Maps on Boundaries of Hyperbolic Metric Spaces

by

Mahan Mitra

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Committee in charge:

Professor Andrew Casson, Chair
Professor John Stallings
Professor Steve Evans

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The dissertation of Mahan Mitra is approved:

Chair Date

Date

Date

University of California at Berkeley

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Abstract

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Let H be a hyperbolic normal subgroup of a hyperbolic group G. Then $i : \Gamma_H \to \Gamma_G$ extends continuously to a map $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$.

Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let v be a vertex of T. Let (X_v, d_v) denote the hyperbolic metric space corresponding to v. Then $i : X_v \to X$ extends continuously to a map $\hat{i} : \widehat{X_v} \to \widehat{X}$. These generalize and give a new proof of a Theorem of Cannon and Thurston. Applications to graphs of hyperbolic groups and local connectivity of limit sets of Kleinian groups are also given.

Professor Andrew Casson
Dissertation Committee Chair

To G.C. and family,

for their not quite anonymous help.

Contents

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Chapter 1

Introduction

The study of hyperbolic groups was started by Gromov in [16]. In this thesis, the main objects of study are hyperbolic subgroups of hyperbolic groups.

1.1 Statement of Results

Let G be a hyperbolic group in the sense of Gromov [16]. Let H be a hyperbolic subgroup of G. In this thesis we study the extrinsic geometry of H in G. Since the objects of study here come under the purview of coarse geometry, asymptotic or 'large-scale' information is of crucial importance. That is to say, one would like to know what happens 'at infinity'. We put this in the more general context of a hyperbolic group H acting freely and properly discontinuously by isometries on a proper hyperbolic metric space X. Then there is a natural map $i : \Gamma_H \rightarrow X$, sending the vertex set of Γ_H to the orbit of a point under H, and connecting images of adjacent vertices in Γ_H by geodesics in X. Let $\widehat{X}$ denote the Gromov compactification of X.

The main question addressed here is the following:

Question: Does the continuous proper map $i : \Gamma_H \rightarrow X$ extend to a continuous map $\hat{i} : \widehat{\Gamma_H} \rightarrow \widehat{X}$?

Questions along this line have been raised by Bonahon [5]. Related questions in the context of Kleinian groups have been studied by Cannon and Thurston [8], Bonahon [6], Floyd [13] and Minsky [24]. In, [8], [13] or [24], explicit metrics were used. So though some of their results can be thought of as 'coarse', the techniques of proof are not.

Much of this work was motivated by work of Cannon and Thurston [8], who gave a positive answer to the above question for examples coming from closed hyperbolic 3-manifolds fibering over the circle.

Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber F. Let $\widetilde{F}$ and $\widetilde{M}$ denote the universal covers of F and M respectively. Then $\widetilde{F}$ and $\widetilde{M}$ are quasi-isometric to $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. Let $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ denote the standard compactifications. In [8] Cannon and Thurston show that the usual inclusion of $\widetilde{F}$ into $\widetilde{M}$ extends to a continuous map from $\mathbb{D}^2$ to $\mathbb{D}^3$.

In this thesis we use coarse methods to answer the above question affirmatively for two classes of examples (see below). Both can be regarded as generalizations of Cannon and Thurston's work on 3 manifolds fibering over the circle.

Theorem : *Let G be a hyperbolic group and let H be a hyperbolic subgroup that is normal in G. Let $i : \Gamma_H \rightarrow \Gamma_G$ be the continuous proper embedding of Γ_H in Γ_G described above. Then i extends to a continuous map $\hat{i}$ from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$.*

Theorem : *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let v be a vertex of T. Let (X_v, d_v) denote the hyperbolic metric space corresponding to v. If X is hyperbolic, Then $i : X_v \rightarrow X$ extends to a continuous map $\hat{i}$ from $\widehat{X_v}$ to $\widehat{X}$.*

A direct consequence is the following:

Corollary : *Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let H be the stabilizer of a vertex or edge of T. Then $i : \Gamma_H \rightarrow \Gamma_G$ extends continuously to a map $\hat{i}$ from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$.*

In [8] Cannon and Thurston give an explicit description of the continuous extension in terms of 'ending laminations' [See [33] for definitions]. The explicit description depends on Thurston's theory of stable and unstable laminations for pseudo-anosov diffeomorphisms of surfaces [12]. Some aspects of Thurston's theory of ending laminations are also generalized here to the context of normal hyperbolic subgroups of hyperbolic groups. Using this we give an explicit description of the map $\hat{i} : \widehat{\Gamma_H} \rightarrow \widehat{\Gamma_G}$ for H a normal hyperbolic subgroup of a hyperbolic group G.

1.2 Background in Geometric Group Theory

To a group G with finite generating set X, one associates the **Cayley Graph** $\Gamma = \Gamma(G,X)$ of (G,X), which is a directed graph whose vertex set consists of elements of G, with a directed edge from g to gx for each $g \in G$ and $x \in X$. The Cayley graph is given the **word metric** where each edge has length one and distance between two vertices is the length of the shortest path between them.

A geodesically complete proper path metric space X is δ-hyperbolic if geodesic triangles in X are uniformly δ-thin, i.e. if any side is contained in the δ-neighborhood of the other two. A path metric space is hyperbolic if it is δ-hyperbolic for some $\delta > 0$. A finitely generated group is hyperbolic if its Cayley graph with respect to some finite generating set is a hyperbolic metric space. Most facts about hyperbolic groups used in this thesis can be found in [16], [9], [14].

Let (X,d) be a hyperbolic metric space.

Definitions: *A subset Z of X is said to be k-***quasiconvex** *if any geodesic joining $a,b \in Z$ lies in a k-neighborhood of Z. A subset Z is* **quasiconvex** *if it is k-quasiconvex for some k. A map f from one metric space (Y,d_Y) into another metric space (Z,d_Z) is said to be a (K,ϵ)-***quasi-isometric embedding** *if*

$$\frac{1}{K}(d_Y(y_1,y_2)) - \epsilon \leq d_Z(f(y_1),f(y_2)) \leq K d_Y(y_1,y_2) + \epsilon$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a **quasi-isometry***. A (K,ϵ)-quasi-isometric embedding that is a quasi-isometry will be called a (K,ϵ)-quasi-isometry.*

*A (K,ϵ)-***quasigeodesic** *is a (K,ϵ)-quasi-isometric embedding of a closed interval in $\mathbb{R}$. A $(K,0)$-quasigeodesic will also be called a K-quasigeodesic.*

The **Gromov boundary** of X, denoted by ∂X, is the collection of equivalence classes of geodesic rays $r : [0,\infty) \rightarrow \Gamma$ with $r(0) = x_0$ for some fixed $x_0 \in X$, where rays r_1 and r_2 are equivalent if $sup\{d(r_1(t), r_2(t))\} < \infty$. $\widehat{X}$=$X \cup \partial X$ will denote the natural compactification of X topologized the usual way(cf.[14] pg. 124).

The **Gromov inner product** of elements a and b relative to c is defined by

$$(a,b)_c = 1/2[d(a,c) + d(b,c) - d(a,b)].$$

1.3 Cannon-Thurston Maps

Let (X, d_X) be a hyperbolic metric space and Y be a subspace that is hyperbolic with the inherited path metric d_Y. By adjoining the Gromov boundaries ∂X and ∂Y to X and Y, one obtains their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively.

Let $i : Y \to X$ denote inclusion.

Definition: *Let X and Y be hyperbolic metric spaces and $i : Y \to X$ be an embedding. A* **Cannon-Thurston map** *$\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of i.*

The following lemma says that a Cannon-Thurston map exists if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if a geodesic λ lies outside an N ball around y in Y then any geodesic in X joining the end-points of λ lies outside the M ball around $i(y)$ in X. For convenience of use later on, we state this somewhat differently.

Lemma 1.3.1 *A Cannon-Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists if the following condition is satisfied:*

Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ lying outside an N-ball around $y_0 \in Y$ any geodesic segment in Γ_G joining the end-points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.

Proof: Suppose $i : Y \to X$ does not extend continuously . Since i is proper, there exist sequences x_m, $y_m \in Y$ and $p \in \partial Y$, such that $x_m \to p$ and $y_m \to p$ in $\widehat{Y}$, but $i(x_m) \to u$ and $i(y_m) \to v$ in $\widehat{X}$, where $u, v \in \partial X$ and $u \neq v$.

Since $x_m \to p$ and $y_m \to p$, any geodesic in Y joining x_m and y_m lies outside an N_m-ball $y_0 \in Y$, where $N_m \to \infty$ as $m \to \infty$. Any bi-infinite geodesic in X joining $u, v \in \partial X$ has to pass through some M-ball around $i(y_0)$ in X as $u \neq v$. There exist constants c and L such that for all $m > L$ any geodesic joining $i(x_m)$ and $i(y_m)$ in X passes through an $(M + c)$-neighborhood of $i(y_0)$. Since $(M + c)$ is a constant not depending on the index m this proves the lemma. $\square$

The above result can be interpreted as saying that a Cannon-Thurston map exists if the space of geodesic segments in Y embeds properly in the space of geodesic segments in X.

The criterion provided by Lemma 1.3.1 above will be used in the proofs of Theorem 2.2.3 and Theorem 4.1.10 .

Chapters 2, 3 and 4 of this thesis overlap substantially with [25], [27] and [26] respectively.

Chapter 2

Hyperbolic Group Extensions

The main theorem of this chapter is the following:

Theorem 2.2.3 *Given a short exact sequence of finitely generated groups*

$$1 \to H \to G \to Q \to 1,$$

such that H and G are hyperbolic, there exists a Cannon-Thurston map from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$.

When H is finite the theorem is vacuously true as $\partial\Gamma_H = \emptyset$. When H is virtually cyclic, H is quasiconvex in G [cf. [14] pg. 155] and the theorem is again trivial. Therefore we shall assume henceforth that H and G are non-elementary.

A brief outline of the proof: Given a geodesic segment $\lambda \subset \Gamma_H$ we construct a set $B_\lambda \subset \Gamma_G$ containing $i(\lambda)$. B_λ can be roughly described as the union of the images of $i(\lambda)$ under *'right action'* by elements of Q. We describe this construction in the next section and show that there exists a map from Γ_G to B_λ which does not increase distances much. This shows that B_λ's are uniformly quasiconvex. In the final section it is shown that if λ lies outside a large ball around the identity in Γ_H, B_λ lies outside a large ball in Γ_G. Combining this with Lemma 1.3.1 the proof is completed.

2.1 Construction of Quasiconvex Sets

Our starting point is a Lemma due to Lee Mosher[28]. Consider a surjective homomorphism $P : G \to Q$ of finitely generated groups. A **quasi-isometric section** is a subset $\Sigma \subset G$ mapping onto Q such that for any $g, g' \in \Sigma$,

$$\frac{1}{\kappa}d_Q(Pg,Pg') - \epsilon_0 \leq d_G(g,g') \leq \kappa d_Q(Pg,Pg') + \epsilon_0$$

where d_G and d_Q are word metrics in G, Q respectively and $\kappa \geq 1$, $\epsilon_0 \geq 0$ are constants. A **single-valued quasi-isometric section** $\sigma : Q \to G$ is defined by choosing a single element of Σ representing each coset of the kernel of P.

Lemma 2.1.1 *(Quasi-isometric section Lemma [28])Given a non-elementary hyperbolic group H and a short exact sequence of finitely generated groups*

$$1 \to H \to G \to Q \to 1,$$

the map $P : G \to Q$ has a quasi-isometric section Σ. In fact, choosing a generating set B for G and letting $P(B)$ be the generating set for Q, we have for all $g, g' \in \Sigma$,

$$d_Q(Pg,Pg') \leq d_G(g,g') \leq \kappa d_Q(Pg,Pg') + \epsilon_0$$

for some constants $\kappa \geq 1$ and $\epsilon_0 \geq 0$.

When the short exact sequence splits, this is clear. The general case is proven in [28].

Given a quasi-isometric section, choose one element from each coset of H to get a single-valued quasi-isometric section. This is still a quasi-isometric section as $Pg = Pg'$ implies $d_G(g,g') \leq \epsilon_0$. Also, using a left translation t_h by an element $h \in H \subset G$, one can assume that Σ contains the identity element of G. $t_h(\Sigma)$ is still a single-valued quasi-isometric section as t_h preserves cosets. Assume therefore that $1 \to H \to G \to Q \to 1$ is an exact sequence of finitely generated groups with H, G hyperbolic and $\sigma : Q \to G$ is a single-valued quasi-isometric section containing the identity element of G.

Corresponding to every element $g \in G$ there exists an automorphism of H taking h to $g^{-1}hg$ for $h \in H$. Such an automorphism induces a bijection ϕ_g of the vertices of Γ_H. This gives rise to a map from Γ_H to itself, sending an edge $[a, b]$ linearly to a shortest edge-path joining $\phi_g(a)$ to $\phi_g(b)$. Abusing notation slightly, we shall also call this map ϕ_g.

Given a geodesic segment $\lambda \subset \Gamma_H$ the construction of a certain set $B_\lambda \subset \Gamma_G$ will now be described. B_λ will turn out to be quasiconvex. Let a, b denote the end-points of λ and let $\lambda_g \subset \Gamma_H$ denote a geodesic joining $\phi_g(a)$ to $\phi_g(b)$ in Γ_H.

Recall that t_g denotes left translation by an element g of G. Then define

$$B_\lambda = \bigcup_{g \in \sigma(Q)} t_g \cdot i(\lambda_g).$$

Mark that B_λ contains $i(\lambda)$ as $1 \in \sigma(Q)$. It is important to note also that λ_g is contained in Γ_H and not in Γ_G. Its only after acting on λ_g by $t_g \cdot i$ that we obtain a subset of Γ_G.

There is a simple informal way to describe B_λ. Suppose λ joins vertices $a, b \in \Gamma_H$. Then ag, bg lie in the same (right or left) coset. Join ag, bg by a shortest edge-path lying entirely in the corresponding coset. The union of all these edge-paths as g ranges over $\sigma(Q)$ is B_λ.

The aim of the rest of this section is to prove that B_λ is C'-quasiconvex for some C' independent of λ. To do this, we shall define a map $\Pi_\lambda : \Gamma_G \to B_\lambda$ and show that it doesn't increase distances much.

There is a natural identification of Γ_H with $t_g(i(\Gamma_H))$ taking h to $t_g(i(h))$. On Γ_H define a map $\pi_{g,\lambda} : \Gamma_H \to \lambda_g$ taking h to one of the points on λ_g closest to h in the metric d_H. Strictly speaking, $\pi_{g,\lambda}$ is defined only on the vertex set, but this is enough for our purposes. Now define

$$\Pi_\lambda \cdot t_g \cdot i(h) = t_g \cdot i \cdot \pi_{g,\lambda}(h) \text{ for } g \in \sigma(Q).$$

For every $g' \in \Gamma_G$, there exists a unique $g \in \sigma(Q)$ such that $g' \in t_g(i(h))$ as σ is a single-valued section. Hence, Π_λ is well-defined on the entire vertex set of Γ_G.

In order to show that Π_λ does not increase distances by more than a bounded factor, one needs to first show that $\pi_{g,\lambda}$ does not increase distances much. More precisely if $h, h' \in \Gamma_H$, it will be shown that $d_H(\pi_{g,\lambda}(h), \pi_{g,\lambda}(h')) \leq C_1 d_H(h, h')$, where $C_1 \geq 1$ is independent of g. This follows from the following lemma.

Lemma 2.1.2 *Let H be a $\delta-$hyperbolic group and let $\mu \subset \Gamma_H$ be a geodesic segment. Let $\pi : \Gamma_H \to \mu$ map $h \in \Gamma_H$ to one of the points on μ nearest to h. Then $d_H(\pi(x), \pi(y)) \leq C_1 d_H(x, y)$ for all $x, y \in H$ where C_1 depends only on δ.*

Proof: Let $[a, b]$ denote a geodesic edge-path joining vertices a, b. It suffices by repeated use of the triangle inequality to prove the Lemma when $d_H(x, y) = 1$. Let u, v, w be points on $[x, \pi(x)]$, $[\pi(x), \pi(y)]$ and $[\pi(y), x]$ respectively such that $d_H(u, \pi(x)) = d_H(v, \pi(x))$, $d_H(v, \pi(y)) = d_H(w, \pi(y))$ and $d_H(w, x) = d_H(u, x)$. Since H is δ-hyperbolic, the diameter of the inscribed triangle with vertices u, v, w is less than or equal to 2δ (See [1] pg. 16).

$$\begin{aligned} d_H(u,x)+d_H(u,v) \geq d_H(x,\pi(x)) &= d_H(u,x)+d_H(u,\pi(x)) \\ \Rightarrow d_H(u,\pi(x)) \leq d_H(u,v) &\leq 2\delta \\ \Rightarrow d_H(x,\pi(x))+d_H(\pi(x),\pi(y))-d_H(x,\pi(y)) &\leq 4\delta. \end{aligned}$$

Similarly $d_H(y,\pi(y))+d_H(\pi(x),\pi(y))-d_H(y,\pi(x)) \leq 4\delta$.
Therefore,

$$\begin{aligned} & 2d_H(\pi(x),\pi(y)) \\ \leq \;& 8\delta + d_H(x,\pi(y)) - d_H(y,\pi(y)) + d_H(y,\pi(x)) - d_H(x,\pi(x)) \\ \leq \;& 8\delta + d_H(x,y) + d_H(x,y) \\ \leq \;& 8\delta + 2 \end{aligned}$$

Hence $d_H(\pi(x),\pi(y)) \leq 4\delta+1$. Choosing $C_1 = 4\delta+1$, we are through. $\Box$

We will now show that if an edge in Γ_G has end-points x, y not lying in the same coset $t_g(i(\Gamma_H))$, then $d_G(\Pi_\lambda(x),\Pi_\lambda(y)) \leq C$, for some constant C independent of x, y, λ.

Lemma 2.1.3 *Let H be a δ-hyperbolic group. Let μ be a geodesic segment in Γ_H with end-points a, b and let c be any vertex in Γ_H. Let y be a vertex on μ such that $d_H(c,y) \leq d_H(c,z)$ for any $z \in \mu$. Then a geodesic path from c to y followed by a geodesic path from y to a is a k-quasigeodesic for some k dependent only on δ.*

Proof: It suffices to show that if $x \in [c,y]$ and $z \in [y,a]$ then $d_H(x,y) + d_H(y,z) \leq k d_H(x,z)$ for some k depending only on δ.

As in Lemma 2.1.2 let u, v, w be points on edges $[x,y]$, $[y,z]$ and $[z,x]$ respectively such that $d_H(u,y) = d_H(v,y)$, $d_H(v,z) = d_H(w,z)$ and $d_H(w,x) = d_H(u,x)$. Then $d_H(u,y) \leq 2\delta$. Hence

$$d_H(x,y) + d_H(y,z) \leq d_H(x,z) + 4\delta$$

Choosing $k = 1 + 4\delta$ we are through. $\Box$

Recall that the Gromov inner product $(a,b)_c = 1/2[d_H(a,c) + d_H(b,c) - d_H(a,b)]$.

Lemma 2.1.4 *Suppose H is δ-hyperbolic. If μ is a (k_0,ϵ_0)-quasigeodesic in Γ_H and p, q, r are 3 points in order on μ then $(p,r)_q \leq k_1$ for some k_1 dependent on k_0, ϵ_0 and δ only.*

Proof: $[a,b]$ will denote a geodesic path joining a,b. Since p,q,r are 3 points in order on μ, $[p,q]$ followed by $[q,r]$ is a (k_0,ϵ_0)-quasigeodesic in the δ-hyperbolic metric space Γ_H. Hence there exists a k_1 dependent on k_0, ϵ_0 and δ alone such that $d_H(q,[p,r]) \leq k_1$. Let s be a point on $[p,r]$ such that $d_H(q,s) = d_H(q,[p,r]) \leq k_1$. Then

$$\begin{aligned}(p,r)_q &= 1/2(d_H(p,q) + d_H(r,q) - d_H(p,r)) \\ &= 1/2(d_H(p,q) + d_H(r,q) - d_H(p,s) - d_H(r,s)) \\ &\leq d_H(q,s) \leq k_1. \square\end{aligned}$$

Recall that $P: G \to Q$ is the natural surjective homomorphism.

Lemma 2.1.5 *Let $1 \to H \to G \to Q \to 1$ be as in the statement of the main theorem and $\sigma{:}Q \to G$ be a single-valued quasi-isometric section obtained from Lemma 2.1.1. Further let κ and ϵ_0 be as in 2.1.1. Then $\{(\sigma{\cdot}P(x))^{-1}(\sigma{\cdot}P(y))\}$ lies in a finite ball S of radius $\kappa_1 = \kappa + \epsilon_0$ around the identity in Γ_G when $d_G(x,y) = 1$.*

Proof: From Lemma 2.1.1,

$$d_G(x,y) = 1 \Rightarrow d_Q(P(x),P(y)) \leq 1$$
$$\text{Hence } d_G(\sigma(P(x)),\sigma(P(y))) \leq \kappa + \epsilon_0 = \kappa_1 (\text{say}).$$

Thus $\{(\sigma{\cdot}P(x))^{-1}(\sigma{\cdot}P(y))\}$ lies in a finite ball S of radius κ_1 around the identity in Γ_G when $d_G(x,y) = 1$. $\square$

Recall that every element g of G gives an automorphism of H (by conjugation)and this in turn gives rise to a map ϕ_g of Γ_H to itself which is a bijection on the vertex set of Γ_H. Each ϕ_g is a quasi-isometry. Since S is finite, there exist $K \geq 1$ and $\epsilon \geq 0$ such that for all $g \in S$, ϕ_g is a (K,ϵ)-quasi-isometry. Then the image of a geodesic under ϕ_g is a (K,ϵ)-quasigeodesic for all $g \in S$. The following Lemma lies at the heart of the proof of our main Theorem.

