Where connectedness loci of polynomials meet Teichmüller spaces

Sabya Mukherjee

Tata Institute of Fundamental Research

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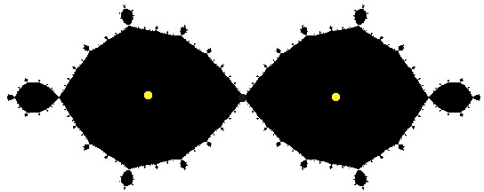
Fatou-Sullivan Dictionary

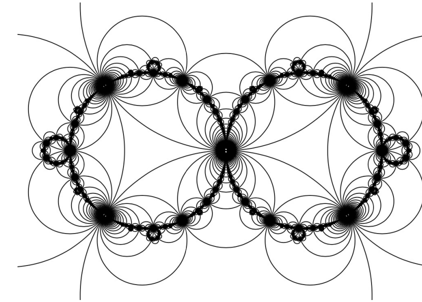
Somewhat parallel worlds

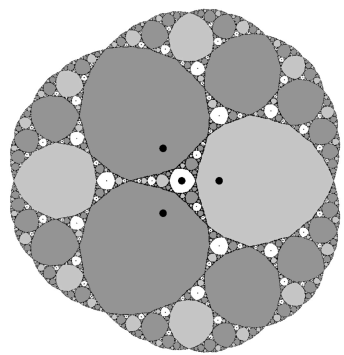
Rational maps		Kleinian groups
Fatou set / Julia set	Normal family arguments	Ordinary set / Limit set
No wandering domain		Ahlfors finiteness
Blaschke / Blaschke Space	QC deformations	Fuchsian / Teichmüller space
Quasi-Blaschke		Quasi-Fuchsian
Polynomial mating	Iteration on Teich. spaces	Double limit
Realizing branched covers as rational maps	$\mathbb{R}$-trees	Hyperbolization of 3-manifolds
Parabolic implosion/ Geometric limits	↑	New parabolics/ Geometric limits
Discontinuity of straightening		Thurston-Kerckhofff discontinuity phenomenon
Canonical decomposition	Common techniques	Torus decomposition

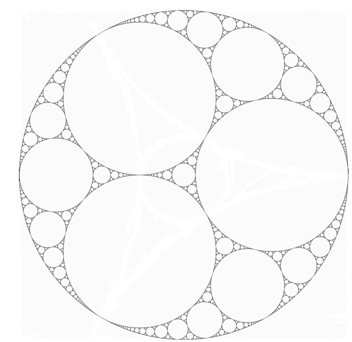
Inter-planetary twin cities:

Critically fixed anti-rational maps $\longleftrightarrow$ Kissing reflection groups









Their deformation spaces have stark resemblances (boundedness, bifurcations, global topology).

[Lodge-Luo-M]

Some features are lost in translation

Rational maps on $\hat{\mathbb{C}}$

Critical points,
Non-invertibility

Positive area Julia sets

Connected, non locally conn. Julia sets

??

No genuine analogue
(Barycentric extension, lamination)

Kleinian groups

Invertible,
many generators

Area(limit set) = 0

Connected limit sets are locally conn.

No invariant line fields

Action on $\mathbb{H}^3$

Fatou (1920s): The similarities are probably not coincidental.
These conformal dynamical systems live inside the galaxy of algebraic correspondences.

A possible general theory?

↓

Probably too ambitious

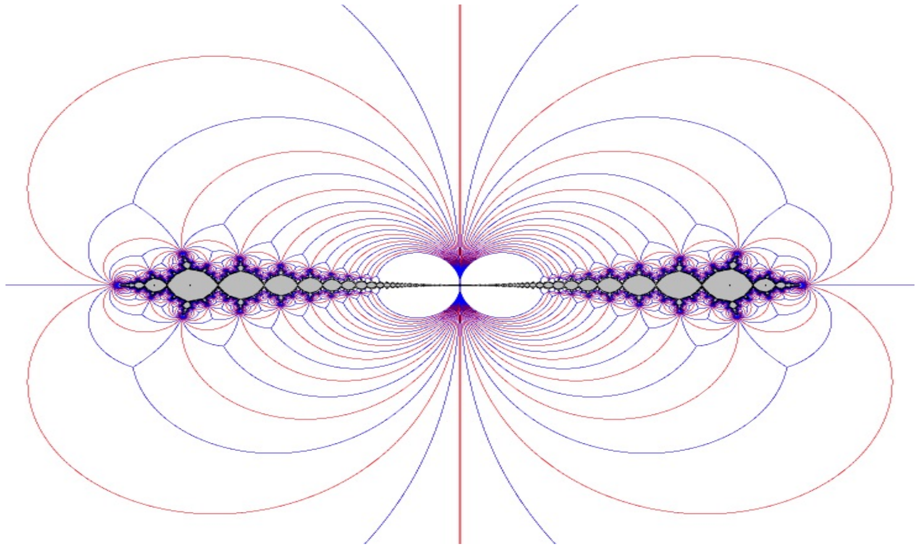
Desire: Want to find a new planet in this galaxy where some of the common phenomena can be observed simultaneously.

