

THE SULLIVAN DICTIONARY AND BOWEN-SERIES MAPS

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ABSTRACT. The Sullivan dictionary between Kleinian groups and rational dynamics describes striking similarities between the fields, both in terms of the objects of study as well as the techniques used. We give an expository account of a recent bridge between the two sides of the dictionary by describing a framework for combining a Fuchsian group with a complex polynomial into a single dynamical system on the Riemann sphere.

To Dennis with admiration and affection.

1. INTRODUCTION

In this expository article, we draw heavily from and build upon two strands of Dennis Sullivan's work:

- (1) The Sullivan dictionary between Kleinian groups and rational dynamics [Sul85].
- (2) The Patterson-Sullivan measure [Sul79, Sul84].

We shall survey these two themes in the light of a recent combination theorem or a bridge between Kleinian groups and polynomial maps discovered by the authors [MM21]. An essential ingredient in the building of this bridge is the Bowen-Series map [Bow79, BS79].

Sullivan's dictionary [Sul85, p. 405] was based on the empirical insight that Kleinian groups and rational dynamics share many common features. For instance, the *limit set* (resp. the *domain of discontinuity*) of a Kleinian group corresponds to the *Julia set* (resp. the *Fatou set*) of a rational map. Sullivan extended these similarities to a deeper similarity between *techniques* by introducing quasiconformal methods into the field of rational dynamics. This led to the proof of his celebrated *no wandering domains* theorem. In fact, using these techniques, Sullivan gave a new proof of Ahlfors' finiteness theorem along the lines of the no wandering domains theorem.

Our focus here is on a line in the Sullivan dictionary that observes the similarity between the following:

- (1) The Bers simultaneous uniformization theorem in Kleinian groups
- (2) Polynomial mating in rational dynamics, introduced by Douady and Hubbard [Dou83].

The first step is to replace the Kleinian group by a single map that captures its dynamics. This brings us to the notion of a mateable map (see Section 2 below

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for details). With the context of mateable maps in place, we address the following question:

Question 1.1. *Which mateable maps and polynomials can be mated in the spirit of Douady and Hubbard?*

It turns out that Bowen-Series maps [Bow79, BS79] for punctured sphere groups provide such examples. Surprisingly, there exists a new class of related maps which we call *higher Bowen-Series maps* that also fit the bill and give rise to combination theorems as well as ‘dynamically natural’ homeomorphisms between limit and Julia sets. As the name suggests, higher Bowen-Series maps are closely related to Bowen-Series maps. Indeed, higher Bowen-Series maps appear as second iterates of suitable Bowen-Series maps. Higher Bowen-Series maps can also be characterized as ‘amalgams’ of several Bowen-Series maps of the same Fuchsian group with overlapping fundamental domains. This part of the story is complex analytic in flavor and is taken largely from [MM21].

It is worth mentioning that examples of dynamically natural homeomorphisms between limit sets of Kleinian reflection groups (i.e., discrete subgroups of $\text{Aut}(\widehat{\mathbb{C}})$ generated by reflections in finitely many Euclidean circles) including the classical Apollonian gasket limit set and Julia sets of anti-holomorphic rational maps were first constructed in [LLMM19], and this phenomenon was studied systematically in a general framework in [LMM22, LLM22]. To the best of our knowledge, [MM21, Theorem 7.16] gives the first example of such an explicit connection between limit sets and Julia sets in the holomorphic setting.

In the last section of this survey, we turn to the measurable dynamics of mateable maps and the resulting matings. From the point of view of group theory, the measure-theoretic framework, naturally and rather appropriately, turns out to be that of Patterson-Sullivan measures. On the other hand, since mateable maps share features of rational maps, the limit set of a mateable map supports a natural dynamically defined measure: the measure of maximal entropy (the existence of a unique maximal entropy measure for a rational map was proved in [Lyu83] and independently in [Mn83, FLMn83]). The fact that a mateable map is an object halfway between groups and polynomials is reflected in close connections between maximal entropy measures of mateable maps and suitable Patterson-Sullivan measures. We conclude the article with some estimates of Hausdorff dimensions of maximal entropy measures of (higher) Bowen-Series maps and related open questions.

The phenomenon of “mating” of rational maps with Fuchsian groups was discovered in the 1990s by Bullett and Penrose in the context of iterated algebraic correspondences [BP94] and was studied comprehensively in [BL20, BL21]. Specifically, they constructed a family of algebraic correspondences of bi-degree $(2, 2)$, and showed that the members of this family can be interpreted in an appropriate sense as matings of the modular group with quadratic rational maps. This is quite different from our mating framework as we extract a non-invertible map (a mateable map) from a Kleinian group (i.e., a semi-group dynamics from the dynamics of a non-commutative group) and then combine this map with the dynamics of a polynomial thereby producing a hybrid dynamical system in one complex variable. It would be quite interesting to know if our mating framework has deeper connections with that of Bullett-Penrose-Lomonaco.

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2. MATEABILITY

Let $\text{Aut}(\mathbb{D})$ denote the group of all conformal automorphisms of the unit disk \mathbb{D} . A Fuchsian group Γ is a discrete subgroup of $\text{Aut}(\mathbb{D})$. The aim of this section is to spell out what it means to mate a Fuchsian group with a polynomial. We provide the definition of mateability at the outset. The definition below will imply that Γ is a lattice (Lemma 2.9).

Definition 2.1. *A continuous map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a mateable map associated with a Fuchsian group Γ if the following are satisfied:*

- (1) *A is orbit equivalent to Γ .*
- (2) *A is piecewise analytic on \mathbb{S}^1 .*
- (3) *A is an expansive covering map of degree greater than one.*
- (4) *A is Markov.*
- (5) *No periodic break-point of A is asymmetrically hyperbolic.*

The failure of any of the conditions in Definition 2.1 provides an obstruction to mateability. Somewhat surprisingly, it turns out that these necessary conditions are also sufficient (see Proposition 2.10).

We elaborate now on the terms used in Definition 2.1. Let $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a (not necessarily continuous) map. The *grand orbit* of a point $x \in \mathbb{S}^1$ under A is defined as

$$\text{GO}_A(x) := \{x' \in \mathbb{S}^1 : A^m(x) = A^n(x'), \text{ for some } m, n \geq 0\}.$$

Let Γ be a Fuchsian group with limit set equal to $\Lambda \subset \mathbb{S}^1$. We say that a (not necessarily continuous) map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is *orbit equivalent* to Γ on Λ if for every $x \in \Lambda$,

$$\Gamma \cdot x = \text{GO}_A(x).$$

A (not necessarily continuous) map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is *piecewise Möbius* if there exist $k \in \mathbb{N}$, closed arcs $I_j \subset \mathbb{S}^1$, and $g_j \in \text{Aut}(\mathbb{D})$ for $j \in \{1, \dots, k\}$, such that

- (1) $\mathbb{S}^1 = \bigcup_{j=1}^k I_j$,
- (2) $\text{int } I_m \cap \text{int } I_n = \emptyset$ for $m \neq n$, and
- (3) $A|_{I_j} = g_j$.

A piecewise Möbius map A as above is called *piecewise Fuchsian* if g_1, \dots, g_k generate a Fuchsian group, which we denote by Γ_A . If the maps g_j are assumed only to be complex-analytic in some small neighborhoods of I_j (without requiring them to be Möbius), then f is said to be *piecewise analytic*.

The maps g_j will be called the *pieces* of A . We shall occasionally refer to the domains I_j of g_j also as *pieces* of A when there is no scope for confusion.

Remark 2.2. We think of the partition of \mathbb{S}^1 into the closed arcs $\{I_j\}$ as a part of the data of the piecewise Möbius/analytic map A . This can be formalized by defining a piecewise Möbius/analytic map A as a pair $(\{g_j\}_{j=1}^k, \{I_j\}_{j=1}^k)$.

Lemma 2.3 below upgrades the regularity of A considerably.

Lemma 2.3. [MM21, Lemma 2.8] *Let $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a (not necessarily continuous) piecewise analytic map that is orbit equivalent to a finitely generated Fuchsian group Γ . Then, A is piecewise Fuchsian, and the pieces of A form a generating set for Γ .*

Suppose that x_1, \dots, x_k are a cyclically ordered collection of k points on \mathbb{S}^1 defining the pieces $I_j = [x_j, x_{j+1}]$ of A ($j+1$ taken modulo k). We shall say that A is *minimal*, if the decomposition of \mathbb{S}^1 given by x_1, \dots, x_k is minimal; i.e., there does not exist i and $h \in \Gamma_A$ such that

- (1) $A|_{[x_i, x_{i+1}]} = h|_{[x_i, x_{i+1}]}$, and
- (2) $A|_{[x_{i-1}, x_i]} = h|_{[x_{i-1}, x_i]}$.

Thus, a minimal A has no superfluous break-points.

Let A be a continuous piecewise Möbius map on the circle. Let \mathbb{D} denote the unit disk. Let I_1, \dots, I_k be a circularly ordered family of intervals with disjoint interiors such that

- (1) $I_j \cap I_{j+1} = \{x_{j+1}\}$ (the indices being taken mod k).
- (2) $A|_{I_j} = g_j$.

Let γ_j be the semi-circular arc in \mathbb{D} between x_j, x_{j+1} meeting \mathbb{S}^1 at right angles at x_j, x_{j+1} , and let $\mathcal{D}_j \subset \overline{\mathbb{D}}$ be the closed region bounded by I_j and γ_j . Then \hat{A} , the *canonical extension of A to a piecewise Möbius map in $\overline{\mathbb{D}}$* is defined on $\cup_j \mathcal{D}_j$ as $\hat{A} = g_j$ on \mathcal{D}_j .

Set $\mathcal{D} := \cup_j \mathcal{D}_j$ and call \mathcal{D} the *canonical domain of definition* of \hat{A} . Let $R = \mathbb{D} \setminus \mathcal{D}$. We shall call R the *fundamental domain* of A , as well as the fundamental domain of \hat{A} . Each bi-infinite hyperbolic geodesic contained in the boundary ∂R will be called an *edge* of R . The ideal vertices of R will be called the *vertices* of R . Let S be the set of vertices of R . A pair of non-adjacent points in S , or equivalently the bi-infinite geodesic joining them in R will be called a *diagonal* of R .

Remark 2.4. We note that the fundamental domain of a piecewise Fuchsian map A may not be a fundamental domain for the Fuchsian group Γ_A generated by the pieces of A (see Subsection 4.1).

Remark 2.5. Let $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a continuous piecewise Möbius map with pieces $\{g_j\}_{j=1}^k$. By continuity, $g_j(x_{j+1}) = g_{j+1}(x_{j+1})$; i.e., $a_j = g_j^{-1} \circ g_{j+1} \in \Gamma_A$ fixes x_{j+1} (indices taken modulo k). Then, $a_k \cdots a_1 = 1$ as a group element, or equivalently, $a_1 \circ \cdots \circ a_k = \text{id}$. Moreover, if A is orbit equivalent to a Fuchsian group Γ on \mathbb{S}^1 , then Γ is generated by $\{g_1, a_1, \dots, a_k\}$ by Lemma 2.3.

A continuous map $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is said to be *expansive* if there exists $\delta > 0$ such that for any $a \neq b \in \mathbb{S}^1$, there exists $n \in \mathbb{N}$ such that $d(f^n(a), f^n(b)) > \delta$.

We endow \mathbb{S}^1 with the counter-clockwise orientation. For $a, b \in \mathbb{S}^1$, we denote the counter-clockwise arc of \mathbb{S}^1 connecting a, b by \widehat{ab} . Suppose that y_0 is a *periodic point* of period n of a piecewise Möbius covering map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Then, A^n is orientation-preserving, and it maps an arc of the form $\widehat{y_1 y_0}$ to an arc of the form

$\widehat{y_2 y_0}$. We define the *one-sided multipliers* of A at y_0 to be the one-sided derivatives of A^n :

$$(A^n)'(y_0^+) = \lim_{\substack{y \rightarrow y_0 \\ y \in \widehat{y_0 \tilde{y}}}} \frac{A^n(y) - y_0}{y - y_0}, \quad (A^n)'(y_0^-) = \lim_{\substack{y \rightarrow y_0 \\ y \in \widehat{\tilde{y} y_0}}} \frac{A^n(y) - y_0}{y - y_0},$$

where $\tilde{y} \neq y_0$ is any point on \mathbb{S}^1 . See [MM21, Section 2] for properties of one-sided multipliers of A .

Let x be a periodic point (of period n) of a piecewise Möbius, expansive circle covering A . Then x is said to be *parabolic on the right* (resp., *on the left*) if $(A^n)'(x^+) = 1$ (resp., $(A^n)'(x^-) = 1$). Likewise, x is *hyperbolic on the right* (respectively, *on the left*) if $(A^n)'(x^+) > 1$ (resp., $(A^n)'(x^-) > 1$). Also, x is *symmetrically parabolic* (respectively, *symmetrically hyperbolic*) if $(A^n)'(x^+) = (A^n)'(x^-) = 1$ (respectively, if $(A^n)'(x^+) = (A^n)'(x^-) > 1$). The point x is called *asymmetrically hyperbolic* if it is hyperbolic on both sides, but $(A^n)'(x^+) \neq (A^n)'(x^-)$. Finally, x is said to be a *periodic point of mixed type* if it is hyperbolic on one side, but parabolic on the other.

Lemma 2.6. [MM21, Lemma 2.15] *Let $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a piecewise Fuchsian expansive covering map having x_1, \dots, x_k as the break-points of its piecewise definition. Further, let x_j be a periodic point of A . Then, x_j is not of mixed type.*

Definition 2.7. *Let X be a topological space and $f : X \rightarrow X$ be a continuous map. A collection of closed subsets $\{X_1, X_2, \dots, X_n\}$ of X is called a Markov partition for (X, f) if the following properties are satisfied:*

- (1) $X = \bigcup_{i=1}^n X_i$,
- (2) $\text{int } X_i \cap \text{int } X_j = \emptyset$ for $i \neq j$,
- (3) $\overline{\text{int } X_i} = X_i$ for $i \in \{1, 2, \dots, n\}$,
- (4) $f|_{X_i}$ is injective, and
- (5) if $f(\text{int } X_i) \cap \text{int } X_j \neq \emptyset$, then $f(X_i) \supset X_j$.

It is well-known that continuous, open and distance expanding self-maps of compact metric spaces admit Markov partitions (see [PU10, §3]). In particular, the polynomial map $z \mapsto z^d$, restricted to the unit circle \mathbb{S}^1 , admits a Markov partition (in fact, explicit Markov partitions for z^d can be easily constructed).

We call $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a *piecewise Fuchsian Markov map* if it is a piecewise Fuchsian expansive covering map (of degree d at least two) such that the pieces I_j (intervals of definition) of A in \mathbb{S}^1 give a Markov partition for $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. The restrictions $A|_{I_j} = g_j (\in \Gamma_A)$ of A to I_j will be referred to as *pieces* of A .

By the Markov property of A , each interval I_j has exactly d pre-images under A . This gives us a natural transition matrix for A^{-1} given by $a_{jl} = 1$ if there exists a point in the interior of I_l mapped to I_j under A , and $a_{jl} = 0$ otherwise. Further, there is a naturally associated topological Markov chain, which we now describe (compare [Ser86, ECH+92]). We construct a d -regular directed graph \mathcal{G} with k vertices (one for each I_j) and a directed edge from vertex j to vertex l if and only if $a_{jl} = 1$. Further, we label such a directed edge from j to l by g_l^{-1} (since the piece of A on I_l is g_l , the inverse branch from I_j to I_l is g_l^{-1}). Note that there are exactly d branches of A^{-1} at each interior point of an I_j and any such branch is given by the inverse of one of the pieces of A ; i.e., for each piece g_i of A , g_i^{-1} is a label of some edge of \mathcal{G} and each label of an edge of \mathcal{G} is of this form.

We now follow a point $z \in \mathbb{S}^1$ under backward iteration of A . Let $\{z = z_0, z_1, \dots\}$ be a (finite or infinite) sequence of points in \mathbb{S}^1 such that $A(z_{i+1}) = z_i$. Then any such sequence encodes a geodesic in \mathcal{G} ; i.e., an isometric immersion of an interval $[0, a]$, or $[0, \infty)$ into \mathcal{G} such that $[i, i+1]$ maps isometrically to an edge of \mathcal{G} labeled by (the unique) g satisfying by the following:

- (1) $z_i \in I_{j(i)}$.
- (2) $z_{i+1} \in I_{j(i+1)}$.
- (3) A restricted to $I_{j(i+1)}$ equals g^{-1} .
- (4) $g(z_i) = z_{i+1}$.

