Complex anaysis - Problem Set 4

Notation:

- D stands for an open, connected subset of $\mathbb{C}$, and $\mathrm{H}(D)$ is the space of all holomorphic functions defined on D.
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
- $f^{\circ n} := f \circ f \circ \cdots \circ f$ (n times) is the n-th iterate of f.

(1) Prove or disprove: Let $f \in \mathrm{C}(\overline{\mathbb{D}}) \cap \mathrm{H}(\mathbb{D})$. Suppose that for a constant $0 < \delta < 1/10$, $f(e^{i\theta}) = 1 + 2i$ for all $-\delta\pi < \theta < \delta\pi$. Then $f \equiv 1 + 2i$ on $\overline{\mathbb{D}}$.

(2) Show that normality is a local property. More precisely, if $\{f_\alpha\} \subset \mathrm{H}(D)$ be such that every point $z \in D$ has a neighborhood U_z such that the collection $\{f_\alpha|_{U_z}\}$ of restricted maps is a normal family, show (by a diagonal argument) that the family $\{f_\alpha\}$ itself is normal.

(3) (a) Let $\{f_n\}$ be a sequence of rational functions that converges normally to f on the Riemann sphere $\widehat{\mathbb{C}}$. Show that f_n has the same degree as f for n large.

(b) Show that the rational functions $\{\frac{z^3}{z+\varepsilon} : 0 < \varepsilon < 1\}$ form a normal family of meromorphic functions on $\widehat{\mathbb{C}} \setminus \{0\}$, but they do not form a normal family of meromorphic functions on $\mathbb{C}$.

(4) Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree at least two. The Fatou set of R, denoted by $\mathcal{F}(R)$, is the largest open subset of $\widehat{\mathbb{C}}$ on which the iterates $\{R^{\circ n}\}$ form a normal family (equivalently, $\mathcal{F}(R)$ consists of all points $z_0 \in \widehat{\mathbb{C}}$ that have an open neighborhood U_{z_0} such that the restrictions of the iterates of R to U_{z_0} form a normal family. The Julia set of R, denoted by $\mathcal{J}(R)$, is the complement of $\mathcal{F}(R)$ in $\widehat{\mathbb{C}}$. A fixed point z_0 of R is said to be *repelling* if $|f'(z_0)| > 1$. Prove the following assertions.

(a) $R(\mathcal{F}(R)) = R^{-1}(\mathcal{F}(R)) = \mathcal{F}(R)$, and $R(\mathcal{J}(R)) = R^{-1}(\mathcal{J}(R)) = \mathcal{J}(R)$.

(b) $\mathcal{J}(R^{\circ k}) = \mathcal{J}(R)$, for all $k \in \mathbb{N}$.

(c) Each repelling fixed point R belongs to $\mathcal{J}(R)$.

(d) $\mathcal{J}(R) \neq \emptyset$.

(e) If R is a monic polynomial (of degree at least two), then $\mathcal{J}(R) = \partial\mathcal{B}_\infty(R)$ (see Problem Set 2 for the definition of the basin of infinity $\mathcal{B}_\infty(R)$ of a polynomial).

(5) (a) If a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ fixes the origin and is not a rotation, prove that the successive images $f^{\circ n}(z)$ converge to zero for all $z \in \mathbb{D}$.

(b) Prove that this convergence is uniform on compact subsets of $\mathbb{D}$. (The example $f(z) = z^2$ shows that convergence need not be uniform on all of $\mathbb{D}$.)

(6) (a) Show that the group $\mathrm{Aut}(\widehat{\mathbb{C}})$ is generated by the subgroup of affine transformations $z \mapsto az + b$ together with the inversion $z \mapsto -1/z$.

(b) Given four distinct points $z_j \in \widehat{\mathbb{C}}$, show that the *cross-ratio*

$$\chi(z_1, z_2 z_3, z_4) := \frac{(z_3 - z_1)(z_4 - z_2)}{(z_2 - z_1)(z_4 - z_3)} \in \mathbb{C} \setminus \{0, 1\}$$

is the image of z_4 under the unique Möbius map that carries z_1, z_2, z_3 to $1, 0, \infty$ respectively. Conclude that it is invariant under Möbius transformations.

(7) Let $R(z) = \frac{z^d+1}{z^d}$, for some $d \geq 2, d \in \mathbb{N}$. Show that there exists no Möbius map M such that $M \circ R \circ M^{-1}$ is a polynomial.

(8) Show that there exists no bijective holomorphic map from the punctured disk $\mathbb{D} \setminus \{0\}$ onto the annulus $\{z : 1 < |z| < 2\}$.

(9) Does there exist $\in \mathrm{H}(\mathbb{D})$ such that $(f(z))^3 = z$ for all $z \in \mathbb{D}$?