To fulfill such a dream, need to

1) establish combination/mating theorems for rational maps and Kleinian groups,

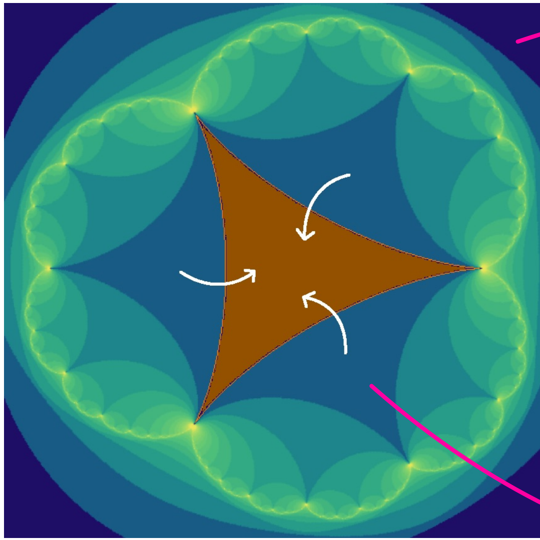
2) study the parameter spaces of such matings.

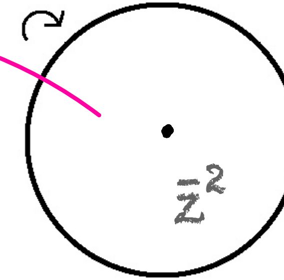
- Bullett and Penrose gave us hope in the 1990s: they discovered algebraic correspondences that combine the actions of certain quadratic rational maps and the modular group.

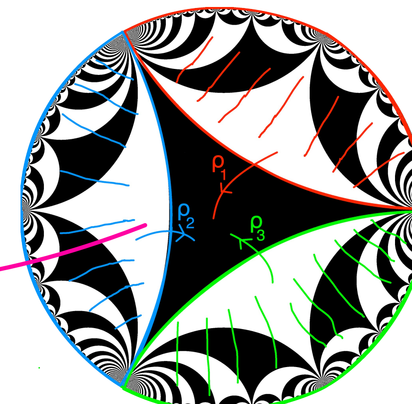


- Bullett and Lomonaco (2020) proved that this family of correspondences contains matings of all quadratic parabolic rational maps and $PSL_2(\mathbb{Z})$.

Dynamics of specific Schwarz reflection maps (in quadrature domains) furnish further examples of such matings.







Schwarz reflection plane

Nielsen of ITRG

[Lee-Lyubich-Makarov-M]