The labeled directed graph \mathcal{G} (also known as a topological Markov chain) imposes a structure akin to that of an automatic group [ECH+92] on backward orbits of points via backward orbits of intervals I_j . Thus, a sequence of backward orbits of an interval I_j may be given by $I_j = I_{j(0)}, I_{j(1)}, \dots, I_{j(n)}, \dots$ such that $I_{j(i)} \subset A(I_{j(i+1)})$. This sequence is also encoded by the same geodesic in \mathcal{G} described above, since the pair $\{I_{j(i)}, I_{j(i+1)}\}$ corresponds to a unique edge in \mathcal{G} , and the label on the edge is the unique $g \in \Gamma$ such that g^{-1} is a piece of A satisfying $I_{j(i)} \subset g^{-1}(I_{j(i+1)})$.

A more concise version of Definition 2.1 can now be furnished by saying that

Definition 2.8. *A piecewise Fuchsian Markov map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is mateable if A is orbit equivalent to the Fuchsian group Γ_A generated by its pieces, and none of the periodic break-points of A are asymmetrically hyperbolic.*

We finally provide the lemma promised before Definition 2.1 guaranteeing that Γ_A is a lattice.

Lemma 2.9. [MM21, Lemma 2.18] *If A is mateable, then Γ_A is a lattice (or equivalently, Γ_A is a finitely generated Fuchsian group such that \mathbb{D}/Γ_A has finite hyperbolic area).*

For a complex polynomial P , its *filled Julia set* $\mathcal{K}(P)$ is the completely invariant set of all points whose forward orbits (under P) stay bounded. A polynomial is said to be *hyperbolic* if each of its critical points converges to an attracting cycle under forward iteration. The set of all hyperbolic polynomials (of a given degree) is open in the parameter space. A connected component of degree d hyperbolic polynomials is called a *hyperbolic component* in the parameter space of degree d polynomials. The hyperbolic component of degree d polynomials containing the map z^d is called the *principal hyperbolic component*, and is denoted by \mathcal{H}_d . The filled Julia set of each map in \mathcal{H}_d is a quasidisk, and the dynamics of such a map on its Julia set is quasimetrically conjugate to the action of z^d on \mathbb{S}^1 .

The next proposition says that the conditions of Definition 2.8 are sufficient to guarantee conformal mateability of piecewise Fuchsian Markov maps and polynomials in principal hyperbolic components.

For a Jordan curve \mathfrak{J} on the Riemann sphere, we denote its complementary components by \mathbf{D}^{in} and \mathbf{D}^{out} . The canonical extension $\hat{A} : \mathcal{D} \rightarrow \overline{\mathbb{D}}$ of a mateable map is said to be *conformally mateable* with a polynomial P in a principal hyperbolic component if there exist a holomorphic map F defined on a subset of $\hat{\mathbb{C}}$, a Jordan curve $\mathfrak{J} \subset \text{Dom}(F)$, and a pair of conformal maps $\phi^{\text{in}} : \overline{\mathbb{D}} \rightarrow \overline{\mathbf{D}^{\text{in}}}$ and $\phi^{\text{out}} : \mathcal{K}(P) \rightarrow \overline{\mathbf{D}^{\text{out}}}$ that conjugate \hat{A} and P (respectively) to F . The following is the first main result of [MM21].

Proposition 2.10 (Mateable maps are mateable). [MM21, Proposition 2.23] *Let $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a mateable map of degree d , and $P \in \mathcal{H}_d$. Then, the maps $\widehat{A} : \mathcal{D} \rightarrow \overline{\mathbb{D}}$ and $P : \mathcal{K}(P) \rightarrow \mathcal{K}(P)$ are conformally mateable.*

Remark 2.11. *A mateable map may have parabolic fixed points on \mathbb{S}^1 , and hence the topological conjugacy between z^d and A is not necessarily quasimetric. This renders classical quasiconformal tools (such as the ones used in the proof of Bers simultaneous uniformization theorem) insufficient for the purpose of conformally mating polynomials with mateable maps associated with Fuchsian groups. However, an appropriate class of ‘generalized quasiconformal maps’, called David homeomorphisms (maps with suitable Sobolev regularity satisfying a quantitative control on the area of the region where the dilatation blows up), allows one to perform the conformal mating construction. Two results that lie at the analytic heart of the proof of Proposition 2.10 are the David integrability theorem (this can be seen as a generalization of the measurable Riemann mapping theorem, see [Dav88], [AIM09, Theorem 20.6.2]) and a David extension theorem for certain circle homeomorphisms (which plays the role of the Ahlfors-Beurling extension theorem in the current setting, see [LMMN20, Theorem 4.9]). In fact, Item 5 in Definition 2.1 is required to guarantee the existence of a David extension of a circle homeomorphism conjugating z^d to a mateable map.*

3. BOWEN-SERIES MAPS OF FUCHSIAN PUNCTURED SPHERE GROUPS

3.1. Bowen-Series maps for General Fuchsian groups. Archetypal examples of piecewise Fuchsian Markov maps of the circle that are orbit equivalent to finitely generated Fuchsian groups are given by *Bowen-Series maps*. These first appeared in the work of Bowen and Series [Bow79, BS79].

A finitely generated Fuchsian group Γ (of the first kind) admits a fundamental domain $R (\subset \mathbb{D})$ that is a (possibly ideal) hyperbolic polygon. Denote the edges of R by $\{s_i\}_{i=1}^n$ (labeled in counter-clockwise order around the circle). Each edge s_i of R is identified with another edge s_j by a corresponding element $h(s_i) \in \Gamma$. The set $\{h(s_i)\}_{i=1}^n$ forms a generating set for Γ .

Let $C(s_i)$ be the Euclidean circular arc in \mathbb{D} containing s_i and meeting \mathbb{S}^1 orthogonally. Further, let \mathcal{N} be the net in \mathbb{D} consisting of all images of edges of R under elements of Γ . The fundamental domain R is said to satisfy the *even corners* property if $C(s_i)$ lies completely in \mathcal{N} , for $i \in \{1, \dots, n\}$.

Definition 3.1 (Bowen-Series map). *Suppose that a fundamental domain R of Γ satisfies the even corners property. Label (following [BS79]) the endpoints of $C(s_i)$ on \mathbb{S}^1 , P_i, Q_{i+1} (with $Q_{n+1} = Q_1$) with P_i occurring before Q_{i+1} in the counter-clockwise order. These points occur along the circle in the order $P_1, Q_1, P_2, Q_2, \dots, P_n, Q_n$ (see Figure 1). The Bowen-Series map $A_{\Gamma, BS} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of Γ (associated with the fundamental domain R) is defined piecewise as $A_{\Gamma, BS} \equiv h(s_i)$, on the sub-arc $[P_i, P_{i+1})$ of \mathbb{S}^1 (traversed in the counter-clockwise order).*

Proposition 3.2. [BS79, Lemma 2.4] *The map $A_{\Gamma, BS}$ is orbit equivalent to Γ , except (possibly) at finitely many points modulo the action of Γ .*

We shall simply denote $A_{\Gamma, BS}$ by A_Γ . The Bowen-Series maps corresponding to Fuchsian groups uniformizing positive genus surfaces (possibly with punctures) are discontinuous. Let us illustrate this with two examples. In the left diagram in

Figure 1, R is a fundamental domain for a (closed) genus two surface where the color coding determines the side-pairings. Note that $h(s_1)(x) = y$, and $h(s_2)(P_2) = Q_5$. Thus, for continuity of the corresponding Bowen-Series map at P_2 , the map $h(s_1)$ must send the geodesic ray from x to P_2 to the geodesic ray from y to Q_5 . But the former ray lies in the net \mathcal{N} (by the even corners property), while the latter ray passes through int R . This is absurd as R is a fundamental domain, proving discontinuity of the Bowen-Series map at P_2 . In the right diagram in the above figure, R is a fundamental domain for a once punctured torus where the sides are paired according to their colors. The side-pairing transformations $h(s_1)$ maps P_2 to P_3 , while $h(s_2)$ carries P_2 to P_1 . This causes discontinuity of the associated Bowen-Series map at P_2 .

Thus, to get continuous Bowen-Series maps, we need to restrict our attention to punctured sphere groups (possibly with orbifold points) equipped with special fundamental domains. In fact, it turns out that the Bowen-Series maps of Fuchsian punctured sphere groups constructed below are coverings of \mathbb{S}^1 with degree at least two.

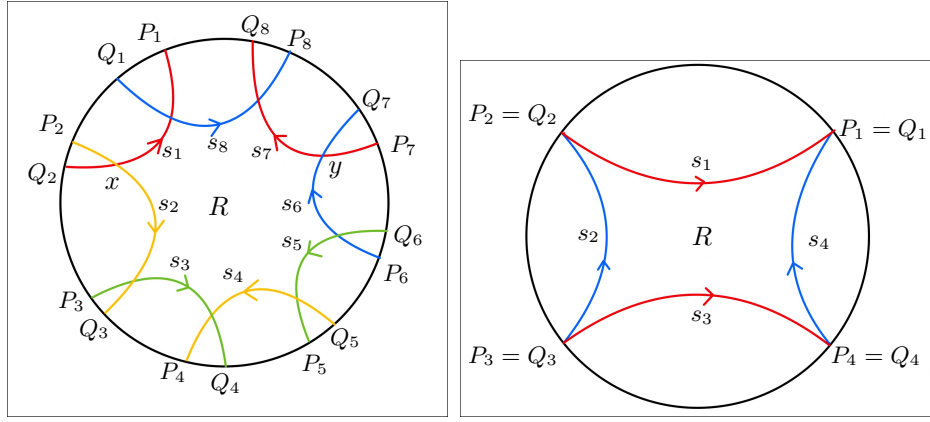


FIGURE 1. Bowen-Series Maps for surfaces of higher genus

3.2. Bowen-Series maps for punctured spheres. We mention at the outset that we always associate Bowen-Series maps with Fuchsian groups decorated with preferred fundamental domains and side-pairing transformations.

We first construct a specific Fuchsian group G_d uniformizing a $(d+1)$ -times punctured sphere equipped with a preferred fundamental domain. The group G_d (equipped with the preferred fundamental domain) will serve as a base-point in the Teichmüller space of $(d+1)$ -times punctured spheres. Since any (marked) group $\Gamma \in \text{Teich}(G_d)$ is conjugate to G_d via a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, the Bowen-Series map of Γ equipped with a marked fundamental domain determined by the quasiconformal conjugacy is easily seen to be a quasiconformal conjugate of the Bowen-Series map of G_d .

Fix $d \geq 2$. For $j \in \{1, \dots, d\}$, let C_j be the hyperbolic geodesic of \mathbb{D} connecting $p_j := e^{\pi i(j-1)/d}$ and $p_{j+1} := e^{\pi i j/d}$, and C_{-j} be the image of C_j under reflection in the real axis. We further denote the complex conjugate of p_j by p_{-j} , $j \in \{2, \dots, d\}$. Choose a Möbius automorphism g_j of \mathbb{D} defined as reflection in C_j followed by

complex conjugation. By construction, g_j carries C_j onto C_{-j} (cf. Figure 2). Note that for $j \in \{1, \dots, d-1\}$, the Möbius map $g_{j+1}g_j^{-1}$ is the composition of reflections in the circular arcs C_{j+1} and C_j . Since C_j and C_{j+1} touch at p_{j+1} , a straightforward computation (using the formula of circular reflections) shows that $g_{j+1}g_j^{-1}$ fixes p_{j+1} and has derivative equal to one at this fixed point. Therefore, $g_{j+1}g_j^{-1}$ is parabolic with its unique fixed point at p_{j+1} . Likewise, the maps g_1, g_d fix p_1, p_{d+1} (respectively), and have derivative equal to one there. Thus, g_1, g_d are also parabolic with their unique fixed points at p_1, p_{d+1} , respectively. Let

$$G_d := \langle g_1, \dots, g_d \rangle.$$

We note that G_d is a Fuchsian group with fundamental domain R having $C_1, \dots, C_d, C_{-d}, \dots, C_{-1}$ as its edges. Moreover, \mathbb{D}/G_d is a $(d+1)$ -times punctured sphere.

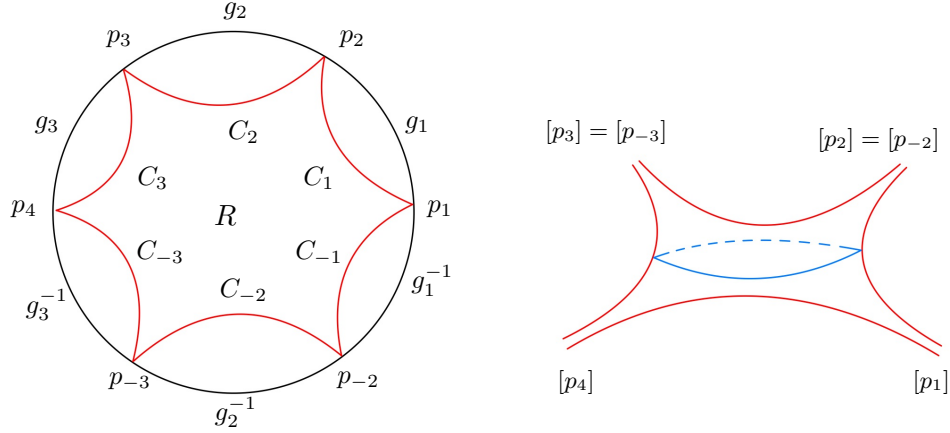


FIGURE 2. The preferred fundamental domain R of G_3 , which uniformizes a four times punctured sphere, is shown. The fundamental domain has all six vertices on \mathbb{S}^1 , and they cut the circle into six arcs. The corresponding Bowen-Series map acts on these arcs by the generators $g_j^{\pm 1}$ displayed next to them.

We refer the reader to Figure 2. For $j \in \{1, \dots, d\}$, let I_j denote the counter-clockwise sub-arc of \mathbb{S}^1 connecting p_j to p_{j+1} . Let I_{-j} denote the image of I_j under reflection in the real axis. Note that the Bowen-Series map A_{G_d} of G_d (equipped with the fundamental domain R) acts on $I_{\pm j}$ by $g_j^{\pm 1}$. The following two properties hold.

Proposition 3.3 (Properties of Bowen-Series maps of punctured spheres). [MM21, Proposition 3.3]

- (1) For $d \geq 2$, the Bowen-Series map A_{G_d} of G_d (equipped with the fundamental domain R) is a C^1 expansive degree $2d-1$ covering of \mathbb{S}^1 , and hence is topologically conjugate to $z^{2d-1}|_{\mathbb{S}^1}$. Moreover, A_{G_d} is a piecewise Fuchsian Markov map.
- (2) A_{G_d} is orbit equivalent to G_d on \mathbb{S}^1 .

We refer the reader to [MM21, Propositions 3.4, 3.5] for an orbifold variant of Proposition 3.3.

Remark 3.4. *In the above examples, the chosen fundamental domains of the groups coincide with those of the corresponding Bowen-Series maps.*

3.3. Mateability of Bowen-Series maps. We note now that Bowen-Series maps for punctured spheres fit into our mating framework. Recall that \mathcal{H}_k stands for the principal hyperbolic component in the space of degree k polynomials.

Theorem 3.5 (Fuchsian punctured sphere Bowen-Series maps are mateable). [MM21, Theorem 3.7] *Let $\Gamma \in \text{Teich}(G_d)$, and $P \in \mathcal{H}_{2d-1}$. Then, the map $\hat{A}_\Gamma : \mathcal{D}_{A_\Gamma} \rightarrow \mathbb{D}$ and $P : \mathcal{K}(P) \rightarrow \mathcal{K}(P)$ are conformally mateable.*

4. FOLDING AND HIGHER BOWEN-SERIES MAPS FOR FUCHSIAN GROUPS

The aim of this section is to describe a new class of piecewise Fuchsian Markov maps (following [MM21]), beyond the Bowen-Series examples that are mateable with polynomials. We start with classes of maps that we shall be considering in this section. Recall that the fundamental domain of a piecewise Fuchsian Markov map A is denoted by R . The set $\mathcal{D} = \mathbb{D} \setminus R$ is the canonical domain of definition of \hat{A} in \mathbb{D} , and a bi-infinite geodesic in R joining a pair of non-adjacent vertices of R is called a *diagonal* of R .

It is instructive to go through the following two definitions in conjunction with the two explicit examples of piecewise Fuchsian Markov maps given in Subsection 4.1 (cf. Figure 3).

