}$ defined on $T_{S,L}(\varepsilon)$ is also independent of the choice of $\widetilde{\eta}$. Finally, the restriction $\Pi \circ \iota_L : L \setminus \bigcup_{S \subset \partial L} T_{S,L}(\varepsilon) \to L \subset U$ is a smooth map, therefore $\iota_L^*\Pi^*\widetilde{\eta} = \iota_L^*\Pi^*\eta_L$ is independent of the choice of $\widetilde{\eta}$ as well. This shows that $\iota_L^*\Pi^*\widetilde{\eta}$ is independent of the choice of $\widetilde{\eta}$, concluding the proof. □

In light of the previous corollary, we can and will unambiguously denote the smooth differential form $\Pi^*\widetilde{\eta}$ along X as $\Pi^*\eta$.

Proposition 3.10. *$\Pi^* : \Omega^\bullet(X) \to \Omega^\bullet_{co}(X)$ is a chain homotopy equivalence.*

Proof. Given a smooth differential form η on X, we can choose a smooth extension $\widetilde{\eta}$ to some regular neighborhood U of $X \subset M$. For any stratum $S \in \Sigma$ of X,

$$\iota_S^*\Pi^* d\widetilde{\eta} = d\iota_S^*\Pi^*\widetilde{\eta}$$

Therefore, $d\Pi^*\eta = \Pi^* d\eta$; hence Π^* is indeed a chain map. Next, observe that as $\Pi : U \to U$ is homotopic to the identity, $\Pi^* : \Omega^\bullet(U) \to \Omega^\bullet(U)$ is chain homotopic to the identity. By the natural identification $\Omega^\bullet(U) \cong \Omega^\bullet(X)$, we have that $\Pi^* : \Omega^\bullet(X) \to \Omega^\bullet(X)$ is chain homotopic to the identity. The image of this map lands inside the subcomplex $\Omega^\bullet_{co}(X) \subset \Omega^\bullet(X)$, therefore we obtain the desired chain homotopy equivalence. □

4. Symplectic stratified spaces

4.1. Definition and examples.

Definition 4.1. *Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space equipped with a smooth structure $j : X \to M$. A smooth 2-form $\omega = \{\omega_S : S \in \Sigma\} \in \Omega^2(X)$ shall be called a* **(stratified) symplectic form** *if for every stratum $S \in \Sigma$, ω_S is a symplectic 2-form on S.*

We shall use the term *symplectic stratified space* to denote a pair (X, ω) where X is an abstractly stratified space equipped with a smooth structure, and $\omega \in \Omega^2(X)$ is a symplectic 2-form. The smooth 2-form ω admits an extension $\widetilde{\omega}$ to a smooth 2-form defined in a neighborhood of $X \subset M$ (cf. Remark 3.1). Note that we do not require $\widetilde{\omega}$ to be closed and nondegenerate, only that its restrictions to each stratum of X be so. Definition 4.1 has appeared in the literature in the context of Marsden-Weinstein reduction theory, and more generally, in the context of symplectic reduction [DJ96, SL91]. Examples of symplectic stratified spaces include (possibly singular) complex projective varieties with the Kähler form.

The primary topological obstruction a symplectic 2-form imposes on a *closed manifold* is the associated nonzero cohomology class. For a *stratified space* X, a priori there is no such cohomology class available in $H^2_{sing}(X;\mathbb{R})$ associated to the stratified symplectic form ω. This is because, as discussed in the beginning of Section 3, there are two flavors of de Rham cohomology available to us, one is smooth de Rham cohomology arising from the complex of $(\Omega^\bullet, d)$ of germs of smooth differential forms along $X \subset M$ and the other is controlled cohomology arising from the subcomplex $(\Omega^\bullet_{co}, d) \subset \bigoplus_{S\in\Sigma}(\Omega^\bullet_S, d_S)$ of controlled differential forms on $(X, \Sigma, \mathscr{T})$. However,

(1) ω is rarely a controlled differential form on X as ω_S is nondegenerate for all $S \in \Sigma$, and controlled differential forms necessarily degenerate near strata of lower depth.
(2) Despite ω being a smooth differential form on X, it is not necessarily a closed smooth differential form as we do not necessarily have a smooth *closed* extension $\widetilde{\omega}$ to an open neighborhood of X in M.

We nevertheless manage to extract such a cohomology class by using the Compression Lemma (Proposition 2.11) to compress the smooth, non-controlled, non-closed form ω into a controlled closed 2-form $\Pi^*\omega$.

Proposition 4.2. *Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space equipped with a smooth structure $j : X \to M$ inherited from an embedding j into a smooth manifold M. Let $\omega = \{\omega_S : S \in \Sigma\} \in \Omega^2(X)$ be a smooth differential 2-form on X such that ω_S is closed on S for every stratum $S \in \Sigma$. Then $\Pi^*\omega \in \Omega^2_{co}(X)$ is a closed form.*

Proof. This is basically because ω is stratum-wise a closed 2-form, and pulling back commutes with exterior derivative. However, some care must be taken as the compression map Π does not preserve strata (see Remark 2.13).

We proceed by upward induction on strata. Let $S \in \Sigma$ be the deepest stratum of X. Then $\Pi^*\omega_S = \omega_S$ is closed on S. Let $L \in \Sigma$ be a stratum and by induction assume that for all boundary strata $S \in \Sigma$, $S \subset \partial L$ of L, $\Pi^*\omega_S$ is closed on S. As $\Pi^*\omega$ is a controlled form on X by Proposition 3.8, $\Pi^*\omega_L = \pi^*_{S,L}\Pi^*\omega_S$ is a closed form on $T_{S,L}(\varepsilon)$ for all boundary strata $S \subset \partial L$. Finally, $\Pi|_L : L \setminus \bigcup_{S\subset\partial L} T_{S,L}(\varepsilon) \to L \subset U$ is a smooth map, therefore, $\Pi^*\omega_L$ is a closed form on $L \setminus \bigcup_{S\subset\partial L} T_{S,L}(\varepsilon)$ as well. This concludes the inductive step, and the proposition follows. □

Therefore, $\Pi^*\omega$ gives rise to a well-defined cohomology class in $H^2_{co}(X)$ and consequently a singular cohomology class in $H^2_{sing}(X;\mathbb{R})$ by Theorem 3.6. We shall call $[\Pi^*\omega] \in H^2_{sing}(X;\mathbb{R})$ the cohomology class *associated* to the symplectic form ω.