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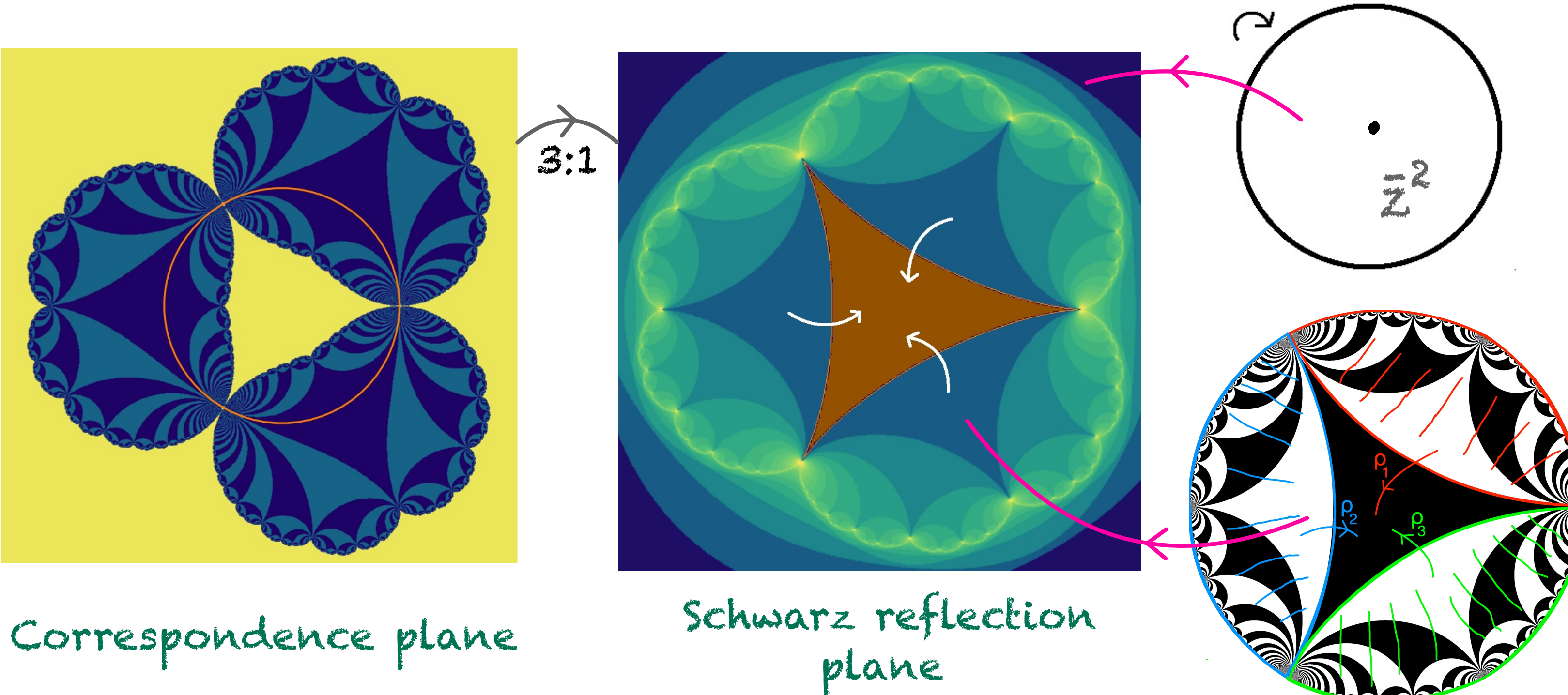


Correspondence plane

Schwarz reflection plane

Nielsen of ITRG

[Mating of $\bar{z}^2$ and the ideal triangle reflection group]

[Lee-Lyubich-Makarov-M]

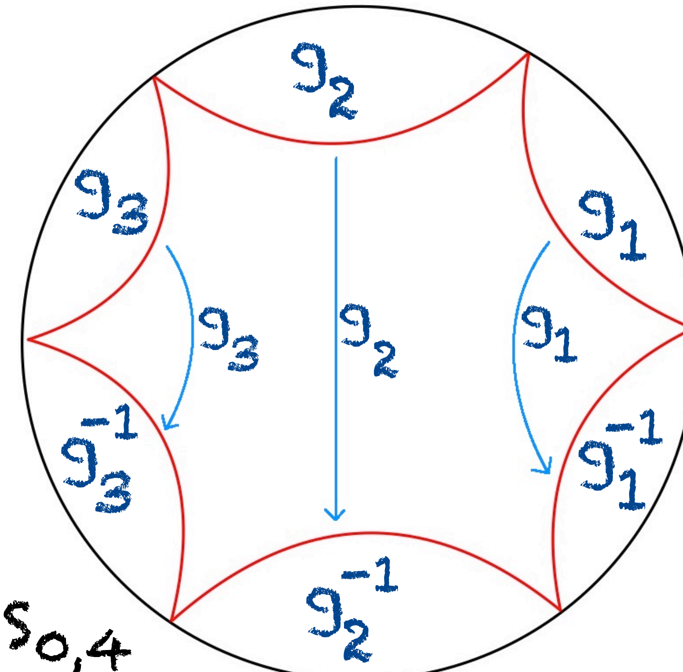
A General Recipe to Combine Genus 0 Orbifolds with Polymomials

Step I: Circle coverings from groups

A. Punctured spheres

$S_{0,d+1} \longrightarrow$ Bowen-Series map,
degree $2d-1$
circle covering

i) piecewise analytic,
ii) expansive circle covering,
iii) orbit equivalent to the group.