Lemma 2.1.6 *Suppose H is δ-hyperbolic. Let μ_1 be some geodesic segment in Γ_H joining a,b and let p be any vertex of Γ_H. Also let q be a vertex on μ_1 such that $d_H(p,q) \leq d_H(p,x)$ for $x \in \mu_1$. Let μ_2 be a geodesic segment in Γ_H joining $\phi_g(a)$ to $\phi_g(b)$ for some $g \in S$. Let r be a point on μ_2 such that $d_H(\phi_g(p),r) \leq d_H(\phi_g(p),x)$ for $x \in \mu_2$. Then $d_H(r,\phi_g(q)) \leq C_2$ for some constant C_2 independent of a,b,g,p.*

Proof: From the Lemma 2.1.5, $\phi_g(\mu_1)$ is a $(K, \epsilon)-$ quasigeodesic joining $\phi_g(a)$ to $\phi_g(b)$ and hence lies in a K'-neighborhood of μ_2 where K' depends only on K, ϵ, δ. Let u be a vertex in $\phi_g(\mu_1)$ lying at a distance at most K' from r. Without loss of generality suppose that u lies on $\phi_g([q, b])$, where $[q, b]$ denotes the geodesic subsegment of μ_1 joining q, b. [See Figure 2.1 below.]

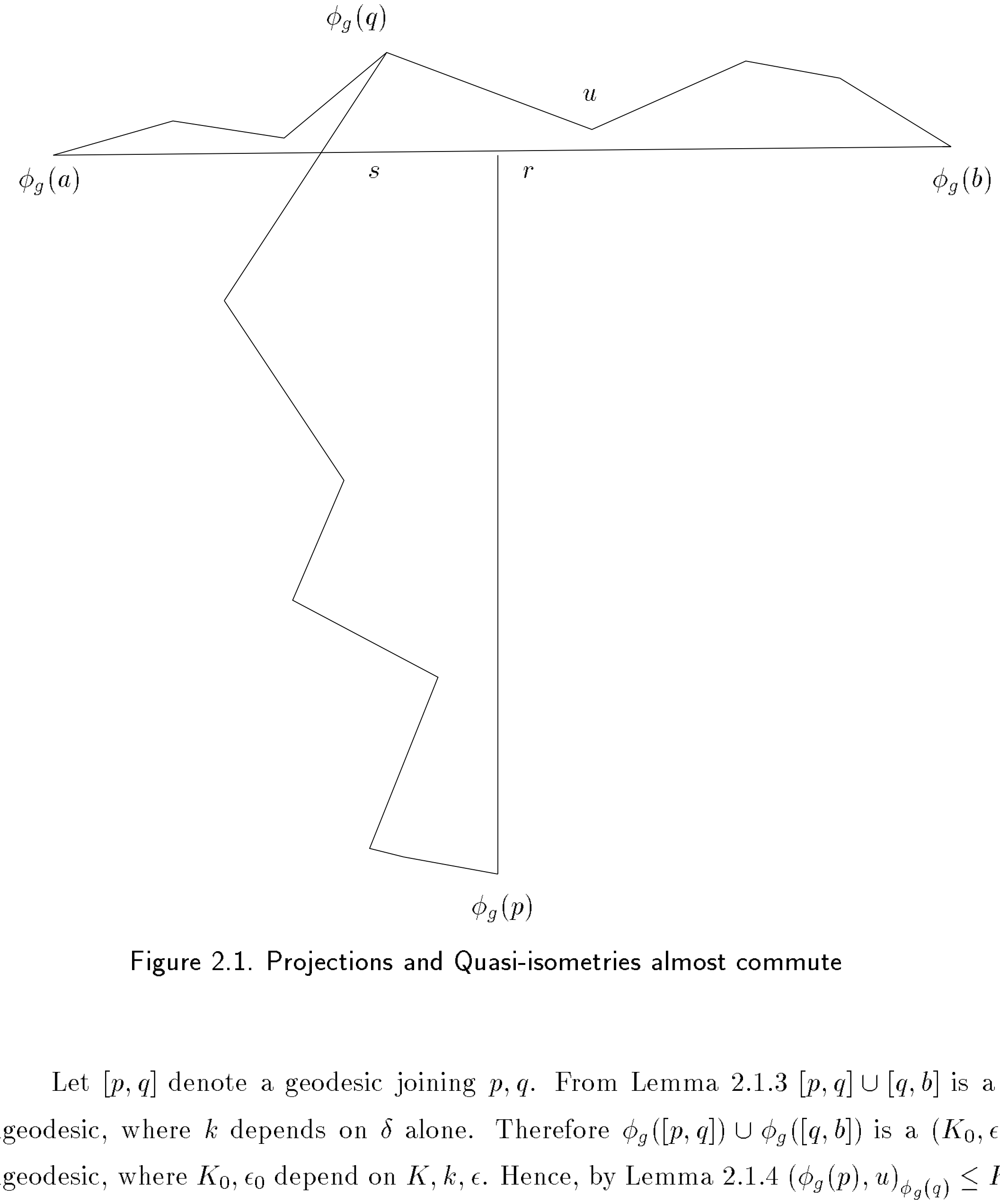

Figure 2.1. Projections and Quasi-isometries almost commute

Let $[p, q]$ denote a geodesic joining p, q. From Lemma 2.1.3 $[p, q] \cup [q, b]$ is a k-quasigeodesic, where k depends on δ alone. Therefore $\phi_g([p, q]) \cup \phi_g([q, b])$ is a (K_0, ϵ_0)-quasigeodesic, where K_0, ϵ_0 depend on K, k, ϵ. Hence, by Lemma 2.1.4 $(\phi_g(p), u)_{\phi_g(q)} \leq K_1$, where K_1 depends on K, k, ϵ and δ alone. Therefore,

$$\begin{aligned}
&(\phi_g(p), r)_{\phi_g(q)} \\
&\quad = 1/2[d_H(\phi_g(p), \phi_g(q)) + d_H(r, \phi_g(q)) - d_H(r, \phi_g(p))] \\
&\quad \leq 1/2[d_H(\phi_g(p), \phi_g(q)) + d_H(u, \phi_g(q)) + d_H(r, u) \\
&\qquad\qquad\qquad\qquad -d_H(u, \phi_g(p)) + d_H(r, u)] \\
&\quad = (\phi_g(p), u)_{\phi_g(q)} + d_H(r, u) \\
&\quad \leq K_1 + K'
\end{aligned}$$

There exists $s \in \mu_2$ such that $d_H(s, \phi_g(q)) \leq K'$

$$\begin{aligned}
(\phi_g(p), r)_s &= 1/2[d_H(\phi_g(p), s) + d_H(r, s) - d_H(r, \phi_g(p))] \\
&\leq 1/2[d_H(\phi_g(p), \phi_g(q)) + d_H(r, \phi_g(q)) - d_H(r, \phi_g(p))] + K' \\
&= (\phi_g(p), r)_{\phi_g(q)} + K' \\
&\leq K_1 + K' + K' \\
&= K_1 + 2K'
\end{aligned}$$

Also, as in the proof of Lemma 2.1.2 $(\phi_g(p), s)_r \leq 2\delta$

$$\begin{aligned}
d_H(r, s) &= (\phi_g(p), s)_r + (\phi_g(p), r)_s \\
&\leq K_1 + 2K' + 2\delta \\
d_H(r, \phi_g(q)) &\leq K_1 + 2K' + 2\delta + d_H(s, \phi_g(q)) \\
&\leq K_1 + 2K' + 2\delta + K'
\end{aligned}$$

Let $C_2 = K_1 + 3K' + 2\delta$. Then $d_H(r, \phi_g(q)) \leq C_2$ and C_2 is independent of a, b, g, p.

□

We will now prove the main theorem of this section. From Lemma 2.1.5, if $d_G(x, y) = 1$ then $d_G(\sigma(P(x)), \sigma(P(y))) \leq \kappa_1$, where κ_1 depends only on the quasi-isometric section σ. Recall that S is the set of elements lying inside the κ_1 ball around the identity. Further recall that Π_λ is defined on the vertex set of Γ_G as

$$\Pi_\lambda \cdot t_g \cdot i(h) = t_g \cdot i \cdot \pi_{g,\lambda}(h) \text{ for } g \in \sigma(Q).$$

Theorem 2.1.7 *There exists a constant $C \geq 1$ such that for all geodesic segments $\lambda \subset \Gamma_H$ and $x, y \in \Gamma_G$,*

$$d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y).$$

Proof: As in Lemma 2.1.2 it suffices (by repeated use of the triangle inequality) to prove the theorem when $d_G(x, y) = 1$.

If $x, y \in t_g(i(\Gamma_H))$ for some $g \in \sigma(Q)$ then let $x = t_g(i(x_1))$ and $y = t_g(i(y_1))$. Assume for simplicity that the generating set for G intersects H in the generating set for H. So in this case, $d_H(x_1, y_1) = 1$.

$$\begin{aligned} d_G(\Pi_\lambda(x), \Pi_\lambda(y)) &\leq d_H(\pi_{g,\lambda}(x_1), \pi_{g,\lambda}(y_1)) \\ &\leq C_1 d_H(x_1, y_1)[\text{Lemma2.1.2}] \\ &= C_1. \end{aligned}$$

If $d_G(x, y) = 1$ but x, y do not lie in the same coset, then $d_G(\sigma(P(x)), \sigma(P(y))) \leq \kappa_1$. Hence $x \in t_{g_0}(i(\Gamma_H))$ for some $g_0 \in \sigma(Q)$ and $y \in t_{g_0 g}(i(\Gamma_H))$ for some $g \in S$. Let $t_{g_0}(x_2) = x$ and $t_{g_0 g}(y_2) = y$ for some $x_2, y_2 \in \Gamma_H$.

Then, unraveling definitions,

$$\begin{aligned} t_{g_0 g} \cdot i \cdot \phi_g \cdot \pi_{g_0,\lambda}(x_2) &= g_0 g g^{-1}(g_0^{-1}\Pi_\lambda(x))g \\ &= \Pi_\lambda(x)g \end{aligned}$$

And

$$\begin{aligned} t_{g_0 g} \cdot i \cdot \pi_{g_0 g,\lambda} \cdot \phi_g(x_2) &= \Pi_\lambda \cdot t_{g_0 g} \cdot i \cdot \phi_g(x_2) \\ &= \Pi_\lambda(g_0 g g^{-1} x_2 g) \\ &= \Pi_\lambda(xg) \end{aligned}$$

Now $\phi_g(\lambda_{g_0})$ is a K-quasigeodesic joining the endpoints of $\lambda_{g_0 g}$. Hence from Lemma 2.1.6 $d_H(\phi_g(\pi_{g_0,\lambda}(x_2)), \pi_{g_0 g,\lambda}(\phi_g(x_2))) \leq C_2$. This implies $d_G(t_{g_0 g} \cdot i \cdot \phi_g \cdot \pi_{g_0,\lambda}(x_2), t_{g_0 g} \cdot i \cdot \pi_{g_0 g,\lambda} \cdot \phi_g(x_2)) \leq C_2$.

Hence $d_G(\Pi_\lambda(xg), \Pi_\lambda(x)g) \leq C_2$.

Now $y, xg \in t_{g_0 g} \cdot i(\Gamma_H)$. Further, $y = t_{g_0 g} \cdot i(y_2)$ and $xg = t_{g_0 g} \cdot i \cdot \phi_g(x_2)$. Since $t_{g_0 g}$ is an isometry

$$d_G(i(y_2), i(\phi_g(x_2))) = d_G(y, xg) \leq d_G(x, xg) + d_G(x, y) \leq \kappa_1 + 1$$

Since Γ_H is properly embedded in Γ_G there exists a constant L independent of g, λ such that

$$\begin{aligned} & d_G(i(y_2), i(\phi_g(x_2))) \leq \kappa_1 + 1 \\ \Rightarrow \quad & d_H(y_2, \phi_g(x_2)) \leq L \\ \Rightarrow \quad & d_H(\pi_{g_0 g,\lambda}(y_2), \pi_{g_0 g,\lambda}(\phi_g(x_2))) \leq C_1 L \quad \text{[Lemma 2.1.2]}. \end{aligned}$$

Hence $d_G(\Pi_\lambda(xg), \Pi_\lambda(y)) \leq C_1 L$.

Therefore,

$$\begin{aligned} & d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \\ \leq \quad & d_G(\Pi_\lambda(x), \Pi_\lambda(x)g) + d_G(\Pi_\lambda(x)g, \Pi_\lambda(xg)) + d_G(\Pi_\lambda(xg), \Pi_\lambda(y)) \\ \leq \quad & \kappa_1 + C_2 + C_1 L. \end{aligned}$$

Taking $C = max\{C_1, \kappa_1 + C_2 + C_1 L\}$ one sees that $d_G(x, y) = 1 \Rightarrow$ $d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C$ and we are through. $\Box$

Remark: The above theorem makes essential use of the hyperbolicity of H. In fact this theorem is false in the case of a $Z + Z$ subgroup of the fundamental group of a closed 3-manifold with Sol-geometry. To see the idea, regard Sol as $\mathbb{R}^3$ equipped with the metric $ds^2 = e^t dx^2 + e^{-t} dy^2 + dt^2$. Then the plane $x = y$ is not quasi-isometrically embedded in Sol.

2.2 Proof of Main Theorem

The hypothesis that G is hyperbolic has not yet been used. Theorem 2.1.7 above shows that $d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y)$ where C is independent of λ. Suppose μ is a geodesic in Γ_G starting and ending in B_λ. Then $\Pi_\lambda(\mu)$ is a C-quasigeodesic. Since quasigeodesics lie in a bounded neighborhood of geodesics in any hyperbolic metric space, this proves:

Lemma 2.2.1 *There exists $C' \geq 0$ such that for all geodesic segments $\lambda \subset \Gamma_H$, B_λ is C'-quasiconvex in Γ_G .*

Thus, there exists C' independent of λ such that every geodesic with end points on B_λ lies in a C'-neighborhood of B_λ. In particular, any geodesic joining the end-points of $i(\lambda)$ lies in a C'-neighborhood of B_λ.

We need one final Lemma.

Lemma 2.2.2 *There exists $A \geq 1$ such that for all $g \in \sigma(Q)$ and $x \in t_g{\cdot}i(\lambda_g)$ there exists $y \in i(\lambda)$ such that $d_G(x,y) \leq Ad_Q(1,Px)$.*

Proof: Let μ be a geodesic path in Γ_Q joining $1 \in \Gamma_Q$ to $Px \in \Gamma_Q$. Order the vertices on μ so that we have a finite sequence $1 = y_0, y_1, \cdots, y_n = Px = Pg$ such that $d_Q(y_i, y_{i+1}) = 1$. and $d_Q(1, Px) = n$. Since σ is a quasi-isometric section, this gives a sequence $\sigma(y_i) = g_i$ such that $d_G(g_i, g_{i+1}) \leq \kappa + \epsilon_0 \leq \kappa_1$. Observe that $g_n = g$.

From Lemma 2.1.5 $g_{i+1}^{-1}g_i = s_i \in S$. Hence ϕ_{s_i} is a (K,ϵ)-quasi-isometry. Note that K, ϵ are independent of the index i.

Let $z = t_{g_{i+1}} \cdot i(u)$ where $u \in \lambda_{g_{i+1}}$.

$$\begin{aligned} t_{g_i}{\cdot}i{\cdot}\phi_{s_i}(u) &= g_i(g_{i+1}^{-1}g_i)^{-1}({g_{i+1}}^{-1}z)(g_{i+1}^{-1}g_i) \\ &= zg_{i+1}^{-1}g_i \\ &= zs_i \end{aligned}$$

Since ϕ_{s_i} is a (K,ϵ)-quasi-isometry, $\phi_{s_i}(\lambda_{g_{i+1}})$ is a (K,ϵ)-quasigeodesic in Γ_H joining the end-points of λ_{g_i}. Therefore there exists v on λ_{g_i} such that $d_H(v, \phi_{s_i}(u)) \leq K'$, where K' depends on K, ϵ and δ. Hence,
$d_G(t_{g_i} \cdot i(v), t_{g_i} \cdot i \cdot \phi_{s_i}(u)) \leq K'$. Let $t_{g_i} \cdot i(v) = w$. Then $d_G(w, zs_i) \leq K'$ and

$$d_G(w,z) \leq d_G(w, zs_i) + d_G(zs_i, z) \leq K' + \kappa_1 = A(say).$$

Thus we have shown that given $z \in t_{g_{i+1}} \cdot i(\lambda_{g_{i+1}})$ there exists $w \in t_{g_i} \cdot i(\lambda_{g_i})$ such that $d_G(z,w) \leq A$ where A is independent of λ.

$x \in t_g{\cdot}i(\lambda_g)$. Let $x = x_n$. Then there exist $x_i \in t_{g_i}{\cdot}i(\lambda_{g_i})$ for $i = 0 \cdots n$ such that $d_G(x_i, x_{i+1}) \leq A$. Choosing $y = x_0$ we have $y \in i(\lambda)$ and $d_G(x,y) \leq An = Ad_Q(1, Px)$. $\Box$

We are now in a position to prove our main theorem.

Theorem 2.2.3 *Given a short exact sequence of finitely generated groups*

$$1 \to H \to G \to Q \to 1$$

such that H, G are hyperbolic, there exists a Cannon-Thurston map from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$.

Proof: When H is elementary this is trivial. So we assume H non-elementary. It suffices by Lemma 1.3.1 to show that if λ is a geodesic segment lying outside an $N-$ball around the identity in Γ_H, then any geodesic joining the end-points of $i(\lambda)$ in Γ_G lies outside an $M(N)-$ball around the identity in Γ_G and $M(N) \rightarrow \infty$ as $N \rightarrow \infty$.

Since Γ_H is properly embedded in Γ_G there exists $f(N)$ such that $i(\lambda)$ lies outside the $f(N)-$ball in Γ_G and $f(N) \rightarrow \infty$ as $N \rightarrow \infty$.

Let x be any point on B_λ. There exists $y \in i(\lambda)$ such that $d_G(y, x) \leq Ad_Q(1, Px)$ by Lemma 2.2.2. Therefore,

$$\begin{aligned} d_G(1, x) &\geq d_G(1, y) - Ad_Q(1, Px) \\ &\geq f(N) - Ad_Q(1, Px) \end{aligned}$$

Recall that in Lemma 2.1.1 a finite generating set B was chosen for G and $P(B)$ was chosen to be a generating set for Q so that

$$d_G(1, x) \geq d_Q(P1, Px) = d_Q(1, Px)$$

Hence

$$\begin{aligned} d_G(1, x) &\geq max(f(N) - Ad_Q(1, Px), d_Q(1, Px)) \\ &\geq \frac{f(N)}{A+1} \end{aligned}$$

Since (Lemma 2.2.1) B_λ is a C'-quasiconvex set containing $i(\lambda)$, any geodesic joining the end-points of $i(\lambda)$ lies in a C'-neighborhood of B_λ.

Hence any geodesic joining end-points of $i(\lambda)$ lies outside a ball of radius $M(N)$ where

$$M(N) = \tfrac{f(N)}{A+1} - C'$$

Since $f(N) \rightarrow \infty$ as $N \rightarrow \infty$ so does $M(N)$. $\Box$

Note: $f(N)$ can be regarded as the inverse of a 'distortion function' (cf. [15]).

There is a certain analogy between the discussion in this paper and the discussion of [8]. Let M be a 3-manifold fibering over the circle with fiber F. Then the universal covers of F, M correspond to H, G in Theorem 2.2.3. M admits a foliation by flow-lines transverse to the fiber F. These lift to quasi-geodesics in the universal cover of M. It might be helpful to think of a quasi-isometric section as a single 'flow-line'. The translates of a quasi-isometric section under the elements of $i(H)$ exhaust Γ_G and are mutually disjoint. Extending the analogy with [8], one can regard the translates of $\sigma(Q)$ as 'flow-lines' foliating Γ_G.

Chapter 3

Ending Laminations

Let G be a hyperbolic group and let H be a normal subgroup that is hyperbolic. We choose a finite symmetric generating set for H and extend it to a finite symmetric generating set for G. Assume also (for simplicity) that the generating set of G intersects H in the generating set of H. Let Γ_H and Γ_G denote the Cayley graphs of H, G respectively with respect to these generating sets. By adjoining the Gromov boundaries $\partial\Gamma_H$ and $\partial\Gamma_G$ to Γ_H and Γ_G, one obtains their compactifications $\widehat{\Gamma_H}$ and $\widehat{\Gamma_G}$ respectively [14].

Label the vertices of Cayley graphs by the corresponding group elements. G (resp. H) acts on Γ_G (resp. Γ_H) by left-translations. Denote the left action of g (resp. h) by t_g (resp. t_h).

There is a natural embedding $i : \Gamma_H \to \Gamma_G$ sending a vertex of Γ_H labeled h to the vertex of Γ_G labeled h. The main theorem of Chapter 2 states that i extends to a continuous map $\hat{i}$ from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$. Let $\partial i : \partial\Gamma_H \to \partial\Gamma_G$ denote the restriction of $\hat{i}$ to $\partial\Gamma_H$. A large part of this chapter is devoted to a study of the continuous map ∂i.

3.1 Preliminaries

Some standard facts about hyperbolic groups will be needed in this Chapter and we mention them here (mostly without proof) for future reference.

