Complex anaysis - Problem Set 5

Notation:

- D stands for an open, connected subset of $\mathbb{C}$, and $\mathrm{H}(D)$ is the space of all holomorphic functions defined on D.
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
- $f^{\circ n} := f \circ f \circ \cdots \circ f$ (n times) is the n-th iterate of f.

(1) (a) Let D be the domain in the Riemann sphere $\widehat{\mathbb{C}}$ exterior to the circles $|z-1| = 1$ and $|z+1| = 1$. Find a conformal isomorphism of D onto the strip $S + \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 2\}$.

(b) Find a conformal map from $\{z \in \mathbb{C} : |z| < 1, \mathrm{Im}(z) > 0\}$ (a semi-disk) onto the unit disk $\mathbb{D}$.

(2) If g maps the quadrant $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$ conformally onto $\mathbb{D}$ with $g(1) = 1$, $g(i) = -1$, and $g(0) = -i$, find $|g'(1+i)$.

(3) If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = f(\frac{1}{2}) = 0$, show that

$$|f(z)| \leq \left| z \cdot \frac{2z-1}{2-z} \right|$$

for all $z \in \mathbb{D}$.

(4) (a) Let $D = \mathbb{C} \setminus \{a_1, \cdots, a_m\}$ be the complex plane with m punctures. Show that any conformal self-map of D is a Möbius map that permutes the punctures.

(b) Compute the group of conformal automorphisms of $\mathbb{C}\setminus\{0\}$, $\mathbb{C}\setminus\{0,1\}$, and $\mathbb{D}\setminus\{0, 1/3\}$.

(5) (a) A conformal map $g : D \to \mathbb{D}$ induces a hyperbolic metric ρ_D on D (called the *Poincaré metric*) as the pullback of the hyperbolic metric on $\mathbb{D}$. Show that it is given by the formula

$$d\rho_D(z) = \frac{2|g'(z)|}{1 - |g(z)|^2} dz,$$

and that ρ_D is independent of the conformal isomorphism g.

(b) Show that the hyperbolic metric of the upper half-plane $\mathbb{H}$ is given by

$$d\rho_{\mathbb{H}}(z) = \frac{|dz|}{y},$$

and the hyperbolic metric on the horizontal strip $S = \{z \in \mathbb{C} : -\pi/2 < \mathrm{Im}(z) < \pi/2\}$ is given by

$$d\rho_S(z) = \frac{|dz|}{\cos y}.$$

(6) The universal covering surface for the punctured disk $\mathbb{D} \setminus \{0\}$ can be identified with the left half-plane $\{w = u + iv; u < 0\}$ under the exponential map $w \mapsto z = e^w$.

(a) Show that the Poincaré metric $\frac{|dw|}{|u|}$ on the left-half plane descends to the Poincaré metric $\frac{|dz|}{|r \ln r|}$ on the punctured disk, where $r = |z|$.

(b) Conclude that the circle $\{|z| = r\}$ has Poincaré length $2\pi/|\ln r|$, which tends to zero as $r \to 0$, although this circle has infinite Poincaré distance from the boundary point at the origin.

(7) Let $B(z) = \frac{3z^2+1}{z^2+3}$. Show that $\{B^{\circ n}(z)\} \longrightarrow 1$ as $n \to \infty$ for all $z \in \mathbb{D}$.

(8) Let $f : \mathbb{H}_R = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \to \mathbb{C}$ be holomorphic such that $f(\mathbb{H}_R) \subset B(f(a), r)$, for some $a \in \mathbb{H}_R$ and $r > 0$. Prove that

$$\frac{|f(z) - f(a)|}{|z - a|} \leq \frac{r}{|z + \overline{a}|} \text{ for all } z \in \mathbb{H}_R \setminus \{a\}, \text{ and } |f'(a)| \leq \frac{r}{2\mathrm{Re}(a)}.$$