Definition 4.3. *Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space. Let $\omega = \{\omega_S : S \in \Sigma\} \in \Omega^2(X)$ be a smooth differential 2-form on X such that ω_S is closed on S for every stratum $S \in \Sigma$. Then ω is said to be* **integral** *if $[\Pi^*\omega] \in H^2(X;\mathbb{R})$ is contained in the image of the (change of coefficients) homomorphism $H^2_{sing}(X;\mathbb{Z}) \to H^2_{sing}(X;\mathbb{R})$.*

In particular, if (X, ω) is a symplectic stratified space, we call ω integral if the associated cohomology class $[\Pi^*\omega] \in H^2_{sing}(X;\mathbb{R})$ is contained in the image of the homomorphism $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$.

We are now in a position to state the main theorem of this paper. We shall use the following convention: given an abstractly stratified space $(X, \Sigma, \mathscr{T})$ equipped

with a smooth structure $j : X \to M$, two k-forms $\eta, \xi \in \Omega^k(X)$ shall be said to be *equal on* X if $\iota_S^* \eta = \iota_S^* \xi$ for all strata $S \in \Sigma$ of X. We stress that this is not the same as saying η, ξ are equal as elements of $\Omega^k(X)$, as smooth differential forms on X are really germs of differential forms on M along X. Observe, η equals ξ on X if and only if $\eta = \xi + \lambda$ for some $\lambda \in \Omega^2(X)$ such that $\iota_S^* \lambda = 0$ for all $S \in \Sigma$.

Theorem 4.4. *Let (X, ω) be a compact stratified symplectic space where $\omega \in \Omega^2(X)$ is integral. There exists $N \geq 1$ and an embedding $f : X \to \mathbb{CP}^N$ such that $f^* \omega_N = \omega$ on X, where ω_N is the standard Kähler form on $\mathbb{CP}^N$.*

More generally, let $(X, \Sigma, \mathscr{T})$ be a compact abstractly stratified space, and $\omega \in \Omega^2(X)$ be integral. Then there exists $N \geq 1$ and an embedding $f : X \to \mathbb{CP}^N$ such that $f^* \omega_N = \omega$ on X, where ω_N is the standard Kähler form on $\mathbb{CP}^N$.

Theorem 4.4 generalizes a theorem of Gromov (the isosymplectic embedding theorem [Gro71]) and Tischler [Tis77]. The proof occupies the rest of this section and consists of the following steps:

Step 1: We would first like to construct a map $f : X \to \mathbb{CP}^N$ such that

$$"[f^*(\omega_N)] = [\omega]",$$

in a suitable sense. Note that by the discussion preceding Theorem 4.4, $[\omega]$ does not make sense literally. Thus, what we really prove is:

$$[f^*(\omega_N)] = [\Pi^* \omega],$$

where Π is the map occurring in the Compression Lemma (Proposition 2.11). This is done in Section 4.2.

Step 2: We start with the function $f : X \to \mathbb{C}P^N$ obtained in Step 1, and consider the 'stabilization' $f_1 : X \to \mathbb{C}P^N \subset \mathbb{C}P^{N+k}$. We would like to homotope f_1 to $g : X \to \mathbb{C}P^{N+k}$, such that $g^*(\omega_N) = \omega$ on X. This is accomplished in two stages. We first write

$$f^* \omega_N = \omega + d\eta + \beta,$$

where

(1) η is a smooth one form in a tubular neighborhood $\mathcal{U} \subset M$ of X, and
(2) β is a smooth 2-form in $\mathcal{U} \subset M$ vanishing on each stratum of X.

Using a technique going back to Nash, we modify f to an embedding $g : \mathcal{U} \to \mathbb{C}P^{N+k}$, so that

$$g^* \omega_{N+k} = \omega + \beta$$

on $\mathcal{U}$. Roughly speaking, we absorb the $d\eta$ term into g. Finally, using the fact that β vanishes on strata of X, we conclude that

$$g^* \omega_N \text{ equals } \omega \text{ on } X$$

in the sense of the convention preceding Theorem 4.4. This is done in Section 4.3.

4.2. **Equating second cohomology classes.** Let X be a smooth stratified symplectic space. Recall that this means there exists a smooth manifold M, a smooth embedding $X \subset M$, an open neighborhood $\mathcal{U}$ of X, and a 2-form ω on $\mathcal{U}$ such that $\omega|_S$ is closed non-degenerate for every stratum S. Equip $\mathcal{U}$ with the stratification given by $(\mathcal{U} \setminus X)$ along with the strata of X. Let Π be the compression map occurring in the Compression Lemma (Proposition 2.11. Without loss of generality, we assume henceforth that $\mathcal{U}$ deformation retracts onto X, so that $\Pi(\mathcal{U} \setminus X) = X$ is

the compression map on the top stratum $(\mathcal{U} \setminus X)$ of $\mathcal{U}$. By Proposition 2.11 , $\Pi^*\omega$ is a smooth closed form on X and hence $[\Pi^*\omega]$ is well-defined. Further, observe that the construction of Π shows that $\Pi^*\omega$ is a controlled closed form on the stratified space $\mathcal{U}$.

Proposition 4.5. *Suppose that ω is integral. There exists $N \in \mathbb{N}$ and an embedding $f : \mathcal{U} \to \mathbb{C}P^N$, such that*

$$[f^*(\omega_N)] = [\Pi^*\omega], \tag{1}$$

on $\mathcal{U}$, where ω_N is the Kähler form on $\mathbb{C}P^N$. In particular Equation (1) *holds on X.*

Proof. Note that $[\Pi^*\omega]$ is a well-defined cohomology class in $H^2_{co}(U) = H^2_{co}(X)$. By Theorem 3.6, $H^2_{co}(U, \mathbb{R}) = H^2_{sing}(U, \mathbb{R})$. Further, since ω is integral, $[\Pi^*\omega]$ lies in the image of $H^2_{sing}(U, \mathbb{Z})$ in $H^2_{sing}(U, \mathbb{R})$.

