Homotopical Height

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Question

Given f.p. group G and a class $\mathcal{C}$ of smooth manifolds (e.g. symplectic, contact, Kähler etc), what is the obstruction to constructing a $K(G,1)$ manifold within the class $\mathcal{C}$?

Definition

G - finitely presented group;
$\mathcal{C}$ - class of smooth manifolds of dimension greater than zero.
$ht_{\mathcal{C}}(G)$ is $-\infty$ if G is not the π_1 of some manifold in $\mathcal{C}$.
If $\pi_2(M) \neq 0$ for all $M \in \mathcal{C}$ with $\pi_1(M) = G$, then $ht_{\mathcal{C}}(G)$ is 2.
$ht_{\mathcal{C}}(G)$ is greater than or equal to n if there exists a manifold $M \in \mathcal{C}$ such that $\pi_1(M) = G$ and $\pi_i(M) = 0$ for every $1 < i < n$.
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G is of type FP if it admits a finite $K(G,1)$ space.
Hardness or softness (à la Gromov): Class $\mathcal{C}$ is of

Type 1: $ht_{\mathcal{C}}(G) = \infty$ for all groups of type FP;

Type 2: if $ht_{\mathcal{C}}(\{1\}) = \infty$ for the trivial group;

Type 3: if $ht_{\mathcal{C}}(G) \geq 0$ for all groups of type FP;

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Type 1 – softest; Type 4 – hardest
Type 2 and Type 3 are classes that exhibit intermediate behavior of different (and not quite comparable) kinds.

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Examples relevant to this talk:

Type 1: Closed Almost Complex Manifolds: $\mathcal{AC}$, $ht_{\mathcal{AC}}(G) = \infty$ for all groups of type *FP*

Type 2: Closed Complex Analytic Manifolds: $ht_{\mathcal{CA}}(\{1\}) = \infty$ for the trivial group (Calabi-Eckmann)

Type 3: Closed Complex Analytic Manifolds: $ht_{\mathcal{CA}}(G) \geq 0$ for all groups of type *FP* (Gompf-Taubes);

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First, we show that the class $\mathcal{AC}$ of Closed Almost Complex Manifolds is soft of Type I.

Proof Idea:

Step 1: X is a finite K(G,1). Embed X in $\mathbb{R}^N$ for large N. Take regular neighborhood of X and let M be its boundary.

Then M has higher homotopy groups vanishing till as far as one likes (taking N larger and larger).

Also M is of codimension one in $\mathbb{R}^N$. Hence TM is stably trivial. Therefore $M \times \mathbb{R}^m$ has trivial tangent bundle for all large enough m. Hence $M \times \mathbb{R}^m$ is almost complex for all large enough m whenever $m + dim(M)$ is even.

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Hence for any non-trivial element $g \in G$, the cyclic group $\langle g \rangle \subset G$ acts freely by complex automorphisms on C_n.
$i_g = [G : \langle g \rangle]$ = index of $\langle g \rangle$ in G. There is an action of G on the Cartesian product $C_n^{i_g}$ by holomorphic automorphisms, such that the action of the subgroup $\langle g \rangle$ is free. Hence diagonal action of G on the product

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Question (Kollar)

Is a projective group G_1 commensurable to a group G, admitting a $K(G, 1)$ space which is a smooth quasi-projective variety?

Dimca, Papadima and Suciu have furnished examples of finitely presented groups giving a negative answer to this Question.

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Tools:

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Ingredient 1: Bieri's Theorem on Poincaré Duality Groups:

Theorem

(Bieri) Let $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ be a short exact sequence of groups, with both G, Q PD groups. Further suppose that N is not a PD group of finite cohomology dimension $cd(G) - cd(Q)$. Then N is not of type FP. Hence, N cannot have a K(G,1) space homotopy equivalent to a finite CW complex. In particular, N cannot have a quasiprojective K(G,1) space.

