Complex anaysis - Problem Set 1

Notation:

- D stands for an open, connected subset of $\mathbb{C}$.
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
- A function $f : D \to \mathbb{C}$ is called $\mathbb{C}$-differentiable if it has continuous first order derivatives that satisfy the Cauchy-Riemann equations.

(1) Show that if f is $\mathbb{C}$-differentiable on D, and if $|f|$ is constant, then f is constant.

(2) Show that $f(x, y) = e^x(\cos y + i \sin y) : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-differentiable.

(3) Show that if f and $\overline{f}$ are both analytic on a domain D, then f is constant.

(4) If $f = u + iv : D \to \mathbb{C}$ is $\mathbb{C}$-differentiable, then its Jacobian matrix J_f (as a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^2$) has determinant

$$\det J_f(z) = |f'(z)|^2.$$

Deduce (using the inverse function theorem for functions of two real variables) that if $f'(z_0) \neq 0$ for some $z_0 \in D$, then f is locally injective near z_0 and the local inverse is also $\mathbb{C}$-differentiable.

(5) Prove that every continuous automorphism of the complex field is either the identity or the conjugation map.

(6) Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = c_n(z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

(7) Suppose that $f, g : \mathbb{D} \to \mathbb{C}$ are holomorphic, and neither f nor g has a zero in $\mathbb{D}$. If

$$\frac{f'}{f}\left(\frac{1}{n}\right) = \frac{g'}{g}\left(\frac{1}{n}\right),$$

find a simple relation between f and g.

(8) Show that if D is a bounded domain with smooth boundary, then

$$\text{Area}(D) = \frac{1}{2i}\int_{\partial D} \overline{z}dz.$$

(9) A function f on the complex plane is doubly periodic if there are two periods ω_0 and ω_1 of f that do not lie on the same line through the origin (that is, ω_0 and ω_1 are linearly independent over the reals, and $f(z + \omega_o) = f(z + \omega_1) = f(z)$ for all $z \in \mathbb{C}$). Prove that the only entire functions that are doubly periodic are the constants.

(10) Compute the integrals in the 1st problem of Exercise IV.4 (page 116) of Gamelin. **(Please do not hand in the solutions.)**