Next, observe that $H^2_{sing}(U, \mathbb{Z}) = [U, \mathbb{C}P^\infty]$, i.e. $H^2_{sing}(U, \mathbb{Z})$ can be identified with the homotopy class of maps from U to the classifying space $\mathbb{C}P^\infty$ of second integral cohomology. Thus, we have the following sequence of maps:

$$[U, \mathbb{C}P^\infty] \xrightarrow{\cong} H^2_{sing}(U, \mathbb{Z}) \to H^2_{sing}(U, \mathbb{R}) \xrightarrow{\cong} H^2_{dR}(U, \mathbb{R}) \xrightarrow{\cong} H^2_{co}(U, \mathbb{R}),$$

where the second arrow is injective and the remaining maps are isomorphisms. The third arrow is the de Rham isomorphism, and the fourth arrow is the controllification sending η to $\Pi^*\eta$ for any closed form η. Let $\alpha \in H^2_{sing}(\mathbb{C}P^\infty, \mathbb{Z})$ denote the generator of the cohomology ring. Then the first arrow sends $f \in [U, \mathbb{C}P^\infty]$ to $f^*\alpha$. Let ω_N denote the Kähler class in $\mathbb{C}P^\infty$, i.e. it denotes the Kähler class in $\mathbb{C}P^N$ for all N (assuming that $\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \cdots$ is standard and that the union is $\mathbb{C}P^\infty$). Hence, $[\omega_0] = \alpha$. Thus, the composition of the first three arrows is given by the map: $f \longrightarrow [f^*\omega_N]$.

On the other hand, the integral cohomology class $[\Pi^*\omega] \in H^2_{sing}(U, \mathbb{Z}) \subset H^2_{sing}(U, \mathbb{R})$ defines a homotopy class of maps $U \to \mathbb{C}P^\infty$. Choosing any f from this homotopy class, we have

$$[f^*\omega_N] = [\Pi^*\omega]$$

. Choosing N sufficiently large, and homotoping f slightly if necessary, we can assume without loss of generality that $f : U \to \mathbb{C}P^N (\subset \mathbb{C}P^\infty)$ is a smooth embedding by compactness of X. This proves the first assertion of the proposition.

Similarly, we also have:

$$[X, \mathbb{C}P^\infty] \xrightarrow{\cong} H^2_{sing}(X, \mathbb{Z}) \to H^2_{sing}(X, \mathbb{R}) \xrightarrow{\cong} H^2_{dR}(X, \mathbb{R}) \xrightarrow{\cong} H^2_{co}(X, \mathbb{R}),$$

and the maps are natural with respect to the inclusion $X \subset U$, proving the second assertion of the proposition. □

4.3. **The Nash-Gromov-Tischler embedding.** In Proposition 4.5 we obtained what can be described as a *formal solution* to the problem of finding an *isosymplectic embedding* of X in $\mathbb{CP}^N$ in the sense of [EM02, Section 12.1.1], i.e. a map $f : X \to \mathbb{CP}^N$ satisfying $f^*[\omega_N] = [\Pi^*\omega]$. In this section we upgrade f to a *holonomic solution*, which in this case would be an honest isosymplectic embedding $g : X \to \mathbb{CP}^m$ for some $m \geq 1$, i.e., $g^*\omega_m = \omega$.

We begin by observing that $f^*\omega_N = \Pi^*\omega + d\alpha'$ for some $\alpha' \in \Omega^1(X)$. Let $\widetilde{\omega}$ be a smooth extension of ω to some regular neighborhood U of $X \subset M$. Recall

$\Pi : U \to X$ is the time-1 map of a homotopy $\mathcal{H} : U \times I \to U$. We use Cartan's magic formula to integrate over the fibers of this homotopy,

$$\begin{aligned}\Pi^*\widetilde{\omega} - \widetilde{\omega} = \mathcal{H}(-,1)^*\widetilde{\omega} - \mathcal{H}(-,0)^*\widetilde{\omega} &= \int_0^1 \partial_t(\mathcal{H}^*\widetilde{\omega})\, dt \\ &= \int_0^1 \mathcal{L}_{\partial_t}\mathcal{H}^*\widetilde{\omega}\, dt \\ &= \int_0^1 (di_{\partial_t}\mathcal{H}^*\widetilde{\omega} + i_{\partial_t}d\mathcal{H}^*\widetilde{\omega})\ dt \\ &= d\alpha'' + \beta\end{aligned}$$

where we have set

$$\alpha'' = \int_0^1 i_{\partial_t}\mathcal{H}^*\widetilde{\omega}\, dt$$
$$\beta = \int_0^1 i_{\partial_t}d\mathcal{H}^*\widetilde{\omega}\, dt$$

Now, observe,

$$\begin{aligned}\beta_p(v_1, \cdots, v_n) &= \int_0^1 d\mathcal{H}^*\widetilde{\omega}_p(\partial_t, v_1, \cdots, v_n)dt \\ &= \int_0^1 d\widetilde{\omega}_q(\mathcal{H}_*(\partial_t), \mathcal{H}_*(v_1), \cdots, \mathcal{H}_*(v_n))dt\end{aligned}$$

where $q = \mathcal{H}(p)$. Suppose $p \in X \times 0 \subset U \times I$ and $v_1, \cdots, v_n \in T_pS$ where $S \in \Sigma$ is the unique stratum containing p. Then $q \in X$ and $\mathcal{H}_*(v_1), \cdots, \mathcal{H}_*(v_n) \in T_qS'$ where $S' \in \Sigma$ is the unique stratum containing q. Moreover $\mathcal{H}_*(\partial_t) \in T_qS'$ as well, as $\mathcal{H}$ restricts to a smooth mapping between sufficiently small neighborhoods of p in S and q in S', respectively. As ω is closed restricted to every stratum, we obtain that the integrand is zero, therefore $\iota_S\beta = 0$ for every stratum $S \in \Sigma$ of X. Combining everything, we obtain

$$f^*\omega_N = \omega + d\alpha + \beta$$

where $\alpha = \alpha' + \alpha'' \in \Omega^1(U)$ is a smooth 1-form and $\beta \in \Omega^2(U)$ is a smooth 2-form which vanishes along X, in the sense that $\iota_S^*\beta = 0$ for every stratum $S \in \Sigma$.

We shall try to modify f to an embedding $g : U \to \mathbb{CP}^m$ such that $g^*\omega_m = \omega + \beta$ on U. Roughly, our strategy is to first *stabilize* f to $f : U \to \mathbb{CP}^{N+1}$ by composing with the canonical inclusion $\mathbb{CP}^N \hookrightarrow \mathbb{CP}^{N+1}$ for some $k \geq 1$ to gain "enough space" so as to *corrugate* f (in the spirit of [EM02]) within a normal neighborhood of $\mathbb{CP}^N$ in $\mathbb{CP}^{N+1}$, and obtain a new map $f_1 : U \to \mathbb{CP}^{N+1}$. The purpose of these corrugations is to compensate, at in least in part, for the "area-defect" $d\alpha$ in $f^*\omega_N - \omega$. We repeat this process many times to finally obtain an isosymplectic embedding $g : U \to \mathbb{CP}^{N+k}$.