Definition: *An* **edge-path** *in Γ representing a group element w is a sequence of vertices $v_0, v_1 \cdots v_n$ in Γ such that for all i, v_i, v_{i+1} are connected by edges in Γ and $v_n = v_0 w$.*

$|w|$ will denote the length of a shortest edge-path representing the group element w.

Unless otherwise stated $[x, y]$ will denote a geodesic segment joining x, y in Γ.

Proposition 3.1.1 *Given $A \geq 0$ there exist $K \geq 1$ and $\epsilon \geq 0$ such that if a, b, c are vertices of Γ_H with $(a, c)_b \leq A$ then $[a, b] \cup [b, c]$ is a (K, ϵ)-quasigeodesic.*

Proof: It suffices to show that if $x \in [a, b]$ and $z \in [b, c]$ then $d_H(x, b) + d_H(b, z) \leq K d_H(x, z) + \epsilon$ for some K, ϵ depending only on δ.

Let u, v, w be points on edges $[x, b]$, $[b, z]$ and $[z, x]$ respectively such that $d_H(u, b) = d_H(v, b)$, $d_H(v, z) = d_H(w, z)$ and $d_H(w, x) = d_H(u, x)$. Then $d_H(u, b) = (x, z)_b \leq A$. Hence

$$d_H(x, b) + d_H(b, z) \leq d_H(x, z) + 2A$$

Choosing $K = 1$ and $\epsilon = 2A$ we are through. $\square$

Proposition 3.1.2 *Given $A_0 \geq 0$ there exist $\beta > 1$, $B > 0$, $K \geq 1$ and $\epsilon \geq 0$ such that if $[x, y], [y, z]$ and $[z, w]$ are geodesics in Γ_H with $(x, z)_y \leq A_0$, $(y, w)_z \leq A_0$ and $d_H(y, z) \geq B$ then*

1) Any path joining x to w and lying outside a D-neighborhood of $[y, z]$ has length greater than or equal to $\beta^D d_H(y, z)$,
where $D = min\{(d_H(x, [y, z]) - 1), (d_H(w, [y, z]) - 1)\}$.
2) $[x, y] \cup [y, z] \cup [z, w]$ is a (K, ϵ)-quasigeodesic.

The first part of the above Proposition states that it is exponentially more inefficient to travel outside large neighborhoods of geodesics (See [33] pg. 5.41 for instance). The second part follows from the first part and Proposition 3.1.1 above.

The following proposition is a 'quasification' of the fact that geodesics converging to the same point on $\partial\Gamma_H$ are asymptotic [See [14], pg. 117].

Proposition 3.1.3 *Given $K \geq 1, \epsilon \geq 0$ there exists α such that if $r_1, r_2 : [0, \infty) \to \Gamma_H$ are two (K, ϵ)-quasigeodesics converging to the same point in $\partial\Gamma_H$, then there exists $T \geq 0$ such that for all $t \geq T$*

$$max\ \{d_H(r_1(t), r_2), d_H(r_2(t), r_1)\} \leq \alpha$$

The following is again a simple 'quasification' of the standard fact that geodesics diverge exponentially in a δ-hyperbolic metric space [1].

Proposition 3.1.4 *Given $K \geq 1, \epsilon \geq 0, D \geq 0$ there exist $b > 1$, $A > 0$ and $C > 0$ such that the following holds:*
If r_1, r_2 are two (K, ϵ)-quasigeodesics with $d(r_1(0), r_2(0)) \leq D$ and there exists $T \geq 0$ with $d(r_1(T), r_2(T)) \geq C$ then any path joining $r_1(T+t)$ to $r_2(T+t)$ and lying outside the union of the $\frac{T+t-1}{K+\epsilon}$-balls around $r_1(0), r_2(0)$ has length greater than Ab^t for all $t \geq 0$.

The following Corollary can be obtained by combining Proposition 3.1.1 and Proposition 3.1.4.

Corollary 3.1.5 *Given $A_0 \geq 0$ there exist $B > 0$, $K \geq 1$ and $\epsilon \geq 0$ such that if $[x, y], [y, z]$ and $[z, w]$ are geodesics in Γ with $(x, z)_y \leq A_0$, $(y, w)_z \leq A_0$ and if there exist u, v in $[x, y]$ and $[z, w]$ respectively with $d(u, x) \geq B$, $d(v, y) = d(u, v) \geq B$ then $[x, y] \cup [y, z] \cup [z, w]$ is a (K, ϵ)-quasigeodesic.*

We need to establish a dictionary between certain objects in negatively curved manifolds and those in Cayley graphs.

Definitions:*An* **edge path representative** *of a word $w = h_1 h_2 \cdots h_n \in H$ (h_i's are in the generating set for H) starting at $v \in \Gamma_H$ is an edge-path $v_0 v_1 v_2 \cdots v_n$ in Γ_H such that $v_i^{-1} v_{i+1} = h_i$ where v_i's are vertices of Γ_H and $v_0 = v$.*

A **geodesic representative** *of $w \in H$ starting at v is a geodesic in Γ_H starting at v and ending at u, where $v^{-1} u = w$ as an element of H.*

A **cyclic conjugate** *of a word w is a word w_0 (not necessarily freely reduced) such that there exist words w_1 and w_2 with $w = w_1 w_2$ and $w_0 = w_2 w_1$ as freely unreduced words.*

A **cyclic conjugate** *of an edge-path μ is an edge-path representative of a cyclic conjugate of the word represented by μ.*

A **free homotopy representative** *of a word $w \in H$ is a geodesic $[a, aw_0]$ in Γ_H where w_0 is a shortest word in the conjugacy class of w in H.*

Edge path representatives, geodesic representatives and free homotopy representatives of words correspond to lifts (to the universal cover of a negatively curved manifold) of loops, based geodesic loops and shortest closed geodesics in free homotopy classes respectively.

In a closed negatively curved manifold a closed quasigeodesic lies close to the shortest closed geodesic in its free homotopy class. The following Proposition is the analog of this result for hyperbolic groups.

Proposition 3.1.6 *Let H be a δ-hyperbolic group. Given $K \geq 1, \epsilon \geq 0$ there exists $M \geq 0$ such that for any word $w \in H$ if all edge-path representatives of cyclic conjugates of w are (K,ϵ)-quasigeodesics then there exists a word c with length $|c| \leq M$ such that geodesic representatives of $c^{-1}wc$ are free homotopy representatives of w.*

Proof: Consider the set of pairs (h_1, h_2), where h_2 is a cyclic conjugate of w and $h_1 h_2 h_1^{-1} = u$ with u a shortest word in the conjugacy class of w. Choose (c, w_0) belonging to this set such that $|c|$ the minimum possible.

Consider a quadrilateral in Γ_H with vertices $1, c, uc, u$ where 1 is the identity. $1, c$ are joined by a geodesic representative $[1, c]$ of c starting at 1. u, uc are joined by $t_u([1, c])$. $1, u$ are joined by a geodesic representative $[1, u]$ of u and c, uc are joined by the edge-path representative ω corresponding to w_0 starting at c.

For $p \in \omega$ and $q \in [1, u]$, $d_H(p, q) \geq |c|$ as otherwise there would be an element $c_1 \in H$ with length less than $|c|$ conjugating a cyclic conjugate of w to a cyclic conjugate of u. In particular $d_H(c, q) \geq |c|$ for any $q \in [1, u]$ and $d_H(uc, q) \geq |c|$ for any $q \in [1, u]$. Let $[c, u]$ be a geodesic from c to u.

Let l, m, n be points on $[1, c]$, $[c, u]$ and $[1, u]$ respectively such that $d_H(1, n) = d_H(1, l)$, $d_H(c, l) = d_H(c, m)$ and $d_H(u, m) = d_H(u, n)$. Then $(u, c)_1 = d_H(l, 1)$. Also, since H is δ-hyperbolic, the diameter of the inscribed triangle with vertices l, m, n is less than or equal to 2δ (See [1] pg. 16).

$$\begin{aligned} d_H(c, l) + d_H(l, n) &\geq d_H(c, 1) = d_H(c, l) + d_H(l, 1) \\ \Rightarrow d_H(l, 1) &\leq d_H(l, n) \leq 2\delta \\ \Rightarrow (u, c)_1 &\leq 2\delta \end{aligned}$$

Similarly, $(uc, 1)_u \leq 2\delta$.

Hence, by Proposition 3.1.2 there exists $\beta > 1$ depending on δ alone such that the length of ω is greater than or equal to $\beta^{|c|}|u|$ for $|u|$ greater than some critical number B depending on δ. For $|u| \leq B$ the proposition is easy as there are only finitely many such elements. Else since ω is a (K, ϵ)-quasigeodesic, there exists $M_0 \geq 0$ depending on δ, K and ϵ alone such that $|c| \leq M_0$.

Also from Proposition 3.1.2 $[c, 1] \cup [1, u] \cup [u, uc]$ is a (k_0, ϵ_0)-quasigeodesic for some k_0, ϵ_0 depending only on δ. Hence ω lies in an M-neighborhood of $[c, 1] \cup [1, u] \cup [u, uc]$ for some M depending only on δ, K and ϵ. The Proposition follows easily. $\square$

The proof above yields the following corollary.

Corollary 3.1.7 *There exists $M \geq 0$ such that the following holds:*
If v, w are words in the same conjugacy class such that all edge-path representatives of w or v are free homotopy representatives then there exists a cyclic conjugate v_1 of v and a word c with length $|c| \leq M$ such that $c^{-1}wc = v_1$.

Since quasi-isometries take geodesics to quasi-geodesics another easy corollary of Proposition 3.1.6 is the following.

Corollary 3.1.8 *Let H be a δ-hyperbolic group. Given $K \geq 1, \epsilon \geq 0$ there exists $M \geq 0$ such that for any element $h \in H$ and any automorphism ϕ of H inducing a (K, ϵ)-quasi-isometry of the vertex set of Γ_H there exists a word c with length $|c| \leq M$ such that a geodesic representative of $c^{-1}\phi(h)c$ is a free homotopy representative.*

3.2 Ending Laminations

In order to motivate the definition of ending laminations for hyperbolic group extensions, we recall (a slight modification of) Thurston's notion of a stable lamination for a pseudo-anosov diffeomorphism ϕ of a closed hyperbolic surface S. (See [12] pg 71 for a detailed discussion of Thurston's notion.) Let c be a simple closed geodesic on S. Let c_n be the shortest geodesic in the free homotopy class of $\phi^n(c)$ for $n \geq 0$. Consider all Hausdorff limits of convergent subsequences of $\{c_n\}$ as $n \to \infty$. Let Λ be the union of all such limits as c ranges over all simple closed geodesics on S. It is this construction of Λ that will be generalized to the case of a hyperbolic normal subgroup of a hyperbolic group.

Remark: Λ constructed above is closely related to laminations appearing in [4] and [12]. In [12] or [34] a limit is extracted in the space of projectivized measured laminations

$\mathcal{PML}(S)$, which gives a (measured) geodesic lamination of S with complementary regions consisting of ideal polygons. However, limits in the Hausdorff topology may introduce diagonals of these ideal polygons and a union of these might contain intersecting diagonals. That this is the only possibility follows from [8] and Theorem 3.3.11. The laminations in [4] (H free) are exactly the same as those appearing here (see [4] for a proof).

A variant of the Hausdorff topology is defined below (see [7]).

Definition: *The* **Chabauty topology** *on the set $C(X)$ of closed subsets of a topological space X has a sub-basis given by sets of the following form:*
i) $O_1(K) = \{A | A \cap K = \emptyset\}$ *where K is compact.*
ii)$O_2(K) = \{A | A \cap U \neq \emptyset\}$ *where U is open.*

For a careful study of the relation between the Hausdorff topology and the Chabauty topology, see [21]. The two coincide for X compact Hausdorff. One advantage that the Chabauty topology has is that $C(X)$ is compact for any topological space X. The following lemma indicates the geometric nature of the topology.

Lemma 3.2.1 *[7] Suppose X is a locally compact metric space. A sequence $\{A_n\}$ of closed subsets of X converges in $C(X)$ (equipped with the Chabauty topology) to the closed subset A iff*
1) If $\{x_{n(k)} \in A_{n(k)}\}$ converges to $x \in X$ then $x \in A$.
2) If $x \in A$ then there exists a sequence $\{x_n\}$ converging to x such that $x_n \in A_n$.

Define $\widehat{\Gamma_H}^{(2)} = (\widehat{\Gamma_H} \times \widehat{\Gamma_H} \setminus \Delta)/(x,y) \sim (y,x)$ where $\Delta = \{(x,x) | x \in \widehat{\Gamma_H}\}$, and let $\partial^2 \Gamma_H = (\partial \Gamma_H \times \partial \Gamma_H \setminus \Delta)/(x,y) \sim (y,x)$.

Two geodesics in Γ_H are defined to be equivalent if they have the same end-points in $\widehat{\Gamma_H}$. So points in $\widehat{\Gamma_H}^{(2)}$ correspond to equivalence classes of geodesics.

Both $\widehat{\Gamma_H}^{(2)}$ and $\partial^2 \Gamma_H$ are locally compact and metrizable.

We need to recall some of the ingredients of Chapter 2. Let $P : G \to Q$ be a surjective homomorphism of finitely generated groups. Abusing notation slightly P will also denote the induced map from the vertex set of Γ_G to the vertex set of Γ_Q.

Definition:*A* (κ, ϵ_0)**-quasi-isometric section** *is a subset $\Sigma \subset G$ mapping onto Q such that for any $g, g' \in \Sigma$,*

$$\frac{1}{\kappa} d_Q(Pg, Pg') - \epsilon_0 \leq d_G(g, g') \leq \kappa d_Q(Pg, Pg') + \epsilon_0$$

where d_G and d_Q are word metrics in G, Q respectively and $\kappa \geq 1$, $\epsilon_0 \geq 0$ are constants.

A **single-valued quasi-isometric section** *is defined by choosing a single element of Σ representing each coset of the kernel of P.*

If a map σ from Γ_Q to Γ_G has a single-valued quasi-isometric section as its image, then σ will also be referred to as a quasi-isometric section. It will be clear from context whether we mean the map or its image.

The above notion is due to Lee Mosher [28]. A crucial lemma of [28] is the following.

Lemma 3.2.2 *(Quasi-isometric section Lemma [28])Given a non-elementary hyperbolic group H and a short exact sequence of finitely generated groups*

$$1 \to H \to G \to Q \to 1,$$

the map $P : G \to Q$ has a single-valued quasi-isometric section Σ. In fact, choosing a generating set B for G and letting $P(B)$ be the generating set for Q, we have for all $g, g' \in \Sigma$,

$$d_Q(Pg, Pg') \leq d_G(g, g') \leq \kappa d_Q(Pg, Pg') + \epsilon_0$$

for some constants $\kappa \geq 1$ and $\epsilon_0 \geq 0$.

Using a left translation t_h by an element $h \in H \subset G$, one can assume that Σ contains the identity element of G. $t_h(\Sigma)$ is still a single-valued quasi-isometric section as t_h preserves cosets.

Definition:*A* **(κ, ϵ_0)-quasi-isometric section** *of a subset U of Γ_Q is a map $\sigma : U \to \Gamma_G$ such that $P \cdot \sigma$ is the identity map on U and*

$$\frac{1}{\kappa} d_Q(k, k') - \epsilon_0 \leq d_G(\sigma(k), \sigma(k')) \leq \kappa d_Q(k, k') + \epsilon_0$$

where d_G and d_Q are word metrics in G, Q respectively and $\kappa \geq 1$, $\epsilon_0 \geq 0$ are constants.

Henceforth assume that H is a non-elementary hyperbolic group.

Corresponding to every element $g \in G$ there exists an automorphism of H taking h to $g^{-1}hg$ for $h \in H$. Such an automorphism induces a bijection ϕ_g of the vertices of Γ_H. This gives rise to a map from Γ_H to itself, sending an edge $[a, b]$ linearly to a shortest edge-path joining $\phi_g(a)$ to $\phi_g(b)$. Since ϕ_g is a quasi-isometry, this map extends to $\widehat{\Gamma_H}$ and acts by homeomorphisms on $\partial\Gamma_H$. Abusing notation slightly, we shall also call the extended map ϕ_g. Note that the action of ϕ_g on $\partial\Gamma_H$ and the extension of $t_{g^{-1}}$ to $\partial\Gamma_H$ coincide.

Fixing $z \in \partial\Gamma_Q$ for the time being (for notational convenience) we shall define the set of ending laminations corresponding to z.

Let $[1, z)$ be a semi-infinite geodesic ray in Γ_Q starting at the identity 1 and converging to $z \in \partial\Gamma_Q$. Let σ be a single-valued quasi-isometric section of Q into G. Let z_n be the vertex on $[1, z)$ such that $d_Q(1, z_n) = n$ and let $g_n = \sigma(z_n)$.

Given $h \in H$ let Σ_n^h be the (H-invariant) collection of all free homotopy representatives of $\phi_{g_n^{-1}}(h)$ in Γ_H. Identifying equivalent geodesics in Σ_n^h one obtains a subset S_n^h of $\widehat{\Gamma_H}^{(2)}$. The intersection with $\partial^2\Gamma_H$ of the union of all subsequential limits (in the Chabauty topology) of $\{S_n^h\}$ will be denoted by Λ_z^h.

Definition: *The set of* **ending laminations corresponding to** $z \in \partial\Gamma_Q$ *is given by*

$$\Lambda_z = \bigcup_{h\in H} \Lambda_z^h.$$

Definition: *The set* Λ *of all* **ending laminations** *is defined by*

$$\Lambda = \bigcup_{z\in\partial\Gamma_Q} \Lambda_z.$$

Note: Σ_n^h and therefore S_n^h are independent of the conjugacy class of h. Hence in the definition of Λ_z above it is enough to consider only smallest elements in conjugacy classes. Also from Corollary 3.1.7 it is enough, in the definition of Λ_z^h, to consider a single free homotopy representative of h and its translates by elements of H.

It needs to be checked that Λ_z is independent of the quasi-isometric section σ and the geodesic ray $[1, z)$.

For any two quasi-isometric sections σ and σ_1 of Q into G, and any $x \in Q$, $\sigma(x)$ and $\sigma_1(x)$ lie in the same coset of H in G. Hence for any $h \in H$, $\sigma(x)^{-1}h\sigma(x)$ and $\sigma_1(x)^{-1}h\sigma_1(x)$ are conjugate in H. Therefore Λ_z is independent of the quasi-isometric section σ.

Lemma 3.2.3 Λ_z *is independent of the geodesic ray* $[1, z)$.

Proof: Let μ_1 and μ_2 be two geodesic rays in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$. Suppose Q is δ_1-hyperbolic and σ is a (K, ϵ)-quasi-isometric section. Then

$\sigma(\mu_1)$ and $\sigma(\mu_2)$ are quasigeodesics lying in a D-neighborhood of each other for some D dependent on K, ϵ, δ_1. Let Λ_z^1 and Λ_z^2 be the ending laminations given by these two quasigeodesics. Let $h \in H$. Let $\{g_n\}$ be a sequence with $g_n \in \sigma(\mu_1)$ such that the set of all free homotopy representatives of $g_n h g_n^{-1}$ converge to a subset S of Λ_z^1. There exist $q_n \in G$ with $|q_n| \leq \delta_1(K+\epsilon)$ such that $g_n {q_n}^{-1} = r_n \in \sigma(\mu_2)$. Therefore $g_n h {g_n}^{-1} = r_n({q_n} h {q_n}^{-1}) r_n^{-1}$ for $h \in H$. Since $|q_n|$ is bounded there exists a subsequence $\{n_i\}$ such that $q_{n_i}{=}q$ for some $q \in G$. Let $h_1 = qhq^{-1}$. Then $g_{n_i} h g_{n_i}^{-1} = r_{n_i} h_1 r_{n_i}^{-1}$ Since the set of all free homotopy representatives (in Γ_H) of $g_n h {g_n}^{-1}$ converges to S, so does every subsequence. Hence the set of free homotopy representatives of $\{r_{n_i} h_1 {r_{n_i}}^{-1}\}$ converges to S. Since Λ_z^1 is obtained by taking a union of all such subsets S over all $h \in H$, $\Lambda_z^1 \subset \Lambda_z^2$. Similarly $\Lambda_z^2 \subset \Lambda_z^1$. Hence Λ_z is independent of the geodesic ray $[1, z)$. $\Box$

We shall now provide some justification for the terminology. The main theorem of Chapter 2 (Theorem 2.2.3) is the following:

Theorem 3.2.4 *Let G be a hyperbolic group and H a hyperbolic normal subgroup. Then $i : \Gamma_H \to \Gamma_G$ extends uniquely to a continuous map $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$.*

Definition:*A* **leaf** *of an ending lamination Λ_z is a bi-infinite geodesic whose set of end-points is an element of Λ_z.*

Lemma 3.2.5 *If λ is a leaf of an ending lamination Λ_z with end-points u, v then $\hat{i}(u) = \hat{i}(v)$.*

Proof: Suppose $(u, v) \in \Lambda_z^h$. Since H is normal in G, $ghg^{-1} {\in} H$ for all $g \in G$. Let $g_n \in \sigma([1, z))$ such that some sequence of free homotopy representatives of $g_n h g_n^{-1}$ in Γ_H converges to λ. Since Λ_z is H-invariant assume without loss of generality that λ passes through the identity. Since Σ_n^h is independent of the conjugacy class of h in H, one can assume that h is a shortest element in its conjugacy class. Choose p_n in the same coset as g_n such that the geodesic representative (in Γ_H) of $p_n h p_n^{-1}$ (regarded as an element of H) is a free homotopy representative. Note also by Corollary 3.1.7, one can assume (after passing to a subsequence if necessary) that a sequence of geodesic representatives of $p_n h p_n^{-1}$ converges to λ. Since geodesic representatives of $p_n h p_n^{-1}$ and $(p_n h) h (p_n h)^{-1}$ coincide therefore $d_G(1, p_n h) \geq d_G(1, p_n)$. Hence $(1, p_n h)_{p_n} \leq \frac{1}{2}|h|$. Similarly, $(1, p_n h p_n^{-1})_{p_n h} \leq \frac{1}{2}|h|$. Also, the length of a geodesic representative (in Γ_H) of $p_n h p_n^{-1}$ tends to infinity as n

tends to infinity. Hence by Corollary 3.1.5 $[1, p_n] \cup [p_n, p_n h] \cup [p_n h, p_n h p_n^{-1}]$ is a (K_h, ϵ_h)-quasigeodesic in Γ_G where K_h, ϵ_h are independent of n. If $\{[u_n, v_n]\}$ is a sequence of geodesic representatives of $p_n h {p_n}^{-1}$ converging to λ then there exists $N \geq 0$ such that for all $n \geq N$, $[u_n, v_n]$ passes through the identity 1. Also $v_n = u_n(p_n h p_n^{-1})$. Then $t_{u_n}([1, p_n] \cup [p_n, p_n h] \cup [p_n h, p_n h p_n^{-1}])$ is a (K_h, ϵ_h)-quasigeodesic in Γ_G lying outside a D_n-ball around the identity, where $D_n \to \infty$ as $n \to \infty$. Hence any geodesic joining $i(u_n), i(v_n)$ in Γ_G lies outside a $(D_n - K'_h)$-ball around the identity in Γ_G where K'_h is independent of n. As $(D_n - K'_h) \to \infty$ as $n \to \infty$, Theorem 3.2.4 shows that $\hat{i}(u) = \hat{i}(v)$. □

3.3 Identification of Boundary Points

Let Λ' be the set of equivalence classes (x, y) of bi-infinite geodesics in Γ_H (regarded as elements of $\partial^2\Gamma_H$) such that $\hat{i}(u)$=$\hat{i}(v)$. A bi-infinite geodesic contained in Γ_H and belonging to such an equivalence class will be called a **leaf** of Λ'. The main aim of this section is to prove the following theorem.