Definition 4.1 (Completely folding map). *A piecewise Fuchsian Markov map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is said to be a completely folding map if there exist finitely many diagonals $\delta_1, \dots, \delta_l$ of R such that the following hold:*

- (1) *For every edge α of R , $\hat{A}(\alpha)$ is one of the diagonals $\delta_1, \dots, \delta_l$.*
- (2) *The ideal endpoints p_i, q_i of δ_i are fixed points of A for all i ; i.e., $A(p_i) = p_i$ and $A(q_i) = q_i$ whenever p_i, q_i are ideal endpoints of δ_i .*
- (3) *For p_i, q_i as above, $q_i = p_{i+1}$.*
- (4) *$\delta_i \cap \delta_j = \emptyset$ for $i \neq j$. Further, $p_1 \neq q_l$; i.e., the sequence of diagonals δ_i forms a chain of non-intersecting bi-infinite geodesics such that, after adjoining the ideal endpoints, one obtains a ‘piecewise geodesic’ embedding of the closed interval $[0, 1]$ in the closed disk \mathbb{D} .*

Definition 4.2 (Higher degree map without folding). [MM21, Definition 4.2] *A piecewise Fuchsian Markov map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is said to have a diagonal fold if there exist consecutive edges α_1, α_2 of ∂R and a diagonal δ of R such that $\hat{A}(\alpha_i) = \delta$ for $i = 1, 2$. Note that if a_1, a_2 (resp. a_2, a_3) are the endpoints of α_1 (resp. α_2) and p, q are the endpoints of δ , then $A(a_1) = p = A(a_3)$ and $A(a_2) = q$ by continuity of A on \mathbb{S}^1 .*

A piecewise Fuchsian Markov map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is said to be a higher degree map without folding if

- (1) *there exists an (open) ideal polygon $D \subset R$ such that all the edges $\delta_1, \dots, \delta_l$ of D are (necessarily non-intersecting) diagonals of R . We assume further that $\delta_1, \dots, \delta_l$ are cyclically ordered along ∂D . We shall call D the inner domain of A .*
- (2) *If p is an ideal vertex of D , then $A(p) = p$.*
- (3) *For every edge α of R , $\hat{A}(\alpha)$ is one of the diagonals $\delta_1, \dots, \delta_l$.*
- (4) *A has no diagonal folds.*

Cyclically ordering the edges $\alpha_1, \dots, \alpha_k$ of R , it follows from Definition 4.2, that under a higher degree map without folding A , consecutive edges α_i, α_{i+1} of R go to consecutive edges of D . Note however that an counter-clockwise cyclic ordering of edges of R may be taken to a clockwise cyclic ordering of edges of D under A . In any case we have a continuous map $\hat{A} : \partial R \rightarrow \partial D$. Adjoining the ideal endpoints of R and D , \hat{A} has a well-defined degree d . Further, each edge of D has exactly $|d|$ pre-images under \hat{A} since there are no folds. Also, since each δ_i is a diagonal of R , we have $|d| > 1$. We call $|d|$ the *polygonal degree* of A . (Since $|d| > 1$, we call A a higher degree map without folding.)

Remark 4.3. *A piecewise Fuchsian Markov map with a diagonal fold need not be a completely folding map; see Subsection 4.4.2 for an example.*

4.1. A completely folding map and a higher degree map without folding for the sphere with three punctures. We now give two simple examples: a completely folding map and a higher degree map without folding which are orbit equivalent to Γ_0 corresponding to a sphere with three punctures. Then Γ_0 is isomorphic to F_2 , the fundamental group of $S_{0,3}$ (see Figure 3). We will denote a bi-infinite hyperbolic geodesic in \mathbb{D} having its (ideal) endpoints at $a, b \in \mathbb{S}^1$ by \overline{ab} .

Fix a (closed) fundamental domain W of Γ_0 , given by an ideal quadrilateral with its ideal vertices at the fourth roots of unity (the quadrilateral 1236 in the figure). The generators of Γ_0 are given by h, g , where h takes the edge $\overline{12}$ to $\overline{16}$, g takes $\overline{32}$ to $\overline{36}$, and $g^{-1}h$ is parabolic. The combinatorics in this case is relatively simple and the case-by-case analysis for proving orbit equivalence in Proposition 4.4 is easy.

4.1.1. A completely folding map for $S_{0,3}$. We shall first construct a completely folding map, and then modify the construction slightly to obtain a higher degree map without folding. We define the fundamental domain R of the completely folding map $A_{\Gamma_0, \text{cfm}}$ (to be constructed) as

$$R = \text{int}(W \cup h.W \cup g.W).$$

Thus, R is the interior of the octagon 12345678 in Figure 3. We define the pieces of $A_{\Gamma_0, \text{cfm}}$ as follows. In the list below, an arc will be indicated by \widehat{ij} where the pair of numbers i, j are its endpoints, provided there are no other break-points of $A_{\Gamma_0, \text{cfm}}$ in the arc. Otherwise, we will denote the arc by all the break-points it contains. Further the label of the arrow will denote the piece of $A_{\Gamma_0, \text{cfm}}$ that takes the domain arc to the range arc.

- $\widehat{12} \xrightarrow{h} \widehat{123456}$
- $\widehat{23} \xrightarrow{g} \widehat{678123}$
- $\widehat{34} \xrightarrow{g^{-1}} \widehat{3456}$
- $\widehat{45} \xrightarrow{g^{-1}} \widehat{6781}$
- $\widehat{56} \xrightarrow{h \circ g^{-1}} \widehat{123456}$ (we use the convention that \circ indicates composition of maps)
- $\widehat{67} \xrightarrow{g \circ h^{-1}} \widehat{678123}$
- $\widehat{78} \xrightarrow{h^{-1}} \widehat{3456}$
- $\widehat{81} \xrightarrow{h^{-1}} \widehat{6781}$

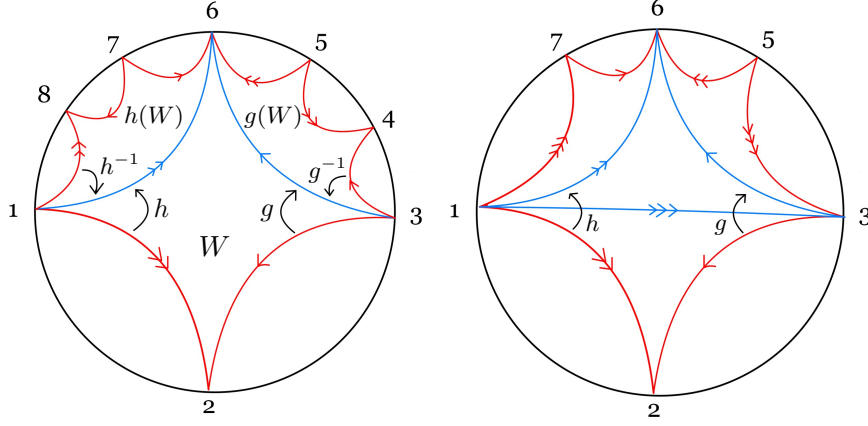


FIGURE 3. Fundamental domains for a completely folding map and a higher degree map without folding: 3 punctures

4.1.2. *A higher degree map without folding for $S_{0,3}$.* There is a higher degree map without folding naturally associated with the completely folding map above. Note that the completely folding map $A_{\Gamma_0, \text{cfm}}$ is not minimal. The pieces of $A_{\Gamma_0, \text{cfm}}$ for the contiguous arcs $\widehat{34}$ and $\widehat{45}$ is g^{-1} . Similarly, the pieces of $A_{\Gamma_0, \text{cfm}}$ for the contiguous arcs $\widehat{78}$ and $\widehat{81}$ is h^{-1} . We define $A_{\Gamma_0, \text{hBS}} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ to be the minimal piecewise Fuchsian Markov map agreeing with $A_{\Gamma_0, \text{cfm}}$ everywhere (here ‘hBS’ is an acronym for ‘higher Bowen-Series’, the reason behind this terminology will be explained in Remark 4.8). Although $A_{\Gamma_0, \text{cfm}}$ and $A_{\Gamma_0, \text{hBS}}$ agree pointwise, they are formally different piecewise Fuchsian Markov maps as $A_{\Gamma_0, \text{cfm}}$ has more pieces (some of which are repeated). Consequently, their canonical extensions $\widehat{A}_{\Gamma_0, \text{cfm}}$ and $\widehat{A}_{\Gamma_0, \text{hBS}}$ have different domains of definition.

It is easy to see that $\widehat{A}_{\Gamma_0, \text{hBS}}$ is a higher degree map without folding. The fundamental domain R' for $\widehat{A}_{\Gamma_0, \text{hBS}}$ is the interior of the ideal hexagon 123567 contained in R . The inner domain of $\widehat{A}_{\Gamma_0, \text{hBS}}$ is given by the ideal triangle 136 (see Definition 4.2). The pieces of $A_{\Gamma_0, \text{hBS}}$ are given by the following list (note that 4, 8 are not break-points of $A_{\Gamma_0, \text{hBS}}$ and hence we omit them from the notation):

- $\widehat{12} \xrightarrow{h} \widehat{12356}$
- $\widehat{23} \xrightarrow{g} \widehat{67123}$
- $\widehat{35} \xrightarrow{g^{-1}} \widehat{35671}$
- $\widehat{56} \xrightarrow{h \circ g^{-1}} \widehat{12356}$
- $\widehat{67} \xrightarrow{g \circ h^{-1}} \widehat{67123}$
- $\widehat{71} \xrightarrow{h^{-1}} \widehat{35671}$

The polygonal degree of $\widehat{A}_{\Gamma_0, \text{hBS}}$ is 2.

4.1.3. Orbit equivalence.

Proposition 4.4. *Let $A_{\Gamma_0, \text{cfm}}$, $A_{\Gamma_0, \text{hBS}}$ be as above. Then $A_{\Gamma_0, \text{cfm}}$, $A_{\Gamma_0, \text{hBS}}$ are orbit equivalent to Γ_0 .*

Proof. Since $A_{\Gamma_0, \text{cfm}}$ and $A_{\Gamma_0, \text{hBS}}$ agree as maps on \mathbb{S}^1 , it suffices to check this for $A_{\Gamma_0, \text{hBS}}$. It is easy to see that $A_{\Gamma_0, \text{hBS}}$ -grand orbits are contained in Γ_0 -orbits simply because the pieces of $A_{\Gamma_0, \text{hBS}}$ are elements of Γ_0 . It therefore suffices to show that if x, y are in the same Γ_0 -orbit then they lie in the same $A_{\Gamma_0, \text{hBS}}$ -grand orbit. It suffices to check this for the generators g, h and their inverses.

Let $y = g.x$. We want to show that x, y lie in the same grand orbit under $A_{\Gamma_0, \text{hBS}}$.

Case 1: $y \in \widehat{32176}$. Then $x \in \widehat{32}$ and the piece of $A_{\Gamma_0, \text{hBS}}$ restricted to $\widehat{32}$ is g . Hence $y = A_{\Gamma_0, \text{hBS}}(x)$.

Case 2: $y \in \widehat{345}$. The branch of $A_{\Gamma_0, \text{hBS}}$ restricted to $\widehat{345}$ is g^{-1} . Rewriting $y = g(x)$ as $g^{-1}(y) = x$, we see that $A_{\Gamma_0, \text{hBS}}(y) = x$.

Case 3: $y \in \widehat{56}$. Then $x \in \widehat{12}$. Note that the branch of $A_{\Gamma_0, \text{hBS}}$ restricted to $\widehat{56}$ is $h \circ g^{-1}$, and the branch of $A_{\Gamma_0, \text{hBS}}$ restricted to $\widehat{12}$ is h . Hence,

$$A_{\Gamma_0, \text{hBS}}(y) = h(g^{-1}(y)) = h(g^{-1}(g(x))) = h(x) = A_{\Gamma_0, \text{hBS}}(x).$$

This shows that x and y are grand orbit equivalent under $A_{\Gamma_0, \text{hBS}}$.

Next, if $y = g^{-1}.x$, then $x = g.y$ and exchanging the roles of x, y in the previous paragraph shows that x, y are grand orbit equivalent under $A_{\Gamma_0, \text{hBS}}$. Finally, by the symmetry of the setup, the same argument applies to h, h^{-1} . \square

As a circle covering, the degree of $A_{\Gamma_0, \text{hBS}}$ is equal to 4. This can be easily seen from the actions of the pieces of $A_{\Gamma_0, \text{hBS}}$ (along with their range) listed in Section 4.1.2. Thus, we have now exhibited two different examples of piecewise Fuchsian Markov maps that are orbit equivalent to a thrice punctured sphere Fuchsian group; namely, the Bowen-Series map (of degree 3) and the higher degree map without folding $A_{\Gamma_0, \text{hBS}}$ defined above (of degree 4). Moreover, the polygonal degree of $\hat{A}_{\Gamma_0, \text{hBS}}$ is 2, while the Bowen-Series map induces a self-homeomorphism on the boundary of its fundamental domain.

4.2. Folding and higher degree maps for general punctured spheres. We follow the scheme of Section 4.1 above and generalize it to the case of $S_{0,k}$ -a sphere with k punctures, $k > 3$. We shall use Figure 4 below as an illustration for the general case. Fix a (closed) fundamental domain of $\Gamma_0 = G_{k-1}$ (see Subsection 3.2 for the definition of G_{k-1}), given by an ideal $(2k-2)$ -gon W (the figure illustrates the $k = 4$ case). For definiteness, let us assume that the ideal vertices of W are the $(2k-2)$ -th roots of unity. To make the book-keeping a little easier, we modify the notation as follows.

- (1) The vertices of W on the bottom semi-circle are numbered $1 = 1_-, 2_- \dots, k_- = k$ in counter-clockwise order.
- (2) The vertices of W on the top semi-circle are numbered $1, 2, \dots, k$ in clockwise order.
- (3) Between vertices $i, i+1$ (and including $i, i+1$) on the top semi-circle, there are $2k-2$ vertices given by the vertices of $g_i.W$ (noting that $g_i.W \cap W$ equals the bi-infinite geodesic $\overline{i(i+1)}$). We label the $2k-4$ vertices strictly between $i, i+1$ as $\{i, 2\}, \{i, 3\}, \dots, \{i, 2k-3\}$ in clockwise order.

The generators of Γ_0 are given by g_1, \dots, g_{k-1} , where g_i takes the edge $\overline{i_-(i+1)_-}$ to the bi-infinite geodesic $\overline{i(i+1)}$.

4.2.1. A completely folding map for $S_{0,k}$. Define R as

$$R = \text{int} \left(W \cup \bigcup_{i=1, \dots, k-1} g_i \cdot W \right),$$

so that $\overline{i(i+1)}$ are diagonals of R .

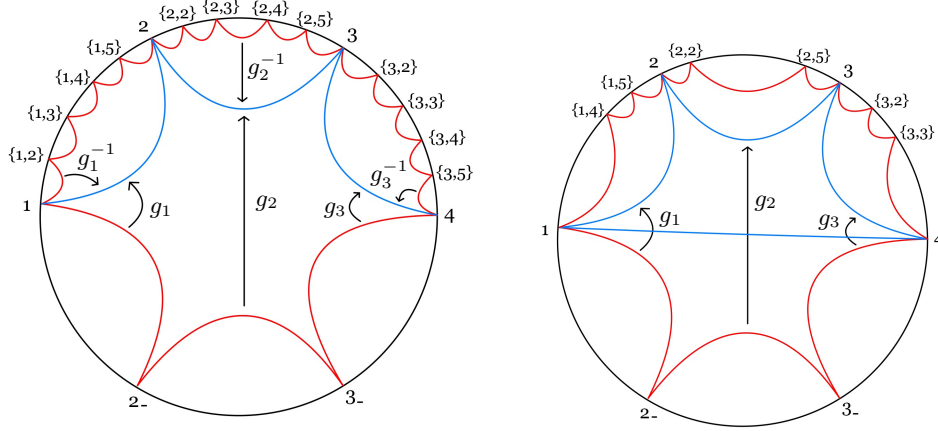


FIGURE 4. Fundamental domains for a completely folding map and a higher degree map without folding: 4 punctures

As in Section 4.1, we define $A_{\Gamma_0, \text{cfm}}$ in terms of its pieces as follows. Recall that \widehat{ij} stands for an arc with its endpoints at the break-points i, j such that there are no other break-points of $A_{\Gamma_0, \text{cfm}}$ in the arc.