Let us carry out the above sketch in detail. First, select a finite open cover $\{V_1, \cdots, V_\ell\}$ of $\mathbb{CP}^N$ such that the normal bundle $\mathcal{O}(1)$ of $\mathbb{CP}^N \subset \mathbb{CP}^{N+1}$ trivializes over each element of the open cover. We pullback this open cover to U by f, and pass to a finer collection $\{W_1, \cdots, W_k\}$ of charts of M covering X such that $f(W_i) \subset V_j$ for some $1 \leq j \leq \ell$, using compactness of X. *This is the only place where we need compactness of X.* By passing to an even finer finite collection covering X if

necessary, we can assume that there exist compactly supported smooth functions $p_i, q_i : U \to \mathbb{R}$ with $\mathrm{supp}(p_i), \mathrm{supp}(q_i) \subset W_i$ such that,

$$\alpha = p_1 dq_1 + \cdots + p_k dq_k$$

Indeed, to see that this refinement is possible, note that by using a partition of unity subordinate to $\{W_1, \cdots, W_k\}$, it suffices to check that such a decomposition is possible for a 1-form defined on any one of the charts $W_i \subset U$. Equivalently, it suffices to check this for any 1-form on an open subset of a Euclidean space, where it is clear. We conclude the following:

$$f^*\omega_N = \omega + \sum_{i=1}^{k} dp_1 \wedge dq_1 + \beta$$

Next, let us stabilize f by a single extra dimension to $f : U \to \mathbb{CP}^N \subset \mathbb{CP}^{N+1}$. Focus on a single chart W_i, and suppose that $f(W_i) \subset V_j$, where $1 \leq j \leq \ell$. By our choice of V_j, we have a trivialization of the normal bundle, $\mathcal{O}(1)|_{V_j} \cong V_j \times \mathbb{C}$. By the symplectic tubular neighborhood theorem, there is a symplectic embedding of an appropriate disk bundle of $\mathcal{O}(1)$ in $\mathbb{CP}^{N+1}$ as a normal neighborhood of $\mathbb{CP}^N$. In particular we obtain a symplectic embedding $V_j \times \mathbb{D}(\varepsilon) \hookrightarrow \mathbb{CP}^{N+1}$ as a piece of the normal neighborhood lying over $V_j \subset \mathbb{CP}^N$, for some $\varepsilon > 0$.

Using Darboux coordinates, there exists a smooth map $h_i : W_i \to \mathbb{D}(\varepsilon)$ such that

$$h_i^*(dz \wedge d\bar{z}) = dp_i \wedge dq_i, \tag{2}$$

where $\mathbb{D}(\varepsilon)$ is equipped with a complex coordinate z. Using Equation (2), we can construct a "corrugation" of f as follows. Define $f_1 : U \to \mathbb{CP}^{N+1}$ by $f_1(x) = (f(x), -h_1(x))$ for all $x \in W_1$ and $f_1 \equiv f$ outside of W_1. Observe,

$$f_1^*(\omega_{n+1}) = f^*\omega_N - h_i^*(dz \wedge d\bar{z}) = \omega + \sum_{i=2}^{k} dp_1 \wedge dq_1 + \beta.$$

We repeat this process k times to obtain $g := f_k : U \to \mathbb{CP}^{N+k}$ with

$$g^*\omega_{n+k} = \omega + \beta.$$

Since $\beta|_S = 0$ for every stratum $S \in \Sigma$, this completes the proof of Theorem 4.4. □

The *proof* of Theorem 4.4 above gives the following h-principle result for symplectic embeddings in complex projective spaces.

Corollary 4.6. *Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space equipped with a smooth structure $j : X \to M$. Let $\omega \in \Omega^2(X)$ be a smooth 2-form and $f : X \to \mathbb{CP}^N$ be an embedding such that $f^*[\omega_N] = [\Pi^*\omega]$. Then there exists $k \geq 1$ and an embedding $g : X \to \mathbb{CP}^{N+k}$ such that $g^*\omega_{n+k} = \omega$ on X, and moreover g is ε-close to $f : X \to \mathbb{CP}^N \subset \mathbb{CP}^{N+k}$ in the C^0-topology.*

In particular, if $[\Pi^*\omega] = 0$, then the constant map $f : X \to \mathbb{CP}^0$ satisfies the hypothesis of Corollary 4.6 above. Hence, by Corollary 4.6, we obtain a symplectic embedding $g : X \to \mathbb{CP}^k$ such that g is ε-close to the constant map to a point $p := \mathbb{CP}^0 \subset \mathbb{CP}^k$, and therefore has image contained in the affine open chart $\mathbb{R}^{2n} \subset \mathbb{CP}^0$ containing p. We conclude the following.

Corollary 4.7. *Let $(X, \Sigma, \mathscr{T})$ be an abstractly stratified space equipped with a smooth structure $j : X \to M$. Suppose $\omega \in \Omega^2(X)$ is a smooth 2-form such that $[\Pi^*\omega] = 0$. Then there exists an embedding $f : X \to \mathbb{R}^{2N}$ such that $f^*\omega_N^0 = \omega$ where ω_N^0 is the standard symplectic form on $\mathbb{R}^{2N}$.*

5. Algebraic and geometric notions of symplectic stratified spaces

5.1. **Sub and quotient algebras of functions.** We start with the following theorem of Schwarz [Sch75] and Mather [Mat77] as summarized in [HP19, Theorem 1.1].

Theorem 5.1. *Let $G \to O(V)$ be a finite dimensional orthogonal representation of the compact Lie group G. Let $\rho_1 \cdots, \rho_l \in \mathbb{R}[V]^G$ be a minimal complete system of polynomial invariants. Let $\rho = (\rho_1 \cdots, \rho_l) : V \to \mathbb{R}^l$ be the corresponding Hilbert map. Then the pullback $\rho^* : C^\infty(\mathbb{R}^l) \to C^\infty(V)^G$ is split surjective, i.e. there exists a continuous map $\lambda : C^\infty(V)^G \to C^\infty(\mathbb{R}^l)$ such that $\rho^* \circ \lambda : C^\infty(V)^G \to C^\infty(V)^G$ is the identity.*

The map ρ factors through a quotient map $i : V/G \to \mathbb{R}^l$, which is a topological embedding as shown in [Mat77]. The embedding i is referred to as the *Hilbert embedding* in [HP19]. We shall follow this terminology. Let $X = i(V/G)$. Then X is a smooth Whitney stratified space (see for instance, [PS85]). As explained in [HP19], the orbit space V/G can thus be equipped with a smooth structure in two different ways:

(1) By setting $C^\infty(V/G) = C^\infty(V)^G$ as an algebra of smooth functions.
(2) By setting $C^\infty(X) = \{f \in C^0(X) | \exists F \in C^\infty(\mathbb{R}^l) : F|_X = f\}$.