Theorem 3.3.11 Λ'=$\bigcup_{z\in\partial\Gamma_Q}\Lambda_z$

Lemma 3.2.5 shows that $\bigcup_{z\in\partial\Gamma_Q}\Lambda_z \subset \Lambda'$. Hence it is enough to show that if $(u, v) \in \Lambda'$ then $(u, v) \in \Lambda_z$ for some $z \in \partial\Gamma_Q$.

There are two main ingredients of the proof. First, one observes that leaves of Λ' are aperiodic and consequences of this observation are obtained. The second ingredient is a coarse separation property (a notion introduced in [11]). These two properties are dealt with in the two subsections of this section. Theorem 3.3.11 will follow fairly easily.

3.3.1 Aperiodicity and its Consequences

For any $w \in H$ an edge-path representative of the bi-infinite word $w^* = \cdots www \cdots$ converges to some $x_w, y_w \in \partial\Gamma_H$ and includes into Γ_G as a quasi-geodesic ([14], pg. 155).

Definition : *A bi-infinite geodesic $\mu \subset \Gamma_H$ is said to be* **periodic** *if there exists $w \in H$ such that μ lies in a bounded neighborhood of an edge-path representative of the bi-infinite word $w^* = \cdots www \cdots$. A path is* **aperiodic** *if it is not periodic.*

Since leaves of Λ' are not included as quasigeodesics in Γ_G, they are aperiodic.

This simple observation regarding aperiodicity of leaves of Λ' has several consequences which we now explore. First note that Λ' is H-invariant and (by Theorem 3.2.4) closed.

Lemma 3.3.1 *For all $w \in H$ there exists N_w such that if p, pw^n are vertices on a leaf of Λ', then $n \leq N_w$.*

Proof: Suppose not. Then for all n there exist leaves λ_n of Λ' containing geodesic subsegments $[p_n, p_n w^n]$. Since Λ' is H-invariant we can assume that for all n the identity 1 is a vertex in λ_n closest to the mid-point of $[p_n, p_n w^n]$.

An edge-path representative of the bi-infinite word $w^* = \cdots www \cdots$ converges to some $x_w, y_w \in \partial\Gamma_H$ and includes into Γ_G as a quasi-geodesic ([14] pg. 155). In particular, $\hat{i}(x_w) \neq \hat{i}(y_w)$. If x_n, y_n are end-points of λ_n in $\partial\Gamma_H$ then $(x_n, y_n) \to (x_w, y_w)$ in the Chabauty topology on $\partial^2\Gamma_H$. This implies that $(x_w, y_w) \in \Lambda'$ contradicting $\hat{i}(x_w) \neq \hat{i}(y_w)$. This proves the lemma. $\square$

Definition: *A geodesic segment $[r, q]$ in Γ_H is said to be an* **extension** *of a geodesic segment $[a, b]$ in a bi-infinite geodesic λ if $[a, b] \subset [r, q] \subset \lambda$. (Here r, q, a, b are all vertices of Γ_H.)*

Recall that a cyclic conjugate of an edge-path μ is an edge-path representative of a cyclic conjugate of the word represented by μ.

Lemma 3.3.2 *There exist $K \geq 1$, $\epsilon \geq 0$ such that for any leaf λ of Λ' and any geodesic subsegment $[p, q]$ of λ there exists an extension $[r, q]$ of $[p, q]$ in λ with $d_H(p, r)$ equal to 0 or 1 such that edge-path representatives of all cyclic conjugates of $[p, r]$ are (K, ϵ)-quasigeodesics.*

Proof: Suppose not. Then for every pair $K \geq 1$, $\epsilon \geq 0$ there exists a leaf λ of Λ' and a geodesic subsegment $[p, q]$ of λ such that both extensions $[r, q]$ of $[p, q]$ in λ with $d_H(p, r) \leq 1$ have cyclic conjugates that are not (K, ϵ)-quasigeodesics. Since Λ' is H-invariant, we can assume that $q = 1$.

A cyclic conjugate of any geodesic $[r, 1]$ is of the form $[r_1, 1] \cup [1, r^{-1}r_1]$ where $r_1 \in H$ is the label of a vertex on $[r, 1]$.

Hence from Proposition 3.1.1 for all $j \geq 0$ there exist leaves λ_j of Λ', $[p_j, 1] \subset [q_j, 1] \subset \lambda_j$ and $s_j \in [p_j, 1]$, $t_j \in [q_j, 1]$ such that $d_H(p_j, q_j) = 1$, $(s_j, p_j^{-1}s_j)_1 \geq j$ and $(t_j, q_j^{-1}t_j)_1 \geq j$.

Also as in Proposition 3.1.1 there exist $w_j \in [s_j, 1] \cap [t_j, 1]$, $u_j \in [1, p_j^{-1}s_j]$, $v_j \in [1, q_j^{-1}t_j]$ such that $d_H(1, w_j) = j$, $d_H(w_j, u_j) \leq 2\delta$ and $d_H(w_j, v_j) \leq 2\delta$. Then $d_H(u_j, v_j) \leq 4\delta$ and hence $||u_j| - |v_j|| \leq 4\delta$.

Let $u_j{=}v_j g_j$ for some g_j with $|g_j| \leq 4\delta$. Then $x_j{=}p_j u_j \in [p_j, s_j] \subset \lambda_j$, $y_j{=}q_j v_j \in [q_j, t_j] \subset \lambda_j$ and $d_H(p_j, q_j) = 1$.

Since $||u_j| - |v_j|| \leq 4\delta$ $d_H(x_j, y_j) \leq 4\delta + 1$. Hence $[p_j, p_j u_j]$ and $[q_j, q_j v_j]$ are geodesic subsegments of λ_j starting at a distance one apart and ending at a distance less than or equal to $(4\delta{+}1)$. Let $[q_j, q_j u_j]$ be a geodesic segment from q_j to $q_j u_j$. Then $[p_j, p_j u_j]$ and $[q_j, q_j u_j]$ are geodesic segments of length greater than or equal to $(j - 2\delta)$ starting at a distance one apart and ending at a distance less than or equal to $8\delta + 1$. Furthermore, (See [14] pg. 101) $[p_j, p_j u_j]$ and $[q_j, q_j u_j]$ lie in a $c(\delta)$-neighborhood of each other. Hence for all vertices h_j of $[1, u_j]$, $d_H(p_j h_j, q_j h_j) \leq c(\delta) + 1$.

This shows, (by the pigeon-hole principle) that for all $j \geq 0$, there exist leaves λ'_j of Λ', $[p'_j, 1] \subset [q_j', 1] \subset \lambda'_j$ and $l_j, m_j \in H$ with $d_H(l_j, m_j) \geq j$ and $d_H(p'_j, q'_j) = 1$ such that

$$\begin{aligned} p'_j l_j, p'_j m_j &\in [p'_j, 1] && \text{and} \\ (p_j' l_j)^{-1}(q_j' l_j) &= (p_j' m_j)^{-1}(q_j' m_j) && \text{and} \\ d_H(p_j' l_j, q_j' m_j) &\leq c(\delta) + 1. \end{aligned}$$

Hence $l_j m_j^{-1}$ and $(p_j')^{-1} q_j'$ commute.

$(p_j')^{-1} q_j'$ is a generator of H. Since there are only finitely many of these, after passing to a subsequence if necessary we can assume that $l_j m_j{}^{-1}$ commutes with a generator h_0 of H for all j. Since H is hyperbolic, any abelian subgroup is cyclic. Hence (after passing to a subsequence again if necessary) $l_j m_j{}^{-1} = h^{n_j}$ for some fixed $h \in H$ and $n_j \geq j$. Choosing $j \geq N_h{+}1$ (where N_h is as in Lemma 3.3.1) this contradicts Lemma 3.3.1, as some geodesic edge-path $[p_j' l_j, p_j' m_j]$ is contained in λ'_j. This contradiction proves the lemma. $\square$

Let $\lambda \subset \Gamma_H$ be a geodesic (finite, semi-infinite or bi-infinite). Let a, b denote the end-points of λ and let $\lambda_g \subset \Gamma_H$ denote a geodesic joining $\phi_g(a)$ to $\phi_g(b)$ in Γ_H. Note that $a, b \in \widehat{\Gamma_H}$ and may lie in Γ_H or $\partial\Gamma_H$.

Lemma 3.3.3 *Given $K \geq 1$, $\epsilon \geq 0$ there exists $C \geq 0$ such that for any geodesic ray $[a, z) \subset \Gamma_Q$ starting at a, converging to $z \in \partial\Gamma_Q$ and passing through $1 \in \Gamma_Q$ the following* *holds:*

If λ is a geodesic segment in Γ_H with end-points $1, h \in \Gamma_H$ all whose cyclic conjugates are (K, ϵ)-quasigeodesics then there exists a $(C, 0)$-quasi-isometric section σ_0 of $[a, z)$ into Γ_G containing $1 \in \Gamma_G$ such that for all $g \neq 1$ in $\sigma[a, z)$, λ_g is a free homotopy representative (i.e. all cyclic conjugates of λ_g are geodesics for $g \neq 1$).

Proof: Firstly, by Corollary 3.1.7 there exist $M \geq 0$ (depending on K, ϵ) and $c \in H$ with $|c| \leq M$ such that any geodesic representative of $c^{-1}hc$ is a free homotopy representative. Let λ' be a geodesic segment from c to hc. Then λ' is a free homotopy representative. If we can construct a $(C_0, 0)$-quasi-isometric section σ_0 of $[a, z)$ through c such that for all $g \in \sigma_0([a, z))$, λ'_g is a free homotopy representative, then by taking $C = C_0 + M$ and changing $\sigma_0(1)$ from c to 1 we would have a quasi-isometric section satisfying the conclusions of the Lemma.

Hence assume for the sake of simplicity that λ is a free homotopy representative. Let z_n be a point in $[1, z) \subset [a, z)$ such that $d_Q(1, z_n) = n$ and let z_{-n} be a point on $[a, 1] \subset [a, z)$ such that $d_Q(1, z_{-n}) = n$.

Let σ be a (κ, ϵ_0)-quasi-isometric section of Γ_Q obtained from Lemma 3.2.2. Let

$$\begin{array}{lrcllr} & (\sigma(z_n))^{-1}(\sigma(z_n)) & = & s_n & \text{for} & n \geq 0, \\ \text{and} & (\sigma(z_n))^{-1}(\sigma(z_n)) & = & s_{n-1} & \text{for} & n \leq 0. \end{array}$$

Then for all j, $|s_j| \leq \kappa + \epsilon_0$ and ϕ_{s_j} is a (K_1, ϵ_1)-quasi-isometry for some K_1, ϵ_1 depending only on κ, ϵ_0. The quasi-isometric section σ_0 of $[a, z)$ will now be constructed inductively. Put $\sigma(z_0) = 1$.

Assume that $\sigma(z_j) = g_j$ has been constructed for $-m \leq j \leq n$ satisfying the conclusions of the lemma.

Since ϕ_{s_n} is a (K_1, ϵ_1)-quasi-isometry, and λ has been assumed to be a free homotopy representative, $\lambda_{g_n s_n}$ is a geodesic all whose cyclic conjugates are (K_1, ϵ_1)-quasi-geodesics. Hence from Corollary 3.1.8 there exist M_1 (depending on K_1, ϵ_1) and $h_n \in H$ such that $|h_n| \leq M_1$ and $\lambda_{g_n s_n h_n}$ is a free homotopy representative. Put $g_{n+1} = \sigma(z_{n+1}) = g_n s_n h_n$. Similarly, one can construct $g_{-m-1} = \sigma(z_{-m-1})$.

Note that $d_G(z_n, z_{n+1}) \leq \kappa + \epsilon_0 + M_1$. Choose $C_0 = \kappa + \epsilon_0 + M_1$. Then $\sigma(z_j) = g_j$ gives a $(C_0, 0)$-quasi-isometric section and the Lemma is proven. $\square$

Remark: Note that C in Lemma 3.3.3 above is greater than or equal to $(\kappa + \epsilon_0)$.

Translating the quasi-isometric section obtained in Lemma 3.3.3 above by an element of G, one obtains the following Corollary.

Corollary 3.3.4 *Given $K \geq 1$ and $\epsilon \geq 0$ there exists $C \geq (\kappa + \epsilon_0)$ such that for any geodesic ray $[1, z)$ in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$, and any $g \in P^{-1}([1, z))$ the following holds :*

If λ is a geodesic segment in Γ_H with end-points $1, h \in \Gamma_H$ all whose cyclic conjugates are (K, ϵ)-quasigeodesics then there exists a $(C, 0)$-quasi-isometric section σ_0 of $[1, z)$ into Γ_G containing $g \in \Gamma_G$ such that for all $g' \neq g$ in $\sigma_0([1, z))$ $\lambda_{g^{-1}g'}$ is a free homotopy representative (i.e. all cyclic conjugates of $\lambda_{g^{-1}g'}$ are geodesics for $g' \neq g$).

From Lemma 3.3.2 and Corollary 3.3.4 above we obtain the following Lemma needed to prove Theorem 3.3.11.

Lemma 3.3.5 *There exists $C' \geq \kappa + \epsilon_0$ such that for any leaf λ of Λ', any geodesic ray $[1, z)$ in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$ and any geodesic subsegment $[p, q]$ of λ_g for some $g \in P^{-1}([1, z))$ the following holds:*
There exists an extension $[r, q] = \mu$ of $[p, q]$ in λ_g with $d_H(p, r)$ equal to 0 or 1 and a $(C', 0)$-quasi-isometric section $\sigma:[1, z) \to \Gamma_G$ such that $gr \in \sigma([1, z))$ and $\mu_{r^{-1}g^{-1}g'}$ is a free homotopy representative for all $g' \neq gr$ in $\sigma([1, z))$ (i.e. all cyclic conjugates of $\mu_{r^{-1}g^{-1}g'}$ are geodesics for $g' \neq g$).

3.3.2 Coarse Separation and its Consequences

Recall that if $\lambda \subset \Gamma_H$ is a geodesic (finite, semi-infinite or bi-infinite) with end-points a, b, then $\lambda_g \subset \Gamma_H$ denotes a geodesic joining $\phi_g(a)$ to $\phi_g(b)$ in Γ_H. Note that $a, b \in \widehat{\Gamma_H}$ and may lie in Γ_H or $\partial\Gamma_H$. Let $\sigma : \Gamma_Q \to \Gamma_G$ be a quasi-isometric section through the identity. Define

$$B(\lambda, \sigma) = \bigcup_{g \in \sigma(Q)} t_g \cdot i(\lambda_g).$$

Mark that $B(\lambda, \sigma)$ contains $i(\lambda)$ as $1 \in \sigma(Q)$. It is important to note also that λ_g is contained in Γ_H and not in Γ_G. Its only after acting on λ_g by $t_g \cdot i$ that we obtain a subset of Γ_G. Note that if λ is bi-infinite, $B(\lambda, \sigma)$ can be assumed to be independent of the quasi-isometric section σ, as for any $h \in H$ and $g \in \sigma$, $t_g \cdot t_h \cdot i \cdot t_{h^{-1}} \cdot \phi_g(\lambda) = t_g \cdot i \cdot \phi_g(\lambda)$, and $t_{gh} \cdot i \cdot \phi_{gh}(\lambda)$ and $t_g \cdot t_h \cdot i \cdot t_{h^{-1}} \cdot \phi_g(\lambda)$ lie in a δ-neighborhood of each other.

There is a natural identification of Γ_H with $t_g(i(\Gamma_H))$ taking h to $t_g(i(h))$. On Γ_H define a map $\pi_{g,\lambda} : \Gamma_H \to \lambda_g$ taking h to one of the points on λ_g closest to h in the metric d_H. Strictly speaking, $\pi_{g,\lambda}$ is defined only on the vertex set, but this is enough for our purposes. Now define

$$\Pi^\sigma_\lambda \cdot t_g \cdot i(h) = t_g \cdot i \cdot \pi_{g,\lambda}(h) \text{ for } g \in \sigma(Q).$$

For every $g' \in \Gamma_G$, there exists a unique $g \in \sigma(Q)$ such that $g' \in t_g(i(h))$ as σ is a single-valued section. Hence, Π^σ_λ is well-defined on the entire vertex set of Γ_G.

Recall the following theorem (Theorem 2.1.7) from Chapter 2:

Theorem 3.3.6 *There exists a constant C independent of λ such that for $x, y \in \Gamma_G$,*

$$d_G(\Pi^\sigma_\lambda(x), \Pi^\sigma_\lambda(y)) \leq C d_G(x, y).$$

Note that the proof did not require λ to be finite.

If σ is a quasi-isometric section, then from Theorem 3.3.6, $\Pi^\sigma_\lambda \cdot \sigma$ is also a quasi-isometric section. Moreover, $\Pi^\sigma_\lambda \cdot \sigma(Q) \subset B(\lambda, \sigma)$.

The following Lemma is an easy consequence of Lemma 2.2.2

Lemma 3.3.7 *For all $K \geq 1$ and $\epsilon \geq 0$ there exists $A \geq 1$ such that if σ is a (K, ϵ)-quasi-isometric section then for all $p, q \in \sigma(Q)$ and $x \in t_p \cdot i(\lambda_p)$ there exists $y \in t_q \cdot i(\lambda_q)$ such that $d_G(x, y) \leq A d_Q(Px, Py) = A d_Q(Pp, Pq)$.*

Suppose now that λ is a bi-infinite geodesic in Γ_H joining $u, v \in \partial\Gamma_H$. Let $a \in i(\lambda)$ and let $\sigma(Q) \subset B(\lambda, \sigma)$ be a (K, ϵ)-quasi-isometric section through $i(a)$. Let λ^- and λ^+ denote the semi-infinite geodesics contained in λ starting at a and converging to u, v respectively. Then

$$\begin{aligned} B(\lambda, \sigma) &= B(\lambda^-, \sigma) \cup B(\lambda^+, \sigma) \\ \text{and} \qquad \sigma(Q) &= B(\lambda^-, \sigma) \cap B(\lambda^+, \sigma) \end{aligned}$$

Definition : *Let X, Y, Z be geodesically complete metric spaces such that $X \subset Y \subset Z$. X is said to coarsely separate Y into Y_1 and Y_2 if*

(1) $Y_1 \cup Y_2 = Y$

(2) $Y_1 \cap Y_2 = X$

(3) For all $M \geq 0$*, there exist* $y_1 \in Y_1$ *and* $y_2 \in Y_2$ *such that* $d(y_1, Y_2) \geq M$ *and* $d(y_2, Y_1) \geq M$

(4) There exists $C \geq 0$ *such that for all* $y_1 \in Y_1$ *and* $y_2 \in Y_2$ *any geodesic in* Z *joining* y_1, y_2 *passes through a* C*-neighborhood of* X*.*

We will show that σ coarsely separates $B(\lambda, \sigma)$ into $B(\lambda^-, \sigma)$ and $B(\lambda^+, \sigma)$. The first three conditions follows easily from the construction. The following Lemma verifies condition (4) above.