$S_{0,4}$

B. A class $\mathcal{S}$ of genus zero orbifolds:

hyperbolic orbifolds Σ of genus zero with
(1) at least one puncture,
(2) at most one order 2 orbifold point,
(3) at most one order $n \geqslant 3$ orbifold point.

$$\begin{tikzcd} \tilde{\Sigma} \arrow[r, rightsquigarrow] \arrow[d, "n:1"'] & A^{BS}_{\tilde{\Sigma}} \arrow[d, "n:1"] \\ \Sigma \arrow[r, rightsquigarrow] & A_{\Sigma} \end{tikzcd}$$

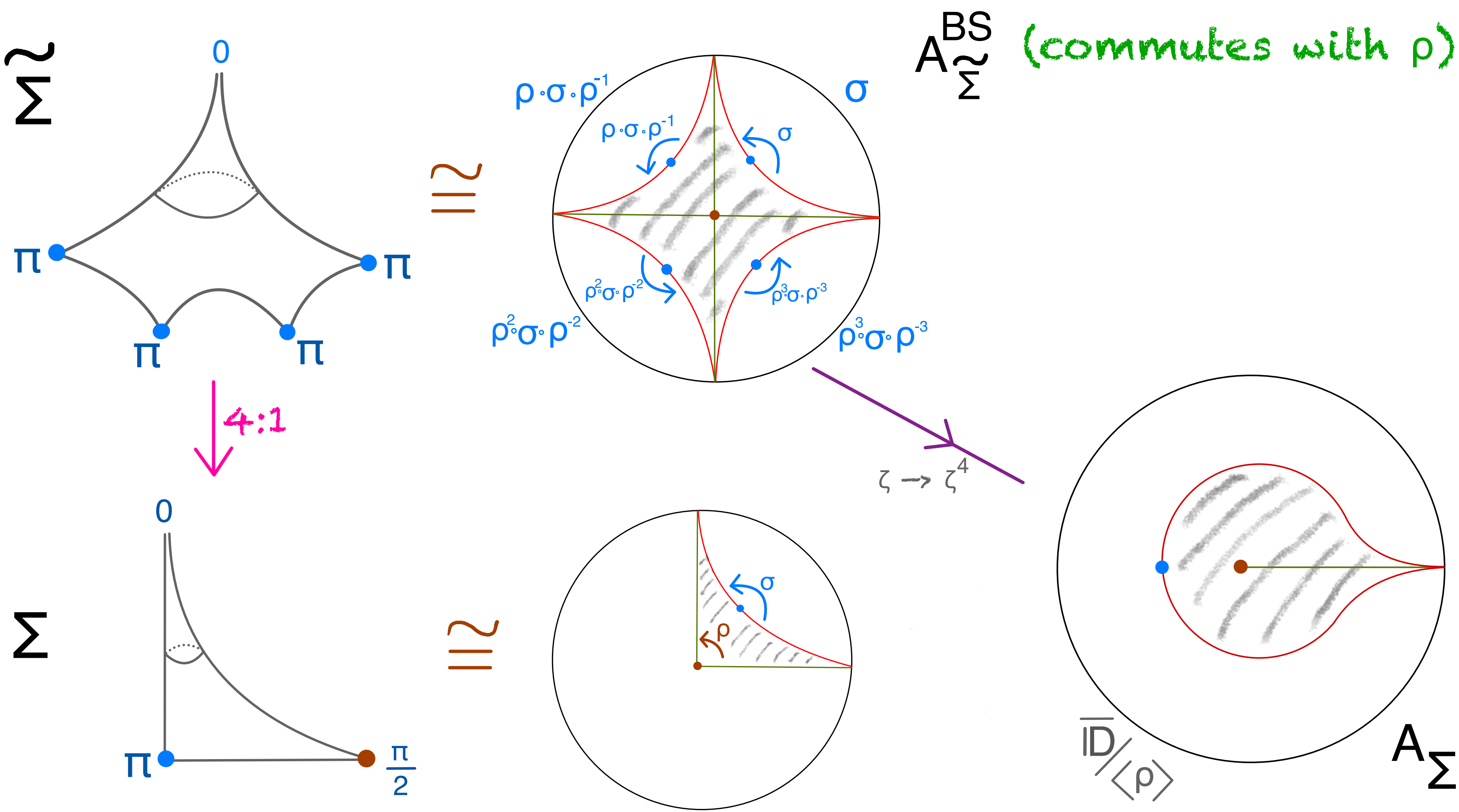
(discontinuous, but commutes with the order n elliptic)

This yoga is required to obtain a continuous circle map

i) piecewise analytic,
ii) expansive circle covering,
iii) $\boxed{\text{virtually}}$ orbit equivalent to the group.