Corollary 5.2. [HP19, p. 2] *The Hilbert embedding i induces an isomorphism of algebras $i^* : C^\infty(X) \to C^\infty(V)^G$.*

The same holds if V is replaced by a $G-$invariant closed subanalytic subset $Z \subset V$, i.e. $i^* : C^\infty(\rho(Z)) \to C^\infty(Z)^G$ is an isomorphism of algebras.

A word about the last statement of Corollary 5.2 above is in order (see Remark 4.1 and Proposition 4.4 of [HP19] for a more general statement). This is because $C^\infty(\rho(Z))$ can be identified with the quotient of $C^\infty(X)$ by the ideal $\mathcal{I}_{\rho(Z)}$ of functions that vanish on $C^\infty(\rho(Z))$. But $i^*(\mathcal{I}_{\rho(Z)}) \subset C^\infty(V)^G$ consists of precisely the $G-$invariant functions that vanish on Z.

Remark 5.3. *The hypothesis on subanalyticity is used here only to maintain consistency with the notion of Whitney functions in the literature, in particular* [HP19, Section2]. *This is never used in the proof.*

The following useful fact allows us to upgrade smooth Hamiltonian $G-$actions to analytic Hamiltonian $G-$actions.

Theorem 5.4. [KL00] *Let G be a compact Lie group (with its natural real analytic structure). Then any symplectic $G-$manifold has a unique real analytic $G-$structure. More precisely, let M be a smooth $G-$manifold equipped with a $G-$invariant symplectic form ω. Then the following hold:*

(1) *There exists a real analytic manifold N equipped with a $G-$invariant real analytic symplectic form η.*
(2) *G acts real analytically on N.*
(3) *There exists a $G-$equivariant symplectomorphism $\Phi : N \to M$.*

Further, if (N_1, η_1) and (N_2, η_2) are real analytic $G-$manifolds and if $\Phi : N_1 \to N_2$ is a smooth equivariant diffeomorphism, then there also exists a real analytic equivariant diffeomorphism between N_1 and N_2.

Proposition 5.5. *Let (M, ω) be a smooth symplectic $G-$manifold equipped with a Hamiltonian $G-$action. Let $\mu : M \to \mathfrak{g}^*$ be the associated moment map. Let $Z = \mu^{-1}(0)$ and let $\phi : M \to \mathbb{R}^n$ be a $G-$equivariant smooth embedding into $\mathbb{R}^n$ equipped with a linear $G-$action. Let $\rho : \mathbb{R}^n/G \to \mathbb{R}^l$ be the Hilbert embedding, and let $X = \rho(\phi(Z)/G)$ be the embedded image of Z/G in $\mathbb{R}^l$. Let $C^\infty(X)$ denote the algebra of restrictions of $C^\infty(\mathbb{R}^l)$ to X. Then $C^\infty(X)$ is isomorphic to the algebra $C^\infty(Z)^G$ of $G-$invariant smooth functions on Z (as a subset of M).*

Proof. By Theorem 5.4, we can assume without loss of generality, that M is a real analytic manifold equipped with a $G-$invariant real analytic symplectic form η, and further that the $G-$action is real analytic.

Hence, the moment map μ is locally real analytic by the local normal form theorem of Marle and Guillemin-Sternberg [Ler05, Section 2]. Hence $Z = \mu^{-1}(0)$ is a closed subanalytic subset of M. Corollary 5.2 now furnishes the conclusion. □

5.2. **Two notions.** We first recall Definition 4.1 giving a *geometric* notion of a symplectic stratified space: let $(X, \Sigma, \mathscr{T})$ be a stratified space equipped with an embedding $j : X \to M$ to a smooth manifold M defining a smooth structure on X. Then, as per Definition 4.1, a smooth 2-form $\omega = \{\omega_S : S \in \Sigma\} \in \Omega^2(X)$ is a (stratified) symplectic form if for every stratum $S \in \Sigma$, ω_S is a symplectic 2-form on S. Note that ω admits an extension $\widetilde{\omega}$ to an open neighborhood of X in M. For the purposes of this section, we shall refer to X equipped with a (stratified) symplectic form as a *geometric symplectic stratified space*.

In [SL91, Definition 1.12], Sjamaar and Lerman (see also [LMS93, Pfl01, SLV15]) provided a dual notion of a symplectic stratified space specifically in the context of symplectic reduction. Thus, we have the following notion of a Poisson-symplectic stratified space, where we adjoin the prefix 'Poisson' to distinguish it from the geometric notion of Definition 4.1.

Definition 5.6. *Let $(X, \Sigma, \mathscr{T})$ be a stratified space equipped with an embedding $j : X \to M$ to a smooth manifold M defining a smooth structure on X. Let $C^\infty(X)$ denote the algebra of restrictions of $C^\infty(M)$ to X. Then X is said to be a Poisson-symplectic stratified space if*

1. *Each stratum S is equipped with a symplectic 2-form ω_S. Let $\{\cdot\,,\cdot\}_S$ denote the induced Poisson bracket on $C^\infty(S)$.*
2. *$C^\infty(X)$ is a Poisson algebra. Let $\{\cdot\,,\cdot\}$ denote the Poisson bracket on $C^\infty(X)$.*
3. *The embeddings $S \to X$ are Poisson, i.e. the Poisson bracket induced on the restrictions $\{f|_S, f \in C^\infty(X)\}$ is given by $\{\cdot\,,\cdot\}_S$. Equivalently, the restriction map*

$$\big(C^\infty(X), \{\cdot\,,\cdot\}\big) \to \big(C^\infty(S), \{\cdot\,,\cdot\}_S\big)$$

is a Poisson map.