Lemma 3.3.8 *For all* $K \geq 1$ *and* $\epsilon \geq 0$ *there exists* $M \geq 0$ *such that the following holds:*

If λ *is a bi-infinite geodesic in* Γ_H*,* a *is a vertex on* λ *splitting* λ *into semi-infinite geodesics* λ^- *and* λ^+ *and* $\sigma(Q) \subset B(\lambda, \sigma)$ *is a* (K, ϵ)*-quasi-isometric section through* $i(a)$ *then any geodesic joining a point in* $B(\lambda^-, \sigma)$ *to a point in* $B(\lambda^+, \sigma)$ *passes through an* M*-neighborhood of* $\sigma(Q)$*.*

Proof: Let $[p, q]$ be a geodesic in Γ_G joining $p \in B(\lambda^-, \sigma)$ to $q \in B(\lambda^+, \sigma)$. Let $\mu = \Pi_\lambda^\sigma[p, q]$. Then μ is a C-quasigeodesic contained in $B(\lambda, \sigma)$. Let $p = p_0, p_1, \cdots, p_n = q$ be the sequence of vertices (in order) on μ.

There exists j such that $p_j \in B(\lambda^-, \sigma)$ and $p_{j+1} \in B(\lambda^+, \sigma)$. Then $d_G(p_j, p_{j+1}) \leq C$. By Lemma 3.3.7 there exists $q_j \in B(\lambda^+, \sigma)$ in the same coset as p_j such that $d_G(q_j, p_{j+1}) \leq AC$ where A depends only on K, ϵ. Hence, $d_G(q_j, p_j) \leq (A+1)C$.

Let $x_j \in \sigma(Q)$ be in the same coset as p_j. Let $[p_j, q_j]$ be the subsegment of $t_{x_j} \cdot i(\lambda_{x_j})$ joining p_j and q_j. Since Γ_H is properly embedded in Γ_G there exists a constant M depending only on A, C (and hence on K, ϵ alone) such that the length of the edge-path $[p_j, q_j]$ is less than or equal to M. Since x_j lies on this edge-path, $d_G(p_j, x_j) \leq M$ and the Lemma is proven. $\square$

Let λ be a leaf of Λ' joining $u, v \in \partial\Gamma_H$. Let p_n, q_n be vertices on λ such that $p_n \to u$ and $q_n \to v$. Recall that $B(\lambda, \sigma)$ can be assumed to be independent of the quasi-isometric section. Let σ be a (K, ϵ)-quasi-isometric section contained in $B(\lambda, \sigma)$. Then, by Lemma 3.3.8 there exist $k_n \in \Gamma_Q$ and $M \geq 0$ such that any geodesic in Γ_G joining $i(p_n)$, $i(q_n)$ passes through an M-neighborhood of $\sigma(k_n)$. Since $\hat{i}(u) = \hat{i}(v)$, and $\hat{i}$ is continuous (Theorem 3.2.4), the sequences $\{i(p_n)\}, \{i(q_n)\}$ and $\{\sigma(k_n)\}$ converge to the same point in

$\partial\Gamma_G$. Since σ is a quasi-isometric section, it extends to an embedding of $\partial\Gamma_Q$ into $\partial\Gamma_G$. Hence there exists $z \in \partial\Gamma_Q$ such that $\{k_n\} \to z$. Also $\sigma([1,z))$ is a quasi-geodesic contained in $B(\lambda, \sigma)$.

If σ' is another quasi-isometric section contained in $B(\lambda, \sigma)$ then the above argument shows that there exists a sequence ${k_n}'$ such that $\{i(p_n)\}$, $\{i(q_n)\}$ and $\{\sigma'({k_n}')\}$ converge to the same point in $\partial\Gamma_G$ and ${k_n}' \to z'$ for some $z' \in \partial\Gamma_Q$. So $\sigma([1,z))$ and $\sigma'([1,z'))$ are quasi-geodesics converging to the same point in $\partial\Gamma_G$. Moreover, by Proposition 3.1.3, they are asymptotic.

Since σ and σ' are quasi-isometric sections $z = z'$.

The above discussion proves the following.

Lemma 3.3.9 *Given $K \geq 1, \epsilon \geq 0$ there exists α such that if λ is a leaf of Λ' and σ a quasi-isometric section then there exists $z \in \partial\Gamma_Q$ satisfying the following:*
If σ_1 and σ_2 are (K,ϵ)-quasi-isometric sections contained in $B(\lambda,\sigma)$ then there exists $z_n \in [1,z)$ such that

$$max\ \{d_G(\sigma_1([z_n,z)), \sigma_2([1,z))), d_G(\sigma_2([z_n,z)), \sigma_1([1,z)))\} \leq \alpha$$

For any ray $[1,z)$ in Γ_Q starting at 1 and converging to $z \in \partial\Gamma_Q$, let z_n be the point on $[1,z)$ such that $d_Q(1,z_n) = n$.

Combining Lemma 3.3.9 above with Lemma 3.3.7 we obtain the following Corollary.

Corollary 3.3.10 *Given $K \geq 1, \epsilon \geq 0$ there exists α such that if λ is a leaf of Λ' then there exists $z \in \partial\Gamma_Q$ satisfying the following:*
If σ and σ' are (K,ϵ)-quasi-isometric sections such that $B(\lambda,\sigma) = B(\lambda,\sigma')$ and σ, σ' are contained in $B(\lambda,\sigma)$ then there exists $N \geq 0$ such that for all $n \geq N$

$$d_G(\sigma(z_n), \sigma'(z_n)) \leq \alpha$$

We are now in a position to prove the main Theorem of this section.

Theorem 3.3.11 $\Lambda' = \bigcup_{z\in\partial\Gamma_Q} \Lambda_z$

Proof: Lemma 3.2.5 shows that $\bigcup_{z\in\partial\Gamma_Q}\Lambda_z \subset \Lambda'$. Hence it is enough to show that if λ is a leaf of Λ' then λ is a leaf of Λ_z for some $z \in \partial\Gamma_Q$. Since Λ' is invariant under H, λ can be assumed to be a leaf of Λ' passing through $1 \in \Gamma_H$.

Let σ_0 be a (κ, ϵ_0)-quasi-isometric section obtained from Lemma 3.2.2. Also suppose $\sigma_0(1) = 1$. Let $z \in \partial\Gamma_Q$ be as in Corollary 3.3.10 above and let $A_\lambda = B(\lambda, \sigma)\cap P^{-1}([1, z))$. ($A_\lambda$ can be thought of as a 'quasi-horodisk'.)

From Lemma 3.3.5, there exists $C' \geq \kappa+\epsilon_0$ such that for any geodesic subsegment $[p, q]$ of λ_g (for some $g \in \sigma_0([a, z))$) there exists an extension $[r, q] = \mu$ of $[p, q]$ in λ_g with $d_H(p, r)$ equal to 0 or 1 and a $(C', 0)$-quasi-isometric section σ:$[1, z) \to \Gamma_G$ such that $gr \in \sigma([1, z))$ and $\mu_{r^{-1}g^{-1}g'}$ is a free homotopy representative for all $g' \neq gr$ in $\sigma([1, z))$.

If σ is a $(C', 0)$-quasi-isometric section of $[1, z)$ into Γ_G then from Theorem 3.3.6, $\Pi^{\sigma}_{\lambda}\cdot\sigma$ is a $(CC', 0)$-quasi-isometric section of $[1, z)$ into A_λ. Let $C_2 = CC'$.

For σ_0 the above-mentioned (κ, ϵ_0)-quasi-isometric section let σ_e be the restriction of $\Pi^{\sigma_0}_{\lambda}\cdot\sigma_0$ to $[1, z)$. Let $g_n = \sigma_e(z_n)$ (where z_n is the point on $[1, z)$ such that $d_Q(1, z_n) = n$). Then σ_e is a $(C_2, 0)$-quasi-isometric section of $[1, z)$ into A_λ.

From Corollary 3.3.10 above there exists α depending on C_2 such that for σ' any $(C_2, 0)$-quasi-isometric section of $[1, z)$ contained in A_λ there exists $N \geq 0$ such that for all $n \geq N$

$$d_G(g_n, \sigma'(z_n)) \leq \alpha$$

From Proposition 3.1.4 there exists $b > 1$, $A > 0$ and $\eta > 0$ depending on α such that if $\sigma'([1, z))$ is (the image of) a $(C_2, 0)$-quasi-isometric section of $[1, z)$ contained in A_λ with $d_G(\sigma'(z_n), g_n) \geq \eta$ then any path in $i(\Gamma_H)$ joining $\sigma'(1)$ and $\sigma(1) = 1$ has length greater than or equal to Ab^n.

Let λ^+ and λ^- denote the closures of the two components of $\lambda\backslash\{1\}$. Recall that the generating set of G intersects H in the generating set for H. Hence for all $n > 0$,there exist $p_n \in t_{g_n} \cdot i(\lambda^-_{g_n})$ and $q_n \in t_{g_n} \cdot i(\lambda^+_{g_n})$ such that

$$\begin{aligned} d_G(p_n, g_n) &= \eta + 1 \\ d_G(q_n, g_n) &= \eta \end{aligned}$$

From Lemma 3.3.5, there exists $r_n \in t_{g_n} \cdot i(\lambda^-_{g_n})$ with $d_G(r_n, p_n)$ equal to 0 or 1, such that there exists a $(C', 0)$-quasi-isometric section σ_n of $[1, z)$ into Γ_G satisfying the following two conditions:

(1) $r_n = \sigma_n(z_n)$

(2) If $\mu^{(n)}$ is the subsegment of λ_{g_n} joining $(t_{g_n} \cdot i)^{-1}(r_n)$ and $(t_{g_n} \cdot i)^{-1}(q_n)$ then $\mu^{(n)}_{r_n^{-1} g_n}$ is a free homotopy representative.

${\sigma_n}' = \Pi^{\sigma_e}_{\lambda} \cdot \sigma_n$ is a $(C_2, 0)$-quasi-isometric section. And $d_G({\sigma_n}'(z_n), \sigma_e(z_n)) = d_G(r_n, g_n) \geq \eta + 1 - 1 = \eta$.

Hence from, Proposition 3.1.4 the length of the subsegment of $i(\lambda)$ joining $\sigma'_n(1)$ and $\sigma_e(1) = 1$ is greater than or equal to Ab^n.

Let $\tau_n{=}t_{q_n r_n^{-1}} \cdot \sigma_n$ - a $(C', 0)$-quasi-isometric section with $\tau_n(z_n) = q_n$. As in the case of ${\sigma_n}'$, ${\tau_n}' = \Pi^{\sigma_e}_{\lambda} \cdot \tau_n$ is a $(C_2, 0)$-quasi-isometric section such that $d_G({\tau_n}'(z_n), \sigma_e(z_n)) = d_G(q_n, g_n) \geq \eta$.

Hence from, Proposition 3.1.4 (as before) the length of the subsegment of $i(\lambda)$ joining $\tau'_n(1)$ and $\sigma_e(1) = 1$ is greater than or equal to Ab^n.

Note that $\sigma_n(1)$, ${\sigma_n}'(1)$, $\tau_n(1)$, ${\tau_n}'(1)$ all lie in $i(\Gamma_H)$ for all n. Let $[{\sigma_n}'(1), {\tau_n}'(1)]$ denote the subsegment of λ joining $i^{-1} \cdot {\sigma_n}'(1)$ and $i^{-1} \cdot {\tau_n}'(1)$. Then the sequence $\{[{\sigma_n}'(1), {\tau_n}'(1)]\}$ converges to λ in the Chabauty topology on Γ_H.

Since $d_G(r_n, q_n) \leq 2\eta + 1$, there exists $\rho > 0$ such that $r_n^{-1} q_n$ is an element of H with $|r_n^{-1} q_n| \leq \rho$. Since there are only finitely many of these, pass to a subsequence n_j such that $r_{n_j}^{-1} q_{n_j} = h$ where h is some element of H.

Since the sequence $\{[{\sigma_n}'(1), {\tau_n}'(1)]\}$ converges to λ in the Chabauty topology on Γ_H, so does the subsequence $\{[{\sigma_{n_j}}'(1), {\tau_{n_j}}'(1)]\}$.

Let $[\sigma_n(1), {\sigma_n}'(1)]$ denote a geodesic segment in Γ_H joining $i^{-1} {\cdot} \sigma_n(1)$ and $i^{-1} {\cdot} {\sigma_n}'(1)$ and let $[{\tau_n}'(1), \tau_n(1)]$ denote a geodesic segment in Γ_H joining $i^{-1} {\cdot} {\tau_n}'(1)$ and $i^{-1} {\cdot} \tau_n(1)$.

Let $\pi_\lambda = i^{-1} \cdot \Pi^{\sigma_e}_{\lambda} \cdot i$. Since $\pi_\lambda {\cdot} i^{-1} \sigma_n(1) = i^{-1} {\sigma_n}'(1)$ and $\pi_\lambda {\cdot} i^{-1} \tau_n(1) = i^{-1} {\tau_n}'(1)$ therefore by Proposition 3.1.2 and using the fact that quasi-geodesics lie in a bounded neighborhood of geodesics there exists $B \geq 0$ such that $[\sigma_n(1), {\sigma_n}'(1)] \cup [{\sigma_n}'(1), {\tau_n}'(1)] \cup [{\tau_n}'(1), \tau_n(1)]$ is a quasi-geodesic lying in a B-neighborhood of the geodesic $[\sigma_n(1), \tau_n(1)]$ in Γ_H joining $i^{-1} {\cdot} \sigma_n(1)$ and $i^{-1} {\cdot} \tau_n(1)$.

The sequence $\{[{\sigma_{n_j}}'(1), {\tau_{n_j}}'(1)]\}$ converges to λ in the Chabauty topology. Therefore the sequence $\{[\sigma_{n_j}(1), {\sigma_{n_j}}'(1)] \cup [{\sigma_{n_j}}'(1), {\tau_{n_j}}'(1)] \cup [{\tau_{n_j}}'(1), \tau_{n_j}(1)]\}$ also converges to λ in the Chabauty topology. In particular the sequences

$\{i^{-1}\cdot\sigma_{n_j}(1)\}$ and $\{i^{-1}\cdot\tau_{n_j}(1)\}$ converge to the end-points of λ in $\partial\Gamma_H$. Since $[\sigma_n(1),\tau_n(1)] = \mu^{(n)}_{r_n^{-1}g_n}$ is a free homotopy representative for all $n > 0$, λ is a leaf of Λ_z^h and hence of Λ_z. This proves the theorem. $\Box$

Remark: Note that the proof of Theorem 3.3.11 above shows that in the definition of Λ_z it is enough to consider only finitely many elements h of H. That is $\Lambda_z = \bigcup_{h\in H}\Lambda_z^h$ is actually equal to $\bigcup_{h\in B_{(2\eta+1)}}\Lambda_z^h$ where $B_{(2\eta+1)}$ is the ball of radius $(2\eta+1)$ around the identity in Γ_H. Since η can be seen to be dependent on δ alone (by unravelling definitions of the constants involved) η works equally well for all $z\in\partial\Gamma_Q$. Since Λ_z^h is closed for all h, Λ_z is also closed, being a finite union of closed sets.

3.4 Properties of Ending Laminations

The purpose of this section is to investigate some basic properties of the ending laminations Λ_z. If Λ_s and Λ_u are stable and unstable laminations of a pseudo-anosov diffeomorphism acting on a closed hyperbolic surface, then leaves of Λ_s and Λ_u intersect (if they do) at an angle bounded away from 0 or π (See [12] pg. 71). The following proposition generalizes this fact to the case of normal hyperbolic subgroups of hyperbolic groups.

Proposition 3.4.1 *Given $D\geq 0$ and a pair of distinct points p,q in $\partial\Gamma_Q$ there exists $L>0$ such that if l_p and l_q are leaves of Λ_p and Λ_q then $l_q\cap N_D(l_p)$ has diameter less than L, where $N_D(l_p)$ is the D-neighborhood of l_p in Γ_H.*

Proof: Suppose not. Then for some $D\geq 0$ and $p\neq q$ in $\partial\Gamma_Q$ and all positive integers n, there exist leaves l_p^n and l_q^n such that the set $l_q^n\cap N_D(l_p^n)$ has diameter greater than n.

Let u_n, v_n be points in $l_q^n\cap N_D(l_p^n)$ such that $d_H(u_n,v_n)\geq n$. Since both Λ_p and Λ_q are H-invariant, assume without loss of generality that the geodesic segment $[u_n,v_n]\subset l_q^n$ has the identity 1 as its mid-point (or a vertex closest to its mid-point). After passing to a subsequence if necessary, l_p^n (resp. l_q^n) converges to a leaf l_p^0 (resp. l_q^0) of Λ_p (resp. Λ_q), since Λ_p and Λ_q are closed, by the Remark following Theorem 3.3.11.

After passing to a further subsequence if necessary, the segments $[u_n,v_n]$ converge to l_q^0 in the Chabauty topology.

Since $[u_n,v_n]\subset(N_Dl_p^n)$, $l_q^0\subset N_D(l_p^0)$. Hence l_p^0 and l_q^0 are geodesics with the same end-points in $\partial\Gamma_Q$. Let u,v be these end-points.

Let σ be a quasi-isometric section through the identity contained in $B(l_p^0, \sigma)$. σ extends to an embedding $\partial\sigma$ of Γ_Q into Γ_G. Then from the proof of Lemma 3.3.9, $\hat{i}(u) = \hat{i}(v) = \partial\sigma(p)$.

Since l_q^0 too has end-points u, v, $\hat{i}(u) = \hat{i}(v) = \partial\sigma(q)$. Hence $\partial\sigma(p) = \partial\sigma(q)$. Since $\partial\sigma$ is an embedding, this implies that $p = q$, contradicting the hypothesis and proving the Proposition. $\Box$

Since Λ_z depends only on $z \in \partial\Gamma_Q$, there is a well-defined map from $\partial\Gamma_Q$ to $C(\partial^2\Gamma_H)$ - the set of closed subsets of $\partial^2\Gamma_H$ equipped with the Chabauty topology. Let F denote this map. Then $F(z) = \Lambda_z$. Proposition 3.4.1 above shows in particular that F is injective. It would be interesting to know if F is continuous. Proposition 3.4.3 below provides some weak positive evidence.

The following Lemma will be required for the proof:

Lemma 3.4.2 *Let l_i be a sequence of bi-infinite geodesics converging (in the Chabauty topology) to a bi-infinite geodesic l. If $\sigma(Q)$ is a (K, ϵ)-quasi-isometric section contained in $B(l, \sigma)$ and $\sigma_i = \Pi_{l_i}^{\sigma} \cdot \sigma$, then for any $x \in \partial\Gamma_Q$, $\partial\sigma_i(x) \to \partial\sigma(x)$ in $\partial\Gamma_G$, where $\partial\sigma_i$ (resp. $\partial\sigma$) denotes the continuous extension of σ_i (resp. σ) to $\partial\Gamma_Q$.*

Proof: Assume first that for any two points in $\partial\Gamma_H$ there exists a unique bi-infinite geodesic joining them in Γ_H. Then $l_i \to l$ implies that $\phi_g(l_i) \to \phi_g(l)$ for any $g \in G$. Recall that $t_g \cdot i \cdot \phi_g(l_i)$ and $t_{gh} \cdot i \cdot \phi_{gh}(l_i)$ are asymptotic in both directions for $h \in H$. Using the assumption that there exists a unique geodesic joining a pair of points in $\partial\Gamma_H$, $t_g \cdot i \cdot \phi_g(l_i) = t_{gh} \cdot i \cdot \phi_{gh}(l_i)$. Hence $B(l_i, \sigma_i) = B(l_i, \sigma)$ converges to $B(l, \sigma)$ in the Chabauty topology on Γ_G.

Let $[1, x)$ denote a geodesic ray in Γ_Q starting at 1 and converging to $x \in \partial\Gamma_Q$. Then for all $u \in [1, x)$, there exists $N \geq 0$ such that $\sigma[1, u] \subset B(l_i, \sigma_i)$ for all $i \geq N$. Hence $\sigma[1, u] = \sigma_i[1, u]$ for all $i \geq N$.

Since σ_i's are $(CK, C\epsilon)$-quasi-isometric sections (from Theorem 3.3.6), $\sigma_i([1, x))$ converges to $\sigma([1, x))$ in the Chabauty topology on Γ_G. Hence $\partial\sigma_i(x) \to \partial\sigma(x)$ in $\partial\Gamma_G$.

In general, if Γ_H is δ-hyperbolic, any geodesic bigon is δ-thin. Then the above argument shows that there exists $c = c(\delta)$, such that for all $u \in [1, x)$ the following holds : There exists $N \geq 0$ such that $\sigma[1, u]$ lies in a $c(\delta)$-neighborhood of $B(l_i, \sigma_i)$ for all $i \geq N$. Therefore $\partial\sigma_i(x) \to \partial\sigma(x)$. $\Box$

Proposition 3.4.3 *If $z_i \to z$ in $\partial\Gamma_Q$ and l_i's are leaves of Λ_{z_i} converging to a bi-infinite geodesic leaf l then l is a leaf of Λ_z.*

Proof: Let u_i, v_i be the end-points of l_i in $\partial\Gamma_H$ and let u, v be the end-points of l in $\partial\Gamma_H$. Then $\hat{i}(u_i) = \hat{i}(v_i)$ for all i. Hence by Theorem 3.2.4 $\hat{i}(u) = \hat{i}(v)$. Moreover, by Theorem 3.3.11 l is a leaf of Λ_{z_0} for some $z_0 \in \partial\Gamma_Q$. It is enough to show that $z = z_0$.