- On the arc $\overline{i_-(i+1)_-}$, define $A_{\Gamma_0, \text{cfm}}$ to be g_i for $i = 1, \dots, k-1$. Then $A_{\Gamma_0, \text{cfm}}(\overline{i_-(i+1)_-})$ equals the complement of (the interior of) the arc $\widehat{i(i+1)}$ in \mathbb{S}^1 .
- For every $i = 1, \dots, k-1$, and on each of the $k-1$ short arcs $\widehat{\{i, j\}\{i, j+1\}}$ for $i \leq j \leq i+k-2$ between $i, i+1$, define $A_{\Gamma_0, \text{cfm}}$ to be g_i^{-1} . Then $A_{\Gamma_0, \text{cfm}}(\bigcup_{j=i}^{i+k-2} \widehat{\{i, j\}\{i, j+1\}})$ equals the upper semi-circle between 1 and k . (Here, for notational convenience, we identify $\{i, 1\}$ with i and $\{i, i+2k-2\}$ with $i+1$.) Also, for $i \leq j \leq i+k-2$, $A_{\Gamma_0, \text{cfm}}$ maps the clockwise arc from $\{i, j\}$ to $\{i, j+1\}$ onto the clockwise arc from j to $j+1$. We refer to the clockwise arcs from $\{i, j\}$ to $\{i, j+1\}$ (for $i \leq j \leq i+k-2$) as *short folding arcs under $A_{\Gamma_0, \text{cfm}}$* .
- For $i \in \{2, \dots, k-1\}$ and $1 \leq j \leq i-1$, set $j = i-s$, so that $1 \leq s \leq i-1$. We define $A_{\Gamma_0, \text{cfm}}$ to be $g_s \circ g_i^{-1}$ on $\widehat{\{i, j\}\{i, j+1\}}$. Thus, for $j \leq i-1$, $A(\widehat{\{i, j\}\{i, j+1\}})$ equals the counter-clockwise (long) arc from s to $s+1$.
- For $i \in \{1, \dots, k-2\}$ and $i+k-1 \leq j \leq 2k-3$, let $j = i+k-1+t$, so that $0 \leq t \leq k-2-i$. We define $A_{\Gamma_0, \text{cfm}}$ to be $g_{k-1-t} \circ g_i^{-1}$ on $\widehat{\{i, j\}\{i, j+1\}}$. Thus, for $i+k-1 \leq j \leq 2k-3$, $A(\widehat{\{i, j\}\{i, j+1\}})$ equals the counter-clockwise (long) arc from $k-1-t$ to $k-t$.

We refer to the clockwise arcs from $\{i, j\}$ to $\{i, j + 1\}$ (for $j \leq i - 1$ or $i + k - 1 \leq j$) as *long folding arcs under A* .

- Note that $A_{\Gamma_0, \text{cfm}}(i) = i$ for all $i = 1, \dots, k$.

It is easy to see from the above definition that $A_{\Gamma_0, \text{cfm}} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a completely folding map. As any (marked) group $\Gamma \in \text{Teich}(\Gamma_0)$ is conjugate to Γ_0 via a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ that preserves \mathbb{S}^1, \mathbb{D} and respects the markings, we define the associated completely folding map $A_{\Gamma, \text{cfm}}$ to be the conjugate of $A_{\Gamma_0, \text{cfm}}$ under such a quasiconformal homeomorphism.

Remark 4.5. *It is not hard to cook up other examples of completely folding maps. However, we do not know of any other completely folding map that is orbit equivalent to the Fuchsian group generated by its pieces.*

4.2.2. A higher degree map without folding for $S_{0,k}$. Again, as in Section 4.1, define $A_{\Gamma_0, \text{hBS}}$ to be the minimal piecewise Fuchsian Markov map coinciding with $A_{\Gamma_0, \text{cfm}}$ on \mathbb{S}^1 . Denote the canonical extension of $A_{\Gamma_0, \text{hBS}}$ by $\widehat{A}_{\Gamma_0, \text{hBS}}$, its canonical domain of definition in $\overline{\mathbb{D}}$ by $\mathcal{D}_{\Gamma_0, \text{hBS}}$, and the fundamental domain of $\widehat{A}_{\Gamma_0, \text{hBS}}$ by $R_{\Gamma_0, \text{hBS}}$. Further, let D be the open ideal polygon bounded by the bi-infinite geodesics $\overline{12}, \overline{23}, \dots, \overline{(k-1)k}, \overline{k1}$. Evidently, all the edges of D are (non-intersecting) diagonals of $R_{\Gamma_0, \text{hBS}}$, each ideal vertex of D is fixed by $A_{\Gamma_0, \text{hBS}}$, each edge of $R_{\Gamma_0, \text{hBS}}$ is mapped by $\widehat{A}_{\Gamma_0, \text{hBS}}$ to an edge of D , and $\widehat{A}_{\Gamma_0, \text{hBS}}$ has no diagonal folds. Therefore, $\widehat{A}_{\Gamma_0, \text{hBS}}$ is a higher degree map without folding having D as its inner domain.

Definition 4.6 (Higher Bowen-Series map). *We call the piecewise Fuchsian Markov map $A_{\Gamma_0, \text{hBS}}$ the higher Bowen-Series map of Γ_0 (associated with the fundamental domain W). For any (marked) group $\Gamma \in \text{Teich}(\Gamma_0)$, we define the higher Bowen-Series map of (the marked group) Γ to be the conjugate of $A_{\Gamma_0, \text{hBS}}$ under a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ that conjugates Γ_0 to Γ (and respects the marking), and denote it by $A_{\Gamma, \text{hBS}}$.*

Clearly, the higher Bowen-Series map of each $\Gamma \in \text{Teich}(\Gamma_0)$ is a higher degree map without folding. We refer the reader to [MM21, Proposition 5.2] for a characterization of higher Bowen-Series maps among all higher degree maps without folding.

4.2.3. Connections between Bowen-Series and higher Bowen-Series maps. The next two propositions are about the relationship between Bowen-Series maps and higher Bowen-Series maps (for $\Gamma \in \text{Teich}(\Gamma_0)$). In fact, Proposition 4.7 will give an alternative, more direct construction of the higher Bowen-Series map of Γ in terms of the Bowen-Series maps of Γ associated with various overlapping fundamental domains.

Proposition 4.7 (Characterizing higher Bowen-Series maps as piecewise Bowen-Series maps). [MM21, Proposition 4.5] *Let W be a (closed) fundamental domain for a Fuchsian group $\Gamma \in \text{Teich}(\Gamma_0)$ (uniformizing a k -times punctured sphere) which is an ideal $(2k - 2)$ -gon. We label the ideal vertices of W as $1 = 1_-, 2_-, \dots, (k-1)_-, k_- = k, k-1, \dots, 2$ in counterclockwise order, and assume that the side-pairing transformations of W (generating Γ) are given by g_1, \dots, g_{k-1} , where g_i takes the edge $\overline{i_-(i+1)_-}$ to the edge $\overline{i(i+1)}$.*

Further, let D be the interior of the ideal polygon bounded by the bi-infinite geodesics $\overline{12}, \overline{23}, \dots, \overline{(k-1)k}, \overline{k1}$, and P the interior of the ideal polygon bounded

by the bi-infinite geodesics $\overline{1-2-}, \overline{2-3-}, \dots, \overline{(k-1)-k-}, \overline{k-1-}$. Then the following hold.

- (1) $W = \overline{D} \cup \overline{P}$, and for each $j \in \{1, \dots, k-1\}$, $\overline{D} \cup \overline{g_j(P)}$ is a (closed) fundamental domain for Γ .
- (2) On the clockwise arc from j to $j+1$, the higher Bowen-Series map $A_{\Gamma, \text{hBS}}$ equals the Bowen-Series map of Γ associated with the (closed) fundamental domain $\overline{D} \cup \overline{g_j(P)}$ ($j \in \{1, \dots, k-1\}$), and on the counterclockwise arc from 1 to k , $A_{\Gamma, \text{hBS}}$ equals the Bowen-Series map of Γ associated with the fundamental domain $W = \overline{D} \cup \overline{P}$.

Conversely, a map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ defined as in condition (2) above is a higher Bowen-Series map orbit equivalent to Γ , and the fundamental domain of A is given by $R = \text{int} \left(W \cup \bigcup_{i=1, \dots, k-1} g_i \cdot W \right)$.

Remark 4.8. The preceding description of $A_{\Gamma, \text{hBS}}$ shows that $A_{\Gamma, \text{hBS}}$ is made up of Bowen-Series maps corresponding to various (overlapping) fundamental domains of Γ . This justifies the terminology ‘higher Bowen-Series maps’.

Higher Bowen-Series maps also arise as second iterates of suitable Bowen-Series maps.

Proposition 4.9 (Higher Bowen-Series as second iterate of Bowen-Series). [MM21, Corollary 5.6] *Let $d \geq 2$.*

- (1) *For $\Gamma \in \text{Teich}(G_d) = \text{Teich}(S_{0,d+1})$ (respectively, $\Gamma \in \text{Teich}(G_{d,2})$), we have $A_{\Gamma, \text{BS}}^2 = A_{\Gamma', \text{hBS}}$, where Γ' is an index-two subgroup of Γ with $\mathbb{D}/\Gamma' \cong S_{0,2d}$.*
- (2) *For $\Gamma \in \text{Teich}(G_{d,1})$, we have $A_{\Gamma, \text{BS}}^2 = A_{\Gamma', \text{hBS}}$, where Γ' is an index-two subgroup of Γ with $\mathbb{D}/\Gamma' \cong S_{0,2d-1}$.*

In all cases, the second iterate of the Bowen-Series map of Γ is orbit equivalent to an index-two subgroup of Γ .

The degree of the higher Bowen-Series map as a self-covering of \mathbb{S}^1 is $(\chi - 1)^2$, where $\chi = 2 - k$ is the Euler characteristic of $S_{0,k}$.

The last statement may be found in [MM21, Section 4.3.2]. We do not know if higher iterates of Bowen-Series maps produce further examples of mateable maps (see Question 4.16 below). With careful combinatorial book-keeping, the arguments of the proof of Proposition 4.4 can be adapted for the general case.

Proposition 4.10 (Orbit equivalence). [MM21, Proposition 4.7] *Let $\Gamma \in \text{Teich}(\Gamma_0)$, and $A_{\Gamma, \text{cfm}}$, $A_{\Gamma, \text{hBS}}$ be as above. Then $A_{\Gamma, \text{cfm}}$, $A_{\Gamma, \text{hBS}}$ are orbit equivalent to Γ .*

4.3. Consequences. We now discuss some consequences.

4.3.1. Interpolating between completely folding maps and higher degree maps without folding. The completely folding map $A_{\Gamma, \text{cfm}}$ and the higher Bowen-Series map $A_{\Gamma, \text{hBS}}$ described in Sections 4.1 and 4.2 agree on \mathbb{S}^1 . We denote the interior of the polygon in Section 4.2 with vertices $1, \dots, k$ by D . Note that \overline{D} is ‘half’ the (closed) fundamental domain W in the sense that doubling \overline{D} along the bi-infinite geodesic $\overline{1k}$ gives W . Choose $1 = i_1 < i_2 < \dots < i_{l+1} = k$ to be a selection of vertices in clockwise cyclic order along the upper semi-circle. Let $\cup_{1 \leq j \leq l} (i_j, i_{j+1}) = \mathcal{L}$ denote

a finite union of edges and diagonals of W contained in \overline{D} . Let W_0 denote the part of W contained above \mathcal{L} and let $W_{\mathcal{L}} = W_0 \cup \mathcal{L}$. Set

$$R_{\mathcal{L}} = \text{int} \left(W_{\mathcal{L}} \cup \bigcup_{i=1, \dots, k} g_i \cdot W_{\mathcal{L}} \right).$$

Then $R_{\mathcal{L}}$ is the fundamental domain of the piecewise Fuchsian Markov map $A_{\mathcal{L}}$ whose canonical extension $\widehat{A}_{\mathcal{L}}$ has domain $\mathcal{D}_{\mathcal{L}} = \mathbb{D} \setminus R_{\mathcal{L}}$.

Note that, for all \mathcal{L} , the map $A_{\mathcal{L}}$ equals $A_{\Gamma, \text{cfm}}$ on \mathbb{S}^1 . The map $A_{\Gamma, \text{hBS}}$ is the unique minimal representative and corresponds to the case $1 = i_1 < i_2 = k$. The map $A_{\Gamma, \text{cfm}}$ lies at the other end of the spectrum, with $l + 1 = k$, and $i_j = j$ for $j = 1, \dots, k$. The maps $A_{\mathcal{L}}$ are non-minimal representatives whenever $l > 1$.

4.3.2. Mateability of completely folding maps and higher Bowen-Series maps. We now record the fact that higher Bowen-Series maps satisfy the conditions of Definition 2.1, and hence can be conformally mated with hyperbolic complex polynomials (of appropriate degree) with Jordan curve Julia sets.

Theorem 4.11 (Fuchsian higher Bowen-Series maps are mateable). [MM21, Theorem 4.8] *Let $\Gamma \in \text{Teich}(\Gamma_0)$, and $P \in \mathcal{H}_{(k-1)^2}$ (where \mathcal{H}_d stands for the principal hyperbolic component in the space of degree d polynomials). Then, $\widehat{A}_{\Gamma, \text{hBS}} : \mathcal{D}_{\Gamma, \text{hBS}} \rightarrow \overline{\mathbb{D}}$ (respectively, $\widehat{A}_{\Gamma, \text{cfm}} : \mathcal{D}_{\Gamma, \text{cfm}} \rightarrow \overline{\mathbb{D}}$) and $P : \mathcal{K}(P) \rightarrow \mathcal{K}(P)$ are conformally mateable.*

In light of Proposition 2.10, we make the following definition.

Definition 4.12 (Moduli space of matings). *The moduli space of matings between a topological surface Σ and complex polynomials in principal hyperbolic components consists of triples (Γ, A_{Γ}, P) , where*

- (1) Γ is a Fuchsian group uniformizing Σ ,
- (2) A_{Γ} is a minimal mateable map orbit equivalent to Γ on \mathbb{S}^1 , and
- (3) P is a polynomial in a principal hyperbolic component with $\deg(P) = \deg(A_{\Gamma} : \mathbb{S}^1 \rightarrow \mathbb{S}^1)$.

An immediate implication of Theorems 3.5 and 4.11 is that the moduli space of matings between the topological surface $S_{0,k}$ ($k \geq 3$) and complex polynomials in principal hyperbolic components is disconnected. Specifically, it contains at least two components corresponding to

- Bowen-Series maps associated to groups in $\text{Teich}(S_{0,k})$ and polynomials in \mathcal{H}_{2k-3} , and
- higher Bowen-Series maps associated to groups in $\text{Teich}(S_{0,k})$ and polynomials in $\mathcal{H}_{(k-1)^2}$.

We refer the readers to [MM21, §6.4] for further details.

Yet another application of orbit equivalence between higher Bowen-Series maps and Fuchsian punctured sphere groups is the failure of orbit equivalence rigidity for Fuchsian groups (see [FW03] for general background on orbit equivalence rigidity and positive results, and [MM21, §8] for a precise statement of its failure in the Fuchsian case).

4.4. Two non-examples. In this subsection, we will consider two modifications of higher Bowen-Series maps and show that the resulting piecewise Fuchsian Markov maps are not orbit equivalent to the groups generated by their pieces.

4.4.1. A non-example without folding. The following description of the higher Bowen-Series map $A_{\Gamma_0, \text{hBS}}$ on \mathbb{S}^1 is straightforward to check from its construction (see Subsection 4.2):

$$A_{\Gamma_0, \text{hBS}} = \begin{cases} A_{\Gamma_0, \text{BS}}, & \text{on } \left(\bigcup_{i=1}^{k-1} \overline{i_-(i+1)_-} \right) \cup \left(\bigcup_{i=1}^{k-1} \bigcup_{j=i}^{i+k-2} \overline{\{i, j\}\{i, j+1\}} \right), \\ A_{\Gamma_0, \text{BS}}^2, & \text{otherwise,} \end{cases}$$

where $A_{\Gamma_0, \text{BS}}$ denotes the Bowen-Series map of Γ_0 associated with the fundamental domain W .

In fact, the agreement of $A_{\Gamma_0, \text{hBS}}$ and $A_{\Gamma_0, \text{BS}}$ on the arcs $\overline{\{i, j\}\{i, j+1\}}$ ($i \in \{1, \dots, k-1\}, j \in \{i, \dots, i+k-2\}$) played an important role in the proof of orbit equivalence of Γ_0 and $A_{\Gamma_0, \text{hBS}}$ (see Proposition 4.4). However, if one replaces $A_{\Gamma_0, \text{BS}}$ by $A_{\Gamma_0, \text{BS}}^2$ on these arcs as well, the resulting minimal piecewise Fuchsian Markov map

$$B := \begin{cases} A_{\Gamma_0, \text{BS}} & \text{on } \mathbb{S}^1 \cap \{z : \text{Im}(z) \leq 0\}, \\ A_{\Gamma_0, \text{BS}}^2 & \text{on } \mathbb{S}^1 \cap \{z : \text{Im}(z) \geq 0\}, \end{cases}$$

is *not* orbit equivalent to Γ_0 .