[Mj-M]

Special case: Hecke group



Step II: Mating hybrid classes with external maps

☆ Let $\Sigma \in \mathbb{S}$, and A_Σ be the associated circle covering.

☆ Let f be a complex polynomial of degree $d := \deg(A_\Sigma : S^1 \to S^1)$ with connected Julia set.

☆ Following the theory of polynomial-like maps, we want to mate $f : K(f) \to K(f)$ with the external map A_Σ.

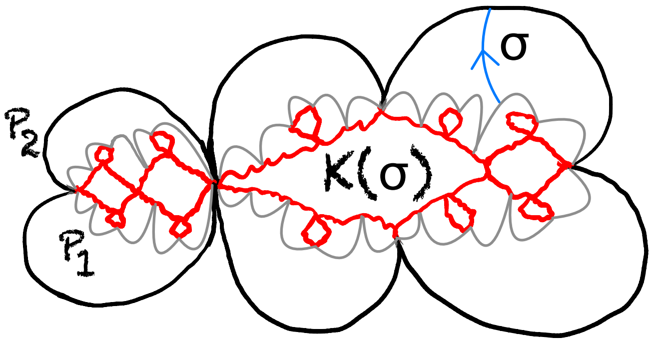
☆ Two obstacles:

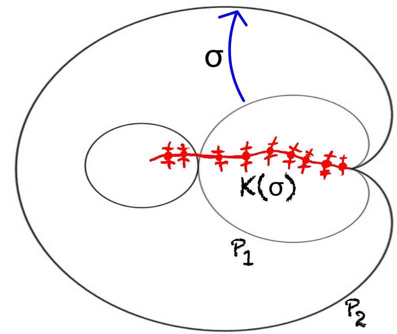
a) A_Σ has parabolics, so the external map z^d of f is topologically, but not quasisymmetrically, conjugate to A_Σ
$\Rightarrow$ qc surgery is inadequate.

b) A_Σ is not analytic in a neighborhood of S^1, the resulting mating would be a $\boxed{\text{`degenerate/pinched' polynomial-like map.}}$

(related objects were considered earlier by Makienko, Lomonaco, and others)

Theorem. Let f be geometrically finite.
Then there exists a degenerate polynomial-like map σ whose internal class is f and external map is A_Σ.





- The circle conjugacy between z^d and A_Σ admits a David extension to the disk.
- Use the above extension to glue the action of A_Σ outside of $K(f)$, and appeal to the David integrability theorem to uniformize.

[Lyubich-Merenkov-M-Ntalampekos, Mj-M]

Theorem. Let f be periodically repelling and at most finitely renormalizable.

Then there exists a degenerate polynomial-like map σ whose internal class is f and external map is A_Σ.

- The desired degenerate polynomial-like map is constructed as a limit of PCF maps. This involves a combination of puzzle machinery, combinatorial continuity, and combinatorial rigidity arguments.

- Such a limit exists due to compactness of the space of degenerate polynomial-like maps having A_Σ as their external map.

  (Some care is needed to topologize and to prove compactness as a degenerate poly-like map is defined on a pinched disk and has no fundamental annulus.)