Remark 5.7. *Let $S \in \Sigma$ be a stratum of dimension k. Recall from Theorem 2.9 that for every $p \in S$, the link $Lk(p)$ is a fixed stratified space (up to a natural isomorphism given by isotopy). Hence, for every $p \in S$, there exists a basis of neighborhoods*

homeomorphic to $\mathbb{R}^k \times c\ Lk(p)$, where the first factor gives a neighborhood of p in S, and $c\ Lk(p)$ denotes the cone on $Lk(p)$.

5.3. **Algebraic implies geometric.** We spell out the Poisson-symplectic stratified space condition of Definition 5.6 in local co-ordinates using the structure given in Remark 5.7. Let $p \in S$ as in Remark 5.7, and let $V \cong D^m$ denote a closed ball contained in a top-dimensional stratum of $Lk(p)$. Then $\mathbb{R}^k \times cD^m \subset \mathbb{R}^k \times c\ Lk(p)$ gives a co-ordinate chart homeomorphic to $\mathbb{R}^k \times \mathbb{H}^{m+1}$, where $\mathbb{H}^{m+1}$ is the half space in $\mathbb{R}^{m+1}$ given by $x_1 \geq 0$. There exists an embedding of $\mathbb{R}^k \times c\ Lk(p)$ in some $\mathbb{R}^N$, such that the first k co-ordinates of $\mathbb{R}^N$ coincide with the co-ordinates of the $\mathbb{R}^k-$factor, and $\mathbb{H}^{m+1}$ is embedded as a half space of the next $m+1$ co-ordinates.

Let H_i denote the Hamiltonian vector field corresponding to the co-ordinate function x_i in $\mathbb{R}^k \times \mathbb{H}^{m+1}$, and $Y_i = d/dx_i$. Then, the third condition of Definition 5.6 furnishes the following:

$$\omega_L(H_i, Y_j) = dx_i(x_j) = \delta_{ij}, \tag{3}$$

where ω_L is the symplectic form on a top-dimensional stratum L. Let $\omega_L = \sum_{rl} \omega_{rl} dx_r \wedge dx_l$ denote the local co-ordinate description of ω_L (thought of as a $(k+m+1) \times (k+m+1)$ matrix in the first $(k+m+1)$ co-ordinates of $\mathbb{R}^N$ with coefficients the smooth functions ω_{rl}). Then Equation (3) gives (the first $(k+m+1)$ co-ordinates of) H_i according to the following formula:

$$H_i = \omega_L^{-1}(Y_i). \tag{4}$$

Thus, by conditions 2, 3 of Definition 5.6, and restricting to the first $(k+m+1)$ co-ordinates, the $(k+m+1) \times (k+m+1)$ functions given by

$$\omega_L(H_i, H_j) = \omega_L(\omega_L^{-1}(Y_i), \omega_L^{-1}(Y_j)) = (Y_i, \omega_L^{-1}(Y_j))$$

extend to smooth functions in a neighborhood of the closure $\overline{L}$. Equivalently, we say that ω_L^{-1} extends smoothly. We summarize this as follows:

Lemma 5.8. *Let $(X, \Sigma, \mathscr{T})$ be a Poisson-symplectic stratified space. Let ω_L denote a symplectic form on a top-dimensional stratum $L \in \Sigma$ giving an isomorphism $\omega : TL \to T^*L$, and let $\omega^{-1} : T^*L \to TL$ denote the adjoint. Then ω^{-1} extends smoothly to an open neighborhood of the closure $\overline{L}$ in the above sense: the smooth functions ω_{ij}^{-1} which are the entries of ω^{-1} individually extend to smooth functions in a neighborhood of the closure $\overline{L}$.*

As a consequence we conclude:

Proposition 5.9. *With the notation of Lemma 5.8, let the stratum-wise symplectic 2-forms be given by $\omega = \{\omega_S : S \in \Sigma\}$. Then (each component of) ω^{-1} extends smoothly to a 2-form (not necessarily symplectic) an open neighborhood of X in M.*

Thus, X equipped with ω^{-1} is a geometric symplectic stratified space (in the sense of Definition 4.1).

Proof. The first assertion is a consequence of Lemma 5.8. The second assertion follows from the Whitney extension theorem [Mal67, Chapter 1], since on each lower dimensional stratum S, the limiting values of ω^{-1} on higher dimensional strata L abutting S agree. □

Example 5.1. Consider $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ as a stratified space with strata $S = \mathbb{R}^2 \times \{0\}$ and $L = \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$. Let $\omega_1 = dx_1 \wedge dy_1$ on the first $\mathbb{R}^2$ factor, and $\omega_2 = dx_2 \wedge dy_2$ on the second $\mathbb{R}^2$ factor. Equip $\mathbb{R}^4$ with the 2-form $\omega = \omega_1 + r^2\omega_2$, where $r^2 = x_2^2 + y_2^2$. Then

(1) $\omega_S = \omega_1$, $\omega_L = \omega$ does furnish a geometric symplectic stratified space structure on $X = \mathbb{R}^4$ stratified as above.
(2) $\omega_S = \omega_1, \omega_L = \omega$ does not furnish a Poisson-symplectic stratified space structure on $X = \mathbb{R}^4$ stratified as above.

Poisson stratified symplectic structures on symplectic reductions are geometric: Recall the setup of Proposition 5.5:

(1) (M, ω) is a smooth symplectic $G-$manifold equipped with a Hamiltonian $G-$action.
(2) $\mu : M \to \mathfrak{g}^*$ is the associated moment map, and $Z = \mu^{-1}(0)$.
(3) $\phi : M \to \mathbb{R}^n$ is a $G-$equivariant smooth embedding of M into $\mathbb{R}^n$ equipped with a linear $G-$action.
(4) $\rho : \mathbb{R}^n/G \to \mathbb{R}^l$ is the Hilbert embedding and $X = \rho(\phi(Z)/G)$ is the embedded image of Z/G in $\mathbb{R}^l$.
(5) $C^\infty(X)$ is the algebra of restrictions of $C^\infty(\mathbb{R}^l)$ to X.

Then, by Proposition 5.5, $C^\infty(X)$ is isomorphic to the algebra $C^\infty(Z)^G$ of $G-$invariant smooth functions on Z (as a subset of M). Hence, the Poisson structure in [SL91, Section 6] is the same as that on $C^\infty(X)$ obtained via a stratified symplectic form ω_X furnished by [SL91].

Let $P : Z \to X$ denote the quotient map by G (under the identification $X = \rho(\phi(Z)/G)$). Thus,

(1) $P^*\omega_X$ is a 2-form ω_Z on Z.
(2) ω_Z is the restriction of the smooth symplectic 2-form ω on M.