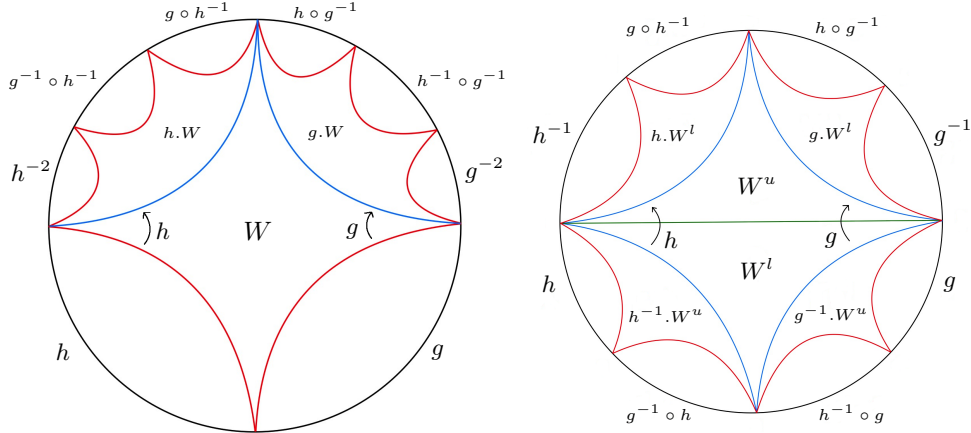


FIGURE 5. The two piecewise Fuchsian Markov maps B (left) and C (right) are depicted. They are not orbit equivalent to the Fuchsian groups generated by their pieces.

Proposition 4.13. [MM21, Proposition 4.9] *The map $B : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is not orbit equivalent to the Fuchsian group Γ_0 generated by its pieces.*

Remark 4.14. *The map B has no diagonal fold, but it is not a higher degree map without folding. This is because some edges of the boundary of the fundamental domain of B are not mapped to diagonals, and hence Condition (3) of Definition 4.2 is violated (see Figure 5 (left)).*

4.4.2. *A non-example with folding.* Yet another example of a piecewise Fuchsian Markov map that is not orbit equivalent to the Fuchsian group generated by its pieces is given by the following symmetric version of higher Bowen-Series maps. For simplicity, we illustrate the thrice punctured sphere case.

Consider the Fuchsian group Γ_0 of Section 4.1, the (closed) fundamental domain W with vertices at the fourth roots of unity, and the generators g, h that pair the sides of W (as in Figure 3). We set

$$W^u := W \cap \{\operatorname{Im}(z) \geq 0\}, \quad W^l := W \cap \{\operatorname{Im}(z) \leq 0\}.$$

The piecewise Fuchsian Markov map $C : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ depicted in Figure 5 (right) has degree 5. It has diagonal folds in the sense of Definition 4.2, but it is not a completely folding map since Condition (4) of Definition 4.1 fails. Moreover,

Proposition 4.15. *The piecewise Fuchsian Markov map $C : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of Figure 5 (right) is not orbit equivalent to the Fuchsian group Γ_0 generated by its pieces.*

Proof. Observe that $g(-i) = i$, and thus the points $\pm i$ lie in the same Γ_0 -orbit. But both these points are fixed by C , and hence they cannot lie in the same grand orbit of C . \square

The proofs of orbit equivalence for Bowen-Series and higher Bowen-Series maps only involve looking at the first iterates of the maps. On the other hand, the orbit equivalence property is ruled out for the above two piecewise Fuchsian Markov maps simply by furnishing suitable fixed points of the maps. In general, we ask the following question.

Question 4.16. *Is there a general recipe to test whether a piecewise Fuchsian Markov map is orbit equivalent to the Fuchsian group generated by its pieces?*

5. INVARIANT LAMINATIONS AND BERS BOUNDARY GROUPS

The existence of mateable maps orbit equivalent to Fuchsian punctured sphere groups leads one to the hunt for groups on boundaries of Teichmüller spaces (of punctured spheres) that can be conformally mated with complex polynomials. Since Fuchsian realizations of Teichmüller spaces are non-compact, the aforementioned pursuit ought to be carried out on boundaries of Bers slices of Fuchsian punctured sphere groups (note that such a realization of the Teichmüller space is precompact in a suitable topology). As in the Fuchsian case, the first challenge one encounters in this program is to come up with the correct notion of ‘mateable maps’ for Bers boundary groups.

Let us fix a Fuchsian punctured sphere group Γ_0 equipped with a (higher) Bowen-Series map A_{Γ_0} . We denote the Bers slice of Γ_0 by $\mathcal{B}(\Gamma_0)$. The map A_{Γ_0} defines, for each (marked) group Γ' in $\mathcal{B}(\Gamma_0)$, a piecewise Möbius Markov covering map $A_{\Gamma'}$ (via quasiconformal conjugation) of the limit set $\Lambda(\Gamma')$ such that $A_{\Gamma'}$ is orbit equivalent to Γ' on $\Lambda(\Gamma')$. The map $A_{\Gamma'}$ is the (higher) Bowen-Series map associated with the map Γ' . Now let $\Gamma \in \partial\mathcal{B}(\Gamma_0)$. Guided by the Fuchsian situation, we call a continuous self-map $A_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)$ the *(higher) Bowen-Series map of $\Gamma \in \partial\mathcal{B}(\Gamma_0)$* if

- (1) A_Γ is orbit equivalent to Γ , and
- (2) A_Γ is the uniform limit of the (higher) Bowen-Series maps $A_{\Gamma'}$, as $\Gamma' \in \mathcal{B}(\Gamma_0)$ converges to Γ in the strong topology.

The Bers density conjecture, now a theorem due to Brock-Canary-Minsky [Min10, BCM12] (see also [BB04]) states that the Bers slice is dense in the space of all Kleinian surface groups with one end carrying a fixed conformal structure. Thus, the closure of the Bers slice gives all such Kleinian surface groups. For any such group Γ on the Bers boundary, there is an end-invariant called the *ending lamination*—a geodesic lamination supporting a transverse measure. Further, there exists a topological semiconjugacy [Mj14a, Mj17], called a *Cannon-Thurston map* from the circle onto the limit set $\Lambda(\Gamma)$ of Γ . It was shown in [Mj14b, Mj17] that the Cannon-Thurston map identifies precisely the end-points of the ending lamination.

It follows from [Min10, BCM12] (see also [BB04]) that any Bers boundary group is a strong limit of groups in the Bers slice. Let $\Gamma_n = \phi_n \circ \Gamma_0 \circ \phi_n^{-1}$ be a sequence of groups in $\mathcal{B}(\Gamma_0)$ (where ϕ_n is a quasiconformal homeomorphism inducing the representation $\Gamma_0 \rightarrow \Gamma_n$) converging strongly to $\Gamma \in \partial\mathcal{B}(\Gamma_0)$. By [MS13, MS17], [Mj22, Section 4.2], Cannon-Thurston maps of Γ_n converge uniformly to the Cannon-Thurston map of Γ . Since the (higher) Bowen-Series map of Γ_n is equal to $\phi_n \circ A_{\Gamma_0} \circ \phi_n^{-1}$, by definition, the (higher) Bowen-Series map A_Γ , if it exists, must be given by $\phi_\infty \circ A_{\Gamma_0} \circ \phi_\infty^{-1}$, where $\phi_\infty : \mathbb{S}^1 \rightarrow \Lambda(\Gamma)$ is the Cannon-Thurston map of Γ . Thus, the (higher) Bowen-Series map $A_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)$, if it exists, must be semi-conjugate to the base (higher) Bowen-Series map $A_{\Gamma_0} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ via the Cannon-Thurston map of Γ (see [MM21, §7.1] for details):

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{A_{\Gamma_0}} & \mathbb{S}^1 \\ \text{C.T.} \downarrow & & \downarrow \text{C.T.} \\ \Lambda(\Gamma) & \xrightarrow{A_\Gamma} & \Lambda(\Gamma) \end{array}$$

Put differently, in the light of [Mj14b], the existence of a (higher) Bowen-Series map A_Γ requires the ending lamination \mathcal{L} of Γ (where $\mathbb{S}^1/\mathcal{L} \cong \Lambda(\Gamma)$) to be invariant under the action of A_{Γ_0} ([MM21, Lemma 7.3]). On the other hand, such laminations are necessarily invariant under the action of Γ_0 . Simultaneous invariance of \mathcal{L} under the group Γ_0 and the covering map A_{Γ_0} can be thought of as a compatibility condition between Kleinian group dynamics and polynomial dynamics, which turns out to be very restrictive.

Theorem 5.1 (Bers boundary (higher) Bowen-Series maps are sparse). [MM21, Propositions 7.6, 7.8] *Let Γ_0 be a punctured sphere Fuchsian group. Then, there are only finitely many quasiconformal conjugacy classes of groups $\Gamma \in \partial\mathcal{B}(\Gamma_0)$ for which the Cannon-Thurston map of Γ semi-conjugates the (higher) Bowen-Series map of Γ_0 to a self-map A_Γ of $\Lambda(\Gamma)$ that is orbit equivalent to Γ . These Kleinian groups arise out of pinching finitely many disjoint, simple, closed curves (on the surface \mathbb{D}/Γ_0) out of an explicit finite list. In particular, all such groups Γ are geometrically finite.*

Remark 5.2. *Consider the Bowen-Series map A_{G_d} associated with the Fuchsian group G_d equipped with the preferred fundamental domain R given by the ideal polygon with vertices at the $2d$ -th roots of unity (see Subsection 3.2). In this case, the explicit finite list of Theorem 5.1 is*

$$\mathcal{S}_d := \{g_2, \dots, g_{d-1}\} \cup \{g_i^{-1} \circ g_j : i, j \in \{1, \dots, d\}, i - j > 1\}$$

(see [MM21, Proposition 7.6]), and hence every A_{G_d} -invariant geodesic lamination on $\mathbb{D}/G_d \cong S_{0,d+1}$ is a subset of S_d . For $d = 3$, this gives exactly two invariant laminations: $\{\{g_2\}, \{g_3^{-1} \circ g_1\}\}$. The curve corresponding to g_2 (respectively, $g_3^{-1} \circ g_1$) on the four times punctured sphere depicted in Figure 2 is the ‘vertical’ (respectively, ‘horizontal’) curve which separates the punctures $[p_3], [p_4]$ from $[p_1], [p_2]$ (respectively, $[p_1], [p_4]$ from $[p_2], [p_3]$).

The (higher) Bowen-Series map of a Bers boundary group (when it exists) is piecewise Möbius and hence admits a canonical extension \hat{A}_Γ to a subset of the filled limit set $K(\Gamma)$ of the group (i.e., the complement of the completely invariant component of its domain of discontinuity).

Now let P be a complex polynomial in the principal hyperbolic component \mathcal{H}_k , where $k = \deg\{A_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)\}$. Then, the action of P on its Julia set $\mathcal{J}(P)$ is topologically conjugate to $z^k|_{\mathbb{S}^1}$. On the other hand, $A_\Gamma|_{\Lambda(\Gamma)}$ is a factor of $A_{\Gamma_0}|_{\mathbb{S}^1}$, which is in turn topologically conjugate to $z^k|_{\mathbb{S}^1}$. One can now glue the filled Julia set $\mathcal{K}(P)$ (which is a closed Jordan disk) outside the filled limit set $K(\Gamma)$ using a semi-conjugacy between $P|_{\mathcal{J}(P)}$ and $A_\Gamma|_{\Lambda(\Gamma)}$, and this produces a topological 2-sphere. Moreover, the existence of this semi-conjugacy implies that the action of P on $\mathcal{K}(P)$ and the action of \hat{A}_Γ on a subset of $K(\Gamma)$ paste together to yield a continuous map on the copy of \mathbb{S}^2 just defined. This map is called the *topological mating* of \hat{A}_Γ and P . We say that the canonical extension \hat{A}_Γ of the (higher) Bowen-Series map of a Bers boundary group is *conformally mateable* with a polynomial P in the principal hyperbolic component \mathcal{H}_k if the above topological 2-sphere admits a complex structure that turns the topological mating into a holomorphic map (cf. [MM21, §7.5]).

A sophisticated surgery procedure involving David homeomorphisms yields the following conformal mateability theorem.

Theorem 5.3 (Bers boundary (higher) Bowen-Series maps are mateable).

[MM21, Theorem 7.19] *Let $\Gamma \in \partial\mathcal{B}(\Gamma_0)$ be a group that admits a (higher) Bowen-Series map A_Γ . Then the canonical extension \hat{A}_Γ can be conformally mated with polynomials lying in the principal hyperbolic component \mathcal{H}_k , where $k = \deg\{A_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)\}$.*

For a group $\Gamma \in \partial\mathcal{B}(\Gamma_0)$ admitting a (higher) Bowen-Series map A_Γ , the corresponding geodesic lamination \mathcal{L} is invariant under the base (higher) Bowen-Series map A_{Γ_0} . The associated equivalence relation \mathcal{L} on \mathbb{S}^1 satisfies the following properties.

- (1) \mathcal{L} is closed in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$.
- (2) Each equivalence class X of \mathcal{L} is a finite subset of \mathbb{R}/\mathbb{Z} .
- (3) \mathcal{L} -equivalence classes are pairwise *unlinked*; i.e., if X and Y are two distinct equivalence classes of \mathcal{L} , then there exist disjoint intervals $I_X, I_Y \subset \mathbb{R}/\mathbb{Z}$ such that $X \subset I_X$ and $Y \subset I_Y$.
- (4) If X is an \mathcal{L} -equivalence class, then $A_{\Gamma_0}(X)$ is also an \mathcal{L} -equivalence class.
- (5) If X is an \mathcal{L} -equivalence class, then $X \mapsto A_{\Gamma_0}(X)$ is a cyclic order preserving bijection.

On the other hand, the lamination associated with a complex polynomial P with connected Julia set also enjoys analogues of the properties listed above (where the role of A_{Γ_0} is played by the base polynomial z^d). Roughly speaking, the lamination

associated with P is a z^d -invariant closed equivalence relation on \mathbb{S}^1 such that the quotient of \mathbb{S}^1 by the equivalence relation yields a topological model of the Julia set of P (cf. [Kiw04]). Remarkably, the topological conjugacy between $A_{\Gamma_0}|_{\mathbb{S}^1}$ and $z^d|_{\mathbb{S}^1}$ (for some $d \geq 2$) provides us with a tool to pass from laminations in the group world to those in the polynomial world. This combinatorial link allows one to invoke standard realization results from polynomial dynamics and conclude that the limit set $\Lambda(\Gamma)$ is indeed homeomorphic to the Julia set of a complex polynomial in a ‘dynamically natural’ way.

Theorem 5.4 (Equivariant homeomorphism between limit and Julia set). [MM21, Theorem 7.16] *Let $\Gamma \in \partial\mathcal{B}(\Gamma_0)$ be a group that admits a (higher) Bowen-Series map A_Γ . Then there exists a complex polynomial P_Γ (of degree equal to that of $A_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)$) such that the action of A_Γ on the limit set $\Lambda(\Gamma)$ is topologically conjugate to the action of P_Γ on its Julia set.*

6. MEASURES OF MAXIMAL ENTROPY AND PATTERSON-SULLIVAN MEASURES

In this section, we study the measure-theoretic dynamics of Bowen-Series and higher Bowen-Series maps associated with Fuchsian punctured sphere groups, thus linking this theme to another seminal piece of work by Sullivan—the Patterson-Sullivan measure. Specifically, we show that measures of maximal entropy of (higher) Bowen-Series maps acting on the circle are push-forwards of appropriate Patterson-Sullivan measures supported on Gromov boundaries of free groups.

Informally speaking, the Sullivan-Patterson measure on the Gromov boundary of a group is the weak limit of a sequence of atomic measures supported on the words of length n , appropriately weighted by the distances of the group elements from a fixed base point (for the free group with the standard generating set, the sequence reduces to Formula (1)). Although we will not use the general theory of Patterson-Sullivan measures, we encourage the reader to consult [Pat76, Sul79, Coo93] for the construction and basic properties of these measures in the context of Fuchsian groups, Kleinian groups, and hyperbolic groups, respectively. For background on symbolic dynamics and topological/measure-theoretic entropy, we refer the reader to [Wal82, BS02].

6.1. Maximal entropy measure for Bowen-Series maps. For definiteness, let us fix the Fuchsian $(d+1)$ -times punctured sphere group $\Gamma_0 = G_d$ of Section 3.2 and the fundamental domain R given by the ideal polygon with vertices at the $2d$ -th roots of unity ($d \geq 2$). Further let $A \equiv A_{\Gamma_0, \text{BS}} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the Bowen-Series map of Γ_0 associated with the fundamental domain R .