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(Bieri) Let $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ be a short exact sequence of groups, with both G, Q PD groups. Further suppose that N is not a PD group of finite cohomology dimension $cd(G) - cd(Q)$. Then N is not of type FP. Hence, N cannot have a K(G,1) space homotopy equivalent to a finite CW complex. In particular, N cannot have a quasiprojective K(G,1) space.

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Suppose
a) M is a closed orientable $2n$–dimensional manifold,
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(e.g. any smooth complex projective variety M realizing $ht_{\mathfrak{S}}(G)$ or a hyperplane section of M) Then

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Ingredient 3: Topological Lefschetz fibration:

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A topological Lefschetz fibration on a smooth, closed, oriented $2n-$manifold M consists of the following data:

- *a closed orientable 2-manifold S,*
- *a finite set of points $K = \{b_i\} \subset M$ called the critical set,*
- *a smooth map $f : M \longrightarrow S$ whose differential df is surjective outside K,*
- *for each critical point x of f, there are orientation preserving coordinate charts about x and $f(x)$ (into $\mathbb{C}^n$ and $\mathbb{C}$, respectively) in which f is given by $f(z_1, \cdots, z_n) = \sum_{i=1\cdots n} z_i^2$, and*
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Ingredient 4: (Complex) Morse Theory gives the follwoing topological generalization of a Theorem of Dimca-Papadima-Suciu:

Theorem

Let $f : M \longrightarrow S$ be an irrational topological Lefschetz fibration that is not a Kodaira fibration, with $\dim M = 2n+2$, $n \geq 2$. *Let K be the finite critical set of f. Further suppose that $\widetilde{M}$ is contractible. Let F denote the regular fiber and $N = \pi_1(F)$. Then*

a) $\pi_k(F) = 0$ for $1 < k < n$,

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b) $\pi_n(F)$ is a free $\mathbb{Z}N$-module,
with generators in one-to-one correspondence with $K \times \pi_1(S)$,
c) $\widetilde{F}$ is homotopy equivalent to a wedge of n–spheres.

Ingredient 4: (Complex) Morse Theory gives the follwoing topological generalization of a Theorem of Dimca-Papadima-Suciu:

Theorem

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Theorem

d) N cannot be of type FP; in particular, there does not exist a quasiprojective $K(N,1)$ space.

Sketch of Proof:
Note that $\pi_1(M)$ and $\pi_1(S)$ are PD groups of dimension $(2n+2)$ and 2 respectively.
To show that N cannot be of type FP, it suffices (by Theorem 4) to show that N cannot be a PD(2n) group.
Spectral Sequence Proposition gives

$$0 \longrightarrow H^n(N, \mathbb{Z}N) \longrightarrow H^n(F, \mathbb{Z}N) \longrightarrow H^n(\widetilde{F}, \mathbb{Z}N)^N \longrightarrow H^{n+1}(N, \mathbb{Z}N) \longrightarrow \cdots$$

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If N is PD(2n) group, then $H^n(N, \mathbb{Z}N) = H^{n+1}(N, \mathbb{Z}N) = 0$ because $n \geq 2$.
Hence $H^n(F, \mathbb{Z}N) = H^n(\widetilde{F}, \mathbb{Z}N)^N$.
Now, by Poincaré Duality and the Hurewicz' Theorem, we have,

$$H^n(F, \mathbb{Z}N) = H^n_c(\widetilde{F}) = H_n(\widetilde{F}) = \pi_n(\widetilde{F}) = \bigoplus_I \mathbb{Z},$$

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$$H^n(\widetilde{F}, \mathbb{Z}N)^N = (Hom_{\mathbb{Z}}(H_n(\widetilde{F}), \mathbb{Z}N))^N = (Hom_{\mathbb{Z}}(\pi_n(\widetilde{F}), \mathbb{Z}N))^N =$$

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Hence

$$\bigoplus_I \mathbb{Z} = \prod_I \mathbb{Z}N\,.$$

Since $f : M \longrightarrow S$ is irrational, I is countably infinite.
Therefore, $\bigoplus_I \mathbb{Z}$ is countable and $\prod_I \mathbb{Z}N$ is uncountable and the two cannot be equal.
A contradiction.

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