Since Λ_{z_i}'s and Λ_z are H-invariant, assume (after passing to a subsequence if necessary) that l_i's and l pass through the identity. Let $\sigma(Q)$ be a (K, ϵ)-quasi-isometric section through 1 contained in $B(l, \sigma)$. Also, let $\sigma_i = \Pi^{\sigma}_{l_i} \cdot \sigma$. Then σ and σ_i's are $(CK, C\epsilon)$-quasi-isometric sections (Theorem 3.3.6) and extend continuously to embeddings - $\partial\sigma$ and $\partial\sigma_i$ respectively - of $\partial\Gamma_Q$ to Γ_G. As in the proof of Lemma 3.3.9, $\hat{i}(u_i) = \hat{i}(v_i) = \partial\sigma_i(z_i)$ and $\hat{i}(u) = \hat{i}(v) = \partial\sigma(z_0)$. By Theorem 3.2.4, $\partial\sigma_i(z_i) \to \partial\sigma(z_0)$. From Lemma 3.4.2, $\partial\sigma_i(x) \to \partial\sigma(x)$ for any $x \in \partial\Gamma_Q$. Since in addition σ_i's are all $(CK, C\epsilon)$-quasi-isometric embeddings, $\partial\sigma_i(z_i) \to \partial\sigma(z)$. Hence $\partial\sigma(z) = \partial\sigma(z_0)$. Since $\partial\sigma$ is an embedding, $z = z_0$ and l is a leaf of Λ_z. $\Box$

Application : Irreducible automorphisms of fre groups have been studied by Bestvina, Feighn and Handel in [4] . Such automorphisms give examples of hyperbolic groups G with a finitely generated free normal subgroup H and quotient group $\mathbb{Z}$. It follows from [4] that the ending laminations in this case are the same as those defined in this chapter. Let H_1 be a finitely generated subgroup of infinite index in H. Since H_1 is quasiconvex in H and there exists a Cannon Thurston map from $\widehat{\Gamma_H}$ to $\widehat{\Gamma_G}$, there exists a Cannon Thurston map from $\widehat{\Gamma_{H_1}}$ to $\widehat{\Gamma_G}$. It is shown in [4] that no leaf of Λ is contained in H_1. Hence from Theorem 3.2.4 and Theorem 3.3.11 $i : \Gamma_{H_1} \to \Gamma_H \to \Gamma_G$ extends to an embedding on the boundary of Γ_{H_1}. Hence H_1 is quasiconvex in G. We have thus shown

Proposition 3.4.4 *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence induced by an irreducible automorphism of the free group H. Let H_1 be a finitely generated subgroup of infinite index in H. Then H_1 is quasiconvex in G.*

This is analogous to a theorem of Scott and Swarup [31]:

Proposition 3.4.5 *Let $1 \to H \to G \to \mathbb{Z} \to 1$ be an exact sequence induced by an irreducible automorphism of the a closed surface group H. Let H_1 be a finitely generated subgroup of infinite index in H. Then H_1 is quasiconvex in G.*

In fact the argument given above can be modified suitably to give an alternate proof of Proposition 3.4.5 .

Concluding Remarks : Laminations Λ_z constructed in this chapter are parametrized by $z \in \partial\Gamma_Q$. $\partial\Gamma_Q$ can also be used to parametrize limiting actions of H on $\mathbb{R}$-trees [29]. Every direction in Q gives rise to a sequence of automorphisms of H. As in [29] this gives a sequence of actions of H on δ_i-hyperbolic metric spaces, where $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. From this, one can extract a subsequence [29] converging (in the sense of Gromov) to a small isometric action on a 0-hyperbolic metric space, i.e. an $\mathbb{R}$-tree. The laminations of this chapter and small actions on $\mathbb{R}$-trees can be regarded as dual objects.

Chapter 4

Trees of Hyperbolic Metric Spaces

4.1 Cannon Thurston Maps

We start with a notion closely related to one introduced in [3].

Definition: *A tree (T) of hyperbolic metric spaces satisfying the q(uasi) i(sometrically) embedded condition is a metric space (X, d) admitting a map $P : X \to T$ onto a simplicial tree T, such that there exist δ,ϵ and $K > 0$ satisfying the following:*

1. *For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ with the induced path metric d_v is a δ-hyperbolic metric space. Further, the inclusions $i_v : X_v \to X$ are uniformly proper, i.e. for all $M > 0$, $v \in T$ and $x, y \in X_v$, there exists $N > 0$ such that $d(i_v(x), i_v(y)) \leq M$ implies $d_v(x, y) \leq N$.*

2. *Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point of e. Then X_e with the induced path metric is δ-hyperbolic.*

3. *There exist maps $f_e : X_e \times [0, 1] \to X$, such that $f_e|_{X_e \times (0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric.*

4. *$f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ are (K, ϵ)-quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e \times \{0\}}$ and $f_e|_{X_e \times \{1\}}$ will occassionally be referred to as f_{v_1} and f_{v_2} respectively.*

d_v and d_e will denote path metrics on X_v and X_e respectively. i_v, i_e will denote inclusion of X_v, X_e respectively into X.

The main theorem of this section can now be stated:

Theorem: 4.1.10 *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let v be a vertex of T. If X is hyperbolic, there exists a Cannon-Thurston map for (X_v, X).*

Some aspects of the proof of the main theorem of this section are similar to the proof of the main theorem of Chapter 2 (Theorem 2.2.3). Given a geodesic segment $\lambda \subset X_v$, we construct a quasi-convex set $B_\lambda \subset X$ containing λ. It follows from the construction that if λ lies outside a large ball around $y_0 \in X_v$, B_λ lies outside a large ball around $i_v(y_0) \in X$, i.e. for all $M \geq 0$ there exists $N \geq 0$ such that if λ lies oustside the N-ball around $y_0 \in X_v$, B_λ lies outside the M-ball around $i_v(y_0) \in X$. Combining this with Lemma 1.3.1 , the proof of Theorem 4.1.10 is completed.

For convenience of exposition, T shall be assumed to be rooted, i.e. equipped with a base vertex v_0. Let $v \neq v_0$ be a vertex of T. Let v_- be the penultimate vertex on the geodesic edge path from v_0 to v. Let e denote the directed edge from v_- to v. Define $\phi_v : f_{e_-}(X_{e_-}\times\{0\}) \rightarrow f_{e_-}(X_{e_-}\times\{1\})$ as follows:
If $p \in f_{e_-}(X_e\times\{0\}) \subset X_{v_-}$, choose $x \in X_e$ such that $p = f_{e_-}(x\times\{0\})$ and define

$$\phi_v(p) = f_{e_-}(x\times\{1\}).$$

Note that in the above definition, x is chosen from a set of bounded diameter.

Let μ be a geodesic in X_{v_-}, joining $a, b \in f_{e_-}(X_{e_-}\times\{0\})$. $\Phi_v(\mu)$ will denote a geodesic in X_v joining $\phi_v(a)$ and $\phi_v(b)$. Let $X_{v_0} = Y$.

For convenience of exposition, we shall modify X, X_v, X_e by quasi-isometric perturbations. Given a geodesically complete metric space (Z, d) of bounded geometry, choose a maximal disjoint collection $\{N_1(z_\alpha)\}$ of disjoint 1-balls. Then by maximality, for all $z \in Z$ there exist z_α in the collection such that $d(z, z_\alpha) < 2$. Construct a graph Z_1 with vertex set $\{z_\alpha\}$ and edge set consisting of distinct vertices z_α, z_β such that $d(z_\alpha, z_\beta) \leq 4$. Then Z_1 equipped with the path-metric is quasi-isometric to (Z, d). All metric spaces in this section will henceforth be assumed to be graphs of edge length 1 and maps between them will be assumed to be cellular.

We start with a general lemma about hyperbolic metric spaces. This follows easily from the fact that local quasigeodesics in a hyperbolic metric space are quasigeodesics [14]. If x, y are points in a hyperbolic metric space, $[x, y]$ will denote a geodesic joining them.

Lemma 4.1.1 *Given $\delta > 0$, there exist D, C_1 such that if a, b, c, d are vertices of a δ-hyperbolic metric space (Z, d), with $d(a, [b, c]) = d(a, b)$, $d(d, [b, c]) = d(c, d)$ and $d(b, c) \geq D$ then $[a, b] \cup [b, c] \cup [c, d]$ lies in a C_1-neighborhood of any geodesic joining a, d.*

Given a geodesic segment $\lambda \subset Y$, we now construct a quasi-convex set $B_\lambda \subset X$ containing λ.

Construction of B_λ

Choose $C_2 \geq 0$ such that for all $e \in T$, $f_e(X_e \times \{0\})$ and $f_e(X_e \times \{1\})$ are C_2-quasiconvex in the appropriate vertex spaces. Let $C = C_1 + C_2$, where C_1 is as in Lemma 4.1.1.

For $Z \subset X_v$, let $N_C(Z)$ denote the C-neighborhood of Z, that is the set of points at distance less than or equal to C from Z.

<u>*Step 1 :*</u> Let $\mu \subset X_v$ be a geodesic segment in (X_v, d_v). Then $P(\mu) = v$. For each edge e incident on v, but not lying on the geodesic (in T) from v_0 to v, choose p_e, $q_e \in N_C(\mu) \cap f_v(X_e)$ such that $d_v(p_e, q_e)$ is maximal. Let $v_1, \cdots, v_n$ be terminal vertices of edges e_i for which $d_v(p_{e_i}, q_{e_i}) > D$, where D is as in Lemma 4.1.1 above. Observe that there are only finitely many v_i's as μ is finite. Define

$$B^1(\mu) = i_v(\mu) \cup \bigcup_{k=1 \cdots n} \Phi_{v_i}(\mu),$$

where μ_i is a geodesic in X_v joining p_{e_i}, q_{e_i}.

Note that $P(B^1(\mu)) \subset T$ is a finite tree.

<u>*Step 2 :*</u> Step 1 above constructs $B^1(\lambda)$ in particular. We proceed inductively. Suppose that $B^m(\lambda)$ has been constructed such that the convex hull of $P(B^m(\lambda)) \subset T$ is a finite tree. Let $\{w_1, \cdots, w_n\} = P(B^m(\lambda)) \backslash P(B^{m-1}(\lambda))$. (Note that n may depend on m, but we avoid repeated indices for notational convenience.) Assume further that $P^{-1}(v_k) \cap B^m(\lambda)$ is a path of the form $i_{v_k}(\lambda_k)$, where λ_k is a geodesic in (X_{v_k}, d_{v_k}). Define

$$B^{m+1}(\lambda) = B^m(\lambda) \cup \bigcup_{k=1 \cdots n} (B^1(\lambda_k)),$$

where $B^1(\lambda_k)$ is defined in Step 1 above.

Since each λ_k is a finite geodesic segment in Γ_H, the convex hull of $P(B^{m+1}\lambda)$ is a finite subtree of T. Further, $P^{-1}(v) \cap B^{m+1}(\lambda)$ is of the form $i_v(\lambda_v)$ for all $v \in P(B^{m+1}(\lambda))$. This enables us to continue inductively. Define

$$B_\lambda = \cup_{m \geq 0} B^m \lambda.$$

Note finally that the convex hull of $P(B_\lambda)$ in T is a locally finite tree T_1.

Quasiconvexity of B_λ

We shall now show that there exists $C' \geq 0$ such that for every geodesic segment $\lambda \subset Y$, $B_\lambda \subset X$ is C'-quasiconvex. To do this we construct a retraction Π_λ from (the vertex set of) X onto B_λ and show that there exists $C_0 \geq 0$ such that $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d_X(x, y)$. Let $\pi_v : X_v \to \lambda_v$ be a nearest point projection of X_v onto λ_v. Π_λ is defined on $\bigcup_{v \in T_1} X_v$ by

$$\Pi_\lambda(x) = i_v \cdot \pi_v(x) \text{ for } x \in X_v.$$

If $x \in P^{-1}(T \setminus T_1)$ choose $x_1 \in P^{-1}(T_1)$ such that $d(x, x_1) = d(x, P^{-1}(T_1))$ and define $\Pi'_\lambda(x) = x_1$. Next define $\Pi_\lambda(x) = \Pi_\lambda \cdot \Pi'_\lambda(x)$.

The following Lemma says nearest point projections in a δ-hyperbolic metric space do not increase distances much. This is similar to Lemma 2.1.2.

Lemma 4.1.2 *Let (Y, d) be a δ-hyperbolic metric space and let $\mu \subset Y$ be a geodesic segment. Let $\pi : Y \to \mu$ map $y \in Y$ to a point on μ nearest to y. Then $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$ for all $x, y \in Y$ where C_3 depends only on δ.*

Proof: Let $[a, b]$ denote a geodesic edge-path joining vertices a, b. Recall that the Gromov inner product $(a, b)_c = 1/2[d(a, c) + d(b, c) - d(a, b)]$. It suffices by repeated use of the triangle inequality to prove the Lemma when $d(x, y) = 1$. Let u, v, w be points on $[x, \pi(x)]$, $[\pi(x), \pi(y)]$ and $[\pi(y), x]$ respectively such that $d(u, \pi(x)) = d(v, \pi(x))$, $d(v, \pi(y)) = d(w, \pi(y))$ and $d(w, x) = d(u, x)$. Then $(x, \pi(y))_{\pi(x)} = d(u, \pi(x))$. Also, since Y is δ-hyperbolic, the diameter of the inscribed triangle with vertices u, v, w is less than or equal to 2δ (See [1]).

$$\begin{aligned} d(u,x)+d(u,v) &\geq d(x,\pi(x)) = d(u,x)+d(u,\pi(x)) \\ \Rightarrow d(u,\pi(x)) &\leq d(u,v) \leq 2\delta \\ \Rightarrow (x,\pi(y))_{\pi(x)} &\leq 2\delta \end{aligned}$$

Similarly, $(y,\pi(x))_{\pi(y)} \leq 2\delta$.

$$\begin{aligned} &\text{i.e. } d(x,\pi(x))+d(\pi(x),\pi(y))-d(x,\pi(y)) \leq 4\delta \\ &\text{and } d(y,\pi(y))+d(\pi(x),\pi(y))-d(y,\pi(x)) \leq 4\delta \end{aligned}$$

Therefore,

$$\begin{aligned} &2d(\pi(x),\pi(y)) \\ &\leq 8\delta + d(x,\pi(y)) - d(y,\pi(y)) + d(y,\pi(x)) - d(x,\pi(x)) \\ &\leq 8\delta + d(x,y) + d(x,y) \\ &\leq 8\delta + 2 \end{aligned}$$

Hence $d(\pi(x),\pi(y)) \leq 4\delta+1$. Choosing $C_3 = 4\delta+1$, we are through. $\square$

The following Lemma is similar to Lemma 2.1.3 .

Lemma 4.1.3 *Let (Y,d) be a δ-hyperbolic metric space. Let μ be a geodesic segment in Y with end-points a,b and let x be any vertex in Y. Let y be a vertex on μ such that $d(x,y) \leq d(x,z)$ for any $z \in \mu$. Then a geodesic path from x to y followed by a geodesic path from y to z is a k-quasigeodesic for some k dependent only on δ.*

Proof: As in Lemma 4.1.2, let u,v,w be points on edges $[x,y]$, $[y,z]$ and $[z,x]$ respectively such that $d(u,y) = d(v,y)$, $d(v,z) = d(w,z)$ and $d(w,x) = d(u,x)$. Then $d(u,y) = (z,x)_y \leq 2\delta$ and the inscribed triangle with vertices u,v,w has diameter less than or equal to 2δ (See [1]). $[x,y] \cup [y,z]$ is a union of 2 geodesic paths lying in a 4δ neighborhood of a geodesic $[x,z]$. Hence a geodesic path from x to y followed by a geodesic path from y to z is a $k-$quasigeodesic for some k dependent only on δ. $\square$

The following Lemma is similar to Lemma 2.1.4 .

Lemma 4.1.4 *Suppose (Y,d) is a δ-hyperbolic metric space. If μ is a (k_0,ϵ_0)-quasigeodesic in Y and p,q,r are 3 points in order on μ then $(p,r)_q \leq k_1$ for some k_1 dependent on k_0, ϵ_0 and δ only.*

Proof: $[a,b]$ will denote a geodesic path joining a,b. Since p,q,r are 3 points in order on μ, $[p,q]$ followed by $[q,r]$ is a (k_0,ϵ_0)-quasigeodesic in the δ-hyperbolic metric space Y. Hence there exists a k_1 dependent on k_0, ϵ_0 and δ alone such that $d(q,[p,r]) \leq k_1$. Let s be a point on $[p,r]$ such that $d(q,s) = d(q,[p,r]) \leq k_1$. Then

$$\begin{aligned}(p,r)_q &= 1/2(d(p,q)+d(r,q)-d(p,r)) \\ &= 1/2(d(p,q)+d(r,q)-d(p,s)-d(r,s)) \\ &\leq d(q,s) \leq k_1.\Box\end{aligned}$$

The following Lemma says that nearest point projections and quasi-isometries in hyperbolic metric spaces 'almost commute'. The proof is similar to that of Lemma 2.1.6 .

Lemma 4.1.5 *Suppose (Y,d) is δ-hyperbolic. Let μ_1 be some geodesic segment in Y joining a,b and let p be any vertex of Y. Also let q be a vertex on μ_1 such that $d(p,q) \leq d(p,x)$ for $x \in \mu_1$. Let ϕ be a (K,ϵ) - quasiisometry from Y to itself. Let μ_2 be a geodesic segment in Y joining $\phi(a)$ to $\phi(b)$ for some $g \in S$. Let r be a point on μ_2 such that $d(\phi(p),r) \leq d(\phi(p),x)$ for $x \in \mu_2$. Then $d(r,\phi(q)) \leq C_4$ for some constant C_4 depending only on K,ϵ and δ.*

Proof: Since $\phi(\mu_1)$ is a $(K,\epsilon)-$ quasigeodesic joining $\phi(a)$ to $\phi(b)$, it lies in a K'-neighborhood of μ_2 where K' depends only on K,ϵ,δ. Let u be a point in $\phi(\mu_1)$ lying at a distance at most K' from r. Without loss of generality suppose that u lies on $\phi([q,b])$, where $[q,b]$ denotes the geodesic subsegment of μ_1 joining q,b. [See Figure 4.1 below.]

Let $[p,q]$ denote a geodesic joining p,q. From Lemma 4.1.3 $[p,q]\cup[q,b]$ is a k-quasigeodesic, where k depends on δ alone. Therefore $\phi([p,q])\cup\phi([q,b])$ is a (K_0,ϵ_0)-quasigeodesic, where K_0,ϵ_0 depend on K,k,ϵ. Hence, by Lemma 4.1.4 $(\phi(p),u)_{\phi(q)} \leq K_1$, where K_1 depends on K, k, ϵ and δ alone. Therefore,

$$\begin{aligned}&(\phi(p),r)_{\phi(q)} \\ &= 1/2[d(\phi(p),\phi(q))+d(r,\phi(q))-d(r,\phi(p))] \\ &\leq 1/2[d(\phi(p),\phi(q))+d(u,\phi(q))+d(r,u) \\ &\qquad\qquad -d(u,\phi(p))+d(r,u)] \\ &= (\phi(p),u)_{\phi(q)}+d(r,u) \\ &\leq K_1+K'\end{aligned}$$

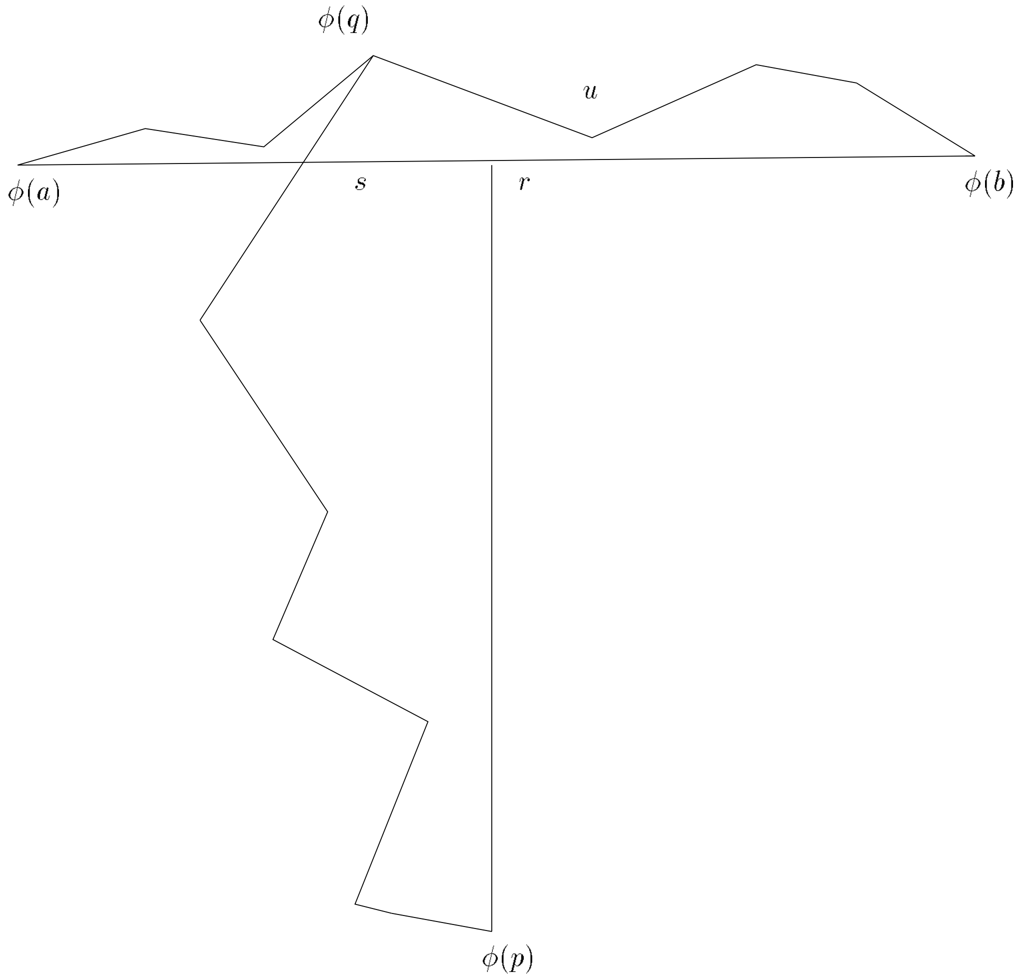

Figure 4.1. Quasi-isometries and Projections Almost Commute

There exists $s \in \mu_2$ such that $d(s, \phi(q)) \leq K'$

$$\begin{aligned}
(\phi(p), r)_s &= 1/2[d(\phi(p), s) + d(r, s) - d(r, \phi(p))] \\
&\leq 1/2[d(\phi(p), \phi(q)) + d(r, \phi(q)) - d(r, \phi(p))] + K' \\
&= (\phi(p), r)_{\phi(q)} + K' \\
&\leq K_1 + K' + K' \\
&= K_1 + 2K'
\end{aligned}$$

Also, as in the proof of Lemma 4.1.2 $(\phi(p), s)_r \leq 2\delta$

$$\begin{aligned}
d(r, s) &= (\phi(p), s)_r + (\phi(p), r)_s \\
&\leq K_1 + 2K' + 2\delta \\
d(r, \phi(q)) &\leq K_1 + 2K' + 2\delta + d(s, \phi(q)) \\
&\leq K_1 + 2K' + 2\delta + K'
\end{aligned}$$

Let $C_4 = K_1 + 3K' + 2\delta$. Then $d(r, \phi(q)) \leq C_4$ and C_4 is independent of a, b, p. $\Box$

Let C_1, D be as in Lemma 4.1.1. Recall that each $f_v(X_e)$ is C_2-quasiconvex and $C = C_1 + C_2$. $[x, y]$ will denote a gedoesic joining x, y.