The topological entropy of a dynamical system is a numerical topological conjugacy invariant that measures the complexity of the system. Roughly, it represents the exponential growth rate of the number of essentially different orbit segments of length n . Since A is topologically conjugate to z^{2d-1} , the topological entropy of the A -action on \mathbb{S}^1 is equal to $\ln(2d-1)$. We are interested in studying the measure of maximal entropy (MME for short) for A ; i.e., the unique A -invariant measure on \mathbb{S}^1 whose measure-theoretic entropy is equal to the topological entropy $\ln(2d-1)$ (see [AKU21a, AKU21b] for computation of topological entropy of Bowen-Series maps associated with cocompact Fuchsian groups and results regarding their measures of maximal entropy).

6.1.1. *MME of A in terms of topological dynamics.* By Proposition 3.3, there exists a homeomorphism

$$\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$$

that conjugates $p : z \mapsto z^{2d-1}$ to A (this homeomorphism can be thought of as a generalization of the Minkowski question-mark function $(?)$; see [LLMM22, §4.4.2] for the analogy in the anti-holomorphic context).

We denote the Haar (normalized Lebesgue) measure on \mathbb{S}^1 by m . Note that m is the unique measure of maximal entropy for the action of p on \mathbb{S}^1 (a straightforward computation shows that the measure-theoretic entropy of $p|_{\mathbb{S}^1}$ with respect to m is equal to the topological entropy $\ln(2d-1)$, and the uniqueness of this measure follows for instance from [Lyu83, Theorem 9]). Since the homeomorphism ϕ is a conjugacy, we have the following.

Proposition 6.1. $\nu = \phi_* m$, where ν is the unique measure of maximal entropy for the A -action on \mathbb{S}^1 .

6.1.2. *MME of A in terms of symbolic dynamics.* The partition of \mathbb{S}^1 determined by the $2d$ -th roots of unity form a Markov partition for A . We denote this partition by $\{I_1, I_{-1}, \dots, I_d, I_{-d}\}$, where I_j is the counter-clockwise arc of \mathbb{S}^1 connecting $e^{2\pi i \frac{(j-1)}{2d}}$ and $e^{2\pi i \frac{j}{2d}}$, and I_{-j} is the complex conjugate of I_j , for $j \in \{1, \dots, d\}$. The transition matrix for this Markov partition is

$$M := \begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \end{bmatrix}.$$

The above transition matrix gives rise to a one-sided subshift of finite type

$$\sigma : \Sigma_M^+ \rightarrow \Sigma_M^+.$$

Here Σ_M^+ is the collection of M -admissible infinite words in $\{\pm 1, \pm 2, \dots, \pm d\}^{\mathbb{N}}$; i.e.,

$$\Sigma_M^+ := \{(i_1, i_2, \dots) \in \{\pm 1, \pm 2, \dots, \pm d\}^{\mathbb{N}} : A(I_{i_j}) \supset I_{i_{j+1}} \text{ for all } j \geq 1\},$$

and σ is the left-shift map. A *cylinder set of rank $k \geq 1$* in Σ_M^+ is a set of the form

$$[r_1, \dots, r_k] := \{(i_1, i_2, \dots) \in \Sigma_M^+ : i_j = r_j, \text{ for } j \in \{1, \dots, k\}\},$$

where $(r_1, \dots, r_k) \in \{\pm 1, \dots, \pm d\}^k$. We metrize Σ_M^+ with the usual ultra-metric (in base e).

Since A is expansive, one obtains a continuous surjection

$$\psi : \Sigma_M^+ \rightarrow \mathbb{S}^1$$

that semi-conjugates σ to A . We may and will assume that ψ carries the cylinder set $[\pm j] \subset \Sigma_M^+$ to the Markov partition piece of A connecting $e^{\pm \pi i (j-1)/d}$ to $e^{\pm \pi i j/d}$.

Remark 6.2. See [Sta04] for Markov partitions of Bowen-Series maps associated with more general Fuchsian punctured surface groups. These maps, however, are not continuous if the genus of the surface is greater than zero.

The unique measure of maximal entropy for the σ -action on Σ_M^+ (which is called the *Parry measure* in symbolic dynamics) is given by the ‘uniform’ Markov measure μ that assigns mass $\frac{1}{2d \cdot (2d-1)^n}$ to each cylinder set of rank $n+1$ ($n \geq 0$). The corresponding topological entropy is also $\ln(2d-1)$ (note that $2d-1$ is the largest eigenvalue of M). The existence of the semi-conjugacy ψ now implies that

Proposition 6.3. *The measure of maximal entropy of A , which we denote by ν , is the push-forward of the Parry measure μ under ψ ; i.e., $\nu = \psi_*\mu$.*

6.1.3. *MME of A in terms of Patterson-Sullivan measure.* Since the Bowen-Series map A is cooked up from the Fuchsian group Γ_0 , it is natural to ask whether the measure of maximal entropy ν of A is related to the Patterson-Sullivan measure class of Γ_0 . The following proposition gives a negative answer to this question (recall that a Patterson-Sullivan measure of Γ_0 lies in the class of the Haar measure m).

Proposition 6.4. *The measure ν is not mutually absolutely continuous with respect to the Haar measure m ; i.e., ν and m do not lie in the same measure class.*

Proof. We learned this from Caroline Series. The proposition follows from the facts that

- (1) the action of the Fuchsian group Γ_0 on $S^1 = \partial\mathbb{D}$ is of type III_1 .
- (2) the action of the polynomial z^k on $S^1 = \partial\mathbb{D}$ is of type $III_{\ln(k)}$.

See [Spa87] for details. □

Fortunately, the free group on d generators $F_d \cong \Gamma_0$ provides us with a Patterson-Sullivan measure (supported on the Gromov boundary of F_d) that is intimately related to ν .

We denote the Cayley tree of F_d by X , and equip it with the word metric. The group acts on the tree by isometries. The Gromov boundary of F_d is denoted by ∂X . Note that we can naturally identify ∂X with the shift space Σ_M^+ . Visualizing the Cayley tree X as dual to the Γ_0 -tessellation of \mathbb{D} (associated with the fundamental domain R), one sees in light of the identification $\partial X \cong \Sigma_M^+$ that the map ψ is the (Floyd-)Cannon-Thurston map from ∂X to \mathbb{S}^1 (cf. [Flo80]).

Definition 6.5. *Let X denote a Cayley graph of a group Γ . Let $g \in \Gamma$ (thought of as a vertex of X). The cone of g consists of the vertices $h \in X$ such that any geodesic $[1, g]$ followed by any geodesic $[g, h]$ is a geodesic $[1, h]$ in X joining $1, h$.*

The next result enables us to connect ν to a suitable Patterson-Sullivan measure on ∂X .

Lemma 6.6. *The Patterson-Sullivan measure on ∂X (with respect to the base point 1 and the standard generating set) is given by the Parry measure μ .*

Proof. Note that the number of words in F_d of length r is $2d \cdot (2d-1)^{r-1}$, for $r \geq 1$. Hence, the Patterson-Sullivan measure on ∂X (with respect to the base point 1 and the standard generating set) is a weak limit of the measures

$$(1) \quad \mu_n := \frac{\delta_1 + \sum_{j=1}^n \frac{1}{(2d-1)^j} \left(\sum_{|g|=j} \delta_g \right)}{1 + \sum_{j=1}^n \frac{2d \cdot (2d-1)^{j-1}}{(2d-1)^j}} = \frac{\delta_1 + \sum_{j=1}^n \frac{1}{(2d-1)^j} \left(\sum_{|g|=j} \delta_g \right)}{1 + \frac{2dn}{2d-1}}.$$

A straightforward computation now shows that the μ_{n+r} -mass of the cone at a group element of length r is:

$$\frac{1}{1 + \frac{2d(n+r)}{2d-1}} \cdot \frac{n+1}{(2d-1)^r},$$

which tends to $\frac{1}{2d(2d-1)^{r-1}}$ as $n \rightarrow +\infty$. It follows that the Patterson-Sullivan measure on ∂X assigns mass $\frac{1}{2d(2d-1)^{r-1}}$ to each cylinder set (in ∂X) of rank r . In view of the definition of μ , the proof is now complete. \square

Since $\nu = \psi_*\mu$, we conclude the following result.

Proposition 6.7. *The measure of maximal entropy ν of the Bowen-Series map A is the push-forward of the Patterson-Sullivan measure μ on ∂X (with respect to the base point 1 and the standard generating set) under the (Floyd-) Cannon-Thurston map ψ .*

6.1.4. Γ_0 -invariance of the MME of A . We now exploit the connection between ν and Patterson-Sullivan measures to exhibit Γ_0 -invariance of the measure class of ν .

Proposition 6.8. *For each $\gamma \in \Gamma_0$, the measures ν and $\gamma_*\nu$ are mutually absolutely continuous.*

Proof. First note that the (Floyd-)Cannon-Thurston map ψ semi-conjugates the F_d -action on ∂X to the Γ_0 -action on \mathbb{S}^1 [Flo80]. We will denote the element of F_d corresponding to $\gamma \in \Gamma_0$ by $\tilde{\gamma}$.

By Proposition 6.7 and the previous paragraph, the measure $\gamma_*\nu$ on \mathbb{S}^1 is the push-forward of the measure $\tilde{\gamma}_*\mu$ on ∂X under ψ . Moreover, as μ is a Patterson-Sullivan measure on ∂X , it follows that $\tilde{\gamma}_*\mu$ and μ are mutually absolutely continuous (see [Coo93, Theorem 5.4, Theorem 8.2]). It is now easy to see using the definition of push-forward of a measure that the measures $\nu = \psi_*(\mu)$ and $\gamma_*\nu = \psi_*(\tilde{\gamma}_*\mu)$ are mutually absolutely continuous. \square

Remark 6.9. *For $\gamma \in \Gamma_0$, the Radon-Nikodym derivative $d(\gamma_*\nu)/d\nu$ can be written in terms of ψ and the Radon-Nikodym derivative $d(\tilde{\gamma}_*\mu)/d\mu$, which in turn can be computed from measures of cylinder sets (see [Coo93, §8] for a general method of describing such Radon-Nikodym derivatives in terms of Busemann functions).*

Now observe that the Bowen-Series map A does not depend only on the group Γ_0 , but also on the choice of the fundamental domain R . The translation of R by an element $\gamma \in \Gamma_0$ is a different fundamental domain $\gamma \cdot R$ for Γ_0 . We denote the Bowen-Series map of Γ_0 associated with the fundamental domain $\gamma \cdot R$ by A^γ . Clearly, $A^\gamma = \gamma \circ A \circ \gamma^{-1}$. Moreover, the unique measure of maximal entropy for the A^γ -action on \mathbb{S}^1 is given by $\gamma_*\nu$. Proposition 6.8 now implies the following.

Corollary 6.10. *The measures of maximal entropy for the Bowen-Series maps associated with the fundamental domains $\gamma \cdot R$ (for $\gamma \in \Gamma_0$) are mutually absolutely continuous. In particular, all these measures have the same Hausdorff dimension.*

6.1.5. *MME for matings of Bowen-Series maps and polynomials.* Recall that Theorem 3.5 provides us with a conformal mating of the canonical extension \hat{A} of the Bowen-Series map A (associated with the fundamental domain R of Γ_0) and the polynomial map z^{2d-1} . Also note that the restriction of this conformal mating on

its Jordan curve limit set is topologically conjugate to $A|_{\mathbb{S}^1}$. The following description of the measure of maximal entropy of the conformal mating now follows from Propositions 6.1 and 6.7.

Proposition 6.11. *The unique measure of maximal entropy of the conformal mating of \hat{A} and z^{2d-1} restricted to the limit set is equal to the push-forward of the normalized Lebesgue measure m (which is the unique measure of maximal entropy of $z^{2d-1}|_{\mathbb{S}^1}$) as well as the push-forward of the Patterson-Sullivan measure μ on ∂X (with respect to the base point 1 and the standard generating set) under appropriate conjugacies. In particular, the corresponding topological entropy is $\ln(2d-1)$.*

6.1.6. *Topological entropy of A from a group-theoretic perspective.* The topological entropy $\ln(2d-1)$ of A can be related to the *volume entropy* of the group F_d , which measures the exponential growth rate of the number of words of length n in a group (equivalently, the exponential growth rate of the number of group elements in a ball of radius n around identity).

Lemma 6.12. *The volume entropy of F_d with respect to the standard (symmetric) set of generators and the critical exponent for the F_d -action on X are both equal to $\ln(2d-1)$.*

Proof. Recall that the number of words in F_d of length r is $2d \cdot (2d-1)^{r-1}$, for $r \geq 1$. Hence,

$$\#\{g \in F_d : |g| \leq n\} = 1 + 2d \sum_{r=1}^n (2d-1)^{r-1} = 1 + 2d \frac{(2d-1)^n - 1}{2d-2},$$

from which it follows that the volume entropy is $\ln(2d-1)$.

Now consider the Poincaré series with exponent s :

$$\zeta_{F_d}(s) := \sum_{g \in F_d} e^{-s|g|} = \sum_{n=1}^{\infty} \sum_{\substack{g \in F_d \\ |g|=n}} e^{-sn} = 2d \sum_{n=1}^{\infty} \frac{(2d-1)^{n-1}}{e^{sn}} = \frac{2d}{2d-1} \sum_{n=1}^{\infty} \left(\frac{2d-1}{e^s} \right)^n.$$

Clearly, the series converges if and only if $2d-1 < e^s \iff s > \ln(2d-1)$. In particular, the critical exponent is $\ln(2d-1)$. \square

Remark 6.13. *A connection between the topological entropy of Bowen-Series maps associated with cocompact Fuchsian groups and the volume entropy of suitable hyperbolic groups was established in [Los14].*

In [Sul84, Theorem 1], Sullivan proved equality of critical exponents and Hausdorff dimensions of limit sets for geometrically finite Kleinian groups. While the analogous result for hyperbolic groups follows from general consideration (cf. [Coo93, Theorem 8.3] [KB02, Theorem 15.8]), we can give a simple proof in the present setting.

Lemma 6.14. *The Hausdorff dimension of the Gromov boundary of F_d equipped with the visual metric (in base e) is equal to $\ln(2d-1)$. Moreover, the $\ln(2d-1)$ -dimensional Hausdorff measure $\mathcal{H}^{\ln(2d-1)}$ (on ∂X) and μ are mutually absolutely continuous.*

Proof. The visual metric (in base e) on ∂X is bi-Lipschitz to the ultra-metric given by $d(a, b) = e^{-|c|}$, where c is the bifurcation point for the geodesic rays $[1, a)$ and

$[1, b)$. Hence, it suffices to compute the Hausdorff dimension of ∂X with respect to this ultra-metric.

We first note that the μ -measure of a cylinder of rank n is equal to $\frac{1}{2d(2d-1)^{n-1}}$. On the other hand, the diameter of a cylinder of rank n is e^{-n} . Thus,

$$\mu(B(a, e^{-n})) = \frac{1}{2d(2d-1)^{n-1}} \implies \mu(B(a, r)) \sim r^{\ln(2d-1)}.$$

The result now follows from standard results on Hausdorff dimension (for instance, see [Fal03, Proposition 4.9]). In fact, we have shown that the $\ln(2d-1)$ -dimensional Hausdorff measure is positive and finite. The second statement is obvious from the above proof. \square

6.2. Maximal entropy measure for higher Bowen-Series maps. We now carry out a similar analysis for the measure of maximal entropy of a higher Bowen-Series map of a Fuchsian punctured sphere group. For simplicity of exposition, we work with the thrice punctured sphere case.

Let us fix the Fuchsian thrice punctured sphere group Γ_0 of Section 4.1 and a (closed) fundamental domain W given by the quadrilateral with vertices at the fourth roots of unity. Further let $A \equiv A_{\Gamma_0, \text{hBS}} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the associated higher Bowen-Series map of Γ_0 . As $A|_{\mathbb{S}^1}$ is topologically conjugate to $z^4|_{\mathbb{S}^1}$, the topological entropy of A is equal to $\ln(4)$. We denote the unique measure of maximal entropy for $A|_{\mathbb{S}^1}$ by ν .

6.2.1. Topological dynamics. As A is an expansive circle covering of degree 4, there exists a homeomorphism

$$\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$$

that conjugates $p : z \mapsto z^4$ to A . Using the conjugacy ϕ , one can write the measure of maximal entropy ν for $A|_{\mathbb{S}^1}$ as the push-forward measure $\phi_* m$.