Since ω_Z does not blow up, nor does ω_X. Hence Proposition 5.9 shows that X equipped with ω^{-1} (and *not* ω) is a geometric symplectic stratified space (in the sense of Definition 4.1).

5.4. **Geometric implies algebraic.** In the converse direction to Proposition 5.9 we have a stronger conclusion.

Proposition 5.10. *Let $(X, \Sigma, \mathscr{T})$ be a geometric symplectic stratified space (in the sense of Definition 4.1). Then X is a Poisson-symplectic stratified space (in the sense of Definition 5.6).*

Proof. Let $\omega = \{\omega_S : S \in \Sigma\}$ be a collection of symplectic 2-forms on the strata $S \in \Sigma$, defining a geometric symplectic stratified space structure on X. Let $H^2(X, \mathbb{R}) = \mathbb{R}^M$. Let Π denote the compression map occurring in Proposition 3.8. Let $[\Pi^*\omega] \in H^2(X, \mathbb{R})$ be given by $(c_1, \cdots, c_M) \in \mathbb{R}^M$, where we assume without loss of generality, that $c_i \geq 0$ for all i (by choosing appropriate identifications with the co-ordinate $\mathbb{R}$ factors). Let $\omega_1, \cdots, \omega_M$ be closed stratified smooth 2-forms on X, such that

(1) $\omega = \sum_i \omega_i$,
(2) $[\Pi^*\omega] \in H^2(X, \mathbb{R}) = \mathbb{R}^M$ is given by $(0, \cdots, 0, c_i, 0, \cdots, 0)$, i.e. c_i in the $i-$th place and zero elsewhere.

Let $\mathbb{C}P^n_i, i = 1, \cdots, M$ denote M copies of $\mathbb{C}P^n$, and let η_i denote the Kähler form on $\mathbb{C}P^n_i$. Then, by Theorem 4.4, there exists $n \geq 1$ and embeddings $f_i : X \to \mathbb{C}P^n_i$ such that $c_i f_i^* \eta_i = \omega_i$. (This follows from the fact that $\frac{1}{c_i}\omega_i$ is integral and by choosing n sufficiently large.)

Next, let $\mathbb{C}P_\pi = (\mathbb{C}P^n)^M$ denote the product of M copies of $\mathbb{C}P^n$, let $\pi_i : \mathbb{C}P_\pi \to \mathbb{C}P^n$ denote the projection to the $i-$th factor, and let $\theta_i = \pi_i^* \eta_i$. Let $f_\pi : X \to \mathbb{C}P_\pi$ be given by $f_\pi = (f_1, \cdots, f_M)$. Then

(1) $f_\pi : X \to \mathbb{C}P_\pi$ is an embedding.
(2) $f_\pi^*(\sum_i c_i \theta_i) = \sum_i \omega_i = \omega$.
(3) $\Theta = \sum_i c_i \theta_i$ is a symplectic form on $\mathbb{C}P_\pi$.

Let $\{\cdot, \cdot\}_\Theta$ denote the Poisson structure on $C^\infty(\mathbb{C}P_\pi)$ given by the symplectic form Θ. Then $\{\cdot, \cdot\}_\Theta$ induces a Poisson structure on $C^\infty(X)$ such that the embeddings of each stratum S in X is Poisson. Thus, X is a Poisson-symplectic stratified space (in the sense of Definition 5.6). □

6. Universal stratified symplectic reduction

In [GT89, GT91], Gotay and Tuynman proved, what in hindsight, is really an analog of the Gromov-Tischler theorem in the context of reductions. They argued that in the symplectic category, this is a more natural construction than embeddings. We begin with their definition of reduction from [GT89]:

Definition 6.1. [GT89] *Let (M, ω) be a symplectic manifold and $N \subset M$ be a submanifold. Suppose $\omega_N := \iota_N^* \omega$ has constant rank on N. Further assume that the characteristic distribution $\ker \omega_N \subset TN$ is fibrating, i.e.*

(1) *$\ker \omega_N \subset TN$ integrable with associated foliation $\mathscr{F}$,*
(2) *the leaf space $M/\mathscr{F}$ is a (Hausdorff) smooth manifold,*
(3) *$M \to M/\mathscr{F}$ is a smooth fiber bundle.*

Then $M/\mathscr{F}$ is called the reduction of M by N.

The main point of this definition is that it is a purely topological version of Marsden-Weinstein reduction, where $N \subset M$ turns out to be a generic level set of the moment map of some Hamiltonian Lie group action on M. With this definition at hand, [GT89] proves that $\mathbb{R}^{2n}$ equipped with the standard symplectic form ω_n is the universal symplectic manifold for reduction in the following sense:

Theorem 6.2. [GT89] *Let (M, ω) be a symplectic manifold such that $[\omega] \in H^2(M; \mathbb{R})$ lies in the image of $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})$. Then there exists $n \geq 1$ such that (M, ω) can be realized as a reduction of $(\mathbb{R}^{2n}, \omega_n)$.*

The hypothesis of the theorem is easily seen to be satisfied for any compact symplectic manifold. We give a generalization of the theorem in the setup of compact symplectic stratified spaces. Our proof follows a different route than the one taken by Gotay and Tuynman, and uses ingredients of Theorem 4.4 and Section 4.3 in a crucial way.

Theorem 6.3. *Let (X, ω) be a compact symplectic stratified space. Then there exists $n \geq 1$ and a Whitney stratified set $(Y, \Sigma) \subset \mathbb{R}^{2n}$ equipped with a map $f : Y \to X$ such that*

(1) *(Y, X, f) is a fiber bundle with smooth manifold fibers.*
(2) *For every stratum $S \in \Sigma$, $\ker \iota_S^* \omega_n = \ker df|_S$*

Proof. Let $(X, \Sigma_X, \mathscr{T}_X)$ denote the underlying abstractly stratified space of the symplectic stratified space (X, ω). Let Π_X denote the compression map for X occurring in Proposition 3.8.

First, let us assume $[\Pi_X^*\omega] \in H^2_{sing}(X;\mathbb{Z}) \subset H^2_{sing}(X;\mathbb{R})$, i.e. ω is integral. There exists a complex line bundle L over X corresponding to the class $[\Pi_X^*\omega] \in H^2(X;\mathbb{Z})$; let Y be the associated principal $U(1)$-bundle corresponding to L, and $f : Y \to X$ be the bundle projection. We equip Y with an abstract stratification by defining the collection of strata as $\Sigma_Y := \{f^{-1}(S) : S \in \Sigma_X\}$ and the tube system as simply a pullback $\mathscr{T}_Y := f^*\mathscr{T}_X$ (see Remark 2.5).