Lemma 4.1.6 *Let $\mu_1 = [a, b] \subset X_v$ be a geodesic and let e be an edge of T incident on v. Let $p, q \in N_C(\mu_1) \cap f_v(X_e)$ be such that $d_v(p, q)$ is maximal. Let μ_2 be a geodesic in X_v joining p, q. If $r \in N_C(\mu_1) \cap f_v(X_e)$, then $d_v(r, \mu_2) \leq D_1$ for some constant D_1 depending only on C, D, δ.*

Proof: Let π denote a nearest point projection onto μ_1. Since μ_2 and $[\pi(p), \pi(q)] \subset \mu_1$ are geodesics whose end-points lie at distance at most C apart, there exists C' such that $[\pi(p), \pi(q)] \subset N_{C'}(\mu_2)$. If $\pi(r) \in [\pi(p), \pi(q)]$, then

$$d(r, \mu_2) \leq C + C'.$$

If $\pi(r) \notin [\pi(p), \pi(q)]$, then without loss of generality, assume $\pi(r) \in [a, \pi(p)] \subset [a, \pi(q)]$. Then

$$
\begin{aligned}
d(p,q) &\geq d(r,q) \\
&\geq d(\pi(r),\pi(q)) - 2C \\
&= d(\pi(r),\pi(p)) + d(\pi(p),\pi(q)) - 2C \\
&\geq d(\pi(r),\pi(p)) + d(p,q) - 4C \\
\Rightarrow \quad d(\pi(r),\pi(p)) &\leq 4C \\
\Rightarrow \quad d(r,p) &\leq 6C \\
\Rightarrow \quad d(r,\mu_2) &\leq 6C.
\end{aligned}
$$

Choosing $D_1 = max\{C + C', 6C\}$, we are through. $\square$

Lemma 4.1.7 *Let μ_1, μ_2 be as in Lemma 4.1.6 above. Let π_i denote nearest point projections onto μ_i $(i = 1, 2)$. If $p \in f_v(X_e)$, then $d(\pi_1(p), \pi_2(p)) \leq C_6$ for some constant C_6 depending on δ alone.*

Proof: If $d(\pi_1(p), \pi_1{\cdot}\pi_2(p)) \leq D$, then $d(\pi_1(p), \pi_2(p)) \leq C + D$.

If not, there exists $r \in f_v(X_e)$ such that $d(r, \pi_1(p)) \leq C$, by Lemma 4.1.1.

Then, by Lemma 4.1.6 above, there exists $s \in \mu_2$ such that $d(s, \pi_1(p)) \leq C + D_1$.

As in the proof of Lemma 4.1.2, $(p, s)_{\pi_2(p)} \leq 2\delta$. Hence, $(p, \pi_1(p))_{\pi_2(p)} \leq 2\delta + C + D_1$.

Similarly, $(p, \pi_1{\cdot}\pi_2(p))_{\pi_1(p)} \leq 2\delta$. Hence, $(p, \pi_2(p))_{\pi_1(p)} \leq 2\delta + C$.

Therefore,
$d(\pi_1(p), \pi_2(p)) \leq (p, \pi_1{\cdot}\pi_2(p))_{\pi_1(p)} + (p, \pi_2(p))_{\pi_1(p)} \leq 4\delta + 2C + D_1$. Choosing $C_6 = 4\delta + 2C + D_1$ we are through. $\square$

d_T will denote the metric on T. We are now in a position to prove:

Theorem 4.1.8 *There exists $C_0 \geq 0$ such that $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d_G(x, y)$ for x, y vertices of Γ_G.*

Proof: It suffices to prove the theorem when $d_G(x, y) = 1$.

Case (a): $x, y \in P^{-1}(v)$ for some $v \in T_1$. From Lemma 4.1.2, there exists C_3 such that $d_v(\pi_v{\cdot}i_v^{-1}(x), \pi_v{\cdot}i_v^{-1}(y)) \leq C_3$. Since embeddings of X_v in X are cellular, $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_3$.

Case (b): $x \in P^{-1}(w)$ and $y \in P^{-1}(v)$ for some $v, w \in T_1$.

Since $d_G(x, y) = 1$, v, w are adjacent in T_1. Assume, without loss of generality, $w = v_-$.

Recall that

$$B_\lambda \cap P^{-1}(v) = i_v(\lambda_v)$$
$$B_\lambda \cap P^{-1}(w) = i_w(\lambda_w)$$

Also, $\lambda_v = \Phi_v(\mu_w)$, for some geodesic μ_w contained in X_w, such that end-points of μ_w lie in a C-neighborhood of λ_w.

Let $z \in X_w$ denote a nearest point projection of $i_w^{-1}(x)$ onto μ_w. Then, by Lemma 4.1.5,

$$d(i_w(z), \Pi_\lambda \cdot \phi_v(x)) \leq d(i_w(z), \phi_v \cdot i_w(z)) + d(\phi_v \cdot i_w(z), \Pi_\lambda \cdot \phi_v(x)) \leq 1 + C_4.$$

Since, $d(x, y) = 1 = d(x, \phi_v(x))$ and i_v's are uniformly proper embeddings, there exists $C_5 > 0$ such that $d_v(\phi_v(x), y) \leq C_5$ and
$d(\Pi_\lambda(\phi_v(x)), \Pi_\lambda(y)) \leq C_3 C_5$.

Since the end-points of μ_w lie in a C-neighborhood of λ_w, there exists C_6 from Lemma 4.1.7, depending on δ and C such that $d(z, \Pi_\lambda(x)) \leq C_6$.

Finally, by the triangle inequality,

$$d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_6 + 1 + C_4 + C_3 C_5 = C_7(say)$$

Case (c): $P([x, y])$ is not contained in T_1.

Since $d(x, y) = 1$ $P(x)$ and $P(y)$ belong to the closure T_2 of the same component of $T \backslash T_1$. Then $P \cdot \Pi'_\lambda(x) = P \cdot \Pi'_\lambda(y) = v$ for some $v \in T$.

Also $d(\Pi_\lambda(x), \Pi_\lambda(y)) = d(\Pi_\lambda \cdot \Pi'_\lambda(x), \Pi_\lambda \cdot \Pi'_\lambda(y))$

Let $x_1 = \Pi'_\lambda(x)$ and $y_1 = \Pi'_\lambda(y)$.

Let D and C_1 be as in Lemma 4.1.1. If $d(\Pi_\lambda(x_1), \Pi_\lambda(y_1)) \geq D$, let

$$\begin{aligned} {i_v}^{-1}(x_1) &= u_1, \\ {i_v}^{-1}(\Pi_\lambda(x_1)) &= u_2, \\ {i_v}^{-1}(y_1) &= v_1, \\ {i_v}^{-1}(\Pi_\lambda(y_1)) &= v_2. \end{aligned}$$

Then by Lemma 4.1.1 $[u_1,u_2]\cup[u_2,v_2]\cup[v_2,v_1]$ is a quasigeodesic lying in a C_1-neighborhood of $[u_1,v_1]$.

Also, $x_1, y_1 \in i_v(X_v)$. Since the image of an edge space in a vertex space is C_2-quasiconvex, there exist $e \in T$ and $x_2, y_2 \in f_e(X_e \times \{0\})$ such that $d(x_2, \Pi_\lambda(x_1)) \leq C_1 + C_2 = C$ and $d(y_2, \Pi_\lambda(y_1)) \leq C_1 + C_2 = C$.

By construction $d(\Pi_\lambda(x_2), \Pi_\lambda(y_2)) \leq D$. (Else the edge $P([x,y])$ of T would be in T_1). Therefore,

$$\begin{aligned} d(\Pi_\lambda(x), \Pi_\lambda(y)) &= d(\Pi_\lambda(x_1), \Pi_\lambda(y_1)) \\ &\leq 2C + D + 2C \\ &= 4C + D \end{aligned}$$

Choosing $C_0 = \max\{C_3, C_7, 4C + D\}$, we are through. □

Note that the hyperbolicity of X has not yet been used. To complete the proof of our main Theorem, we need a final Lemma. This is similar to Lemma 2.2.2 .

Lemma 4.1.9 *There exists $A > 0$, such that if $a \in P^{-1}(v) \cap B_\lambda$ for some $v \in T_1$ then there exists $b \in i(\lambda)$ with $d(a,b) \leq A d_T(Pa, Pb)$.*

Proof: Let μ be a geodesic path from v_0 to v in T. Order the vertices on μ so that we have a finite sequence $v_0 = y_0, y_1, \cdots, y_n = v$ such that $d_T(y_i, y_{i+1}) = 1$. and $d_T(v_0, v) = n$. Recall further, $P(B_\lambda) = T_1$. Hence $y_i \in T_1$.

Recall that B_λ is of the form $\bigcup_{v \in T_1} i_v(\lambda_v)$.

It suffices to prove that there exists $A > 0$ independent of v such that if $p \in i_{y_j}(\lambda_{y_j})$, there exists $q \in i_{y_{j-1}}(\lambda_{y_{j-1}})$ with $d(p,q) \leq A$.

By construction, $\lambda_{y_j} = \Phi_{y_j}(\mu)$ for some geodesic μ in $X_{y_{j-1}}$ such that end-points of μ lie in a C-neighborhood of $\lambda_{y_{j-1}}$. Since ϕ_{y_j} is a quasi-isometry, there exists C_1 such that p lies in a C_1 neighborhood of $\phi_{y_j}(q_0)$ for some $q_0 \in \mu$. Therefore, $d(q_0, p) \leq 1 + C$.

Also, since end-points of μ lie in a C-neighborhood of $\lambda_{y_{j-1}}$, there exists $q \in i_{y_j}(\lambda_{y_{j-1}})$ with $d(q_0, q) \leq C_2$ where C_2 depends only on δ and C. Choosing $A = 1 + C + C_2$, we are through. □

The main theorem of this chapter follows:

Theorem 4.1.10 *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let v be a vertex of T. If X is hyperbolic $i_v : X_v \to X$ extends continuously to $\hat{i_v} : \widehat{X_v} \to \widehat{X}$.*

Proof: Without loss of generality, let $v_0 = v$ be the base vertex of T. To prove the existence of a Cannon-Thurston map, it suffices to show (from Lemma 1.3.1) that for all $M \geq 0$ and $x_0 \in X_v$ there exists $N \geq 0$ such that if a geodesic segment λ lies oustside the N-ball around $x_0 \in X_v$, B_λ lies outside the M-ball around $i_v(x_0) \in X$.

To prove this, we show that if λ lies outside the N-ball around $x_0 \in X_v$, B_λ lies outside a certain $M(N)$-ball around $i_v(x_0) \in X$, where $M(N)$ is a proper function from $\mathbb{N}$ into itself.

Since X_v is properly embedded in X there exists $f(N)$ such that $i_v(\lambda)$ lies outside the $f(N)-$ball around x_0 in X and $f(N) \to \infty$ as $N \to \infty$.

Let p be any point on B_λ. There exists $y \in i_v(\lambda)$ such that $d(y,p) \leq Ad_T(Py, Pp)$ by Lemma 4.1.9. Therefore,

$$\begin{aligned} d(x_0, p) &\geq d(x_0, y) - Ad_T(Py, Pp) \\ &\geq f(N) - Ad_T(P(x_0), Pp) \end{aligned}$$

By our choice of metric on X,

$$d(x_0, p) \geq d_T(P(x_0), Pp)$$

Hence

$$\begin{aligned} d(x_0, p) &\geq max(f(N) - Ad_T(P(x_0), Pp), d_T(P(x_0), Pp)) \\ &\geq \frac{f(N)}{A+1} \end{aligned}$$

From Theorem 4.1.8 there exists C' independent of λ such that B_λ is a C'-quasiconvex set containing $i_v(\lambda)$. Therefore any geodesic joining the end-points of $i_v(\lambda)$ lies in a C'-neighborhood of B_λ.

Hence any geodesic joining end-points of $i_v(\lambda)$ lies outside a ball of radius $M(N)$ where

$$M(N) = \tfrac{f(N)}{A+1} - C'$$

Since $f(N) \to \infty$ as $N \to \infty$ so does $M(N)$. $\Box$

The following is a direct consequence of Theorem 4.1.10 above.

Corollary 4.1.11 *Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let H be the stabilizer of a vertex or edge of T. Then there exists a Cannon-Thurston map for (H, G).*

4.2 Geometrically Tame Kleinian Groups

In this section we apply Theorem 4.1.10 to geometrically tame Kleinian groups.

The *convex core* of a hyperbolic 3-manifold N (without cusps) is the smallest convex submanifold $C(N) \subset N$ for which inclusion is a homotopy equivalence. If $C(N)$ has finite volume, N is said to be *geometrically finite*. There exists a compact 3-dimensional submanifold $M \subset N$, the *Scott core* [30] whose inclusion is a homotopy equivalence. The ends of N are in one-to-one correspondence with the components of $N - M$ or, equivalently, the components of ∂M. We say that an end of N is *geometrically finite* if it has a neighborhood missing $C(N)$. An end of N is *simply degenerate* if it has a neighborhood homeomorphic to $S \times \mathbb{R}$, where S is the corresponding component of ∂M, and if there is a sequence of pleated surfaces homotopic in this neighborhood to the inclusion of S, and exiting every compact set. N is called *geometrically tame* if all of its ends are either geometrically finite or simply degenerate. In particular, N is homeomorphic to the interior of M. For a more detailed discussion of pleated surfaces and geometrically tame ends, see [33] or [23].

Let $inj_N(x)$ denote the injectivity radius at $x \in N$. For the purposes of this section, we shall assume that there exists $\epsilon_0 > 0$ such that $inj_N(x) > \epsilon_0$ for all $x \in N$. Further, $\pi_1(N)$ is assumed to be freely indecomposable. By [6], N is geometrically tame. In order to apply Theorem 4.1.10 we need some preliminary Lemmas.

Let E be a simply degenerate end of N. Then E is homeomorphic to $S \times [0, \infty)$ for some closed surface S of genus greater than one.

Lemma 4.2.1 *[33] There exists $D_1 > 0$ such that for all $x \in N$, there exists a pleated surface $g : (S, \sigma) \to N$ with $g(S) \cap B_{D_1}(x) \neq \emptyset$.*

The following Lemma follows easily from the fact that $inj_N(x) > \epsilon_0$:

Lemma 4.2.2 *[6],[33] There exists $D_2 > 0$ such that if $g : (S, \sigma) \to N$ is a pleated surface, then $dia(g(S)) < D_2$.*

The following Lemma due to Minsky [23] follows from compactness of pleated surfaces.

Lemma 4.2.3 *[23] Fix S and $\epsilon > 0$. Given $a > 0$ there exists $b > 0$ such that if $g : (S, \sigma) \to N$ and $h : (S, \rho) \to N$ are homotopic pleated surfaces which are isomorphisms on π_1 and $inj_N(x) > \epsilon$ for all $x \in N$, then*

$$d_N(g(S), h(S)) \leq a \Rightarrow d_{Teich}(\sigma, \rho) \leq b,$$

where d_{Teich} denotes Teichmuller distance.

Definition: *The* **universal curve** *over $X \subset Teich(S)$ is a bundle whose fiber over $x \in X$ is x itself.* [20]

Lemma 4.2.4 *There exist K, ϵ and a homeomorphism h from E to the universal curve over a Lipschitz path in Teichmuller space, such that h is a (K, ϵ)-quasi-isometry.*

Proof: We can assume that $S \times \{0\}$ is mapped to a pleated surface $S_0 \subset N$ under the homeomorphism from $S \times [0, \infty)$ to E. We shall construct inductively a sequence of 'equispaced' pleated surfaces $S_i \subset E$ exiting the end. Assume that $S_0, \cdots, S_n$ have been constructed such that:

1. If E_i be the non-compact component of $E \backslash S_i$, then $S_{i+1} \subset E_i$.

2. Hausdorff distance between S_i and S_{i+1} is bounded above by $3(D_1 + D_2)$.

3. $d_N(S_i, S_{i+1}) \geq D_1 + D_2$.

4. From Lemma 4.2.3 and condition (2) above there exists D_3 depending on D_1, D_2 and S such that $d_{Teich}(S_i, S_{i+1}) \leq D_3$

Next choose $x \in E_n$, such that $d_N(x, S_n) = 2(D_1 + D_2)$. Then by Lemma 4.2.1, there exists a pleated surface $g : (S, \tau) \to N$ such that $d_N(x, g(S)) \leq D_1$. Let $S_{n+1} = g(S)$. Then by the triangle inequality and Lemma 4.2.2, if $p \in S_n$ and $q \in S_{n+1}$,

$$D_1 + D_2 \leq d_N(p,q) \leq 3(D_1 + D_2).$$

This allows us to continue inductively. The Lemma follows. $\Box$

Note that in the above Lemma, pleated surfaces are not assumed to be embedded. This is because immersed pleated surfaces with a uniform lower bound on injectivity radius are uniformly quasi-isometric to the corresponding Riemann surfaces.

Observe that the universal cover $\widetilde{E}$ of E is quasi-isometric to a tree (in fact a ray) of hyperbolic metric spaces by setting $T = [0, \infty)$, with vertex set $\{n : n \in \mathbb{N} \cup \{0\}\}$, edge set $\{[n-1, n] : n \in \mathbb{N}\}$, $X_n = \widetilde{S_n} = X_{[n-1,n]}$. Further, by Lemma 4.2.3 this tree of hyperbolic metric spaces satisfies the quasi-isometrically embedded condition. We shall now describe $\widetilde{C(N)}$ as a tree of hyperbolic metric spaces. Assume $M \subset C(N)$ and $\partial M = \{F_1, \cdots, F_n\}$ where F_i are pleated surfaces in N cutting off ends E_i.

Lemma 4.2.5 *[3] $\pi_1(N)$ is hyperbolic in the sense of Gromov. Also, if $i : E \to N$, denotes inclusion, then $i_* \pi_1(E)$ is a quasiconvex subgroup of $\pi_1(N)$.*

Remark: In fact there exists a geometrically finite hyperbolic manifold homeomorphic to N. This is part of Thurston's monster theorem. See [21] for a different proof of the fact. Also, the limit set of a geometrically finite manifold is locally connected [2]. This shall be of use later.

Recall that $M \subset N$ is the Scott-core of N and that $\pi_1(N)$ is freely indecomposable. Note that $\widetilde{M} \subset \widetilde{N}$ is quasi-isometric to the Cayley graph of $\pi_1(N)$. Hence, $\widetilde{M}$ is a hyperbolic metric space. Let $\widetilde{F_i} \subset \widetilde{N}$ represent a lift of F_i to $\widetilde{N}$. Then, by Lemma 4.2.5 above, $\widetilde{F_i}$ is a word-hyperbolic metric space. If $\widetilde{E_i}$ is a lift of E_i containing $\widetilde{F_i}$ then from our previous discussion, $\widetilde{E_i}$ is a ray of hyperbolic metric spaces. Since there are only finitely many ends E_i, we have thus shown:

Lemma 4.2.6 *The hyperbolic metric space $\widetilde{C(N)}$ is quasi-isometric to a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Further, we can choose a base vertex v_0 of T such that X_{v_0} is homeomorphic to $\widetilde{M}$.*

Applying Theorem 4.1.10, we get

Theorem 4.2.7 *Let Γ be a freely indecomposable Kleinian group, such that $\mathbb{H}^3/\Gamma = M$ has injectivity radius uniformly bounded below by some $\epsilon > 0$. Then there exists a continuous* *map from the Gromov boundary of* Γ *(regarded as an abstract group) to the limit set of* Γ *in* $\mathbb{S}^2_\infty$.

The above theorem has been independently proven by Klarreich [18], where other examples of maps between boundaries of hyperbolic metric spaces are considered.

