Complex anaysis - Problem Set 2

Notation:

- D stands for an open, connected subset of $\mathbb{C}$, and $H(D)$ is the space of all holomorphic functions defined on D.
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
- $f^{\circ n} := f \circ f \circ \cdots \circ f$ (n times) is the n-th iterate of f.

(1) Read Section V.5, and work our problems 1b, 1c and 2 of Exercise V.5 of Gamelin.

(2) Let D be a bounded domain and $f \in H(D)$. Suppose that there exists $M > 0$ such that $\limsup_{z \to a} |f(z)| \leq M$ for all $a \in \partial D$. Show that $|f| \leq M$ on D.

(3) (a) Show that for $a_1, \cdots, a_n \in \mathbb{D}$ and $\theta \in \mathbb{R}$, the finite *Blaschke product*

$$B(z) = e^{i\theta} \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z} \qquad (*)$$

is a proper holomorphic map from $\mathbb{D}$ to itself.

(b) Using Problem (2), show that every proper holomorphic self-map of $\mathbb{D}$ is a finite Blaschke product (i.e., a map of the form $(*)$).

(4) Let p be a monic complex polynomial of degree at least two. Show that there exists at least one point on the unit circle $\{z : |z| = 1\}$ with $|p(z)| \geq 1$. Show, moreover, that if $\sup_{|z|=1} |p(z)| = 1$, then $p(z) = z^n$.

(Hint: Look at $q(z) = z^n p(1/z)$.)

(5) Show that the function $f(z) = \sum_{n \geq 1} n z^n$ is injective on $\mathbb{D}$. Find $f(\mathbb{D})$.

(6) Let f be an entire function. Define

$$M(f, r) := \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \text{ for } r > 0.$$

Find an expression for $M(f, r)$ in terms of the coefficients in the power series of f (around the origin) and r. Conclude that if f satisfies the condition

$$\int\int_{\mathbb{C}} |f(z)|^2 dx dy < +\infty,$$

then $f \equiv 0$.

(7) Let p be a monic polynomial of degree at least two. Prove the following assertions.

(a) There exists $R_0 > 0$ such that $|p(z)| > 2|z|$ for all $|z| > R_0$.

(b) The *basin of attraction of infinity* $\mathcal{B}_\infty(p) := \{z \in \mathbb{C} : p^{\circ n}(z) \longrightarrow \infty\}$ is open.

(c) $\mathcal{B}_\infty(p)$ is connected. (Hint: maximum modulus principle.)

(8) Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$; i.e., $R(z) = \frac{p(z)}{q(z)}$ where p and q are polynomials with $\max(\deg p, \deg q) = d$. Show that

(a) R has $2d - 2$ critical points counted with multiplicities.

(b) The *grand orbit* of a point $x \in \widehat{\mathbb{C}}$ under R is defined as

$$\mathrm{GO}_R(x) := \{y \in \widehat{\mathbb{C}} : R^{\circ m}(x) = R^{\circ n}(y), \text{ for some } m, n \geq 0\}.$$

Show that at most two points in $\widehat{\mathbb{C}}$ can have finite grand orbits under R.