Let Π_Y denote the compression map for Y occurring in Proposition 3.8. As $\Pi_X^*\omega \in \Omega^2_{co}(X)$ is a controlled differential form, we have that $f^*\Pi_X^*\omega = \Pi_Y^* f^*\omega \in \Omega^2_{co}(Y)$ is also a controlled differential form with respect to the abstract stratification on Y defined in the previous paragraph. Then $[\Pi_Y^* f^*\omega] = f^*[\Pi_X^*\omega] \in H^2_{sing}(X;\mathbb{R})$ (by functoriality of the isomorphism in Verona's Theorem 3.6). However as $f : Y \to X$ is the $U(1)$-bundle with Chern class $[\Pi_X^*\omega]$, $f^*[\Pi_X^*\omega] = 0$. Therefore, by Corollary 4.7, we obtain an embedding $h : Y \to \mathbb{R}^{2n}$ such that $h^*\omega_n = f^*\omega$.

Let us identify $h(Y) \subset \mathbb{R}^{2n}$ with Y, so that $f^*\omega$ is simply the restriction of ω_n to the Whitney stratified subset $Y \subset \mathbb{R}^{2n}$. Then $f^*\omega$ is degenerate only along the tangent spaces to the circle fibers of $f : Y \to X$, therefore we conclude the statement of the theorem.

Finally, we indicate the modifications necessary to handle nonintegral forms. The idea is similar to that in Proposition 5.10. We can, in general, write $\omega = \sum_1^k c_i\omega_i$, where each ω_i is integral, and $c_i \in \mathbb{R}$. Let L_i denote the complex line bundle over X corresponding to the class $[\Pi_X^*\omega_i] \in H^2(X;\mathbb{Z})$. Let $L = \oplus_1^k L_i$, and Y be the associated principal $\times_i^k \big(U(1)\big)_i$-bundle corresponding to L. The rest of the argument is identical. □

References

[DJ96] W. Domitrz and S. Janeczko. On Martinet's singular symplectic structures. In *Singularities and differential equations (Warsaw, 1993)*, volume 33 of *Banach Center Publ.*, pages 51–59. Polish Acad. Sci. Inst. Math., Warsaw, 1996.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[GM88] Mark Goresky and Robert MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[Gor78] R. Mark Goresky. Triangulation of stratified objects. *Proc. Amer. Math. Soc.*, 72(1):193–200, 1978.

[Gro71] M. L. Gromov. A topological technique for the construction of solutions of differential equations and inequalities. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 221–225. 1971.

[Gro86] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[GT89] Mark J. Gotay and Gijs M. Tuynman. $\mathbf{R}^{2n}$ is a universal symplectic manifold for reduction. *Lett. Math. Phys.*, 18(1):55–59, 1989.

[GT91] M. J. Gotay and G. M. Tuynman. A symplectic analogue of the Mostow-Palais theorem. In *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, volume 20 of *Math. Sci. Res. Inst. Publ.*, pages 173–182. Springer, New York, 1991.

[HP19] Hans-Christian Herbig and Markus J. Pflaum. Invariant Whitney functions. *Internat. J. Math.*, 30(2):1950009, 26, 2019.

[KL00] Frank Kutzschebauch and Frank Loose. Real analytic structures on a symplectic manifold. *Proc. Amer. Math. Soc.*, 128(10):3009–3016, 2000.

[Ler05] Eugene Lerman. Gradient flow of the norm squared of a moment map. *Enseign. Math. (2)*, 51(1-2):117–127, 2005.

[LMS93] Eugene Lerman, Richard Montgomery, and Reyer Sjamaar. Examples of singular reduction. In *Symplectic geometry*, volume 192 of *London Math. Soc. Lecture Note Ser.*, pages 127–155. Cambridge Univ. Press, Cambridge, 1993.

[Mal67] B. Malgrange. *Ideals of differentiable functions*, volume 3 of *Tata Institute of Fundamental Research Studies in Mathematics*. Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1967.

[Mat77] John N. Mather. Differentiable invariants. *Topology*, 16(2):145–155, 1977.

[Mat12] John Mather. Notes on topological stability. *Bull. Amer. Math. Soc. (N.S.)*, 49(4):475–506, 2012.

[MS99] Eckhard Meinrenken and Reyer Sjamaar. Singular reduction and quantization. *Topology*, 38(4):699–762, 1999.

[MS22] M. Mj and B. Sen. h-principle for stratified spaces. *preprint, TIFR Mumbai*, 2022.

[Nat80] Hiroko Natsume. The realization of abstract stratified sets. *Kodai Math. J.*, 3(1):1–7, 1980.

[Pfl01] Markus J. Pflaum. Smooth structures on stratified spaces. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 231–258. Birkhäuser, Basel, 2001.

[PS85] Claudio Procesi and Gerald Schwarz. Inequalities defining orbit spaces. *Invent. Math.*, 81(3):539–554, 1985.

[Sch75] Gerald W. Schwarz. Smooth functions invariant under the action of a compact Lie group. *Topology*, 14:63–68, 1975.

[SL91] Reyer Sjamaar and Eugene Lerman. Stratified symplectic spaces and reduction. *Ann. of Math. (2)*, 134(2):375–422, 1991.

[SLV15] Petr Somberg, Hông Vân Lê, and Jiři Vanžura. Poisson smooth structures on stratified symplectic spaces. In *Mathematics in the 21st century*, volume 98 of *Springer Proc. Math. Stat.*, pages 181–204. Springer, Basel, 2015.

[Tho69] René Thom. Ensembles et morphismes stratifiés. *Bull. Amer. Math. Soc.*, 75:240–284, 1969.

[Tis77] David Tischler. Closed 2-forms and an embedding theorem for symplectic manifolds. *J. Differential Geometry*, 12(2):229–235, 1977.

[Ver71] Andrei Verona. Le théorème de de Rham pour les préstratifications abstraites. *C. R. Acad. Sci. Paris Sér. A-B*, 273:A886–A889, 1971.

School of Mathematics, Tata Institute of Fundamental Research. 1, Homi Bhabha Road, Mumbai-400005, India

E-mail address: mahan@math.tifr.res.in

School of Mathematics, Tata Institute of Fundamental Research. 1, Homi Bhabha Road, Mumbai-400005, India

E-mail address: balarka@math.tifr.res.in