Lemma 4.2.8 *Let* N *be a geometrically tame 3-manifold with* $inj_N(x) > \epsilon_0 > 0$ *for all* $x \in N$. *Then the Gromov boundary of* $\pi_1(N)$ *is locally connected.*

Proof: This follows from the fact that there exists a geometrically finite manifold $M = \mathbb{H}^3/\Gamma$ homeomorphic to N [21] and that for such an M, the limit set of Γ is locally connected [2]. $\Box$

Since a continuous image of a compact locally connected set is locally connected [17], Lemma 4.2.8 and Theorem 4.2.7 give:

Corollary 4.2.9 *Let* $N = \mathbb{H}^3/\Gamma$ *be a freely indecomposable 3-manifold with* $inj_N(x) > \epsilon_0$ *for all* $x \in N$. *Then the limit set of* Γ *is locally connected.*

Lemma 4.2.4 shows that there exists a quasi-isometry from a lift $\widetilde{E}$ of an end to the universal cover of a universal curve over a Lipschitz path σ in $Teich(S)$. We show further that σ is a Teichmuller quasigeodesic.

It is well known that geodesics in hyperbolic metric spaces diverge exponentially. The following proposition 'quasi-fies' this statement:

Proposition 4.2.10 *Given* δ, $A_0 \geq 0$ *there exist* $\beta>1$, $B > 0$, $K \geq 1$ *and* $\epsilon \geq 0$ *such that if* $[x, y], [y, z]$ *and* $[z, w]$ *are geodesics in a* δ*-hyperbolic metric space* (X, d) *with* $(x, z)_y \leq A_0$, $(y, w)_z \leq A_0$ *and* $d(y, z) \geq B$ *then any path joining* x *to* w *and lying outside a* D*-neighborhood of* $[y, z]$ *has length greater than or equal to* $\beta^D d(y, z)$,
where $D = min\{(d(x, [y, z]) - 1), (d(w, [y, z]) - 1)\}$.

Lemma 4.2.11 σ *is a Teichmuller quasigeodesic.*

Proof: Let $S_0 = \partial E$ be a pleated surface containing a closed geodesic l of N. This can always be arranged by taking a simple closed geodesic sufficiently far out in E and mapping in a pleated surface containing it [33]. Construct a sequence of equispaced pleated surfaces as in Lemma 4.2.4. $\widetilde{E}$ is quasi-isometric to a ray of hyperbolic metric spaces X, with vertex set $\{n : n \in \mathbb{N} \cup \{0\}\}$, edge set $\{[n-1, n] : n \in \mathbb{N}\}$, $X_n = \widetilde{S_n} = X_{[n-1,n]}$.

Fix x_0 in S_0. Inductively, define x_n to be the image of x_{n-1} under the Teichmuller map from S_{n-1} to S_n. Let r denote a quasi-isometric embedding of $[0, \infty)$ sending $[n, n+1]$ to the shortest geodesic from x_n to x_{n+1}. Then r is a quasigeodesic in E. Let $[a, b]$ be a lift of l to $\widetilde{E}$. Let $\lambda \subset X$ be the image of $[a, b]$ under a quasi-isometric homeomorphism h between $\widetilde{E}$ and X, sending $\widetilde{S_n}$ to X_n. Construct $B_\lambda \subset X$, as in the previous section. Lifts of r through a, b diverge exponentially.

From Theorem 4.1.8, B_λ is quasiconvex and hence a hyperbolic metric space with the inherited metric. Let r_1, r_2 be the images of these lifts through the end-points of λ. Then r_1, r_2 are (K, ϵ)-quasigeodesics diverging exponentially in X. Assume, after reparametrization if necessary, $r_1(n)$, $r_2(n) \in X_n$. Let d_n denote the path metric on X_n. Then by Proposition 4.2.10, there exist $C_1 > 1$, $n \geq 1$ such that $d_{N+n}(r_1(N+n), r_2(N+n)) \geq C_1 d_N(r_1(N), r_2(N))$ for all $N \geq 0$. Hence there exists $C_2 > 0$ and $n \in \mathbb{N}$ such that $d_{Teich}(S_N, S_{N+kn}) \geq kC_2$ for all N, $k \in \mathbb{N}$. Since σ was shown to be Lipschitz in Lemma 4.2.4, this proves that σ is a Teichmuller quasigeodesic. $\Box$

So far arguments have been coarse. The argument above circumvents the construction of a model manifold in [24]. At this stage, we need to quote a part of the main theorem of [23], the common ingredient of both proofs.

Theorem 4.2.12 *[23] If N is a geometrically tame hyperbolic 3-manifold with indecomposable fundamental group, such that there exists $\epsilon_0 > 0$ with $inj_N(x) > \epsilon_0$ for all $x \in N$, then each simply degenerate end E of N gives rise to a unique Teichmuller ray r, such that every pleated surface in E lies at a uniformly bounded distance from r. Further, r depends only on the corresponding ending lamination.*

That r depends only on the corresponding ending lamination was proven by Masur [19].

Combining Lemma 4.2.11 and Theorem 4.2.12 we have a proof of the main theorem of [24] : the ending lamination theorem for 3-manifolds with freely indecomposable fundamental group and a uniform lower bound on injectivity radius.

Theorem 4.2.13 *Let N_1 and N_2 be homeomorphic hyperbolic 3-manifolds with freely indecomposable fundamental group. Suppose there exists a uniform lower bound $\epsilon > 0$ on the* *injectivity radii of N_1 and N_2. If the end invariants of corresponding ends of N_1 and N_2 are equal, then N_1 and N_2 are isometric.*

Proof: From Lemma 4.2.11, corresponding simply degenerate ends E_{i1}, E_{i2} of N_1 and N_2 are homeomorphic via quasi-isometries to universal curves over Teichmuller quasi-geodesics l_{i1} and l_{i2} lying in bounded neighbourhoods of Teichmuller geodesics l_i. Hence corresponding ends are homeomorphic via quasi-isometries to each other. Therefore N_1, N_2 are homeomorphic by a quasi-isometry. Finally, by [32] N_1 and N_2 are isometric. $\Box$

4.3 Examples

Let H be a hyperbolic subgroup of a hyperbolic group G.

Definition : [15] [10] *If $i : \Gamma_H \to \Gamma_G$ be an embedding of the Cayley graph of H into that of G, then the distortion function is given by*

$$disto(R) = R^{-1} Diam_{\Gamma_H}(\Gamma_H \cap B(R)),$$

where $B(R)$ is the ball of radius R around $1 \in \Gamma_G$.

All previously known examples of non-quasiconvex hyperbolic subgroups of hyperbolic groups exhibit exponential distortion. We construct in this section some examples exhibiting greater distortion. Some of these will be shown to have Cannon-Thurston maps. For the rest, existence of Cannon-Thurston maps is not yet known. Further, we shall describe certain examples of free subgroups of $PSL_2\mathbb{C}$ and show that they exhibit arbitrarily large distortion. The existence of Cannon-Thurston maps for some of these is not yet known.

Our starting point for constructing distorted subgroups of hyperbolic groups is the following Lemma of Bestvina, Feighn and Handel [4]:

Lemma 4.3.1 *[4] There exists a hyperbolic group G such that $1 \to F \to G \to F \to 1$ is exact, where F is free of rank 3.*

Let $F_1 \subset G$ denote the normal subgroup. Let $F_2 \subset G$ denote a section of the quotient group. Conjugation by generators of F_2 increases lengths of elements of F_1 by at most a multiplicative factor $\lambda > 1$. Hence the distortion of F_1 in G is at most exponential. But since the automorphisms induced by F_2 are hyperbolic [4] the distortion of F_1 in G is exponential.

Let $G_1, \cdots, G_n$ be n distinct copies of G. Let F_{i1} and F_{i2} denote copies of F_1 and F_2 respectively in G_i. Let

$$X_n = G_1 *_{H_1} G_2 * \cdots *_{H_{n-1}} G_n$$

where each H_i is a free group of rank 3, the image of H_i in G_i is F_{i2} and the image of H_i in G_{i+1} is $F_{(i+1)1}$. Then X_n is hyperbolic. This follows inductively from the fact that the image of H_i in G_i is quasiconvex in $G_1 *_{H_1} G_2 * \cdots *_{H_{i-1}} G_i$ and the main combination theorem of [3]. Further, it follows from the preceding paragraph that the distortion of X_m in X_{m+1} is exponential.

Let $H = F_{11} \subset X_n$. Then the distortion of H is superexponential for $n > 1$. In fact, the distortion function is an iterated exponential of height n. To see this one notes that since the distortion of X_m in X_{m+1} is exponential, the distortion of H in X_n is at most an iterated exponential of height n. To see that the distortion is in fact an iterated exponential of height n, we sketch an argument for $n = 2$. Let $G_1 *_{H_1} G_2$ be generated by a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 where a_i's are generators of F_{11}, b_i's are generators of H_1 and c_i's are images of generators of the quotient free group in G_2 under a section. Then $({c_1}^{-m} b_1 {c_1}^m)^{-1} a_1 ({c_1}^{-m} b_1 {c_1}^m)$ is in F_{11} and has length (in F_{11}) an iterated exponential (in m) of height 2. Hence the distortion of H in X_2 is at least an iterated exponential of height 2.

Note further that $G_1 *_{H_1} G_2$ can be regarded as a graph of groups with one vertex and three edges, where the vertex group is G_1 and edge groups are isomorphic to F. Then from Corollary 4.1.11, the pair $(G_1, G_1 *_{H_1} G_2)$ has a Cannon-Thurston map. Proceeding inductively and observing that a composition of Cannon-Thurston maps is a Cannon-Thurston map, we see that (H, G) has a Cannon-Thurston map.

The next class of examples are not known to have Cannon-Thurston maps.

Our starting point is again Lemma 4.3.1. Let a_1, a_2, a_3 be generators of F_1 and b_1, b_2, b_3 be generators of F_2. Then

$$G = \{a_1, a_2, a_3, b_1, b_2, b_3 : b_i^{-1} a_j b_i = w_{ij}\}$$

where w_{ij} are words in a_i's. We add a letter c conjugating a_i's to 'sufficiently random' words in b_j's to get G_1. Thus,

$$G_1 = \{a_1, a_2, a_3, b_1, b_2, b_3, c : b_i^{-1} a_j b_i = w_{ij}, c^{-1} a_i c = v_i\},$$

where v_i's are words in b_j's satisfying a small-cancellation type condition to ensure that G_1 is hyperbolic. See [16], pg. 151 for details on addition of 'random' relations.

Let H be the subgroup of G_1 generated by the a_i's. It can be checked that H has distortion function greater than any iterated exponential. To see this consider the sequence of words given by $w_1 = a_1$ and (inductively) $w_{i+1} = (c^{-1}w_ic)^{-1}w_i(c^{-1}w_ic)$. These are elements of H with length (in H) growing faster than any iterated expoential in i.

The above set of examples were motivated largely by examples of distorted cyclic subgroups in [15], pg. 67.

So far, there is no satisfactory way of manufacturing examples of hyperbolic subgroups of hyperbolic groups exhibiting arbitrarily high distortion. It is easy to see that a subgroup of sub-exponential distortion is quasiconvex [15]. Not much else is known. For instance, one does not know if A^{n^2} can appear as a distortion function.

The situation is far more satisfactory in the case of Kleinian groups. We calculate below the distortion functions for a class of examples appearing in work of Minsky [22]:

Let S be a hyperbolic punctured torus so that the two shortest geodesics a and b are orthogonal and of equal length. Let S_0 denote S minus a neighborhood of the cusp. Let $N_\delta(a)$ and $N_\delta(b)$ be regular collar neighborhoods of a and b in S_0. For $n \in \mathbb{N}$, define $\gamma_n = a$ if n is even and equal to b if n is odd. Let T_n be the open solid torus neighborhood of $\gamma_n \times \{n + \frac{1}{2}\}$ in $S_0 \times [0, \infty)$ given by

$$T_n = N_\delta(\gamma_n) \times (n, n+1)$$

and let $M_0 = (S_0) \times [0, \infty) \setminus \bigcup_{n \in \mathbb{N}} T_n$.

Let $a(n)$ be a sequence of positive integers greater than one. Let $\hat{\gamma_n} = \gamma_n \times \{n\}$ and let μ_n be an oriented meridian for ∂T_n with a single positive intersection with $\hat{\gamma_n}$. Let M denote the result of gluing to each ∂T_n a solid torus $\hat{T_n}$, such that the curve $\hat{\gamma_n}^{a(n)}\mu_n$ is glued to a meridian. Let q_{nm} be the mapping class from S_0 to itself obtained by identifying S_0 to $S_0 \times m$, pushing through M to $S_0 \times n$ and back to S_0. Then $q_{n(n+1)}$ is given by $\Phi_n = D_{\gamma_n}^{a(n)}$, where D_c^k denotes Dehn twist along c, k times. Matrix representations of Φ_n are given by

$$\Phi_{2n} = \begin{pmatrix} 1 & a(2n) \\ 0 & 1 \end{pmatrix}$$

and

$$\Phi_{2n+1} = \begin{pmatrix} 1 & 0 \\ a(2n+1) & 1 \end{pmatrix}$$

Recall that the metric on M_0 is the restriction of the product metric. $\hat{T_n}$'s are given hyperbolic metrics such that their boundaries are uniformly quasi-isometric to $\partial T_n \subset M_0$. Then from [22], M is quasi-isometric to the complement of a rank one cusp in the convex core of a hyperbolic manifold $M_1 = \mathbb{H}^3/\Gamma$. Let σ_n denote the shortest path from $S_0 \times 1$ to $S_0 \times n$. Let $\overline{\sigma_n}$ denote σ_n with reversed orientation. Then $\tau_n = \sigma_n \gamma_n \overline{\sigma_n}$ is a closed path in M of length $2n+1$. Further τ_n is homotopic to a curve $\rho_n = \Phi_1 \cdots \Phi_n(\gamma_n)$ on S_0. Then

$$\Pi_{i=1\cdots n} a(i) \leq l(\rho_n) \leq \Pi_{i=1\cdots n}(a(i)+2)$$

Hence

$$\Pi_{i=1\cdots n} a(i) \leq (2n+1) disto(2n+1) \leq \Pi_{i=1\cdots n}(a(i)+2)$$

Since M is quasi-isometric to the complement of the cusp of a hyperbolic manifold and γ_n's lie in a complement of the cusp, the distortion function of Γ is of the same order as the distortion function above. In particular, functions of arbitraily fast growth may be realised. This answers a question posed by Gromov [15] pg. 66.

Manifolds with unbounded $a(n)$'s are not known to have Cannon-Thurston maps.

A large class of examples (due to I. Kapovich) to which Theorem 4.1.10 applies can be constructed as follows: Let H be a non-elementary hyperbolic group. Let $F = < a_1, a_2 >$ be a free quasiconvex subgroup of rank 2. Let $G = < H, t : t^{-1} a_i t = v_i, i = 1, 2 >$, where v_i's are words in a_1, a_2 satisfying a small cancellation type condition [See [16] pg. 151].

Chapter 5

Conclusion

In this thesis we have focussed on one aspect of the extrinsic geometry of hyperbolic subgroups of hyperbolic groups, viz. Cannon Thurston maps. The general question still remains open : If H is a hyperbolic subgroup of a hyperbolic group G, does there exist a Cannon Thurston map for the pair (H, G) ?

The analysis in Chapter 3 singles out bi-infinite geodesics whose end-points are identified by the Cannon Thurston map. These can be regarded as the bi-infinite geodesics that determine the distortion of H in G.

It might be worthwhile to make this more precise. Let $N_H = g \in G : d_G(g, 1) = d_G(g, \Gamma_H)$. N_H can be regarded as the set of normals to H at $1 \in H$. Then N_H is quasiconvex as it is a union of geodesic segments starting at 1. Quasiconvexity of H is characterized by $t_h(N_H) \cap B_k(N_H) \neq \emptyset$ for at most finitely many $h \in H$. That is to say that translates of N_H diverge away from each other uniformly if and only if H is quasiconvex. The leaves of the ending lamination in Chapter 3 give bi-infinite geodesics along which the opposite phenomenon occurs. To investigate this more thoroughly, one might need a coarse analog of the second fundamental form. For $z \in G$ and $h \in H$ we define $\Pi(z, h)$ to be the length of the shortest path between z and hz lying on the boundary of a $d_G(z, 1)$- neighborhood of Γ_H. One could then study the asymptotic behaviour of $\Pi(z, h)$ and try to single out pairs of directions in N_H and Γ_H which determine the shape of Γ_H in Γ_G. To do this one does not require H or G to be hyperbolic.

Another open problem is to characterize quasiconvexity. Let X_G be a 2-complex with fundamental group G. Let X_H be a cover corresponding to the subgroup H. Let $I(x)$ denote the injectivity radius of X_H at x.

Question : Does $I(x) \to \infty$ as $x \to \infty$ imply that H is quasi-isometrically embedded in G ?

Again, the question above makes sense even when H or G are not hyperbolic.

Bibliography

[1] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short. Notes on word hyperbolic groups. *Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky eds.)*, pages 3–63, 1991.

[2] J. Anderson and B. Maskit. Local connectivity of limit sets of Kleinian groups. *preprint.*

[3] M. Bestvina and M. Feighn. A Combination theorem for Negatively Curved Groups. *J. Diff. Geom., vol 35*, pages 85–101, 1992.

[4] M. Bestvina, M. Feighn, and M. Handel. Laminations, trees and irreducible automorphisms of free groups. *preprint.*

[5] F. Bonahon. Geodesic Currents on Negatively Curved Groups. *in Arboreal Group Theory (ed. R.C. Alperin), MSRI Publ., vol. 19, Springer Verlag, 1991*, pages 143–168.

[6] F. Bonahon. Bouts de varietes hyperboliques de dimension 3. *Ann. Math. vol.124*, pages 71–158, 1986.

[7] R. D. Canary, D. B. A. Epstein, and P. Green. Notes on Notes of Thurston. *in Analytical and Geometric Aspects of Hyperbolic Spaces*, pages 3–92, 1987.

[8] J. Cannon and W. P. Thurston. Group Invariant Peano Curves. *preprint.*

[9] M. Coornaert, T. Delzant, and A.Papadopoulos. Geometrie et theorie des groupes. *Lecture Notes in Math.,vol.1441,Springer Verlag*, 1990.

[10] B. Farb. The extrinsic geometry of subgroups and the generalized word problem. *Proc. LMS (3) 68*, pages 577–593, 1994.

[11] B. Farb and R. E. Schwartz. Quasi-isometric Rigidity of Hilbert Modular Groups. *preprint.*

[12] A. Fathi, M. Laudenbach, and V. Poenaru. Travaux de Thurston sur les surfaces. *Asterisque 66-67*, pages 1–284, 1979.

[13] W. J. Floyd. Group Completions and Limit Sets of Kleinian Groups. *Invent. Math. vol.57*, pages 205–218, 1980.

[14] E. Ghys and P. de la Harpe(eds.). Sur les groupes hyperboliques d'apres Mikhael Gromov. *Progress in Math. vol 83, Birkhauser, Boston Ma.*, 1990.

[15] M. Gromov. Asymptotic Invariants of Infinite Groups. *in Geometric Group Theory,vol.2; Lond. Math. Soc. Lecture Notes 182 (1993), Cambridge University Press.*

[16] M. Gromov. Hyperbolic Groups. *in Essays in Group Theory, ed. Gersten, MSRI Publ.,vol.8, Springer Verlag,1985*, pages 75–263.

[17] J. G. Hocking and G. S. Young. Topology. *Addison Wesley(1961).*

[18] E. Klarreich. PhD Thesis, SUNY, Stonybrook. *in preparation.*

[19] H. Masur. Hausdorff dimension of the set of nonergodic foliations of a quadratic differential. *Duke Math. J.*, 66:387–442, 1992.

[20] C. McMullen. Amenability Poincare Series and Quasiconformal maps. *Invent. math.*, 97:95–127, 1989.

[21] C. McMullen. Iteration on Teichmuller Space. *Invent. math.*, 9:425–454, 1990.

[22] Y. N. Minsky. The Classification of Punctured Torus Groups. *preprint.*

[23] Y. N. Minsky. Teichmuller Geodesics and Ends of 3-Manifolds. *Topology*, pages 1–25, 1992.

[24] Y. N. Minsky. On Rigidity, Limit Sets, and End Invariants of Hyperbolic 3-Manifolds. *J. A.M.S., vol.7*, pages 539–588, 1994.

[25] M. Mitra. Cannon-Thurston Maps for Hyperbolic Group Extensions. *preprint.*

[26] M. Mitra. Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces. *preprint.*

[27] M. Mitra. Ending Laminations for Hyperbolic Group Extensions. *to appear in GAFA*.

[28] L. Mosher. Hyperbolic Extensions of Groups. *preprint.*

[29] F. Paulin. Outer Automorphisms of Hyperbolic Groups and Small Actions on R-trees. *in Arboreal Group Theory (ed.R. C. Alperin),MSRI Publ., vol. 19, Springer Verlag,1991*, pages 331–344.

[30] P. Scott. Compact submanifolds of 3-manifolds. *Journal L.M.S.*, pages 246–250, 1973.

[31] P. Scott and G. Swarup. Geometric Finiteness of Certain Kleinian Groups. *Proc. AMS 109*, pages 765–768, 1990.

[32] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stonybrook Conference, Ann. of Math. Studies 97, Princeton*, 1981.

[33] W. P. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes.*

[34] W. P. Thurston. On the Geometry and Dynamics of Diffeomorphisms of Surfaces. *Bull. A.M.S. vol. 19*, pages 417–431, 1987.