6.2.2. Symbolic dynamics. The pieces of A are given by $g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}$. Their intervals of definition yield a Markov partition (counter-clockwise starting at 1) for A with transition matrix

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The above transition matrix gives rise to a one-sided subshift of finite type

$$\sigma : \Sigma_M^+ \rightarrow \Sigma_M^+,$$

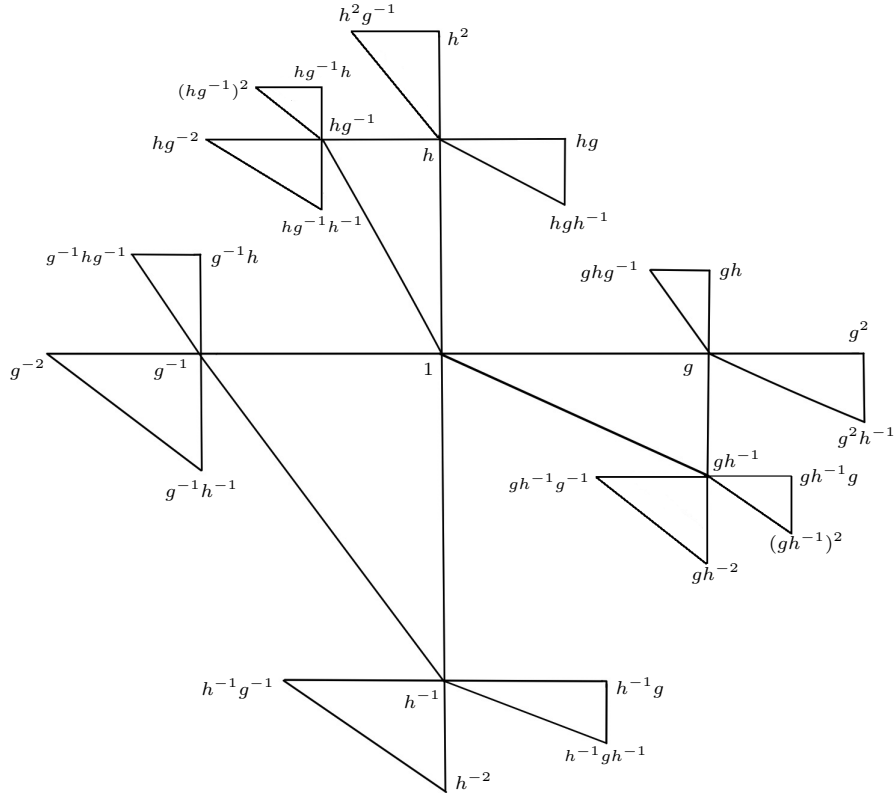
where Σ_M^+ consists of M -admissible infinite words in $\{1, 2, \dots, 6\}^{\mathbb{N}}$, and σ is the left-shift map. As before, we metrize Σ_M^+ with the usual ultra-metric (in base e). Since A is expansive, one obtains a continuous surjection

$$\psi : \Sigma_M^+ \rightarrow \mathbb{S}^1$$

that semi-conjugates σ to A , and sends the cylinders of rank 1 to the Markov partition pieces of A .

The *Parry measure* (i.e., the unique measure of maximal entropy) for the σ -action on Σ_M^+ is given by the ‘uniform’ Markov measure μ , that assigns mass $\frac{1}{6 \cdot 4^n}$ to each

We denote the Cayley graph of F_2 with respect to the above (non-standard) generating set by X , and equip it with the word metric. Note that we can naturally identify the Gromov boundary ∂X with the shift space Σ_M^+ . With this identification, the boundary at infinity of the cone at a generator is the corresponding cylinder set in Σ_M^+ .



Lemma 6.16. *The Patterson-Sullivan measure on ∂X (with respect to the base point 1 and the generating set $\{g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}\}$) is given by the Parry measure μ .*

Proof. It is easy to see from the generators and relations (equivalently, from the Cayley graph depicted in Figure 6) that the number of words of length r in F_2 (with respect to generating set $\{g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}\}$) is $6 \cdot 4^{r-1}$ ($r \geq 1$). A computation similar to the one in the proof of Lemma 6.6 now readily shows that the Patterson-Sullivan measure in question (on ∂X) assigns mass $\frac{1}{6 \cdot 4^{r-1}}$ to each cylinder set (in ∂X) of rank r . Thus, the Patterson-Sullivan measure agrees with μ on each cylinder set. \square

Proposition 6.17. *The measure of maximal entropy ν of the higher Bowen-Series map A is the push-forward of the Patterson-Sullivan measure μ on ∂X (with respect to the base point 1 and the generating set $\{g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}\}$) under ψ .*

6.2.4. Γ_0 –invariance of the class of ν . Proposition 6.17, F_d –invariance of the measure class of the Patterson-Sullivan measure μ (on ∂X) [Coo93, Theorem 5.4, Theorem 8.2], and the fact that the map ψ semi-conjugates the F_d –action on ∂X to the Γ_0 –action on \mathbb{S}^1 together imply the following.

Proposition 6.18. *For each $\gamma \in \Gamma_0$, the measures ν and $\gamma_*\nu$ are mutually absolutely continuous.*

6.2.5. *MME for mating.* According to Theorem 4.11, there exists a conformal mating of the canonical extension \hat{A} of the higher Bowen-Series map A (associated with the closed fundamental domain W of Γ_0) and the polynomial map z^4 such that the restriction of this conformal mating on its Jordan curve limit set is topologically conjugate to $A|_{\mathbb{S}^1}$. The interpretation of the measure ν in terms of m and μ implies the following.

Proposition 6.19. *The unique measure of maximal entropy of the conformal mating of \hat{A} and z^4 restricted to the limit set is equal to the push-forward of the MME of z^4 on \mathbb{S}^1 as well as the push-forward of the Patterson-Sullivan measure on ∂X (with respect to the base point 1 and the generating set $\{g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}\}$) under appropriate conjugacies. In particular, the corresponding topological entropy is $\ln(4)$.*

6.2.6. *Topological entropy, volume entropy, and Hausdorff dimension.* Since there are $6 \cdot 4^{r-1}$ words of length r in F_2 (with respect to generating set $\{g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}\}$), the arguments used in the proof of Lemmas 6.12, 6.14 apply mutatis mutandis to the current setting and prove equality of volume entropy, critical exponent, and Hausdorff dimension of the Gromov boundary. Moreover, this number coincides with the topological entropy of $A|_{\mathbb{S}^1}$.

Lemma 6.20.

- (1) *The volume entropy of F_2 with respect to the generating set $\{g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}\}$ and the critical exponent for the F_2 –action on X are both equal to $\ln(4)$.*
- (2) *The Hausdorff dimension of ∂X equipped with the visual metric (in base e) with respect to the generating set $\{g^{\pm 1}, h^{\pm 1}, g \circ h^{-1}, h \circ g^{-1}\}$ is equal to $\ln(4)$. Moreover, the $\ln(4)$ –dimensional Hausdorff measure $\mathcal{H}^{\ln(4)}$ and μ (on ∂X) are mutually absolutely continuous.*

6.3. Hausdorff dimension of measure of maximal entropy: thrice punctures sphere. Recall from Proposition 6.4 that the measure of maximal entropy

of the Bowen-Series map of a Fuchsian punctured sphere group does not lie in the Lebesgue measure class. In this section, we will prove a sharper version of this fact in the thrice punctured sphere case.

Specifically, we will show that the Hausdorff dimension

$$\text{HD}(\nu) := \inf\{\text{HD}(Y) : Y \subset \mathbb{S}^1, \nu(Y) = 1\}$$

of the MME ν of the (higher) Bowen-Series map of a Fuchsian thrice punctured sphere group is strictly less than 1. This statement should be compared with the analogous result that except for some very special cases, the Hausdorff dimension of the measure of maximal entropy of a rational map is strictly smaller than the Hausdorff dimension of the Julia set [Zdu90].

In what follows, we will cook up a self-map of the interval $[0, 1]$ from the (higher) Bowen-Series map under consideration, and relate the Hausdorff dimension of the MME of the (higher) Bowen-Series map to that of the MME of the associated self-map. This will allow us to obtain the desired upper bounds. We note that while this intermediate step is not essential for the Bowen-Series map (in this case, one can obtain the upper bound of Proposition 6.22 by working directly with the Bowen-Series map), this method yields additional information in the higher Bowen-Series case. Indeed, this reduction step connects the MME of the higher Bowen-Series map to a classical measure from number theory, which makes known results applicable to the current setting and gives a better estimate for the Hausdorff dimension of the MME.

6.3.1. The Bowen-Series case. Since the Teichmüller space of a thrice punctured sphere is a singleton, we may, without loss of generality, work with the Bowen-Series map A_{G_2} introduced in Subsection 3.2. Note that every non-identity element g in the free group G_2 admits a unique shortest representation with respect to the symmetric generating set $\{g_1^{\pm 1}, g_2^{\pm 1}\}$ introduced in Subsection 3.2. The length of this shortest representation is called the *length* of g (the length of the identity element is defined to be zero). Recall that the ideal polygon in \mathbb{D} with vertices at the fourth roots of unity is a fundamental domain for the G_2 -action on \mathbb{D} , and hence its translates under elements of G_2 yield a tiling \mathcal{T}_{G_2} of \mathbb{D} . We call this fundamental domain the *rank 0 tile* and its translate under an element $g \in G_2$ of length k a *rank k tile* for the above tessellation.

We will use a specific symmetric property of the tessellation \mathcal{T}_{G_2} which we now describe. Following Section 3, we denote the ideal polygon in \mathbb{D} with vertices at the fourth roots of unity by R , and its edges by $C_{\pm 1}, C_{\pm 2}$. Let us further denote the anti-Möbius reflections in these edges by $\rho_{\pm 1}, \rho_{\pm 2}$, and the reflection group generated by $\rho_{\pm 1}, \rho_{\pm 2}$ by \mathcal{G} . Note that the polygon R is invariant under the actions of \mathcal{R} and ι , where \mathcal{R} is rotation by angle $\pi/2$ and ι is the reflection in the real axis. It follows that conjugation by \mathcal{R} and ι act as permutations on the generating set $\{\rho_{\pm 1}, \rho_{\pm 2}\}$ of \mathcal{G} , and hence \mathcal{R} and ι conjugate \mathcal{G} to itself. Hence, the \mathcal{G} -tessellation $\mathcal{T}_{\mathcal{G}}$ of \mathbb{D} arising from the fundamental domain \bar{R} (closure taken in \mathbb{D}) is preserved by both \mathcal{R} and ι . Note furthermore that the relations $g_i = \iota \circ \rho_i = \rho_{-i} \circ \iota$, $i \in \{1, 2\}$, and ι -invariance of the tessellation $\mathcal{T}_{\mathcal{G}}$ imply that the tessellations $\mathcal{T}_{\mathcal{G}}$ and \mathcal{T}_{G_2} are the same. It follows that the tessellation \mathcal{T}_{G_2} of \mathbb{D} is symmetric with respect to $\pi/2$ -rotation \mathcal{R} .

For the current purpose, it will be more convenient to work with the upper half-plane model. To this end, consider the Möbius transformation $M(z) = i(1 -$

$z)/(1+z)$ which carries the unit disk onto the upper half-plane such that $M(1) = 0$, $M(i) = 1$, $M(-1) = \infty$, and $M(-i) = -1$. Hence, M sends the ideal polygon in \mathbb{D} with vertices at the fourth roots of unity to the ideal polygon in \mathbb{H} with vertices at $-1, 0, 1$, and ∞ . The map M conjugates G_2 to a discrete subgroup \mathbf{G}_2 of $\mathrm{PSL}_2(\mathbb{R})$, and transports the G_2 -tessellation of \mathbb{D} defined in the previous paragraph to a \mathbf{G}_2 -tessellation of \mathbb{H} . One defines tiles of this tessellation and their ranks as in the previous paragraph. Moreover, M conjugates the Bowen-Series map A_{G_2} to the map

$$\tau : \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}, \quad \tau(t) = \begin{cases} t+2, & t \in [-\infty, -1], \\ \frac{t}{1+2t}, & t \in [-1, 0], \\ \frac{t}{1-2t}, & t \in [0, 1], \\ t-2, & t \in [1, +\infty]. \end{cases}$$

By construction, τ maps $[0, \frac{1}{3}]$ to $[0, 1]$, $[\frac{1}{3}, \frac{1}{2}]$ to $[1, +\infty]$, and $[\frac{1}{2}, 1]$ to $[-\infty, -1]$ (see Figure 7).

Since Euclidean isometric rotation $\mathcal{R} : z \mapsto iz$ (about the origin) respects the G_2 -tessellation of \mathbb{D} , it follows that the conformal rotation $\mathcal{R}_{\mathbb{H}}(w) = M(i \cdot M^{-1}(w)) = \frac{1+w}{1-w} \in \mathrm{PSL}_2(\mathbb{R})$ (about i) respects the corresponding \mathbf{G}_2 -tessellation of \mathbb{H} .

This allows one to construct a self-map of $[0, 1)$ associated with τ :

$$F : [0, 1) \rightarrow [0, 1), \quad F(t) = \begin{cases} \tau(t) & = \frac{x}{1-2x}, & t \in [0, \frac{1}{3}), \\ (\mathcal{R}_{\mathbb{H}}^{-1} \circ \tau)(t) & = \frac{3x-1}{1-x}, & t \in [\frac{1}{3}, \frac{1}{2}), \\ (\mathcal{R}_{\mathbb{H}}^2 \circ \tau)(t) & = \frac{2x-1}{x}, & t \in [\frac{1}{2}, 1). \end{cases}$$

(See Figure 8.) The symmetry of the \mathbf{G}_2 -tessellation of \mathbb{H} under the conformal rotation $\mathcal{R}_{\mathbb{H}}$ implies that F sends the ideal vertices of tiles of a given rank to the ideal vertices of tiles of the previous rank.

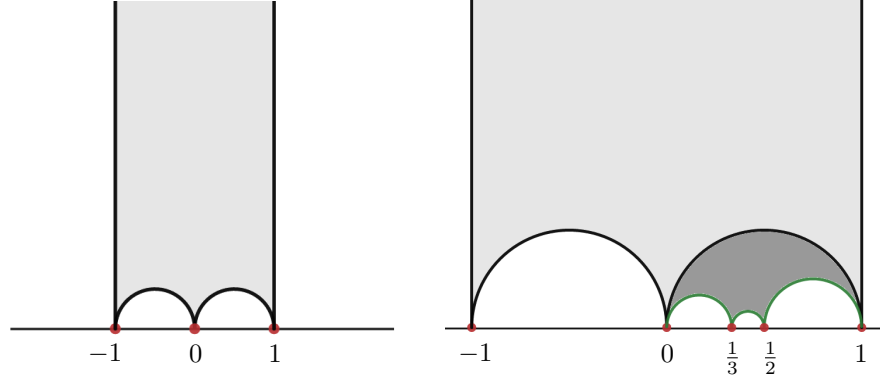


FIGURE 7. Left: A fundamental polygon of the thrice punctured sphere Fuchsian group $\mathbf{G}_2 = M \circ G_2 \circ M^{-1}$ with ideal vertices at $0, \pm 1$, and ∞ . Right: A tile of rank one with ideal vertices at $0, \frac{1}{2}, \frac{1}{3}$, and 1 is shown.

Let $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the homeomorphism conjugating z^3 to the Bowen-Series map A_{G_2} with $\phi(1) = 1$. As the chosen fundamental domain of A_{G_2} is symmetric under rotation by $\pi/2$, one readily sees that the map ϕ commutes with $z \mapsto iz$. Using this, it is straightforward to verify that the tripling map $\times_3 : [0, 1) \rightarrow [0, 1)$

$$\times_3(x) = \begin{cases} 3x, & x \in [0, \frac{1}{3}), \\ 3x - 1, & x \in [\frac{1}{3}, \frac{2}{3}), \\ 3x - 2, & x \in [\frac{2}{3}, 1), \end{cases}$$

is topologically conjugate to F via

$$H : [0, 1] \rightarrow [0, 1], \quad x \mapsto M(\phi(E(x))), \quad \text{where } E(x) = e^{2\pi i \frac{x}{4}}.$$

Due to the conjugation property, the homeomorphism H sends the rational numbers $k/3^n$ (which are the n -th preimages of 0 under \times_3) to the ideal vertices of tiles of rank n (which are the n -th preimages of 0 under F).

Also note that the Lebesgue measure \overline{m} on $[0, 1]$ is the measure of maximal entropy for \times_3 , and hence,

$$(2) \quad \nu' := H_* \overline{m}$$

is the measure of maximal entropy for F .