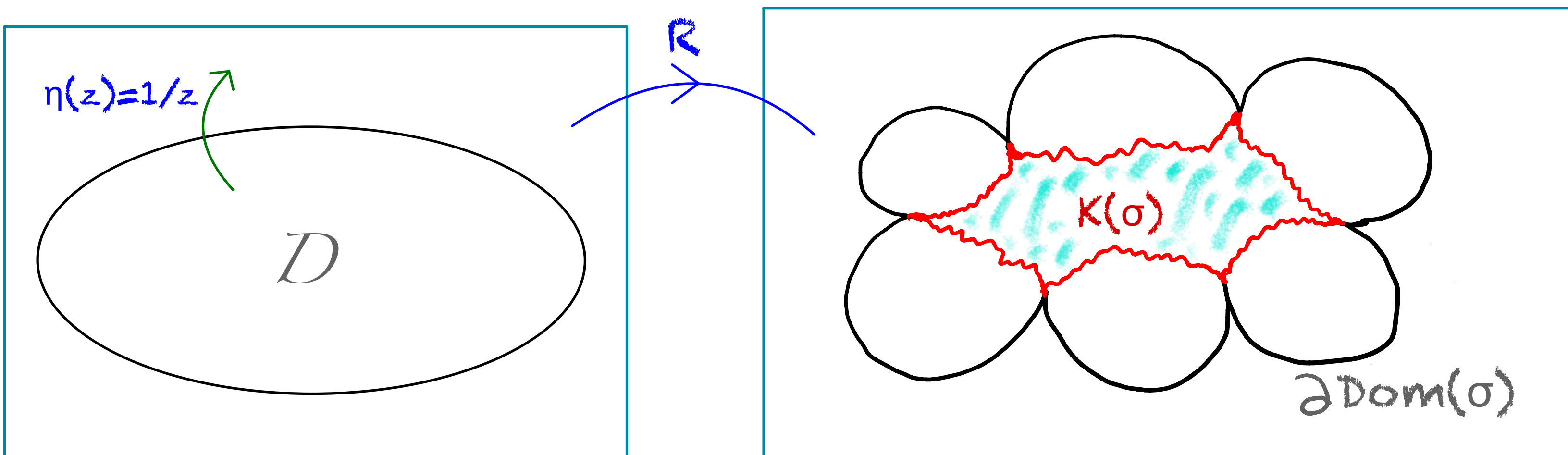
[Luo-Lyubich-M]

Step III: Algebraic description of matings

Q: Can one describe the matings explicitly?

Special case:

- When f lies in the main hyperbolic component, then the domain of definition of σ is a closed Jordan disk.



$$\sigma = R \circ \eta \circ (R|_{\mathcal{D}})^{-1}$$

Theorem. There exist

i) a finite tree of spheres $\mathcal{T}$,

ii) a conformal involution $\eta: \mathcal{T} \to \mathcal{T}$,

iii) a pinched disk $\mathcal{D}$ in $\mathcal{T}$ with $\eta(\mathcal{D}) = \mathcal{T} - \overline{\mathcal{D}}$, and

iv) a rational map $R: \mathcal{T} \to \hat{\mathbb{C}}$ that maps $\overline{\mathcal{D}}$ homeomorphically onto $\mathrm{Dom}(\sigma)$, such that $\boxed{\sigma = R \circ \eta \circ (R|_{\overline{\mathcal{D}}})^{-1}.}$

☆ Such a mating σ restricts to the boundary of its domain of definition as an orientation-reversing involution.

☆ Use the boundary involution to weld two copies of $\mathrm{Dom}(\sigma)$.

☆ Singular points on $\partial\mathrm{Dom}(\sigma)$ necessitate an analysis of the type of the welded Riemann surface.

