Complex anaysis - Problem Set 3

Notation:

- D stands for an open, connected subset of $\mathbb{C}$, and $H(D)$ is the space of all holomorphic functions defined on D.
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

(1) (a) Compute the residue of $f(z) = \frac{\pi}{z^2 \sin(\pi z)}$ at each pole.

(b) Let $f(z) = z + az^{n+1} + bz^{2n+1} + O(z^{2n+2})$ be a holomorphic map defined in a neighborhood of 0, and $\varepsilon > 0$ be such that f has no zero other than the origin in $\overline{B(0, \varepsilon)}$. Show that

$$\frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{dz}{z - f(z)} = \frac{b}{a^2}.$$

(2) Compute the following integrals:

(a) $\oint_{|z|=3} \frac{e^{2iz}}{z^2(z-1)} dz$,

(b) $\oint_{|z|=7} \frac{1+z}{1-\cos z} dz$.

(3) Work out problems 1c, 1e, 6 of Exercise VII.8 of Gamelin.

(4) Let $g \in \mathrm{H}(\mathbb{D} \setminus \{0\})$, and $g_m(z) = g(z/m)$ for each positive integer m. Suppose that $\{g_m\}$ has a subsequence $\{g_{m_k}\}$ such that

$$\max_{|z|=1/2} |g_{m_k}(z)| \leq 1 \ \forall\ k \geq 1.$$

Show that g can be extended to a holomorphic function on $\mathbb{D}$.

(5) (a) Show that the equation $e^z = 3z^n$ has exactly n solutions (counting multiplicity) in $\mathbb{D}$.

(b) Show that the polynomial $z^4 + z^2 + z + 1$ has all four zeroes (counting multiplicity) in $\{z : 1/2 \leq |z| \leq 2\}$.

(6) If f is entire and satisfies $|f''(z) - 3| \geq 0.001$ for all $z \in \mathbb{C}$, $f(0) = 0$, $f(1) = 2$, $f(-1) = 4$, what is $f(i)$?

(7) Let f be entire with $|f'(z)| \leq |z|$ for all $z \in \mathbb{C}$. What can you say about f?

(8) Determine the Laurent series expansions of $f(z) = \frac{1}{z(z-1)(z-2)}$ (centered at 0) on the domains $\{z : 0 < |z| < 1\}$, $\{z : 1 < |z| < 2\}$ and $\{z : |z| > 2\}$.

(9) Let $a > 1$. Show that

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

(10) Let $p(z) = (z^2 + 1)^2$.

(a) Determine the set C of all critical points of f in $\mathbb{C}$.

(b) Set $X := f(C)$. Show that $f : \mathbb{C} \setminus f^{-1}(X) \longrightarrow \mathbb{C} \setminus X$ is a 4-sheeted covering map.

(c) Compute the group of deck transformation of $f : \mathbb{C} \setminus f^{-1}(X) \longrightarrow \mathbb{C} \setminus X$.