Lemma 6.21. *Let ν, ν' be the measures of maximal entropy of the Bowen-Series map A_{G_2} and the map F , respectively. Then, $\text{HD}(\nu') = \text{HD}(\nu)$.*

Proof. We first observe that the Möbius map $M : \{e^{i\theta} : \theta \in [0, \pi/2]\} \rightarrow [0, 1]$ is bi-Lipschitz, and hence preserves Hausdorff dimension (this can, for instance, be deduced from the fact that $M(e^{i\theta}) = \tan(\frac{\theta}{2})$). Thus, by definition of ν' (see Equation 2), we have

$$\text{HD}(\nu') = \text{HD}((\phi \circ E)_* \overline{m}).$$

Note that the measure $E_*(\overline{m})$ is simply the normalized Lebesgue measure on the arc $\{e^{i\theta} : \theta \in [0, \pi/2]\} \subset \mathbb{S}^1$.

Now choose $A \subset \mathbb{S}^1$ with $\nu(A) = 1$, and set $A' := A \cap \{e^{i\theta} : \theta \in [0, \pi/2]\}$. By definition, the set $\phi^{-1}(A)$ has full measure with respect to the Haar measure m on \mathbb{S}^1 , and hence, $m(\phi^{-1}(A')) = 1/4$ (here we have used the fact that ϕ maps the first quadrant of \mathbb{S}^1 to itself). This implies that $\phi^{-1}(A')$ is a full measure set with respect to $E_* \overline{m}$, and thus in turn A' is a full measure set with respect to $(\phi \circ E)_* \overline{m}$. Therefore,

$$\text{HD}((\phi \circ E)_* \overline{m}) \leq \text{HD}(A') \leq \text{HD}(A).$$

Taking the infimum over all full ν -measure subsets A of \mathbb{S}^1 , we conclude that

$$\text{HD}(\nu') = \text{HD}((\phi \circ E)_* \overline{m}) \leq \text{HD}(\nu).$$

For the opposite inequality, pick $A' \subset \{e^{i\theta} : \theta \in [0, \pi/2]\}$ with full $(\phi \circ E)_* \overline{m}$ -measure. Define A to be the symmetrization of A' under rotation by $\pi/2$. As $\phi^{-1}(A')$ has full measure with respect to $E_* \overline{m}$, we have that $m(\phi^{-1}(A')) = 1/4$. Since ϕ commutes with multiplication by i , it now follows that $m(\phi^{-1}(A)) = 1$; i.e., A has ν -measure 1. Therefore,

$$\text{HD}(\nu) \leq \text{HD}(A) = \text{HD}(A')$$

Finally, taking the infimum over all full $(\phi \circ E)_* \bar{m}$ -measure subsets A' of $\{e^{i\theta} : \theta \in [0, \pi/2]\}$, we have that

$$\text{HD}(\nu) \leq \text{HD}((\phi \circ E)_* \bar{m}) = \text{HD}(\nu').$$

□

Proposition 6.22. *Let ν be the measure of maximal entropy of the Bowen-Series map A_{G_2} . Then, $\text{HD}(\nu) < 1$.*

Proof. By Lemma 6.21, it suffices to show that $\text{HD}(\nu') < 1$, where ν' is the measure of maximal entropy of F (defined by Equation 2). The following relation between Hausdorff dimension, entropy, and Lyapunov exponent is standard (see [PU10, §10], [HR92]):

$$\text{HD}(\nu') = \frac{\ln(3)}{\int_0^1 \ln |F'| d\nu'}.$$

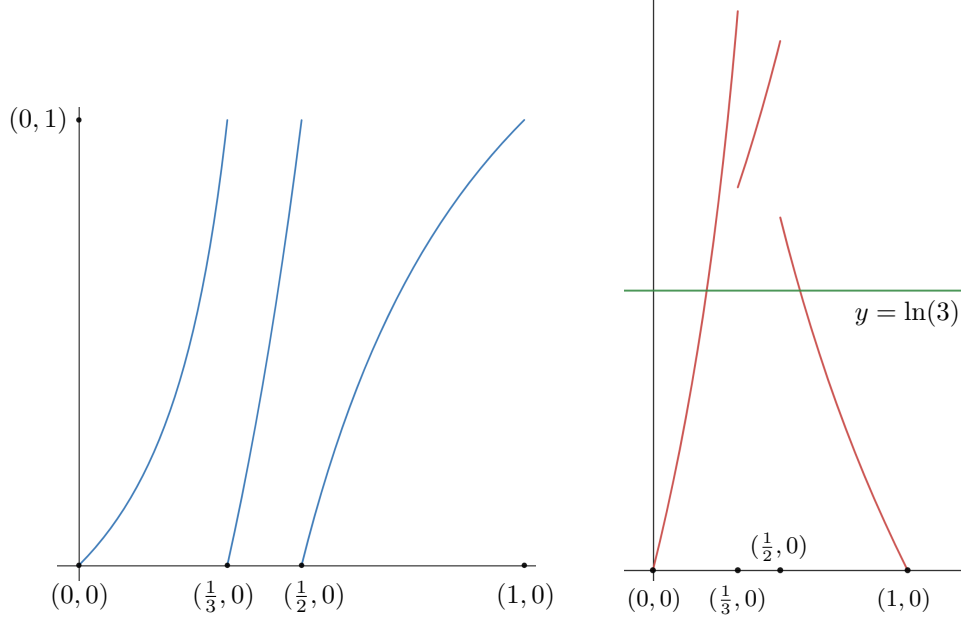


FIGURE 8. Left: The graph of F . Right: The graph of $\ln |F'|$.

We also have the following explicit description of $\ln |F'|$ on $(0, 1) \setminus \{\frac{1}{3}, \frac{1}{2}\}$:

$$\ln |F'| (t) = \begin{cases} \ln \frac{1}{(1-2x)^2}, & t \in (0, \frac{1}{3}), \\ \ln \frac{2}{(1-x)^2}, & t \in (\frac{1}{3}, \frac{1}{2}), \\ \ln \frac{1}{x^2}, & t \in (\frac{1}{2}, 1). \end{cases}$$

(See Figure 8.)

Our goal is to show that $\ln(3)$ is a strict lower bound for the Lyapunov exponent of F . To this end, we first note that by definition of ν' , the ideal vertices (in $[0, 1]$) of the tiles of rank up to three divide the unit interval into 3^3 sub-intervals each of

which has ν' -mass $1/3^3$. The endpoints of these intervals are displayed in Figure 9.

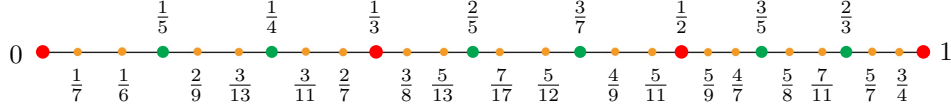


FIGURE 9. The break-points of the piecewise definition of F (which are the ideal vertices of a rank one tile) are marked in red. The new ideal vertices of the rank two, three tiles are displayed in green, orange (respectively). Each of the 27 complementary components has ν' -mass $\frac{1}{27}$.

Since $\ln |F'|$ is increasing on $(0, \frac{1}{3})$ and $(\frac{1}{3}, \frac{1}{2})$, and decreasing on $(\frac{1}{2}, 1)$, we have:

$$\begin{aligned}
& \int_0^1 \ln |F'| d\nu' \\
& \geq \frac{\ln |F'(0) \cdot F'(\frac{1}{7}) \cdot F'(\frac{1}{6}) \cdot F'(\frac{1}{5}) \cdot F'(\frac{2}{9}) \cdot F'(\frac{3}{13}) \cdot F'(\frac{1}{4}) \cdot F'(\frac{3}{11}) \cdot F'(\frac{2}{7})|}{3^3} \\
& \quad + \frac{\ln |F'(\frac{1}{3}) \cdot F'(\frac{3}{8}) \cdot F'(\frac{5}{13}) \cdot F'(\frac{2}{5}) \cdot F'(\frac{7}{17}) \cdot F'(\frac{5}{12}) \cdot F'(\frac{3}{7}) \cdot F'(\frac{4}{9}) \cdot F'(\frac{5}{11})|}{3^3} \\
& \quad + \frac{\ln |F'(\frac{5}{9}) \cdot F'(\frac{4}{7}) \cdot F'(\frac{3}{5}) \cdot F'(\frac{5}{8}) \cdot F'(\frac{7}{11}) \cdot F'(\frac{2}{3}) \cdot F'(\frac{5}{7}) \cdot F'(\frac{3}{4}) \cdot F'(1)|}{3^3} \\
& \approx 1.201 > \ln(3).
\end{aligned}$$

(The number 1.201 above is obtained by explicit numerical computation using the formula of $\ln |F'|$ given above, and is correct up to 3 decimal places.)

Hence, $\text{HD}(\nu') = \frac{\ln(3)}{\int_0^1 \ln |F'| d\nu'} < 1$. \square

6.3.2. The higher Bowen-Series case. We will now show that the Hausdorff dimension of the MME ν of the higher Bowen-Series map A of Section 4.1 is strictly less than 1. To simplify computations, we will first apply a reduction step that will allow us to work with a degree -2 covering of \mathbb{S}^1 (note that A is a degree 4 covering of the circle). This will also relate the Hausdorff dimension of ν to that of a classically studied measure arising naturally from the Minkowski question-mark function $(?)$ (see [Den38, Sal43, Kin60] for details on the question-mark function).

For consistency, we will use the notation employed in Section 4.1. Recall that W is a (closed) ideal quadrilateral in \mathbb{D} with ideal vertices at the fourth roots of unity (the quadrilateral 1236 in Figure 3). The Möbius maps g, h pair the sides of this quadrilateral (as shown in Figure 3), and generate a thrice punctured sphere Fuchsian group Γ_0 . Moreover, W is a (closed) fundamental domain of Γ_0 . The fundamental domain of the higher Bowen-Series map A is given by the ideal hexagon 123567, while the inner domain of A is the ideal triangle 136.

Let us denote reflections in the hyperbolic geodesics $\bar{1}3, \bar{3}6$, and $\bar{6}1$ by r_1, r_2, r_3 . With this notation, the side-pairing transformations g and h are given by $r_2 \circ r_1$ and $r_3 \circ r_1$, respectively. It is also readily checked that the map $A : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is the

second iterate of the piecewise reflection Markov map

$$\mathfrak{R} : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \quad z \mapsto \begin{cases} r_1(z) & \text{if } z \in \widehat{123}, \\ r_2(z) & \text{if } z \in \widehat{356}, \\ r_3(z) & \text{if } z \in \widehat{671}. \end{cases}$$

Hence, the circle endomorphisms A and \mathfrak{R} have the same measure of maximal entropy.

We will now relate the map \mathfrak{R} to a well-studied orientation-reversing double covering of \mathbb{S}^1 . Note that as any pair of hyperbolic ideal triangles are Möbius equivalent, the triangle $\Delta 136$ of Figure 3 is Möbius equivalent to the *regular* ideal polygon $\Pi \subset \mathbb{D}$ with vertices at the third roots of unity. The *Nielsen map* $\rho_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of the *regular ideal triangle reflection group* is defined as anti-Möbius reflections in the three sides of Π on the three corresponding arcs of \mathbb{S}^1 (see Figure 10 for a pictorial illustration and [LLMM22, §2], [LLMM19, §4.1] for the precise definition and properties of this map). The Möbius equivalence of $\Delta 136$ and Π implies that the map \mathfrak{R} is Möbius conjugate to ρ_2 . Moreover, the fact that Möbius maps are bi-Lipschitz tells us that the Hausdorff dimension of the MME ν of the higher Bowen-Series map A is equal to the Hausdorff dimension of the MME of ρ_2 .

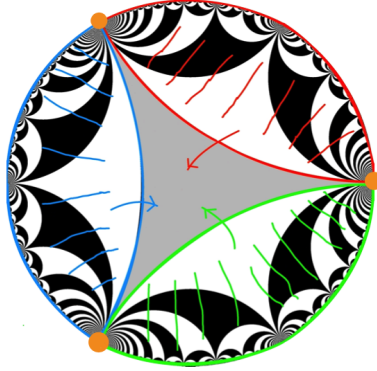


FIGURE 10. The action of the Nielsen map ρ_2 of the ideal triangle group is depicted.

Figure 10: The Nielsen map ρ_2 acts on the arcs $\widehat{1e^{\frac{2\pi i}{3}}}$, $\widehat{e^{\frac{2\pi i}{3}}e^{\frac{4\pi i}{3}}}$, and $\widehat{e^{\frac{4\pi i}{3}}1}$ of \mathbb{S}^1 as reflections in the bi-infinite hyperbolic geodesics $1e^{\frac{2\pi i}{3}}$, $e^{\frac{2\pi i}{3}}e^{\frac{4\pi i}{3}}$, and $e^{\frac{4\pi i}{3}}1$, respectively. It naturally extends as a piecewise anti-Möbius map to the complement of the ideal triangle Π (in grey) in \mathbb{D} .

Applying the construction of Subsection 6.3.1 to the upper half-plane model of ρ_2 (such that the ideal triangle in \mathbb{D} with vertices at the third roots of unity corresponds to the ideal triangle in \mathbb{H} with vertices at $0, 1, \infty$) combined with the arguments of Lemma 6.21, one can show that the Hausdorff dimension of the MME of ρ_2 is equal to the Hausdorff dimension of the MME of the orientation-reversing degree two map

$$F : [0, 1) \rightarrow [0, 1), \quad \tau(t) = \begin{cases} \frac{2t-1}{t-1} \pmod{1} & t \in [0, \frac{1}{2}), \\ \frac{1-t}{t} \pmod{1} & t \in [\frac{1}{2}, 1). \end{cases}$$

We refer the reader to [LLMM19, §9] for the details of this construction. It is also shown there that the map F is topologically conjugate to the orientation-reversing doubling map

$$\times_{-2}(x) = \begin{cases} -2x + 1 \pmod{1} & x \in [0, \frac{1}{2}), \\ -2x + 2 \pmod{1} & x \in [\frac{1}{2}, 1), \end{cases}$$

via the question-mark function. Hence, the MME of F is given by the push-forward of the Lebesgue measure on $[0, 1]$ under $(?)^{-1}$. According to [KS08], the Hausdorff dimension of this measure is strictly less than 1. In fact, it is shown there that the Hausdorff dimension of the push-forward of the Lebesgue measure on $[0, 1]$ under $(?)^{-1}$ is approximately 0.875 (see [KS08, Figure 2, §3]). We collect the upshot of the above analysis in the following proposition.

Proposition 6.23. *Let ν be the measure of maximal entropy of the higher Bowen-Series map A of a Fuchsian thrice punctured sphere group. Then,*

$$\text{HD}(\nu) \approx 0.875 < 1.$$

6.4. Some open questions. We conjecture that the results of Section 6.3 hold in greater generality.

Question 6.24. *Let ν be the measure of maximal entropy of a (higher) Bowen-Series map of a Fuchsian punctured sphere group. Is*

$$\text{HD}(\nu) := \inf\{\text{HD}(Y) : Y \subset \mathbb{S}^1, \nu(Y) = 1\}$$

less than 1?

Henceforth we will assume that $k > 3$, so that the surface $S_{0,k}$ has a non-trivial Teichmüller space.

We believe that the Hausdorff dimension of the limit set of the conformal mating of $\hat{A}_{\Gamma, \text{BS}}$ (respectively, $\hat{A}_{\Gamma, \text{hBS}}$) and P , where $\Gamma \in \text{Teich}(S_{0,k})$ and $P \in \mathcal{H}_{2k-3}$ (respectively, $P \in \mathcal{H}_{(k-1)^2}$), is strictly greater than 1. The next question is motivated by Bowen's theorem on Hausdorff dimension of quasi-Fuchsian limit sets (cf. [Bow79]).

Question 6.25. *Do the Hausdorff dimensions of limit sets of the above class of conformal matings attain its global minimum at a unique point?*

The following questions are motivated by results of McMullen on variation of Hausdorff dimensions of limit sets and naturally associated measures living on them (cf. [McM08]).

Question 6.26. *How does the Hausdorff dimension of the measure of maximal entropy of the (higher) Bowen-Series map vary as the marked group runs over $\text{Teich}(S_{0,k})$?*

Question 6.27. *How does the Hausdorff dimension of the limit set of the conformal mating of $\hat{A}_{\Gamma, \text{BS}}$ (respectively, $\hat{A}_{\Gamma, \text{hBS}}$) and P vary as Γ runs over $\text{Teich}(S_{0,k})$ and P runs over \mathcal{H}_{2k-1} (respectively, over $\mathcal{H}_{(k-1)^2}$)?*

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