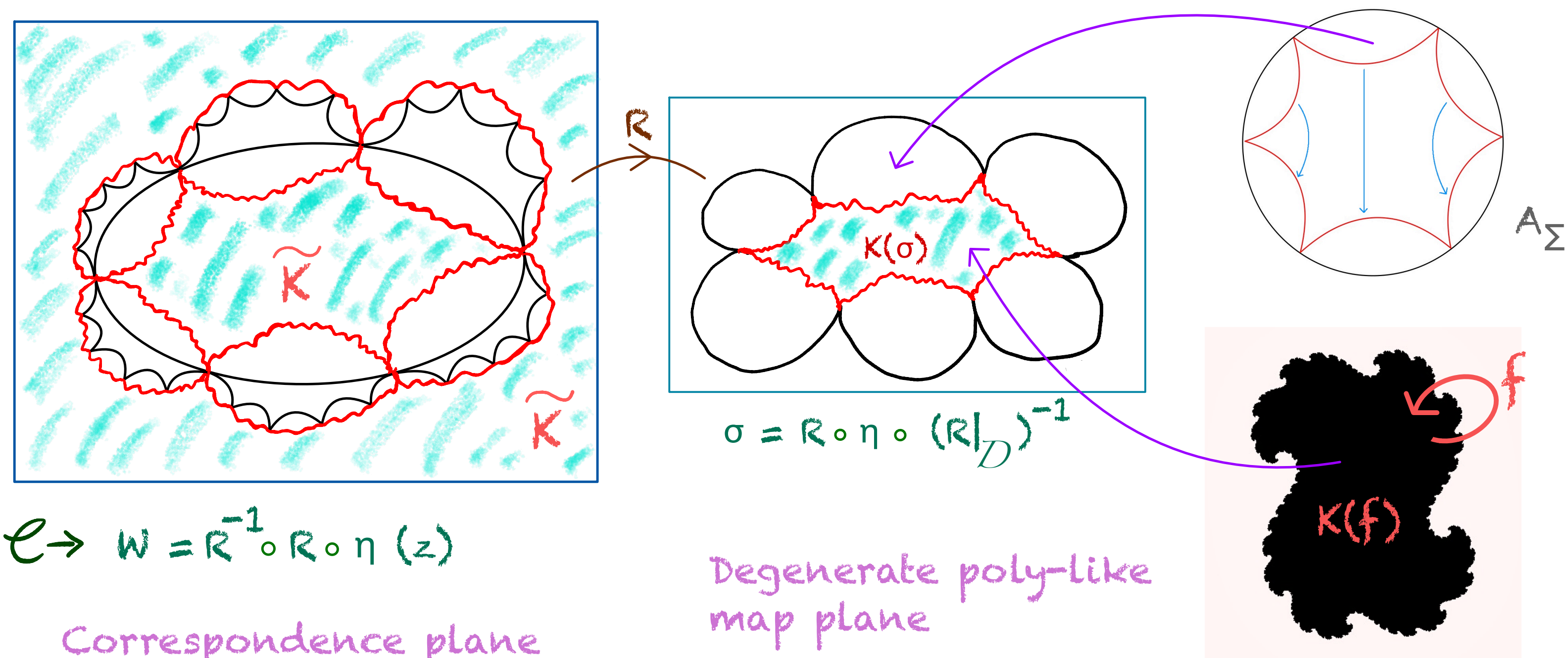
[Luo-Lyubich-M]

Step IV: From degenerate poly-like maps to correspondences

- Lift the σ-dynamics by the rational map R to define a correspondence

$$(z,w)\in \mathcal{C} \iff R(w) = R(\eta(z)).$$

- $\widetilde{K} := R^{-1}(K(\sigma))$, $\widetilde{T} := \mathbb{C} - \widetilde{K}$



Theorem. Let f be

i) geometrically finite, or
ii) periodically repelling and at most finitely renormalizable.

Further, let $\Sigma \in \mathcal{S}$.

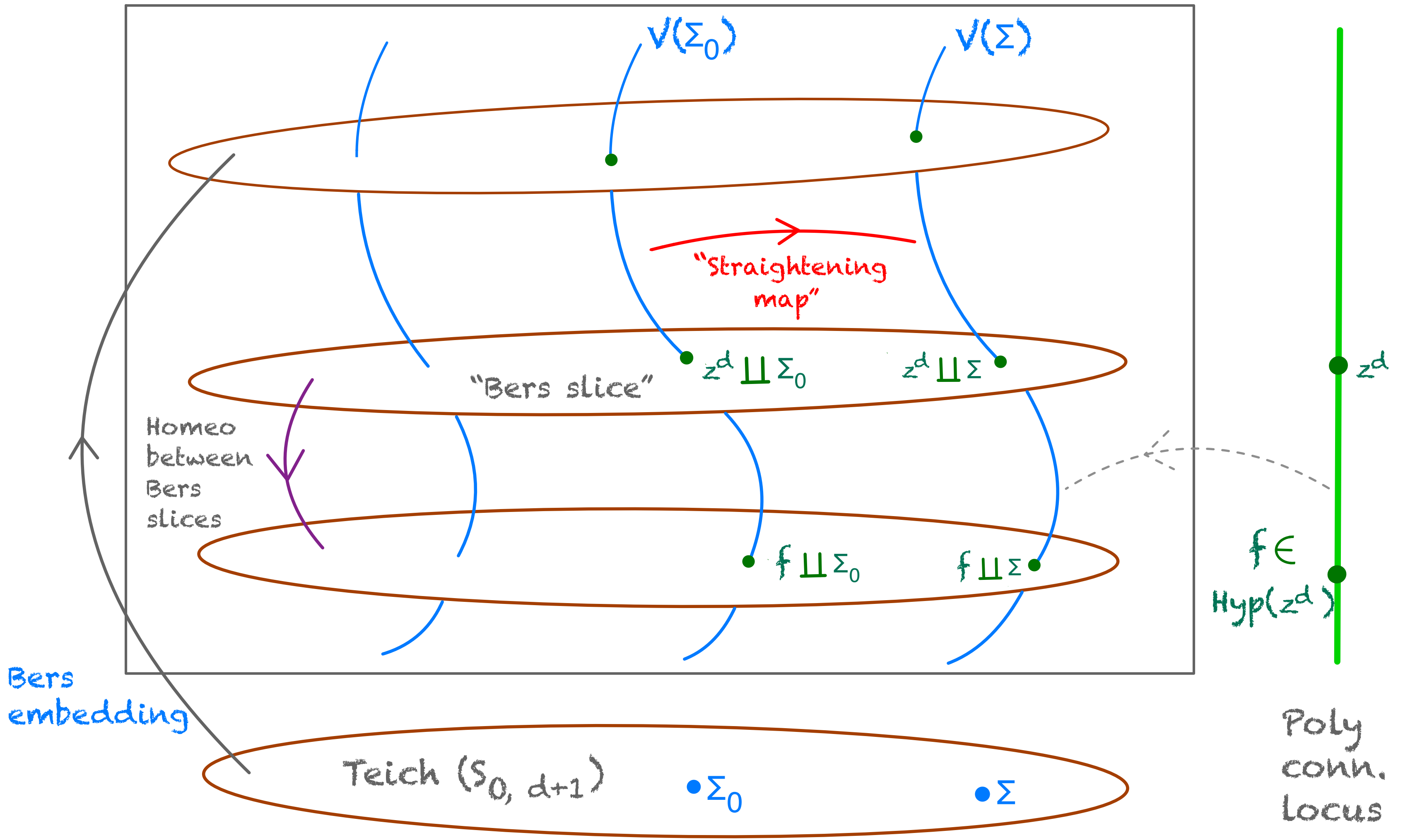
Then there exists an algebraic correspondence $\mathscr{C}$ that combines the actions of the polynomial f and the Fuchsian group uniformizing Σ.

☆ On $\widetilde{K}$, suitable branches of $\mathscr{C}$ are conformally conjugate to $f|_{K(f)}$.

☆ On $\widetilde{T}$, the correspondence $\mathscr{C}$ acts properly discontinuously via conformal automorphisms. Moreover, $\widetilde{T}/\mathscr{C} \cong \Sigma$.

[Luo-Lyubich-M]

Parameter spaces of matings



$V(\Sigma_0)$
$V(\Sigma)$
"Straightening map"
$z^d \amalg \Sigma_0$
$z^d \amalg \Sigma$
"Bers slice"
z^d
Homeo between Bers slices
$f \amalg \Sigma_0$
$f \amalg \Sigma$
$f \in \mathrm{Hyp}(z^d)$
Bers embedding
Teich $(S_{0,\, d+1})$
Σ_0
Σ
Poly conn. locus

Theorem. 1) For each f in the main hyperbolic component, the Bers slice $\{ f \amalg \Sigma : \Sigma \in \mathrm{Teich}(S_{0,\, d+1}) \}$ is pre-compact in the space of algebraic correspondences.

2) For $S_{0,\,4}$, all the above Bers compactifications are naturally homeomorphic.

☆ Rescaling limit arguments.

The compactification cannot be carried out in the space of single-valued maps.

☆ Bers' original argument uses Sullivan's 'no invariant line field' theorem for Kleinian limit sets.

We exploit one-dimensionality of the parameter space and employ a 'Beltrami disk' argument (akin to straightening continuity for the Mandelbrot set).

[Luo-Mj-M]

A few tasks:

1) study the dynamics of the Bers boundary correspondences,

2) study local conn. of the limit sets,

3) classify the Bers boundary correspondences (combinatorial rigidity = "ending lamination"),

4) study geometric limit vs. algebraic limit,

5) study the Thurston-Kerckhoff discontinuity phenomenon,

6) study the topology of the Bers boundaries (non-local conn., self-bumping),

7) construct correspondences with Peano curve lim

Thank You!